4.1.4 Vortex Shedding and Vortex Induced Vibrations
Consider a steady flow $U_o$ over a bluff body with diameter $D$.

We would expect the average forces to be:

However, the measured oscillatory forces are:

- The **measured** drag $F_x$ is found to oscillate about a **non-zero mean** value with frequency $2f$.
- The **measured** lift $F_y$ is found to oscillate about a **zero mean** value with frequency $f$.
- $f = \omega/2\pi$ is the frequency of vortex shedding or Strouhal frequency.
Reason: Flow separation leads to vortex shedding. The vortices are shed in a staggered array, within an unsteady non-symmetric wake called von Karman Street. The frequency of vortex shedding is the Strouhal frequency and is a function of $U_0$, $D$, and $\nu$.

\[ S \equiv \frac{f D}{U_0} \]

The Strouhal number $S$ has a regime dependence on the $Re$ number $S = S(Re)$. For a cylinder:

- Laminar flow $S \sim 0.22$
- Turbulent flow $S \sim 0.3$

i) Strouhal Number

We define the (dimensionless) Strouhal number $S \equiv \frac{f D}{U_0}$.

ii) Drag and Lift

The drag and lift coefficients $C_D$ and $C_L$ are functions of the correlation length.

For ‘$\infty$’ correlation length:

- If the cylinder is fixed, $C_L \sim O(1)$ comparable to $C_D$.
- If the cylinder is free to move, as the Strouhal frequency $f_S$ approaches one of the cylinder’s natural frequencies $f_n$, ‘lock-in’ occurs. Therefore, if one natural frequency is close to the Strouhal Frequency $f_n \sim f_S$, we have large amplitude motions $\Rightarrow$ Vortex Induced Vibration (VIV).
4.2 Drag on a Very Streamlined Body

\[ R_{eL} \equiv \frac{UL}{\nu} \]

\[ C_f \equiv \frac{1}{2\rho U^2} \frac{D}{(Lb)} \]

\[ S = \text{wetted area} \]
\[ \text{one side of plate} \]

\[ C_f = C_f(R_{eL}, L/b) \]

Unlike a bluff body, \( C_f \) is a strong function of \( R_{eL} \) since \( D \) is proportional to \( \nu \left( \tau = \nu \frac{\partial u}{\partial y} \right) \).

See an example of \( C_f \) versus \( R_{eL} \) for a flat plate in the figure below.

- \( R_{eL} \) depends on plate smoothness, ambient turbulence, . . .
- In general, \( C_f \)'s are much smaller than \( C_D \)'s (\( C_f/C_D \sim O(0.1) \) to \( O(0.01) \)). Therefore, designing streamlined bodies allows minimal separation and smaller form drag at the expense of friction drag.
- In general, for streamlined bodies \( C_{\text{Total Drag}} \) is a combination of \( C_D \) (\( R_e \)) and \( C_f \) (\( R_{eL} \)), and the total drag is \( D = \frac{1}{2}\rho U^2 \left( C_{D_{\text{frontal area}}} S + C_f A_{\text{wetted area}} \right) \), where \( C_D \) has a regime dependence on \( R_e \) and \( C_f \) is a continuous function \( R_{eL} \).
4.3 Known Solutions of the Navier-Stokes Equations

4.3.1 Boundary Value Problem

- Navier-Stokes':

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \frac{1}{\rho} \vec{f}
\]

- Conservation of mass:

\[
\nabla \cdot \vec{v} = 0
\]

- Boundary conditions on solid boundaries “no-slip”:

\[
\vec{v} = \vec{U}
\]

Equations very difficult to solve, analytic solution only for a few very special cases (usually when $\vec{v} \cdot \nabla \vec{v} = 0$... )
4.3.2 Steady Laminar Flow Between 2 Long Parallel Plates: Plane Couette Flow

Steady, viscous, incompressible flow between two infinite plates. The flow is driven by a pressure gradient in $x$ and/or motion of the upper plate with velocity $U$ parallel to the $x$-axis. Neglect gravity.

<table>
<thead>
<tr>
<th>Assumptions</th>
<th>Governing Equations</th>
<th>Boundary Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. Steady Flow: $\frac{\partial}{\partial t} = 0$</td>
<td>Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$</td>
<td>$\vec{v} = (0,0,0)$ on $y = 0$</td>
</tr>
<tr>
<td>ii. $(x,z) &gt;&gt; h$: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$</td>
<td>NS: $\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v}$</td>
<td>$\vec{v} = (0,0,0)$ on $y = h$</td>
</tr>
<tr>
<td>iii. Pressure: independent of $z$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Continuity

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v = v(x,z) \Rightarrow v = 0 \quad \text{BC: } v(x,0,z) = 0
\]  

(1)
Momentum $x$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \Rightarrow$$

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho} \frac{\partial p}{\partial x} \label{2}$$

Momentum $y$

$$\frac{\partial v}{\partial t} + \vec{v} \cdot \nabla v + \nu \nabla^2 v = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \Rightarrow$$

$$\frac{\partial p}{\partial y} = 0 \Rightarrow \quad p = p(x) \quad \text{and} \quad \frac{\partial p}{\partial x} = \frac{dp}{dx} \label{3}$$

Momentum $z$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \Rightarrow$$

$$\frac{\partial^2 w}{\partial y^2} = 0 \Rightarrow w = ay + b \Rightarrow w = 0 \quad \text{with} \quad u(x, 0, z) = 0 \quad \text{and} \quad w(x, h, z) = 0 \label{4}$$

From Equations (1), (4)

$$\vec{v} = (u, 0, 0). \quad \text{Also} \quad \Rightarrow \quad u = u(y) \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{du}{dy} \label{5}$$

From Equations (2), (3, and (5))

$$\frac{d^2 u}{dy^2} = \frac{1}{\rho \nu} \frac{d^2 p}{dx^2} \Rightarrow u = -\frac{1}{2\mu} \left( -\frac{dp}{dx} \right) y^2 + C_1 y + C_2 \Rightarrow$$

$$u = -\frac{1}{2\mu} \left( -\frac{dp}{dx} \right) (h - y) y + U \frac{y}{h}$$
• Special cases for Couette flow

\[ u(y) = \frac{1}{2\mu} (h - y) \left( -\frac{dp}{dx} \right) + U \frac{y}{h}, \]

where \( -\frac{dp}{dx} = \frac{P_h - P_{x+L}}{L} \)

<table>
<thead>
<tr>
<th>I. ( U = 0 ), ( (-\frac{dp}{dx}) &gt; 0 )</th>
<th>II. ( U \neq 0 ), ( (-\frac{dp}{dx}) = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Parabolic profile" /></td>
<td><img src="image" alt="Linear profile" /></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Special cases for Couette flow</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(y) = \frac{1}{2\mu} (h - y) \left( -\frac{dp}{dx} \right) + U \frac{y}{h} )</td>
<td>( u(y) = \frac{U y}{h} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Max velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_{max} = u(h/2) = \frac{h^2}{8\mu} \left( -\frac{dp}{dx} \right) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volume flow rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q = \int_0^h u(y)dy = \frac{h^3}{8\mu} \left( -\frac{dp}{dx} \right) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Average velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{u} = \frac{Q}{h} = \frac{h^2}{6\mu} \left( -\frac{dp}{dx} \right) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Viscous stress on bottom plate (skin friction)</th>
</tr>
</thead>
</table>

\[ \tau_w = \mu \left. \frac{du}{dy} \right|_{y=0} = \frac{h}{2} \left( -\frac{dp}{dx} \right) > 0 \]

\[ \tau_w = \mu \left. \frac{du}{dy} \right|_{y=0} = \mu \frac{U}{h} \]
<table>
<thead>
<tr>
<th>III. $U \neq 0$, $(−\frac{dp}{dx}) \neq 0$</th>
</tr>
</thead>
</table>

| $U > 0, (−\frac{dp}{dx}) > 0, G > 0$ | $U > 0, (−\frac{dp}{dx}) < 0, G < 0$ |
|-----------------------------------------------|

| $(−\frac{dp}{dx}) > 0$ | $(−\frac{dp}{dx}) < 0$ |

- Viscous stress on bottom plate (skin friction)

$$\tau_w = \frac{h}{2} (−\frac{dp}{dx}) + \mu \frac{U}{h}$$

$$\tau_w \leq 0 \text{ when } (−\frac{dp}{dx}) \leq −\frac{2\mu U}{h^2}, \text{ in which case the flow is } \left\{ \begin{array}{l} \text{attached} \\
\text{insipient} \\
\text{separated} \end{array} \right.$$
For the general case of $U \neq 0$ and $\left( -\frac{dp}{dx} \right) \neq 0$,

$$\tau_w = \frac{h}{2} \left( -\frac{dp}{dx} \right) + \mu \frac{U}{h}$$

We define a **Dimensionless Pressure Gradient** $G$

$$G \equiv \frac{h^2}{2\mu U} \left( -\frac{dp}{dx} \right)$$

such that

- $G > 0$ denotes a **favorable** pressure gradient
- $G < 0$ denotes an **adverse** pressure gradient
- $G = -1$ denotes an **incipient** flow
- $G < -1$ denotes a **separated or back-flow**

### Lessons learned in § 4.3.2:

1. Reviewed how to simplify the Navier-Stokes equations.
2. Obtained one solution to the Navier-Stokes equations.
3. Realized that once the Navier-Stokes are solved we know **everything**.

In the next paragraph we are going to study one more solution to the Navier-Stokes equation, in polar coordinates.
4.3.3 Steady Laminar Flow in a Pipe: Poiseuille Flow

Steady, laminar pipe flow.
\((r^2 = y^2 + z^2, \vec{v} = (v_x, v_r, v_\theta))\)

KBC: \(v_x(a) = 0\) (no slip) and \(\frac{dv_x}{dr}(0) = 0\) (symmetry).

<table>
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<th>Boundary Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. Steady Flow: (\frac{\partial}{\partial t} = 0)</td>
<td>Continuity: (\frac{1}{r} \frac{\partial}{\partial r} (rv_x) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x} = 0)</td>
<td>(v_x(r = a) = 0) no-slip</td>
</tr>
<tr>
<td>ii. ((x, z) &gt;&gt; h: \frac{\partial v}{\partial x} = \frac{\partial v}{\partial \theta} = 0)</td>
<td>NS: In polar coordinates (see SAH pp.74)</td>
<td>(\frac{dv_x}{dr}</td>
</tr>
<tr>
<td>iii. Pressure: independent of (\theta)</td>
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</tr>
</tbody>
</table>

Following a procedure similar to that for plane Couette flow (left as an exercise) we can show that

\[ v_r = v_\theta = 0, \ v_x = v_x(r), \ p = p(x), \ \text{and} \ \frac{1}{\rho} \frac{dp}{dx} = \nu \left( \frac{1}{r} \frac{d}{dr} \left( r \frac{dv_x}{dr} \right) \right) \]

After applying the boundary conditions we find:

\[ v_x(r) = \frac{1}{4\mu} \left( -\frac{dp}{dx} \right) (a^2 - r^2) \]

Therefore the volume flow rate is given by

\[ Q = \int_0^{2\pi} d\theta \int_0^a r dv_x(r) = \frac{\pi}{8\mu} a^4 \left( -\frac{dp}{dx} \right) \]

and the skin friction evaluates to

\[ \tau_w = \tau_x(-r) = -\tau_{xy} = -\mu \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial r} \right) \bigg|_{r=a} = -\mu \frac{dv_x}{dr} \bigg|_{r=a} \Rightarrow \tau_w = \frac{a}{2} \left( -\frac{dp}{dx} \right) \]
4.4 Boundary Layer Growth Over an Infinite Flat Plate for Unsteady Flow

Boundary layer thickness is related to the area where the viscosity and vorticity effects are diffused.

For a flow over an infinite flat plate, the boundary layer thickness increases unless it is constrained in the $y$ direction and/or by time (unsteady flow).

1. Steady flow, constrained in $y$

   For a steady flow past a flat plate, the boundary layer thickness increases with $x$. If the flow is constrained in $y$, eventually the viscous effects are diffused along the entire cross section and the flow becomes invariant in the streamwise direction.

   In paragraphs 4.3.2 and 4.3.3, we studied two cases of steady laminar viscous flows, where the viscous effects had diffused along the entire cross section.

   Steady (Couette Poiseuille) flow, we assumed that viscous effects diffused through entire \( \left( \frac{h}{a} \right) \).
2. Unsteady flow, unconstrained in $y$

Consider the simplest example of an infinite plate in unsteady motion.

![Diagram showing an infinite plate in unsteady motion](image)

Assumptions $\nabla p = 0$, $\frac{\partial \bar{v}}{\partial x} = \frac{\partial \bar{v}}{\partial z} = 0 \Rightarrow \bar{v} = \bar{v}(y,t)$

Can show $v = w = 0$ and $u = u(y,t)$.

Finally, from $u$ momentum (Navier-Stokes in $x$) we obtain

$$\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial y^2}
\end{align*}$$

Equation (6) is:

* first order PDE in time $\rightarrow$ requires 1 Initial Condition
* second order PDE in $y$ $\rightarrow$ requires 2 Boundary Conditions

- $u(y,t) = U(t)$ at $y = 0$, for $t > 0$
- $u(y,t) \rightarrow 0$ as $y \rightarrow \infty$

From Equation (6), we observe that the flow over a moving flat plate is due to viscous dissipation only.
4.5.1 Sinusoidally Oscillating Plate

i. Evaluation of the Velocity Profile for Stokes Boundary Layer

The flow over an oscillating flat plate is referred to as ‘Stokes Boundary Layer’.

Recall that $e^{i\alpha} = \cos \alpha + i \sin \alpha$ where $\alpha$ is real.

Assume that the plate is oscillating with $U(t) = U_o \cos \omega t = \text{Real} \{U_o e^{i\omega t}\}$. From linear theory, it is known that the fluid velocity must have the form

$$u(y, t) = \text{Real} \{ f(y) e^{i\omega t} \}, \tag{7}$$

where $f(y)$ is the unknown complex (magnitude & phase) amplitude of oscillation.

To obtain an expression for $f(y)$, simply substitute (7) in (6). This leads to:

$$i \omega f = \nu \frac{d^2 f}{dy^2} \tag{8}$$

Equation (8) is a 2nd order ODE for $f(y)$. The general solution is

$$f(y) = C_1 e^{(1+i)(\sqrt{\omega/2\nu})y} + C_2 e^{-(1+i)(\sqrt{\omega/2\nu})y} \tag{9}$$

The velocity profile is obtained from Equations (7), (9) after we apply the Boundary Conditions.

\[
\begin{align*}
\text{Stokes Boundary Layer} & \Rightarrow \left\{ \begin{array}{l}
    u(y, t) = U_o(e^{-y\sqrt{\nu/\omega}}) \cos \left(-y\sqrt{\frac{\omega}{2\nu}} + \omega t\right) \\
    u(y = 0, t) = U(t) \Leftrightarrow f(y = 0) = U_o \Leftrightarrow C_2 = U_o \\
    u(y \to \infty, t) \Rightarrow C_1 = 0
    \end{array} \right. \\
\end{align*}
\]
ii. Some Calculations for the Stokes Boundary Layer

Once the velocity profile is evaluated, we know everything about the flow.

Stokes Boundary Layer. Velocity ratio $\frac{u(y)}{U_o}$ as a function of the distance from the plate $y$.

Observe:

$$\frac{u(y, t)}{U_o} = (e^{-y\sqrt{\frac{2\nu}{\omega} \cos(-y\sqrt{\frac{\omega}{2\nu}} + \omega t)})}$$

(10)

**SBL thickness**

The ratio $\frac{u}{U_o}$ is composed of an exponentially decaying part $\rightarrow$ thickness of SBL decays exponentially with $y$. We define various parameters that can be used as measures of the SBL thickness:

- We define $\delta_{1/e}$ as the distance $y$ from the plate where $\frac{u(\delta_{1/e})}{U_o} = \frac{1}{e}$. Substituting into (10), we find that $\delta \equiv \delta_{1/e} = \sqrt{\frac{2\nu}{\omega}}$

- The oscillating component has wave length $\lambda = 2\pi \sqrt{\frac{2\nu}{\omega}} = 2\pi \delta$. At $\lambda$, $\frac{u(\lambda)}{U_o} \approx 0.002$.

- We define $\delta_{1\%}$ as the distance $y$ from the plate, where $\frac{u(\delta_{1\%})}{U_o} = 1\%$. Substituting into (10), we find that $\delta_{1\%} = -\ln\left(\frac{u(\delta_{1\%})}{U_o}\right)\sqrt{\frac{2\nu}{\omega}} \approx 4.6\delta$. 

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Numerical examples:
For oscillating plate in water \((\nu = 10^{-6} \text{m}^2/\text{s} = 1 \text{mm}^2/\text{s})\) we have

\[
\delta_{1\%} = \frac{4.6}{\sqrt{\pi}} \sqrt{T} \approx 2.6 \sqrt{\frac{T}{\text{in sec}}}
\]

<table>
<thead>
<tr>
<th>(T) (in sec)</th>
<th>(\delta_{1%}) (in mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1s</td>
<td>3mm</td>
</tr>
<tr>
<td>10s</td>
<td>(\leq 1\text{cm})</td>
</tr>
</tbody>
</table>

Excursion length and SBL
The plate undergoes a motion of amplitude \(A\).
\[
X = A \sin(\omega t) \Rightarrow U = \dot{X} = \frac{A \omega}{U_o} \cos(\omega t) \Rightarrow \omega = \frac{U_o}{A}
\]

Comparing the SBL thickness \(\sim \delta\) with \(A\), we find

\[
\frac{\delta}{A} \sim \frac{\sqrt{\nu/\omega}}{A} \Rightarrow \frac{\sqrt{\nu A/U_o}}{A} = \sqrt{\frac{\nu}{U_oA}} \sim \frac{1}{\sqrt{Re_A}}
\]

Skin friction
The skin friction on the plate is given by

\[
\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \ldots = \mu U_o \sqrt{\frac{\omega}{2\nu}} \left( \sin \omega t - \cos \omega t \right)
\]

The maximum skin friction on the wall is

\[
|\tau_w|_{max} = \mu U_o \sqrt{\frac{\omega}{\nu}}
\]

and occurs at \(\omega t = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots\)
4.5.2 Impulsively Started Plate

Recall Equation (6) that describes the flow $u(y, t)$ over an infinite flat plate undergoing unsteady motion.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

For an impulsively started plate, the Boundary Conditions are:

$$\begin{align*}
  u(o, t) &= U_o \quad \text{for } t > 0, \text{i.e. } u(y, 0) = 0 \\
  u(\infty, t) &= 0
\end{align*}$$

Notice that the problem stated by Equation (6) with the above Boundary Conditions has no explicit time scale. In this case it is standard procedure to (a) use Dimensional Analysis to find the similarity parameters of the problem, and (b) look for solution in terms of the similarity parameters:

$$u = f(U_o, y, t, \nu) \quad \Rightarrow \quad \frac{u}{U_o} = f\left(\frac{y}{2\sqrt{\nu t}}\right) \quad \Rightarrow \quad \frac{u}{U_o} = f(\eta)$$

Self similar solution

The velocity profile is thus given by*:

$$\frac{u}{U_o} = \text{erfc}(\eta) = 1 - \text{erf}(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\alpha^2} d\alpha$$
Hints on obtaining the solution:

\[ \eta = \frac{y}{2\sqrt{\nu t}} \]

\[ \frac{\partial}{\partial t} = \frac{\partial\eta}{\partial t} \frac{\partial}{\partial \eta} - \frac{y}{4t\sqrt{\nu t}} \frac{\partial}{\partial \eta} \]

\[ \frac{\partial^2}{\partial y^2} = \left( \frac{\partial\eta}{\partial y} \right)^2 \frac{\partial^2}{\partial \eta^2} = \frac{1}{4\nu t} \frac{\partial^2}{\partial \eta^2} \]

\[ \begin{cases} \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \\ -\eta \frac{d(u/U_o)}{d\eta} = \frac{d^2(u/U_o)}{d\eta^2} \rightarrow \ldots \end{cases} \]

2nd order ODE

Boundary layer thickness

In the same manner as for the SBL, we define various parameters that can be used to measure the boundary layer thickness:

- \( \delta \equiv 2\sqrt{\nu t} \). At \( y = \delta \rightarrow u(\delta)/U_o \approx 0.16. \)
- \( \delta_{1\%} \approx 1.82\delta. \)

Excursion length and boundary layer thickness

At time \( t \), the plate has travelled a distance \( L = U_o t \rightarrow t = \frac{L}{U_o} \).

Comparing the boundary layer thickness \( \sim \delta \) with \( L \), we find

\[ \frac{\delta}{L} \sim \frac{\sqrt{\nu t}}{L} = \frac{\sqrt{\nu L/U_o}}{L} = \frac{\sqrt{\nu}}{U_o L} \sim \frac{1}{\sqrt{Re L}} \]

Skin friction

The skin friction on the plate is given by

\[ \tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \ldots = -\mu \frac{U_o}{\sqrt{\pi \nu t}} \]