3.20 Some Properties of Added-Mass Coefficients

1. \( m_{ij} = \rho \) [function of geometry only]

\[
F, M = [\text{linear function of } m_{ij}] \times [\text{function of instantaneous } U, \dot{U}, \Omega] \quad \text{not of motion history}
\]

2. Relationship to fluid momentum.

\[
L_x(t = T) = \int_B \rho U \Phi n_x dS = U \int_B \int \rho \Phi n_x dS
\]

The force exerted on the fluid from the body is \(-F(t) = -(-m_A \ddot{U}) = m_A \ddot{U}\).
\[
\int_0^T dt \left[ -F(t) \right] = \int_0^T m_A \dot{U} dt = m_A U |_{t=0}^{t=T} - \dot{L}_x (t = T) - L_x (t = 0) = U \int_B \rho \Phi n_x dS
\]

Therefore, \( m_A \) = total fluid momentum for a body moving at \( U = 1 \) (regardless of how we get there from rest) = fluid momentum per unit velocity of body.

K.B.C. \( \frac{\partial \Phi}{\partial n} = \nabla \phi \cdot \hat{n} = (U, 0, 0) \cdot \hat{n} = Un_x, \ \frac{\partial \phi}{\partial n} = Un_x \Rightarrow U n_x \Rightarrow \frac{\partial \Phi}{\partial n} = n_x
\]

\[\therefore \ m_A = \rho \int_B \Phi \frac{\partial \Phi}{\partial n} dS\]

For general 6 DOF:

\[m_{ji} = \rho \int_B \Phi_i n_j dS = \rho \int_B \Phi_i \frac{\partial \Phi_j}{\partial n} dS = j \text{ fluid momentum due to } i \text{ body motion}\]

3. Symmetry of added mass matrix \( m_{ij} = m_{ji} \).

\[m_{ji} = \rho \int_B \Phi_i \left( \frac{\partial \Phi_j}{\partial n} \right) dS = \rho \int_B \Phi_i \left( \nabla \Phi_j \cdot \hat{n} \right) dS = \rho \int_V \nabla \cdot \left( \Phi_i \nabla \Phi_j \right) dV\]

\[= \rho \int_V \left( \nabla \Phi_i \cdot \nabla \Phi_j + \Phi_i \nabla^2 \Phi_j \right) dV\]

Therefore,

\[m_{ji} = \rho \int_V \nabla \Phi_i \cdot \nabla \Phi_j dV = m_{ij}\]
4. Relationship to the kinetic energy of the fluid. For a general 6 DoF body motion \( U_i = (U_1, U_2, \ldots, U_6) \),

\[
\phi = U_i \Phi_i \quad ; \quad \Phi_i = \text{potential for } U_i = 1
\]

\[
\sum \text{ notation}
\]

\[
K.E. = \frac{1}{2} \rho \int \int \int_{V} \nabla \phi \cdot \nabla \phi dV = \frac{1}{2} \rho \int \int \int_{V} U_i \nabla \Phi_i \cdot U_j \nabla \Phi_j dV
\]

\[
= \frac{1}{2} \rho U_i U_j \int \int \int_{V} \nabla \Phi_i \cdot \nabla \Phi_j dV = \frac{1}{2} m_{ij} U_i U_j
\]

K.E. depends only on \( m_{ij} \) and instantaneous \( U_i \).

5. Symmetry simplifies \( m_{ij} \). From 36 \( \rightarrow \) 21 \( \rightarrow \) ‘?’ Choose such coordinate system that some \( m_{ij} = 0 \) by symmetry.

**Example 1** Port-starboard symmetry.

\[
m_{ij} = \begin{bmatrix}
m_{11} & m_{12} & 0 & 0 & 0 & m_{16} \\
m_{22} & 0 & 0 & 0 & m_{26} \\
m_{33} & m_{34} & m_{35} & 0 & 0 \\
m_{44} & m_{45} & 0 & 0 & 0 \\
m_{55} & 0 & m_{55} & 0 & 0 \\
m_{66} & m_{66} & 0 & 0 & 0
\end{bmatrix}
\]

\[
F_x \\
F_y \\
F_z \\
M_x \\
M_y \\
M_z
\]

12 independent coefficients

\[
U_1 \quad U_2 \quad U_3 \quad \Omega_1 \quad \Omega_2 \quad \Omega_3
\]
Example 2 Rotational or axi-symmetry with respect to $x_1$ axis.

$$m_{ij} = \begin{bmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ m_{22} & 0 & 0 & m_{35} \\ m_{22} & 0 & m_{35} & 0 \\ 0 & 0 & 0 \\ m_{55} & 0 \\ m_{55} \end{bmatrix}$$

where $m_{22} = m_{33}, m_{55} = m_{66}$ and $m_{26} = m_{35}$, so 4 different coefficients.

Exercise How about 3 planes of symmetry (e.g. a cuboid); a cube; a sphere?? Work out the details.
3.21 Slender Body Approximation

Definitions

(a) Slender Body = a body whose characteristic length in the longitudinal direction is considerably larger than the body’s characteristic length in the other two directions.

(b) $m_{ij} = \text{the } \text{3D added mass coefficient in the } i^{th} \text{ direction due to a unit acceleration in the } j^{th} \text{ direction. The subscripts } i, j \text{ run from 1 to 6.}$

(c) $M_{kl} = \text{the } \text{2D added mass coefficient in the } k^{th} \text{ direction due to a unit acceleration in the } l^{th} \text{ direction. The subscripts } k, l \text{ take the values 2, 3 and 4.}$

Goal To estimate the added mass coefficients $m_{ij}$ for a 3D slender body.

Idea Estimate $m_{ij}$ of a slender 3D body using the 2D sectional added mass coefficients (strip-wise $M_{kl}$). In particular, for simple shapes like long cylinders, we will use known 2D coefficients to find unknown 3D coefficients.

$$m_{ij} = \sum \left[ M_{kl}(x) \text{ contributions} \right]$$

Discussion If the 1-axis is the longitudinal axis of the slender body, then the 3D added mass coefficients $m_{ij}$ are calculated by summing the added mass coefficients of all the thin slices which are perpendicular to the 1-axis, $M_{kl}$. This means that forces in 1-direction cannot be obtained by slender body theory.
**Procedure** In order to calculate the 3D added mass coefficients $m_{ij}$ we need to:

1. Determine the 2D acceleration of each crosssection for a unit acceleration in the $i^{th}$ direction,

2. Multiply the 2D acceleration by the appropriate 2D added mass coefficient to get the force on that section in the $j^{th}$ direction, and

3. Integrate these forces over the length of the body.

**Examples**

- Sway force due to sway acceleration
  
  Assume a unit sway acceleration $\dot{u}_3 = 1$ and all other $u_j, \dot{u}_j = 0$, with $j = 1, 2, 4, 5, 6$.
  
  It then follows from the expressions for the generalized forces and moments (Lecture 12, JNN §4.13) that the sway force on the body is given by
  
  $$ f_3 = -m_{33}\dot{u}_3 = -m_{33} \Rightarrow m_{33} = -f_3 = -\int L F_3(x)dx $$
  
  A unit 3 acceleration in 3D results to a unit acceleration in the 3 direction of each 2D ‘slice’ $(\dot{U}_3 = \dot{u}_3 = 1)$. The hydrodynamic force on each slice is then given by
  
  $$ F_3(x) = -M_{33}(x)\dot{U}_3 = -M_{33}(x) $$
  
  Putting everything together, we obtain
  
  $$ m_{33} = -\int L M_{33}(x)dx = \int L M_{33}(x)dx $$

- Sway force due to yaw acceleration
  
  Assume a unit yaw acceleration $\dot{u}_5 = 1$ and all other $u_j, \dot{u}_j = 0$, with $j = 1, 2, 3, 4, 6$.
  
  It then follows from the expressions for the generalized forces and moments that the sway force on the body is given by
  
  $$ f_3 = -m_{35}\dot{u}_5 = -m_{35} \Rightarrow m_{35} = -f_3 = -\int L F_3(x)dx $$
  
  For each 2D ‘slice’, a distance $x$ from the origin, a unit 5 acceleration in 3D, results to a unit acceleration in the -3 direction times the moment arm $x$ $(\dot{U}_3 = -x\dot{u}_5 = -x)$. The hydrodynamic force on each slice is then given by
  
  $$ F_3(x) = -M_{33}(x)\dot{U}_3 = xM_{33}(x) $$
Putting everything together, we obtain

\[ m_{35} = - \int_{L} xM_{33}(x)dx \]

- Yaw moment due to yaw acceleration

Assume a unit yaw acceleration \( \dot{u}_5 = 1 \) and all other \( u_j, \dot{u}_j = 0 \), with \( j = 1, 2, 3, 4, 6 \).

It then follows from the expressions for the generalized forces and moments that the yaw force on the body is given by

\[ f_5 = -m_{55}\dot{u}_5 = -m_{55} \leftrightarrow m_{55} = -f_5 = -\int_{L} F_5(x)dx \]

For each 2D ‘slice’, a distance \( x \) from the origin, a unit 5 acceleration in 3D, results to a unit acceleration in the -3 direction times the moment arm \( x (\ddot{U}_3 = -x\dot{u}_5 = -x) \).

The hydrodynamic force on each slice is then given by

\[ F_3(x) = -M_{33}(x)\dot{U}_3 = xM_{33}(x) \]

However, each force \( F_3(x) \) produces a negative moment at the origin about the 5 axis

\[ F_5(x) = -xF_3(x) \]

Putting everything together, we obtain

\[ m_{55} = \int_{L} x^2M_{33}(x)dx \]

In the same manner we can estimate the remaining added mass coefficients \( m_{ij} \) - noting that added mass coefficients related to the 1-axis cannot be obtained by slender body theory.
In summary, the 3D added mass coefficients are shown in the following table. The empty boxes may be filled in by symmetry.

<table>
<thead>
<tr>
<th>$m_{22} = \int L M_{22} dx$</th>
<th>$m_{23} = \int L M_{23} dx$</th>
<th>$m_{24} = \int L M_{24} dx$</th>
<th>$m_{25} = \int L x M_{23} dx$</th>
<th>$m_{26} = \int L x M_{22} dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m_{33} = \int L M_{33} dx$</td>
<td>$m_{34} = \int L M_{34} dx$</td>
<td>$m_{35} = \int L x M_{33} dx$</td>
<td>$m_{36} = \int L x M_{32} dx$</td>
</tr>
<tr>
<td>$m_{44} = \int L M_{44} dx$</td>
<td>$m_{45} = \int L x M_{34} dx$</td>
<td>$m_{46} = \int L x M_{24} dx$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m_{55} = \int L x^2 M_{33} dx$</td>
<td>$m_{56} = \int L x^2 M_{32} dx$</td>
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<td></td>
</tr>
<tr>
<td>$m_{66} = \int L x^2 M_{22} dx$</td>
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<td></td>
</tr>
</tbody>
</table>
3.22 Buoyancy Effects Due to Accelerating Flow

**Example** Force on a stationary sphere in a fluid that is accelerated against it.

\[
\phi (r, \theta, t) = U(t) \left( r + \frac{a^3}{2r^2} \right) \cos \theta
\]

\[
\left. \frac{\partial \phi}{\partial t} \right|_{r=a} = \dot{U} \frac{3a}{2} \cos \theta
\]

\[
\nabla \phi \bigg|_{r=a} = \begin{pmatrix} 0, -\frac{3}{2} U \sin \theta, 0 \end{pmatrix}
\]

\[
\frac{1}{2} |\nabla \phi|^2 \bigg|_{r=a} = \frac{9}{8} U^2 \sin^2 \theta
\]

Then,

\[
F_x = (-\rho) (2\pi r^2) \int_0^\pi d\theta \sin \theta (\cos \theta) \left[ \dot{U} \frac{3a}{2} \cos \theta + \frac{9}{8} U^2 \sin^2 \theta \right]
\]

\[
= \dot{U} 3\pi \rho a^3 \int_0^\pi d\theta \sin \theta \cos^2 \theta + \rho U^2 \frac{9\pi}{4} a^2 \int_0^\pi d\theta \cos \theta \sin^3 \theta
\]

\[
F_x = \dot{U} \rho \left( 2\pi a^3 \right) \frac{\frac{4}{3} \pi a^3 \rho + \frac{2}{3} \pi a^3 \rho}{\frac{4}{3} \pi a^3 \rho + \frac{2}{3} \pi a^3 \rho = \rho V}
\]

9
Part of $F_x$ is due to the **pressure gradient** which must be present to cause the fluid to accelerate:

$$x\text{-momentum, noting } U = U(t): \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} + w \frac{\partial U}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \text{ (ignore gravity)}$$

$$\frac{dp}{dx} = -\rho U \text{ for uniform (1D) accelerated flow}$$

Force on the body due to the pressure field

$$\vec{F} = \iiint_B \rho \hat{n} dS = -\iiint_{V_B} \nabla p dV; \quad F_x = -\iiint_{V_B} \frac{\partial p}{\partial x} dV = \rho \nabla U$$

'Buoyancy' force due to pressure gradient $= \rho \nabla U$
**Analogue:** Buoyancy force due to hydrostatic pressure gradient. Gravitational acceleration $g \leftrightarrow \hat{U} =$ fluid acceleration.

\[
p_s = -\rho g y \\
\nabla p_s = -\rho g \hat{j} \rightarrow \vec{F}_s = -\rho g \hat{j} \quad \text{Archimedes principle}
\]

Summary: Total force on a fixed sphere in an accelerated flow

\[
F_x = \hat{U} \left( \begin{array}{c}
\rho \hat{\nu} \\
\text{Buoyancy}
\end{array} \right) + \left( \begin{array}{c}
m_{(1)} \\
\text{added mass}
\end{array} \right) = \hat{U} \frac{3}{2} \rho \hat{\nu} = 3 \hat{U} m_{(1)}
\]

In general, for any body in an accelerated flow:

\[
F_x = F_{\text{buoyancy}} + \hat{U} m_{(1)},
\]

where $m_{(1)}$ is the added mass in still water (from now on, m)

\[
F_x = -\hat{U} m \text{ for body acceleration with } \hat{U}
\]

**Added mass coefficient**

\[
c_m = \frac{m}{\rho \hat{\nu}}
\]

in the presence of accelerated flow $C_m = 1 + c_m$
Appendix A: More examples on symmetry of added mass tensor

• Symmetry with respect to Y (= “X-Z” plane symmetry)  
  12 non-zero, independent coefficients

• Symmetry with respect to X and Y (= “Y-Z” and “X-Z” plane symmetry)  
  7 non-zero, independent coefficients
• Axisymmetric with respect to X-axis  4 non-zero, independent coefficients

• Axisymmetric with respect to X axis and X (="Y-Z" plane symmetry)  3 non-zero, independent coefficients