3.11 - Method of Images

- Potential for single source: \( \phi = \frac{m}{2\pi} \ln \sqrt{x^2 + y^2} \)

- Potential for source near a wall: \( \phi = \frac{m}{2\pi} \left( \ln \sqrt{x^2 + (y - b)^2} + \ln \sqrt{x^2 + (y + b)^2} \right) \)

Note: Be sure to verify that the boundary conditions are satisfied by symmetry or by calculus for \( \phi(y) = \phi(-y) \).
Vortex near a wall (ground effect): \[ \phi = U_x + \frac{\Gamma}{2\pi} \left( \tan^{-1}(\frac{y-b}{x}) - \tan^{-1}(\frac{y+b}{x}) \right) \]

Verify that \( \frac{d\phi}{dy} = 0 \) on the wall \( y = 0 \).

Circle of radius \( a \) near a wall: \[ \phi \cong U_x \left( 1 + \frac{a^2}{x^2 + (y-b)^2} + \frac{a^2}{x^2 + (y+b)^2} \right) \]

This solution satisfies the boundary condition on the wall \( (\partial \phi / \partial n) = 0 \), and the degree it satisfies the boundary condition of no flux through the circle boundary increases as the ratio \( b/a \gg 1 \), i.e., the velocity due to the image dipole small on the real circle for \( b \gg a \). For a 2D dipole, \( \phi \sim \frac{1}{d}, \nabla \phi \sim \frac{1}{d^2} \).
• More than one wall:

Example 1:

Example 2:

Example 3:

3.12 Forces on a body undergoing steady translation
“D’Alembert’s paradox”

3.12.1 Fixed bodies & translating bodies - Galilean transformation.

\[ x = x' + Ut \]
### Reference system O: \( \vec{v}, \phi, p \) vs. Reference system O': \( \vec{v}', \phi', p' \)

<table>
<thead>
<tr>
<th>Formula</th>
<th>Reference O</th>
<th>Reference O'</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla^2 \phi = 0 )</td>
<td>( \vec{v} \cdot \hat{n} = \frac{\partial \phi}{\partial n} = \vec{U} \cdot \hat{n} = (U, 0, 0) \cdot (n_x, n_y, n_z) )</td>
<td>( \nabla^2 \phi' = 0 )</td>
</tr>
<tr>
<td>( \vec{v} \to 0 ) as (</td>
<td>\vec{x}</td>
<td>\to \infty )</td>
</tr>
<tr>
<td>( \phi \to 0 ) as (</td>
<td>\vec{x}</td>
<td>\to \infty )</td>
</tr>
</tbody>
</table>

#### Galilean transform:

\[
\vec{v}(x, y, z, t) = \vec{v}'(x' = x - Ut, y, z, t) + (U, 0, 0)
\]

\[
\phi(x, y, z, t) = \phi'(x' = x - Ut, y, z, t) + Ux' \Rightarrow
\]

\[
-Ux' + \phi(x = x' + Ut, y, z, t) = \phi'(x', y, z, t)
\]

#### Pressure (no gravity)

\[
p_\infty = -\frac{1}{2} \rho v^2 + C_o = C_o = -\frac{1}{2} \rho v'^2 + C'_o = C'_o - \frac{1}{2} \rho U^2
\]

\[
\therefore C_o = C'_o - \frac{1}{2} \rho U^2
\]

#### In O: unsteady flow

\[
p_s = -\rho \frac{\partial \phi}{\partial t} - \frac{1}{2} \rho \frac{v^2}{U^2} + C_o
\]

\[
\frac{\partial \phi}{\partial t} = (\frac{\partial}{\partial t} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'}) (\phi' + Ux') = -U^2
\]

\[
\therefore p_s = \rho U^2 - \frac{1}{2} \rho U^2 + C_o = \frac{1}{2} \rho U^2 + C_o
\]

\[
p_s - p_\infty = \frac{1}{2} \rho U^2 \text{ stagnation pressure}
\]

#### In O': steady flow

\[
p_s = -\rho \frac{\partial \phi'}{\partial t} - \frac{1}{2} \rho \frac{v'^2}{U'^2} + C'_o = C'_o
\]

\[
p_s - p_\infty = \frac{1}{2} \rho U'^2 \text{ stagnation pressure}
\]
3.12.2 Forces

Total fluid force for ideal flow (i.e., no shear stresses):

\[ \vec{F} = \iint_B p\hat{n}dS \]

For potential flow, substitute for \( p \) from Bernoulli:

\[ \vec{F} = \iint_B -\rho \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + gy + c(t) \right) \hat{n}dS \]

For the hydrostatic case (\( \vec{v} \equiv \phi \equiv 0 \)):

\[ \vec{F}_s = \iint_B (-\rho gy\hat{n}) dS = (-) \iiint_{v_B} \nabla (-\rho gy) dv = \rho g \varpi \hat{j} \quad \text{where } \varpi = \iiint_{v_B} dv \]

We evaluate only the hydrodynamic force:

\[ \vec{F}_d = -\rho \iint_B \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \hat{n}dS \]

For steady motion (\( \frac{\partial \phi}{\partial t} \equiv 0 \)):

\[ \vec{F}_d = -\frac{1}{2} \rho \iint_B v^2 \hat{n}dS \]
3.12.3 Example Hydrodynamic force on 2D cylinder in a steady uniform stream.

\[ F_d = \int_B \left( -\frac{\rho}{2} \right) \left| \nabla \phi \right|^2 \hat{n} d\ell = 2\pi \left( -\frac{\rho}{2} \right) \left| \nabla \phi \right|_{r=a}^2 \hat{n} d\theta \]

\[ F_x = F \cdot \hat{i} = \frac{-\rho a}{2} \int_0^{2\pi} d\theta \left| \nabla \phi \right|_{r=a}^2 \hat{n} \cdot \hat{i} \]

\[ = \frac{\rho a}{2} \int_0^{2\pi} \left| \nabla \phi \right|_{r=a}^2 \cos \theta d\theta \]

Velocity potential for flow past a 2D cylinder:

\[ \phi = Ur \cos \theta \left( 1 + \frac{a^2}{r^2} \right) \]

Velocity vector on the 2D cylinder surface:

\[ \nabla \phi |_{r=a} = (v_r |_{r=a}, v_\theta |_{r=a}) = \left( \frac{\partial \phi}{\partial r} |_{r=a}, \frac{1}{r} \frac{\partial \phi}{\partial \theta} |_{r=a} \right) \]

Square of the velocity vector on the 2D cylinder surface:

\[ \left| \nabla \phi \right|_{r=a}^2 = 4U^2 \sin^2 \theta \]
Finally, the **hydrodynamic force** on the 2D cylinder is given by

\[
F_x = \frac{\rho a}{2} \int_0^{2\pi} d\theta \left( 4U^2 \sin^2 \theta \cos \theta \right) = \left( \frac{1}{2} \rho U^2 \right) \int_0^{2\pi} (2a) \left( \frac{1}{2} \rho U^2 \right) d\theta \sin^2 \theta \cos \theta = 0
\]

Therefore, \( F_x = 0 \Rightarrow \) no horizontal force (symmetry fore-aft of the streamlines). Similarly,

\[
F_y = \left( \frac{1}{2} \rho U^2 \right) (2a) \int_0^{2\pi} d\theta \sin^2 \theta \sin \theta = 0
\]

In fact, *in general* we find that \( \vec{F} \equiv 0 \), on any 2D or 3D body.

**D’Alembert’s “paradox”:**

No hydrodynamic force* acts on a body moving with steady translational (no circulation) velocity in an infinite, inviscid, irrotational fluid.

* The moment as measured in a local frame is not necessarily zero.
3.13 Lift due to Circulation

3.13.1 Example Hydrodynamic force on a vortex in a uniform stream.

\[ \phi = Ux + \frac{\Gamma}{2\pi} \theta = Ur \cos \theta + \frac{\Gamma}{2\pi} \theta \]

Consider a control surface in the form of a circle of radius \( r \) centered at the point vortex. Then according to Newton’s law:

\[ \Sigma \vec{F} = \frac{d}{dt} \vec{L}_{CV} \stackrel{steady \ flow}{\longrightarrow} \]

\[ (\vec{F}_V + \vec{F}_{CS}) + \vec{M}_{NET} = 0 \iff \vec{F} \equiv -\vec{F}_V = \vec{F}_{CS} + \vec{M}_{NET} \]

Where,

- \( \vec{F}_r \) = Hydrodynamic force exerted on the vortex from the fluid.
- \( \vec{F}_V = -\vec{F} \) = Hydrodynamic force exerted on the fluid in the control volume from the vortex.
- \( \vec{F}_{CS} \) = Surface force (i.e., pressure) on the fluid control surface.
- \( \vec{M}_{NET} \) = Net linear momentum flux in the control volume through the control surface.
- \( \frac{d}{dt} \vec{L}_{CV} \) = Rate of change of the total linear momentum in the control volume.

![Diagram](image)

The hydrodynamic force on the vortex is \( \vec{F} = \vec{F}_{CS} + \vec{M}_{IN} \)
a. Net linear momentum flux in the control volume through the control surfaces, $\vec{M}_{NET}$.
Recall that the control surface has the form of a circle of radius $r$ centered at the point vortex.

a.1 The velocity components on the control surface are

\[ u = U - \frac{\Gamma}{2\pi r} \sin \theta \]
\[ v = \frac{\Gamma}{2\pi r} \cos \theta \]

The radial velocity on the control surface is therefore, given by

\[ u_r = U \frac{\partial x}{\partial r} = U \cos \theta = \vec{V} \cdot \hat{n} \]

\[ v_\theta = \frac{\Gamma}{2\pi r} \]

\[ U \]

\[ \theta \]

a.2 The net horizontal and vertical momentum fluxes through the control surface are given by

\[ (M_{NET})_x = -\rho \int_0^{2\pi} d\theta r u_r v_r = -\rho \int_0^{2\pi} d\theta r \left( U - \frac{\Gamma}{2\pi r} \sin \theta \right) U \cos \theta = 0 \]

\[ (M_{NET})_y = -\rho \int_0^{2\pi} d\theta r u_r v_r = -\rho \int_0^{2\pi} d\theta r \left( \frac{\Gamma}{2\pi r} \cos \theta \right) U \cos \theta \]

\[ = -\rho U \Gamma \int_0^{2\pi} \cos^2 \theta d\theta = -\rho U \Gamma \frac{2\pi}{2} = -\rho U \frac{\Gamma}{2} \]
b. Pressure force on the control surface, $\vec{F}_{CS}$.

b.1 From Bernoulli, the pressure on the control surface is

$$p = -\frac{1}{2} \rho |\vec{v}|^2 + C$$

b.2 The velocity $|\vec{v}|^2$ on the control surface is given by

$$|\vec{v}|^2 = u^2 + v^2 = \left( U - \frac{\Gamma}{2\pi r} \sin \theta \right)^2 + \left( \frac{\Gamma}{2\pi r} \cos \theta \right)^2$$

$$= U^2 - \frac{\Gamma}{\pi r} U \sin \theta + \left( \frac{\Gamma}{2\pi r} \right)^2$$

b.3 Integrate the pressure along the control surface to obtain $F_{CS}$

$$(F_{CS})_x = \int_0^{2\pi} d\theta r p(- \cos \theta) = 0$$

$$(F_{CS})_y = \int_0^{2\pi} d\theta r p(- \sin \theta) = \left(- \frac{\rho}{2} \right) \left( -\frac{\Gamma U}{\pi r} \right) \left( -r \right) \int_0^{2\pi} d\theta \sin^2 \theta = -\frac{1}{2} \rho U \Gamma$$


c. Finally, the force on the vortex $\vec{F}$ is given by

$$F_x = (F_{CS})_x + (M_x)_{1N} = 0$$

$$F_y = (F_{CS})_y + (M_y)_{1N} = -\rho U \Gamma$$

i.e., the fluid exerts a downward force $\vec{F} = -\rho U \Gamma$ on the vortex.

Kutta-Joukowski Law

$$2D: \quad \vec{F} = -\rho U \Gamma$$

$$3D: \quad \vec{F} = \rho \vec{U} \times \vec{\Gamma}$$

Generalized Kutta-Joukowski Law:

$$\vec{F} = \rho \vec{U} \times \left( \sum_{i=1}^{n} \vec{\Gamma}_i \right)$$

where $\vec{F}$ is the total force on a system of $n$ vortices in a free stream with speed $\vec{U}$. 10