# Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

# Wavenumber Integration

- Range-independent – Integral Transform solution
- Exact depth-dependent solution
  - Global Matrix Approach
  - Propagator Matrix Approach
  - Invariant Embedding
- Numerical issues:
  - Numerical stability of depth solution
  - Evaluation of inverse transforms

# Global Equations and Unknowns

| Wavefield Unknowns | | Boundary Conditions |
|---|---|---|
| 0 | Vacuum | |
| | | 2 |
| 4 | Elastic Ice Cover | |
| | | 3 |
| 2 | Fluid Water Column | |
| | | 2 |
| 2 | Fluid Sediment Layer | |
| | | 3 |
| 4 | Elastic Sediment Layer | |
| | | 4 |
| 2 | Elastic Halfspace | |
| 14 unknowns | | 14 equations |

## Homogeneous Fluid Layers

$$c = \sqrt{\frac{K}{\rho}}$$
$$k_m(z) = k_m = \omega / c$$

**Depth Solutions**

$$\phi^+(k,z) = e^{ik_z z} \quad \text{Downward Propagating}$$
$$\phi^-(k,z) = e^{-ik_z z} \quad \text{Upward Propagating}$$
$$k_z = \sqrt{k_m^2 - k_r^2} \quad \text{Vertical wavenumber}$$

**Layers without Sources**

$$\phi(r,z) = \int_0^\infty \left[A^- e^{-ik_z z} + A^+ e^{ik_z z}\right] J_0(k_r r)\, k_r\, dk_r\,.$$

**Interface Condition Parameters**

*Vertical Particle Displacements*

$$\begin{aligned} w(r,z) &= \frac{\partial \phi}{\partial z} \\ &= \int_0^\infty \left[-ik_z A^- e^{-ik_z z} + ik_z A^+ e^{ik_z z}\right] J_0(k_r r)\, k_r\, dk_r\,, \end{aligned}$$

*Vertical Normal Stress*

$$\begin{aligned} \sigma_{zz}(r,z) &= -p(r,z) \\ &= K\,\nabla^2 \phi(r,z) \\ &= -\rho\omega^2\, \phi(r,z) \\ &= -\rho\omega^2 \int_0^\infty \left[A^- \boxed{e^{-ik_z z}} + A^+ \boxed{e^{ik_z z}}\right] J_0(k_r r)\, k_r\, dk_r\,. \end{aligned}$$

Local Coefficient Matrix

Wavenumber Coefficient Matrix

Exponential Matrix (diagonal)

# Numerical Solution of the Depth Equation

## Direct Global Matrix Approach

*Layer Degree of Freedom Vector*

$$\mathbf{a}_m(k_r) = \begin{Bmatrix} A_m^-(k_r) \\ B_m^-(k_r) \\ A_m^+(k_r) \\ B_m^+(k_r) \end{Bmatrix}, \qquad m = 1, 2 \ldots N.$$

*Field Parameter Vector*

$$\mathbf{v}_m(k_r,z) = \begin{Bmatrix} w(k_r,z) \\ u(k_r,z) \\ \sigma_{zz}(k_r,z) \\ \sigma_{rz}(k_r,z) \end{Bmatrix}_m, \qquad m = 1, 2 \ldots N,$$

*Matrix Relation*

$$\begin{aligned} \mathbf{v}_m(k_r,z) &= \mathbf{c}_m(k_r,z)\,\mathbf{a}_m(k_r)\,, \qquad m = 1,2\ldots N\,. \\ \mathbf{c}_m(k_r,z) &= \mathbf{d}_m(k_r)\,\mathbf{e}_m(k_r,z)\,, \end{aligned}$$

*Interface Continuity Conditions*

$$\mathbf{v}_m^m(k_r) + \hat{\mathbf{v}}_m^m(k_r) = \mathbf{v}_{m+1}^m(k_r) + \hat{\mathbf{v}}_{m+1}^m(k_r)\,, \quad m = 1,2\ldots N-1\,,$$

$$\Leftrightarrow$$

$$\mathbf{v}_m^m(k_r) - \mathbf{v}_{m+1}^m(k_r) = \hat{\mathbf{v}}_{m+1}^m(k_r) - \hat{\mathbf{v}}_m^m(k_r)\,, \quad m = 1,2\ldots N-1\,,$$

**Discontinuity Cancellation**

# Direct Global Matrix Method

*Local Interface Discontinuity Vectors*

$$\mathbf{v}^m(k_r) = \mathbf{v}^m_m(k_r) - \mathbf{v}^m_{m+1}(k_r)\,, \quad m = 1, 2 \ldots N-1\,,$$
$$\hat{\mathbf{v}}^m(k_r) = \hat{\mathbf{v}}^m_m(k_r) - \hat{\mathbf{v}}^m_{m+1}(k_r)\,, \quad m = 1, 2 \ldots N-1\,,$$

*Local-to-Global Mapping*

$$\mathbf{a}_m(k_r) = \mathbf{S}_m\,\mathbf{A}(k_r)\,, \quad m = 1, 2 \ldots N\,.$$
$$\mathbf{v}^m(k_r) = \left[\mathbf{c}^m_m(k_r)\,\mathbf{S}_m - \mathbf{c}^m_{m+1}(k_r)\,\mathbf{S}_{m+1}\right]\mathbf{A}(k_r)\,, \quad m = 1, 2 \ldots N-1\,.$$

*Global Interface Discontinuity Vector*

Topology Matrices

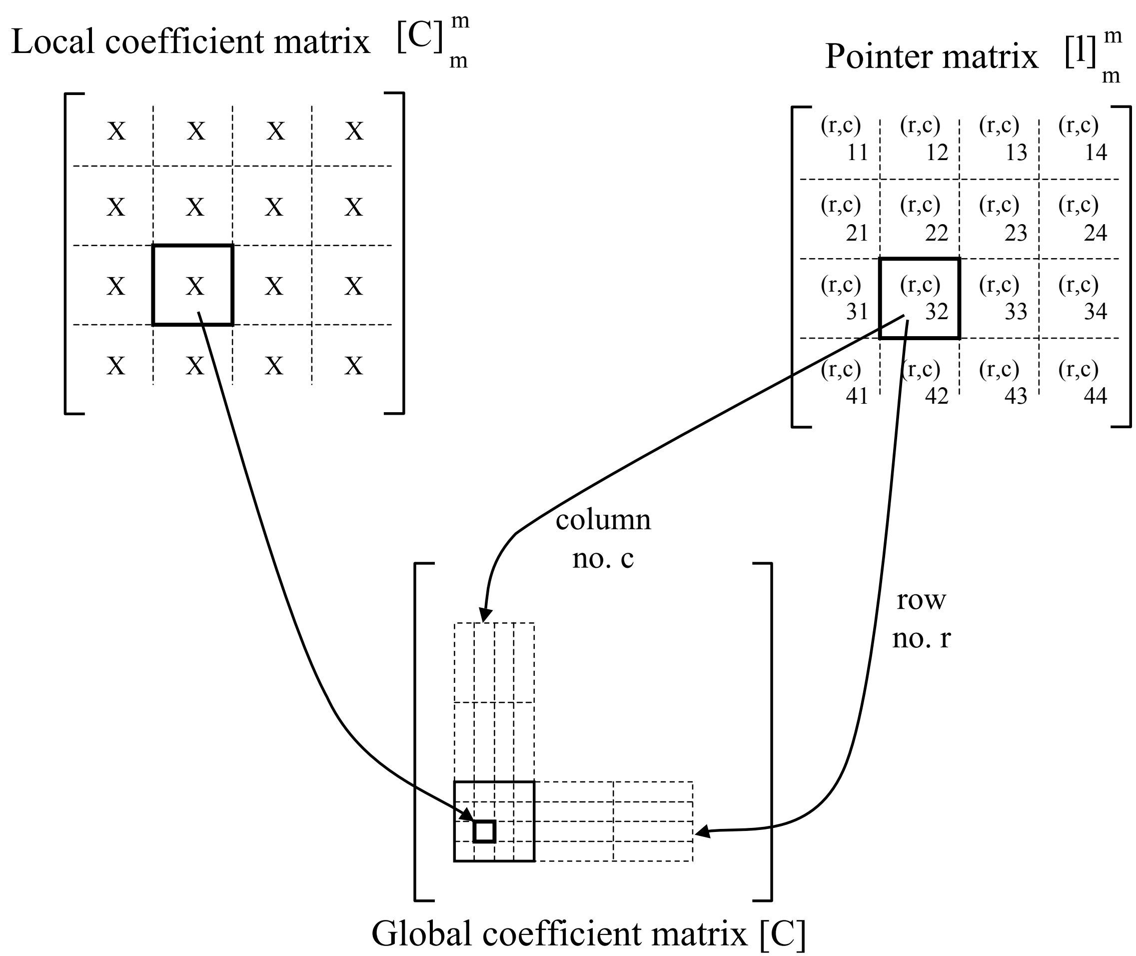
$$\mathbf{V}(k_r) = \sum_{m=1}^{N-1} \mathbf{T}^m\,\mathbf{v}^m(k_r)\,,$$
$$= \sum_{m=1}^{N-1} \mathbf{T}^m\left[\mathbf{c}^m_m(k_r)\,\mathbf{S}_m - \mathbf{c}^m_{m+1}(k_r)\,\mathbf{S}_{m+1}\right]\mathbf{A}(k_r)\,.$$

*Global Source Discontinuity Vector*

$$\widehat{\mathbf{V}}(k_r) = \sum_{m=1}^{N-1} \mathbf{T}^m\left[\hat{\mathbf{v}}^m_m(k_r) - \hat{\mathbf{v}}^m_{m+1}(k_r)\right]\,.$$

*Direct Global Matrix (DGM) Equations*

$$\mathbf{C}(k_r)\,\mathbf{A}(k_r) = -\widehat{\mathbf{V}}(k_r)\,,$$
$$\mathbf{C}(k_r) = \sum_{m=1}^{N-1} \mathbf{T}^m\left[\mathbf{c}^m_m(k_r)\,\mathbf{S}_m - \mathbf{c}^m_{m+1}(k_r)\,\mathbf{S}_{m+1}\right]\,.$$




Mapping between local and global coefficient matrices
by means of row and column pointers.

# Direct Global Matrix Method

## Numerical Stability

*Evanescent Regime*

$$k_z = i\gamma \,,$$

$$\Rightarrow$$

$$\phi_m^+(k_r, z) = e^{-\gamma z} \,,$$

$$\phi_m^-(k_r, z) = e^{+\gamma z} \,,$$

Blows up – $k_r$, z large

*Evanescent Depth Solutions*

$$\phi_m^+(k_r, z) = e^{-\gamma(z - z_{m-1})} \,,$$

$$\phi_m^-(k_r, z) = e^{-\gamma(z_m - z)} \,.$$

< 1 inside layer m

# Direct Global Matrix Method

## Numerical Stability

*Evanescent Regime*

$$k_z \;=\; i\gamma\;,$$

$$\Rightarrow$$

$$\phi_m^+(k_r,z) \;=\; e^{-\gamma z}\;,$$

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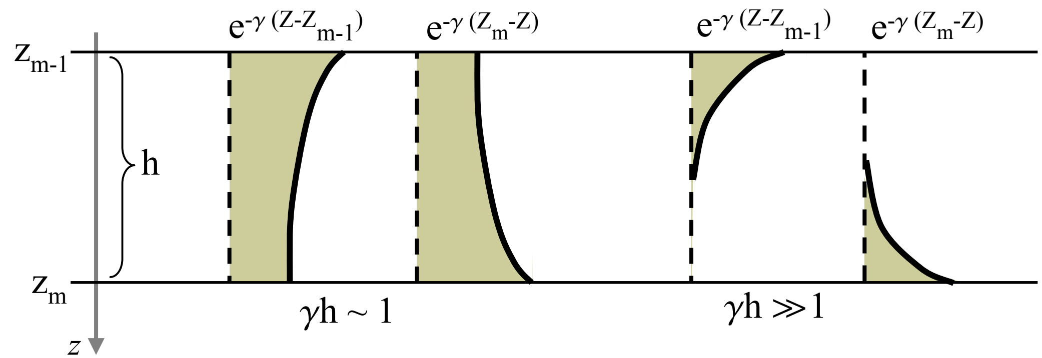
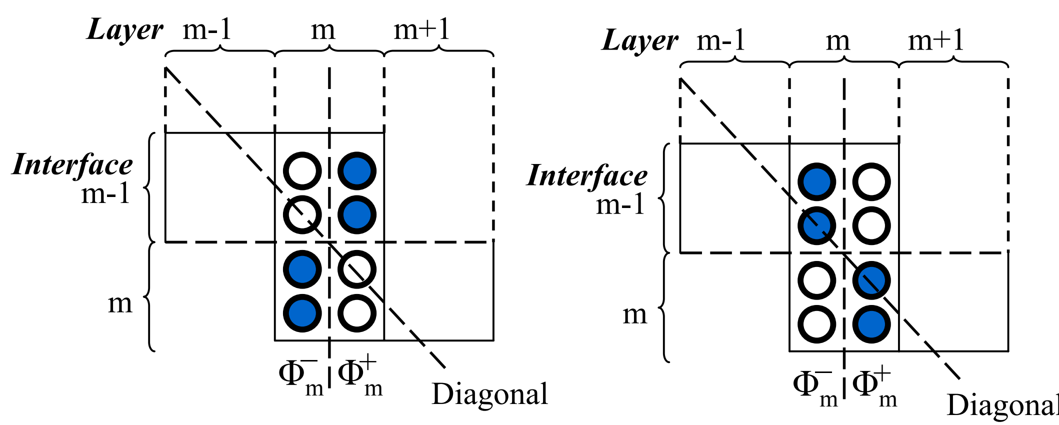
*Evanescent Depth Solutions*

$$\phi_m^+(k_r,z) \;=\; e^{-\gamma(z-z_{m-1})}\;,$$

$$\phi_m^-(k_r,z) \;=\; e^{-\gamma(z_m-z)}\;.$$




Unstable Stable

# DGM - Direct Global Matrix

| | | |
|---|---|---|
| Water | $A_2^+$ | $A_2^-$ |
| Fluid Sediment | $A_3^+$ | $A_3^-$ |
| Elastic | $A_4^+$, $B_4^+$ | |

$k_{z,3}$ real

$$
\begin{array}{c|cc|cc|cc|}
 & \multicolumn{2}{c|}{\text{layer 2}} & \multicolumn{2}{c|}{\text{Layer 3}} & \multicolumn{2}{c}{\text{Lower halfsp.}} \\
\hline
\text{Intfc 1} & x & x & 0 & 0 & 0 & 0 \\
\hline
\text{Intfc 2} & x & x & x & x & 0 & 0 \\
 & x & x & x & x & 0 & 0 \\
\hline
\text{Intfc 3} & 0 & 0 & x & x & x & x \\
 & 0 & 0 & x & x & x & x \\
 & 0 & 0 & x & x & x & x \\
\hline
\end{array}
\left\{ \begin{array}{c} A_2^+ \\ A_2^- \\ \hline A_3^+ \\ A_3^- \\ \hline A_4^+ \\ B_4^+ \end{array} \right\}
=
\left\{ \begin{array}{c} x \\ \hline x \\ x \\ \hline 0 \\ 0 \\ 0 \end{array} \right\}
$$

Block-diagonal. BW = max(|r-c|)

# DGM - Evanescent Layer

## Stable Mapping

| | | |
|---|---|---|
| Water | $A_2^+$ | $A_2^-$ |
| Fluid Sediment | $A_3^+$ | $A_3^-$ |
| Elastic | $A_4^+$ $B_4^+$ | |

$k_{z,3}$ imaginary

2 'separate' systems: No coupling between upper and lower matrices => No error propagation from bottom to top => **Numerically stable**

$$
\begin{array}{c|cc|cc|cc}
 & \multicolumn{2}{c|}{\text{layer 2}} & \multicolumn{2}{c|}{\text{Layer 3}} & \multicolumn{2}{c}{\text{Lower halfsp.}} \\
\hline
\text{Intfc 1} & x & x & 0 & 0 & 0 & 0 \\
\hline
\text{Intfc 2} & x & x & x & & 0 & 0 \\
 & x & x & x & & 0 & 0 \\
\hline
 & 0 & 0 & & x & x & x \\
\text{Intfc 3} & 0 & 0 & & x & x & x \\
 & 0 & 0 & & x & x & x
\end{array}
\begin{Bmatrix} A_2^+ \\ A_2^- \\ \hline A_3^+ \\ A_3^- \\ \hline A_4^+ \\ B_4^+ \end{Bmatrix}
=
\begin{Bmatrix} x \\ \hline x \\ x \\ \hline 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Block-diagonal. BW = max(|r-c|)

# DGM - Evanescent Layer

## Unstable Mapping

| | | |
|---|---|---|
| Water | $A_2^+$ | $A_2^-$ |
| Fluid Sediment | $A_3^+$ | $A_3^-$ |
| Elastic | $A_4^+$ $B_4^+$ | |

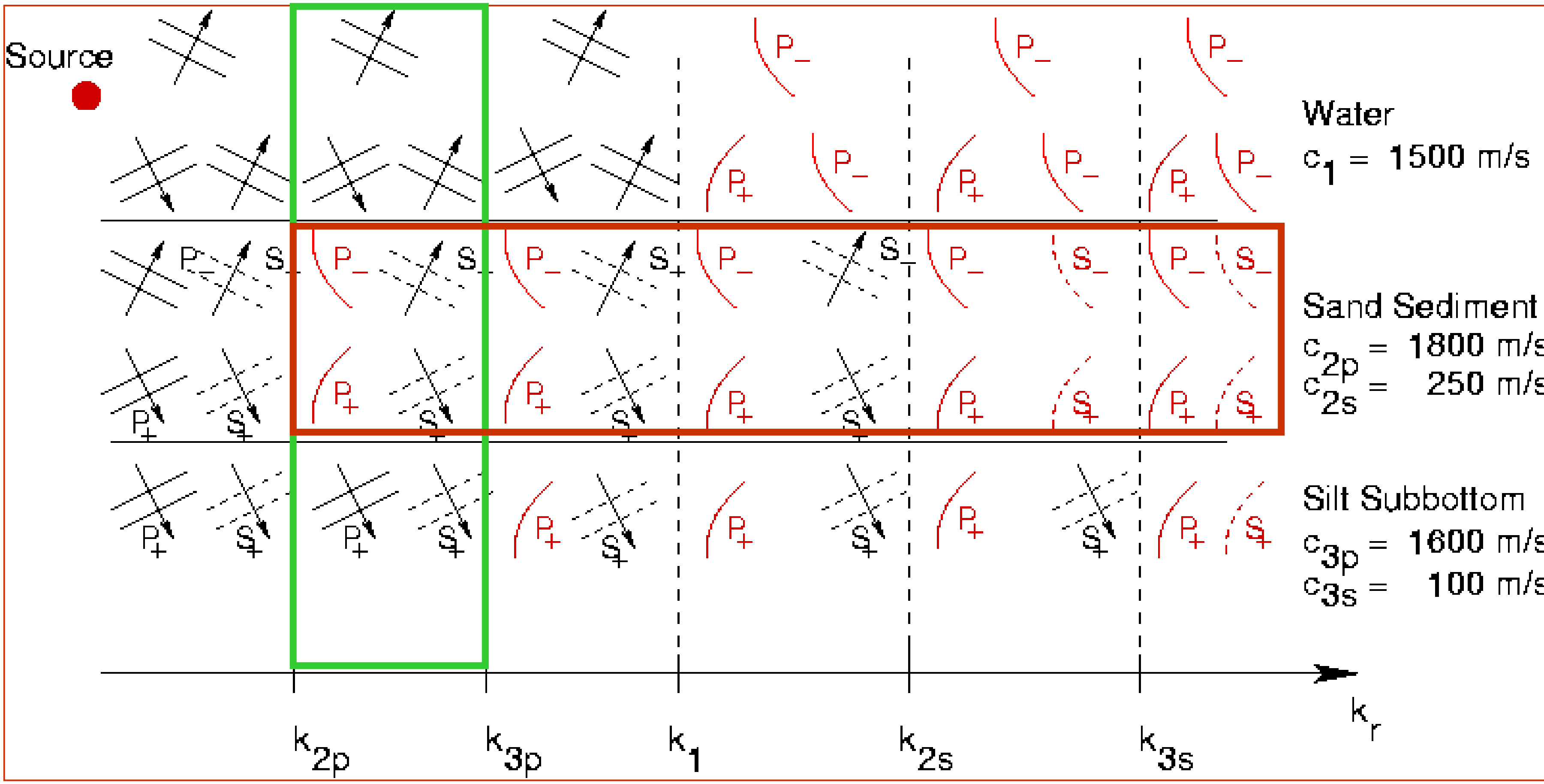
$k_{z,3}$ imaginary

Coupling between upper and lower matrices =>
Error propagation between bottom and top =>
**Numerically unstable**

$$
\begin{array}{l}
 \\
\text{Intfc 1} \\
\text{Intfc 2} \\
 \\
 \\
\text{Intfc 3} \\
 \\
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\text{layer 2} & \text{Layer 3} & \text{Lower halfsp.}
\end{array} \\
\begin{bmatrix}
x & x & 0 & 0 & 0 & 0 \\
x & x &   & x & 0 & 0 \\
x & x &   & x & 0 & 0 \\
0 & 0 & x &   & x & x \\
0 & 0 & x &   & x & x \\
0 & 0 & x &   & x & x
\end{bmatrix}
\end{array}
\begin{Bmatrix}
A_2^+ \\ A_2^- \\ A_3^- \\ A_3^+ \\ A_4^+ \\ B_4^+
\end{Bmatrix}
=
\begin{Bmatrix}
x \\ x \\ x \\ 0 \\ 0 \\ 0
\end{Bmatrix}
$$

Block-diagonal. BW = max(|r-c|)

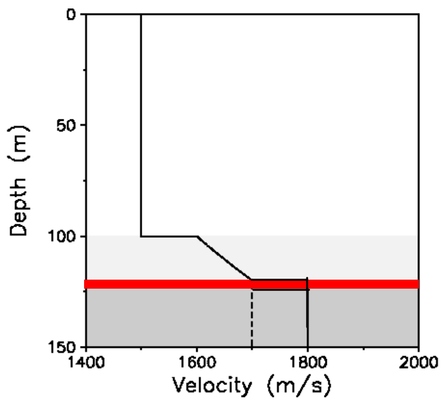
# Stratified Elastic Bottom
# Evanescent Tunneling Regime

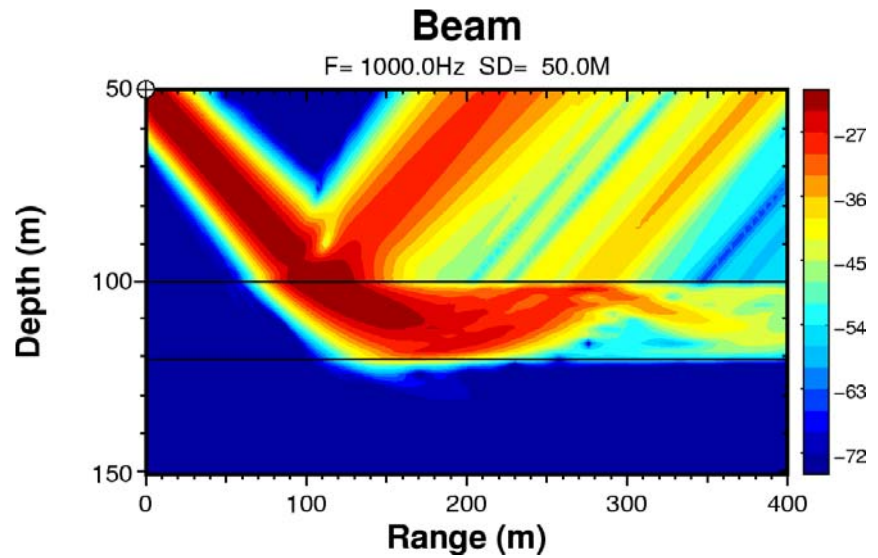


# Evanescent Wave Tunneling

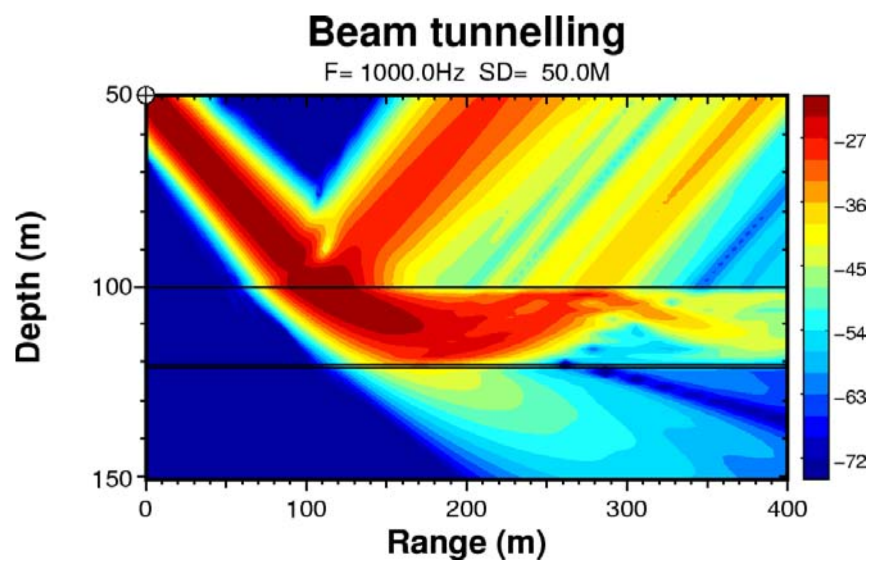
See Fig. 4.14 and 4.15 in Jensen, Kuperman, Porter and Schmidt.
*Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.

# Evanescent Wave Tunneling

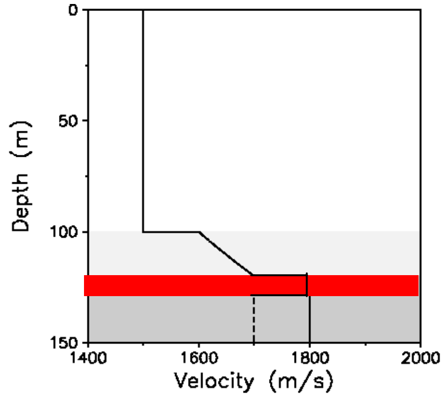



See Fig. 4.14 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.



# Thick Evanescent Layer




[See Fig. 4.14 in Jensen.]