In Lecture 8, paragraph 3.3 we discuss some properties of vortex structures. In paragraph 3.4 we deduce the Bernoulli equation for ideal, steady flow.

3.3 Properties of Vortex Structures

3.3.1 Vortex Structures

- A **vortex line** is a line everywhere tangent to $\vec{\omega}$.

![Vortex Line Diagram]

- A **vortex tube** (filament) is a bundle of vortex lines.

![Vortex Tube Diagram]
• A **vortex ring** is a closed vortex tube.

A sketch and two pictures of the production of vortex rings from orifices are shown in Figures 1, 2, and 3 below.

(Figures 2,3: Van Dyke, *An Album of Fluid Motion* 1982 p.66, 71)

![Side view and cross section diagrams of vortex ring production](image)

*Figure 1: Sketch of vortex ring production*
3.3.2 No Net Flux of Vorticity Through a Closed Surface

Calculus identity, for any vector $\vec{v}$:

\[
\nabla \cdot (\nabla \times \vec{v}) = 0 \Rightarrow \\
\nabla \cdot \vec{\omega} = 0 \Rightarrow
\]

\[
\int \int \int_{V} \nabla \cdot \vec{\omega} = \int \int_{S} \vec{\omega} \cdot \hat{n} \ dS = 0
\]

i.e. The net vorticity flux through a closed surface is zero.

(a) No net vorticity flux through a vortex tube:

\[
(Vorticity \ Flux)_{in} = (Vorticity \ Flux)_{out} \Rightarrow (\vec{\omega} \cdot \hat{n})_{in} \delta A_{in} = (\vec{\omega} \cdot \hat{n})_{out} \delta A_{out}
\]

(b) Vorticity cannot stop anywhere in the fluid. It either traverses the fluid beginning or ending on a boundary or closes on itself (vortex ring).
3.3.3 Conservation of Vorticity Flux

\[ 0 = \Gamma_3 = \oint_{C_3} \vec{v} \cdot d\vec{x} = \iint_{S_3} \vec{\omega} \cdot \hat{n} dS = 0 \]

Therefore, circulation is the same in all circuits embracing the same vortex tube. For the special case of a vortex tube with ‘small’ area:

\[ \Gamma = \omega_1 A_1 = \omega_2 A_2 \]

An application of the equation above is displayed in the figure below:
3.3.4 Vortex Structures are Material Structures

Consider a material patch $A_m$ on a vortex tube at time $t$.

By definition,

$$\vec{\omega} \cdot \hat{n} = 0 \text{ on } A_n$$

Then,

$$\Gamma_{\partial A_m} = \oint_{\partial A_m} \vec{v} \cdot d\vec{x} = \iint_{A_m} \vec{\omega} \cdot \hat{n} ds = 0$$

At time $t + \Delta t$, $A_m$ moves, and for an ideal fluid under the influence of conservative body forces, Kelvin’s theorem states that

$$\Gamma_{\partial A_m} = 0$$

So, $\vec{\omega} \cdot \hat{n} = 0$ on $A_m$ still, i.e., $A_m$ still on the vortex tube. Therefore, the vortex tube is a material tube for an ideal fluid under the influence of conservative forces. In the same manner it can be shown that a vortex line is a material line, i.e., it moves with the fluid.
3.3.5 Vortex stretching

Consider a small vortex filament of length $L$ and radius $R$, where by definition $\vec{\omega}$ is tangent to the tube.

$$\Gamma = \omega A = \text{constant (in time)}$$

But tube is material with volume $= AL = \pi R^2 L = \text{constant in time (continuity)}$

$$\therefore \frac{\Gamma}{\text{Volume}} = \frac{\omega A}{LA} = \frac{\omega}{L} = \text{constant}$$

As a vortex stretches, $L$ increases, and since the volume is constant (from continuity), $A$ and $R$ decrease, and due to the conservation of the angular momentum, $\omega$ increases. In other words,

Vortex stretching $\Leftrightarrow L \uparrow \Rightarrow \omega \uparrow$ (conservation of angular momentum)

$\Rightarrow A$ and $R \uparrow$ (continuity)
3.3.6 Summary on Vortex Structures

**Diagram**: A vortex ring structure with a circle of radius $R$, a line of circulation $\Gamma$, and vorticity $\vec{\omega}$. The cross-sectional area $A$ and vortex ring length $L$ are shown.

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vortex ring length</td>
<td>$L = 2\pi R$</td>
<td>[L]</td>
</tr>
<tr>
<td>Cross sectional area</td>
<td>$A = \pi r^2$</td>
<td>[L$^2$]</td>
</tr>
<tr>
<td>Vortex ring volume</td>
<td>$\forall = AL = \text{const}$</td>
<td>[L$^3$]</td>
</tr>
<tr>
<td>Vorticity</td>
<td>$\vec{\omega} = \nabla \times \vec{v}$</td>
<td>[T$^{-1}$]</td>
</tr>
<tr>
<td>Circulation</td>
<td>$\Gamma = \text{const}$</td>
<td>[L$^2$T$^{-1}$]</td>
</tr>
<tr>
<td></td>
<td>Kelvin’s theorem</td>
<td></td>
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<tr>
<td></td>
<td>$\Gamma = \omega A = \text{const}$</td>
<td>[L$^2$T$^{-1}$]</td>
</tr>
<tr>
<td></td>
<td>vorticity flux through $A$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Gamma \propto U r = \text{const}$</td>
<td>[L$^2$T$^{-1}$]</td>
</tr>
</tbody>
</table>


Continuity relates length ratios

\[ \forall = LA = \text{const} \]

\[ \begin{align*}
A & \propto \frac{\forall}{L} \quad \therefore \text{as } L \uparrow \quad A \downarrow \\
r & \propto \sqrt[\forall]{L} \quad \therefore \text{as } L \uparrow \quad r \downarrow
\end{align*} \]

Kelvin’s theorem + Continuity relate length ratios to \( \Gamma, \omega, U \)

\[ Ur \propto \Gamma = \text{const} \Rightarrow U \propto \frac{\Gamma}{r} \quad \Rightarrow \quad U \propto \Gamma \sqrt{\frac{L}{\Gamma}} \quad \therefore \text{as } L \uparrow \quad U \uparrow \]

\[ \omega A \propto \Gamma = \text{const} \Rightarrow \omega \propto \frac{\Gamma}{A} \quad \Rightarrow \quad \omega \propto \Gamma \sqrt{\frac{L}{\Gamma}} \quad \therefore \text{as } L \uparrow \quad \omega \uparrow \]

Example 1:

Example 2:

\[ A_1 < A_2 \quad L_1 < L_2 \quad \text{Given} \]
\[ \omega_1 > \omega_2 \quad A_1 > A_2 \quad \text{From continuity only} \]
\[ \Gamma_1 = \Gamma_2 \quad \text{From Kelvin’s theorem} \]
\[ \omega_1 < \omega_2 \quad \text{From Kelvin’s theorem + continuity} \]
\[ U_1 < U_2 \quad \text{From Kelvin’s theorem + continuity} \]
3.4 Bernoulli Equation for Steady (\( \frac{\partial}{\partial t} = 0 \)), Ideal(\( \nu = 0 \)), Rotational flow

\[ p = f(\vec{v}) \quad \text{Viscous flow: Navier-Stokes’ Equations (Vector Equations)} \]

\[ p = f(|\vec{v}|) \quad \text{Ideal flow: Bernoulli Equation (Scalar equation)} \]

Steady, inviscid Euler equation (momentum equation):

\[ \vec{v} \cdot \nabla \vec{v} = -\nabla \left( \frac{p}{\rho} + g\vec{y} \right) \quad (1) \]

From Vector Calculus we have

\[ \nabla (\vec{u} \cdot \vec{v}) = (\vec{u} \cdot \nabla) \vec{v} + (\vec{v} \cdot \nabla) \vec{u} + \vec{u} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{u}) \Rightarrow \]

\[ \nabla (\frac{1}{2} |\vec{v}|^2) = \vec{v} \cdot \nabla \vec{v} + \vec{v} \times (\nabla \times \vec{v}) \Rightarrow \]

\[ \vec{v} \cdot \nabla \vec{v} = \nabla \left( \frac{v^2}{2} \right) - \vec{v} \times (\nabla \times \vec{v}) \text{ where } v^2 \equiv \vec{v} \cdot \vec{v} = |\vec{v}|^2 \]

From the previous identity and Equation (1) we obtain

\[ \vec{v} \cdot (1) \rightarrow \vec{v} \cdot \nabla \left( \frac{v^2}{2} \right) - \vec{v} \times (\nabla \times \vec{v}) = -\vec{v} \cdot \nabla \left( \frac{p}{\rho} + g\vec{y} \right) \]

\[ \vec{v} \cdot \text{momentum (1)} \rightarrow \text{energy} \]

Therefore,

\[ \vec{v} \cdot \nabla \left( \frac{v^2}{2} + \frac{p}{\rho} + g\vec{y} \right) = 0 = \frac{D}{Dt} \left( \frac{v^2}{2} + \frac{p}{\rho} + g\vec{y} \right) \]

streamline

pathline

i.e., \( \frac{v^2}{2} + \frac{p}{\rho} + g\vec{y} = \text{constant on a streamline} \)

In general, \( \frac{v^2}{2} + \frac{p}{\rho} + g\vec{y} = F(\Psi) \) where \( \Psi \) is a tag for a particular streamline.

Assumptions: Ideal fluid, Steady flow, Rotational in general.
3.4.1 Example: Contraction in Water or Wind Tunnel

![Diagram of contraction in a tunnel](image)

Contraction Ratio: \( \gamma = \frac{R_1}{R_2} >> 1 \) (\( \gamma = O(10) \) for wind tunnel; \( \gamma = O(5) \) for water tunnel)

Let \( \bar{U}_1 \) and \( \bar{U}_2 \) denote the average velocities at sections 1 and 2 respectively.

1. From continuity: \( \bar{U}_1 \left( \pi R_1^2 \right) = \bar{U}_2 \left( \pi R_2^2 \right) \rightarrow \frac{\bar{U}_2}{\bar{U}_1} = \left( \frac{R_1}{R_2} \right)^2 = \gamma^2 >> 1 \)

2. \( \frac{\partial u}{\partial r} \neq 0, \ \bar{\omega} \neq 0 \rightarrow \) vortex ring.

Since \( \frac{\partial u}{\partial r} \neq 0, \ \bar{\omega} \neq 0 \rightarrow \) vortex ring.
\[
\frac{\omega_1}{2\pi R_1} = \frac{\omega_2}{2\pi R_2} \rightarrow \frac{\omega_2}{\omega_1} = \frac{R_2}{R_1} \sim \frac{1}{\gamma} << 1
\]

since \(\omega \sim \frac{\partial u}{\partial r} \rightarrow \left(\frac{\partial u}{\partial r}\right)_1 << \left(\frac{\partial u}{\partial r}\right)_2\)

i.e.,

3. Near the center, let \(U_1 = U_1 (1 + \varepsilon_1)\) and \(U_2 = U_2 (1 + \varepsilon_2)\) where \(\varepsilon_1\) and \(\varepsilon_2\) measure the relative velocity fluctuations. Apply the Bernoulli equation along a reference average streamline

\[
P_1 + \frac{1}{2} \rho \bar{U}_1^2 = P_2 + \frac{1}{2} \rho \bar{U}_2^2
\]

(2)

Apply Bernoulli Equation to a particular streamline

\[
P_1 + \frac{1}{2} \rho \left[\bar{U}_1 (1 + \varepsilon_1)\right]^2 = P_2 + \frac{1}{2} \rho \left[\bar{U}_2 (1 + \varepsilon_2)\right]^2
\]

From (2) and (3) we obtain

\[
\varepsilon_1 \bar{U}_1^2 = \varepsilon_2 \bar{U}_2^2 + O(\varepsilon^2) \rightarrow \frac{\varepsilon_2}{\varepsilon_1} \sim \frac{\bar{U}_1^2}{\bar{U}_2^2} \sim \frac{1}{\gamma^4} << 1
\]