Chapter 2 - Similitude (Keyword: EQUAL RATIOS)

Similitude: Similarity of behavior for different systems with equal similarity parameters.

Prototype ↔ Model
(real world) (physical/analytical/numerical...experiments)

<table>
<thead>
<tr>
<th>Similitude</th>
<th>Similarity Parameters (SP’s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric Similitude</td>
<td>Length ratios, angles</td>
</tr>
<tr>
<td>Kinematic Similitude</td>
<td>Displacement ratios, velocity ratios</td>
</tr>
<tr>
<td>Dynamic Similitude</td>
<td>Force ratios, stress ratios, pressure ratios</td>
</tr>
<tr>
<td>:</td>
<td></td>
</tr>
<tr>
<td>Internal Constitution Similitude</td>
<td>ρ, ν</td>
</tr>
<tr>
<td>Boundary Condition Similitude</td>
<td></td>
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<tr>
<td>:</td>
<td></td>
</tr>
</tbody>
</table>

For similitude we require that the similarity parameters SP’s (eg. angles, length ratios, velocity ratios, etc) are equal for the model and the real world.

Example 1 Two similar triangles have equal angles or equal length ratios. In this case the two triangles have geometric similitude.

Example 2 For the flow around a model ship to be similar to the flow around the prototype ship, both model and prototype need to have equal angles and equal length and force ratios. In this case the model and the prototype have geometric and dynamic similitude.
2.1 Dimensional Analysis (DA) to Obtain SP’s

2.1.1 Buckingham’s $\pi$ theory
Reduce number of variables $\rightarrow$ derive dimensionally homogeneous relationships.

1. Specify (all) the (say N) relevant variables (dependent or independent): $x_1, x_2, \ldots, x_N$
e.g. time, force, fluid density, distance...
We want to relate the $x_i$’s to each other $\mathcal{I}(x_1, x_2, \ldots, x_N) = 0$

2. Identify (all) the (say P) relevant basic physical units (“dimensions”)
e.g. M, L, T (P = 3) [temperature, charge, …].

3. Let $\pi = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$ be a dimensionless quantity formed from the $x_i$’s. Suppose

$$x_i = C_i M^{m_i} L^{l_i} T^{t_i}, i = 1, 2, \ldots, N$$
where the $C_i$ are dimensionless constants. For example, if $x_1 = KE = \frac{1}{2} MV^2 = \frac{1}{2} M^1 L^2 T^{-2}$ (kinetic energy), we have that $C_1 = \frac{1}{2}, m_1 = 1, l_1 = 2, t_1 = -2$. Then

$$\pi = (C_1^{\alpha_1} C_2^{\alpha_2} \cdots C_N^{\alpha_N}) M^{\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_N m_N} L^{\alpha_1 l_1 + \alpha_2 l_2 + \cdots + \alpha_N l_N} T^{\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_N t_N}$$

For $\pi$ to be dimensionless, we require

$$\begin{cases}
\alpha_i m_i = 0 \\
\alpha_i l_i = 0 \\
\alpha_i t_i = 0 \\
\alpha_i \equiv 0, i = 1, 2, \ldots, N \text{ (i.e. } \pi \text{ is constant)}
\end{cases}$$

There are 2 possibilities:

(a) (1) has no nontrivial solution (only solution is $\pi = \text{constant}$, i.e. independent of $x_i$’s), which implies that the N variable $x_i, i = 1, 2, \ldots, N$ are Dimensionally Independent (DI), i.e. they are ‘unrelated’ and ‘irrelevant’ to the problem.

(b) (1) has $J (J > 0)$ nontrivial solutions, $\pi_1, \pi_2, \ldots, \pi_J$. In general, $J < N$, in fact, $J = N - K$ where $K$ is the rank or ‘dimension’ of the system of equations (1).
2.1.2 Model Law
Instead of relating the $N$ $x_i$’s by $I(x_1, x_2, \ldots x_N) = 0$, relate the $J$ $\pi$’s by

$$F(\pi_1, \pi_2, \ldots \pi_J) = 0, \text{ where } J = N - K < N$$

For similitude, we require

$$(\pi_{\text{model}})_j = (\pi_{\text{prototype}})_j \text{ where } j = 1, 2, \ldots, J.$$  

If 2 problems have all the same $\pi_j$’s, they have similitude (in the $\pi_j$ senses), so $\pi$’s serve as similarity parameters.

Note:
- If $\pi$ is dimensionless, so is $\pi \times \text{const}, \pi^{\text{const}}, 1/\pi$, etc…
- If $\pi_1, \pi_2$ are dimensionless, so is $\pi_1 \times \pi_2, \frac{\pi_1^1}{\pi_2}, \pi_1^{\text{const}1} \times \pi_2^{\text{const}2}$, etc…

In general, we want the set (not unique) of independent $\pi_j$’s, for e.g., $\pi_1, \pi_2, \pi_3$ or $\pi_1, \pi_1 \times \pi_2, \pi_3$, but not $\pi_1, \pi_2, \pi_1 \times \pi_2$.

**Example:** Force on a smooth circular cylinder in steady, incompressible flow

Application of Buckingham’s $\pi$ Theory.

![Diagram of force on a smooth circular cylinder](image)

Figure 1: Force on a smooth circular cylinder in steady incompressible fluid (no gravity)

A Fluid Mechanicinian found that the relevant *dimensional* quantities required to evaluate the force $F$ on the cylinder from the fluid are: the diameter of the cylinder $D$, the fluid velocity $U$, the fluid density $\rho$ and the kinematic viscosity of the fluid $\nu$. Evaluate the *non-dimensional* independent parameters that describe this problem.
\[ x_i : F, U, D, \rho, \nu \rightarrow N = 5 \]
\[ x_i = c_i M^{m_i} L^{l_i} T^{t_i} \rightarrow P = 3 \]

<table>
<thead>
<tr>
<th>( P = 3 )</th>
<th>( m_i )</th>
<th>( l_i )</th>
<th>( t_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 5 )</td>
<td>( F )</td>
<td>( U )</td>
<td>( D )</td>
</tr>
<tr>
<td>( m_i )</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( l_i )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( t_i )</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \pi = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5} \]

For \( \pi \) to be non-dimensional, the set of equations

\[
\begin{align*}
\alpha_i m_i &= 0 \\
\alpha_i l_i &= 0 \\
\alpha_i t_i &= 0
\end{align*}
\]

has to be satisfied. The system of equations above after we substitute the values for the \( m_i \)'s, \( l_i \)'s and \( t_i \)'s assume the form:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & -3 & 2 \\
-2 & -1 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

The rank of this system is \( K = 3 \), so we have \( j = 2 \) nontrivial solutions. Two families of solutions for \( \alpha_i \) for each fixed pair of \((\alpha_4, \alpha_5)\), exists a unique solution for \((\alpha_1, \alpha_2, \alpha_3)\). We consider the pairs \((\alpha_4 = 1, \alpha_5 = 0)\) and \((\alpha_4 = 0, \alpha_5 = 1)\), all other cases are linear combinations of these two.
1. Pair $\alpha_4 = 1$ and $\alpha_5 = 0$.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
4 \\
2
\end{pmatrix}
\]

which has solution

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
2 \\
2
\end{pmatrix}
\]

\[\therefore \pi_1 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5} = \frac{\rho U^2 D^2}{F}\]

Conventionally, $\pi_1 \to 2\pi_1^{-1}$ and $\therefore \pi_1 = \frac{F}{2\rho U^2 D^2} \equiv C_d$, which is the Drag coefficient.

2. Pair $\alpha_4 = 0$ and $\alpha_5 = 1$.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
-2 \\
-1
\end{pmatrix}
\]

which has solution

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
-1 \\
-1
\end{pmatrix}
\]

\[\therefore \pi_2 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5} = \frac{\nu}{UD}\]

Conventionally, $\pi_2 \to \pi_2^{-1}$, $\therefore \pi_2 = \frac{UD}{\nu} \equiv Re$, which is the Reynolds number.

Therefore, we can write the following equivalent expressions for the non-dimensional independent parameters that describe this problem:

\[
F(\pi_1, \pi_2) = 0 \quad \text{or} \quad \pi_1 = f(\pi_2)
\]

\[
F(C_d, Re) = 0 \quad \text{or} \quad C_d = f(Re)
\]

\[
F\left(\frac{F}{\sqrt{1/2 \rho U^2 D^2}}, \frac{UD}{\nu}\right) = 0 \quad \text{or} \quad \frac{F}{\sqrt{1/2 \rho U^2 D^2}} = f\left(\frac{UD}{\nu}\right)
\]
## Appendix A

Dimensions of *some* fluid properties

<table>
<thead>
<tr>
<th>Quantities</th>
<th>Dimensions ((MLT))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle (\theta)</td>
<td>none ((M^0L^0T^0))</td>
</tr>
<tr>
<td>Length (L)</td>
<td>(L)</td>
</tr>
<tr>
<td>Area (A)</td>
<td>(L^2)</td>
</tr>
<tr>
<td>Volume (\forall)</td>
<td>(L^3)</td>
</tr>
<tr>
<td>Time (t)</td>
<td>(T)</td>
</tr>
<tr>
<td>Velocity (V)</td>
<td>(LT^{-1})</td>
</tr>
<tr>
<td>Acceleration (\dot{V})</td>
<td>(LT^{-2})</td>
</tr>
<tr>
<td>Angular velocity (\omega)</td>
<td>(T^{-1})</td>
</tr>
<tr>
<td>Density (\rho)</td>
<td>(ML^{-3})</td>
</tr>
<tr>
<td>Momentum (\mathcal{L})</td>
<td>(MLT^{-1})</td>
</tr>
<tr>
<td>Volume flow rate (Q)</td>
<td>(L^3T^{-1})</td>
</tr>
<tr>
<td>Mass flow rate (Q)</td>
<td>(MT^{-1})</td>
</tr>
<tr>
<td>Pressure (p)</td>
<td>(ML^{-1}T^{-2})</td>
</tr>
<tr>
<td>Stress (\tau)</td>
<td>(ML^{-1}T^{-2})</td>
</tr>
<tr>
<td>Surface tension (\Sigma)</td>
<td>(MT^{-2})</td>
</tr>
<tr>
<td>Force (F)</td>
<td>(MLT^{-2})</td>
</tr>
<tr>
<td>Moment (M)</td>
<td>(ML^2T^{-2})</td>
</tr>
<tr>
<td>Energy (E)</td>
<td>(ML^2T^{-2})</td>
</tr>
<tr>
<td>Power (P)</td>
<td>(ML^2T^{-3})</td>
</tr>
<tr>
<td>Dynamic viscosity (\mu)</td>
<td>(ML^{-1}T^{-1})</td>
</tr>
<tr>
<td>Kinematic viscosity (\nu)</td>
<td>(L^2T^{-1})</td>
</tr>
</tbody>
</table>