Lecture 6.1 - Demand Functions

14.03 Spring 2003

1 The effect of price changes on Marshallian demand

- A simple change in the consumer's budget (i.e., an increase or decrease or I) involves a parallel shift of the feasible consumption set inward or outward from the origin. This economics of this are simple. Since this shift preserves the price ratio $\left(\frac{P_x}{P_Y}\right)$, it typically has no effect on the consumer's marginal rate of substitution (MRS), $\left(\frac{U_x}{U_y}\right)$, unless the chosen bundle is either initially or ultimately at a corner solution.
- A rise in the price of one good holding constant both income and the price of other goods has economically more complex effects:
  - It shifts the budget set inward toward the origin for the good whose price has risen. In other words, the consumer is now effectively poorer. This component is the 'income effect.'
  - It changes the slope of the budget set so that the consumer faces a different set of market trade-offs. This component is the 'price effect.'
  - Although both shifts occur simultaneously, they are conceptually distinct and have potentially different implications for consumer behavior.

1.1 Income effect

First, consider the "income effect." What is the impact of an inward shift in the budget set in a 2-good economy (X_1, X_2):

1. Total consumption?
2. Total utility?
3. Consumption of X_1? [Answer depends on normal, inferior]
4. Consumption of X_2? [Answer depends on normal, inferior]

1.2 Substitution effect

- In the same two good economy, what happens to consumption of X_1 if

$$\frac{p_1}{p_2} \uparrow$$

  but utility is held constant?

- In other words, we want the sign of

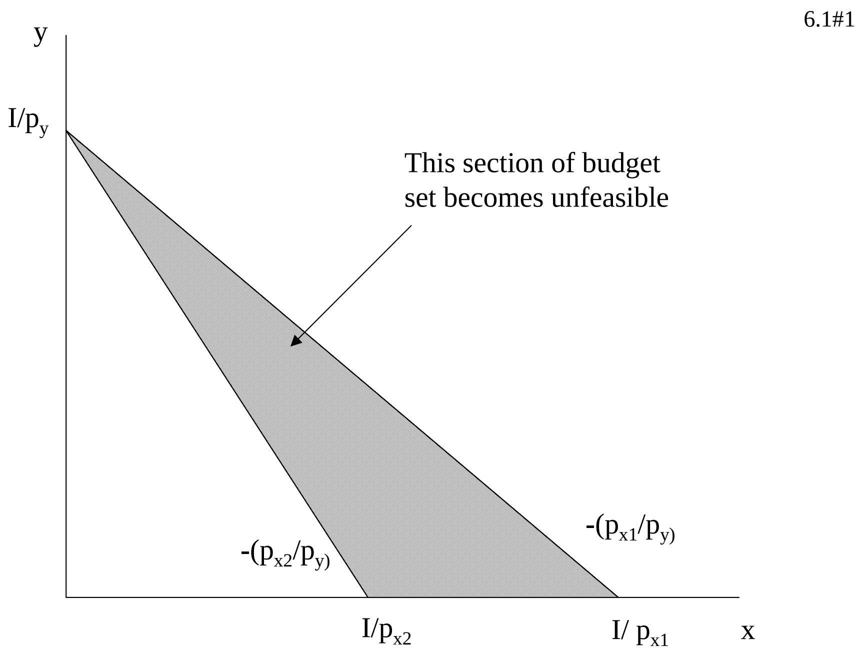
$$\text{Sign}\left\langle \frac{\partial X_1}{\partial p_1}|_{U=U_0} \right\rangle .$$

- Provided that the axiom of diminishing MRS applies, we'll have $\frac{\delta X_1}{\delta p_1}|_{U=U_0} < 0$.

- Holding utility constant, the substitution effect is *always* negative. By contrast, as we established above, the sign of the income effect is ambiguous,

$$\frac{\partial X_1}{\partial I} \gtrless 0,$$

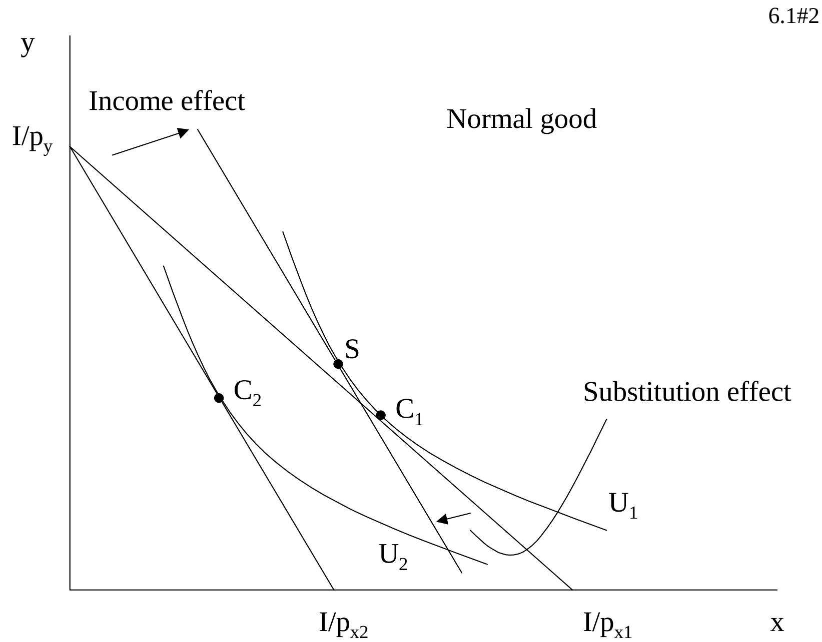
  depending on whether X_1 is a normal or inferior good.



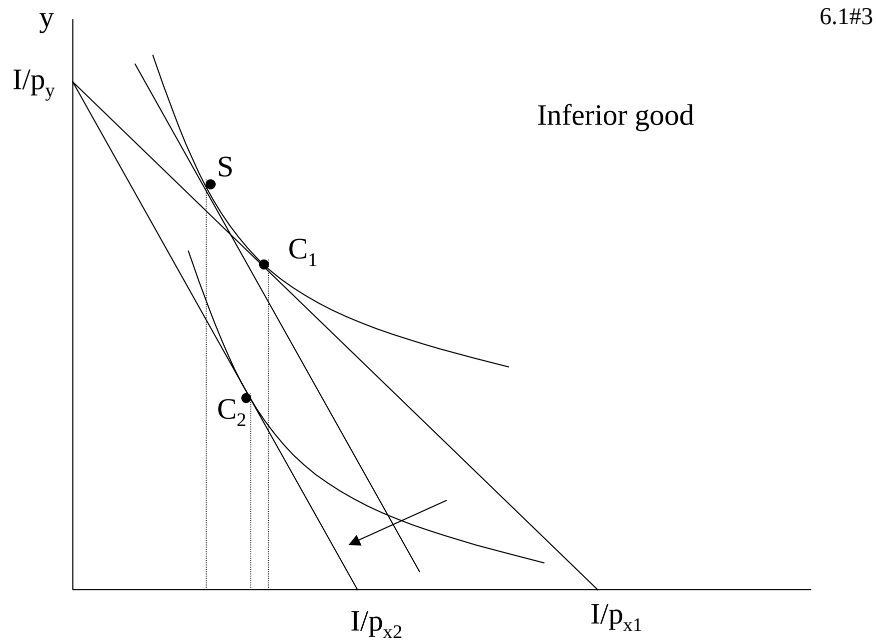
1.3 Types of goods

The fact that the substitution effect is always negative but the income effect has an ambiguous sign gives rise to three types of goods:

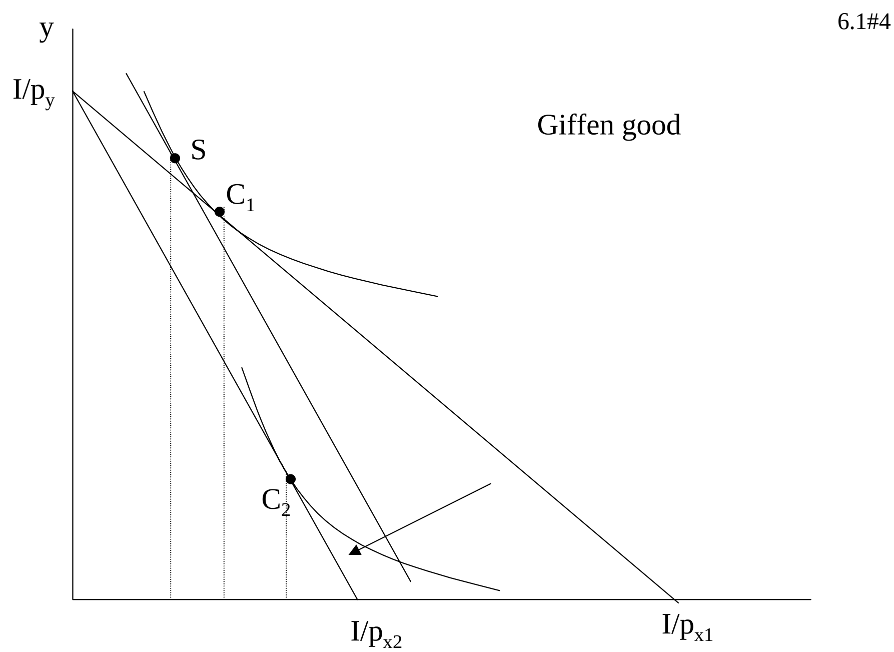
1. Normal good: $\frac{\partial X}{\partial I} > 0$, $\frac{\partial X}{\partial p_x}|_{U=U_0} < 0$. For this type of good, a rise in its price and a decline in income have the same effect–less consumption.



2. Inferior good: $\frac{\partial X}{\partial I} < 0$, $\frac{\partial X}{\partial p_x}|_{U=U_0} < 0$. For this type of good, the income and substitution effects are countervailing. Why countervailing? Even though both derivatives have the same sign, they have opposite effects because a rise in price reduces real income–thereby increasing consumption through the income effect even while reducing it through the substitution effect.



3. Strongly inferior good ('Giffen' good). $\frac{\partial X}{\partial I} < 0, \frac{\partial X}{\partial p_x}|_{U=U_0} < 0$. Similar to a conventional inferior good, the income and substitution effects are countervailing. But what's special about a Giffen good is that the income effect dominates the substitution effect (in some range). Hence, a rise in the price of a Giffen good causes the consumer to buy more of it–demand is upward sloping. Even though a price increase reduces demand due to the substitution effect holding utility constant, the consumer is effectively so much poorer due to the income loss that her demand for the inferior good rises.



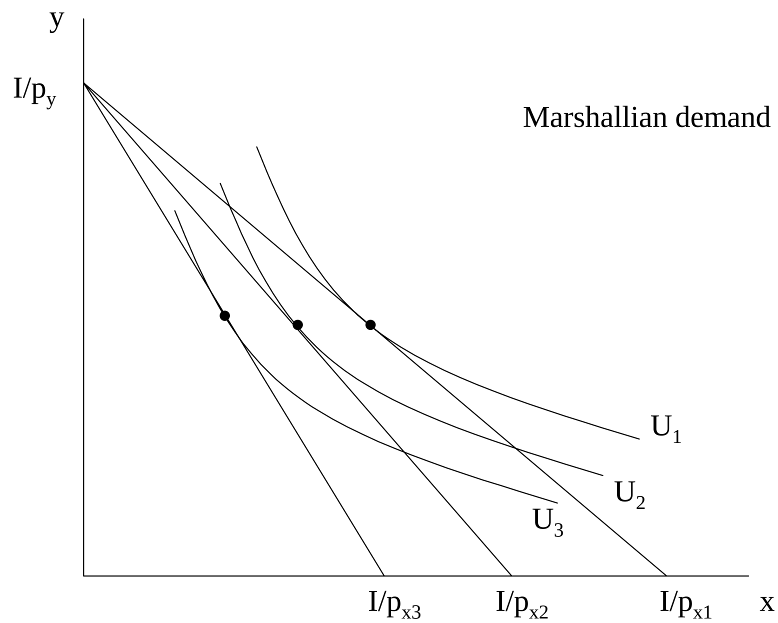
Question: The price of gasoline rises just about every summer, as does the gallons of gas consumed per household. Is gas a Giffen good?

1.4 Marshallian and Hicksian demand

Alfred Marshall was the first economist to draw supply and demand curves. The 'Marshallian cross' is the staple tool of blackboard economics. Marshallian demand curves are simply conventional market or individual demand curves. They answer the question:

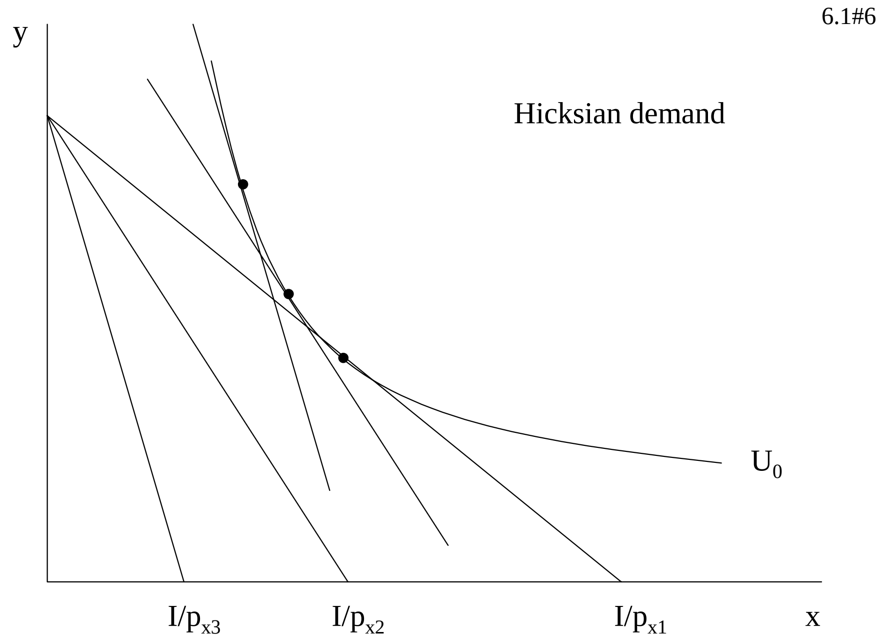
- Holding *income and all other prices constant*, how does the quantity of good X demanded change with P_x? We notate this demand function as $d_x(P_x, P_y, \overline{I})$. Marshallian demand curves implicitly combine income and substitution effects. Hence, they are 'net' demands that sum over these two conceptually distinct behavioral responses to price changes.

6.1#5



One can also conceive of a demand curve that is composed solely of substitution effects. This is called Hicksian demand (after the economist J. R. Hicks) and it answers the question:

- Holding *consumer utility constant*, how does the quantity of good X demanded change with P_x. We notate this demand function as $h_x(P_x, P_y, \overline{U})$. The presence of $\overline{U}$ as a parameter in the Hicksian demand function indicates that this function holds consumer utility constant–on the same indifference curve–as prices change. Hicksian demand is also called 'compensated' demand. This name follows from the fact that to keep the consumer on the same indifference curve as prices vary, one would have to adjust the consumer's income, i.e., compensate them. For the analogous reason, Marshallian demand is called 'uncompensated' demand.



1.5 Relationship between Compensated and Uncompensated demand

- These two demand functions are quite closely related (as show below). But they are not identical.
- Recall from the previous lecture the Expenditure Function,

$$E(P_x, P_y, \overline{U}),$$

which is the function that gives the minimum expenditure necessary to obtain utility $\overline{U}$ given prices P_x, P_y.

- For any chosen level of utility $\overline{U}$, the following identity will hold:

$$h_x(P_x, P_y, \overline{U}) = d_x(P_x, P_y, E(P_x, P_y, \overline{U})).$$

- In other words, for any chosen level of *utility,* compensated and uncompensated demand must equal to one another.
- *But they do not respond identically to a price change.* In particular differentiating this equality with respecting to P_x yields the following equation:

$$\frac{\partial h_x}{\partial P_x} = \frac{\partial d_x}{\partial P_x} + \frac{\partial d_x}{\partial I}\frac{\partial E}{\partial P_x}. \quad (1)$$

Rearranging yields,

$$\frac{\partial d_x}{\partial P_x} = \frac{\partial h_x}{\partial P_x} - \frac{\partial d_x}{\partial I}\frac{\partial E}{\partial P_x}. \tag{2}$$

- In words, the uncompensated demand response to a price change is equal to the compensated demand response minus another term,

$$\frac{\partial d_x}{\partial I}\frac{\partial E}{\partial P_x}.$$

This term deserves closer inspection.

- The $\frac{\partial d_x}{\partial I}$ term should look familiar. It is the income effect on demand for good X. But what is $\frac{\partial E}{\partial P_x}$?

- Recall the expenditure minimization problem that yields $E(P_x, P_y, \overline{U})$. This problem looks as follows:

$$\min_{X,Y} P_x X + P_y Y \ s.t. U(X,Y) \geq \overline{U}.$$

- The Langragian for this problem is:

$$\pounds = P_x X + P_y Y \ + \lambda(\overline{U} - U(X,Y)).$$

- The first order conditions for this problem are:

$$\begin{aligned} \frac{\partial \pounds}{\partial X} &= P_x - \lambda U_x = 0, \\ \frac{\partial \pounds}{\partial y} &= P_X - \lambda U_y = 0, \\ \frac{\partial \pounds}{\partial \lambda} &= \overline{U} - U(X,Y). \end{aligned}$$

- The solutions to this problem will have the following Langragian multipliers:

$$\lambda = \frac{P_x}{U_x} = \frac{P_y}{U_y}.$$

- As we know from the Envelope Theorem, at the solution to this problem,

$$\frac{\partial E}{\partial \overline{\overline{U}}} = \lambda.$$

In other words, relaxing the minimum utility constraint by one unit, raises expenditures by the ratio of prices to marginal utilities.

- But what is $\frac{\partial E}{\partial P_x}$? That is, holding utility constant, how do optimal expenditures respond to a minute change in the price of one good? The answer is:

$$\frac{\partial E}{\partial P_x} = X.$$

- This follows directly from the envelope theorem for constrained problems. Since X and Y are optimally chosen, a minute change in P_x or P_y will not affect the optimal quantity consumed of either good *holding utility constant* (as is always the case with the expenditure function). This result is due to the envelope theorem.

- But a price increase will change *total expenditures* (otherwise utility is not held constant).

- Since the consumer is already consuming X units of the good, a rise in price of 1 raises total expenditures needed to maintain the same level of utility by X. This result is called "Shephard's Lemma."

- This result follows relatively intuitively from a concrete example. If the consumer is currently buying 10 bags of potato chips per day and the price of chips rise by 1 cent per bag, expenditures will rise by about 10 cents. Expenditures rise by the price change times the initial level of consumption.

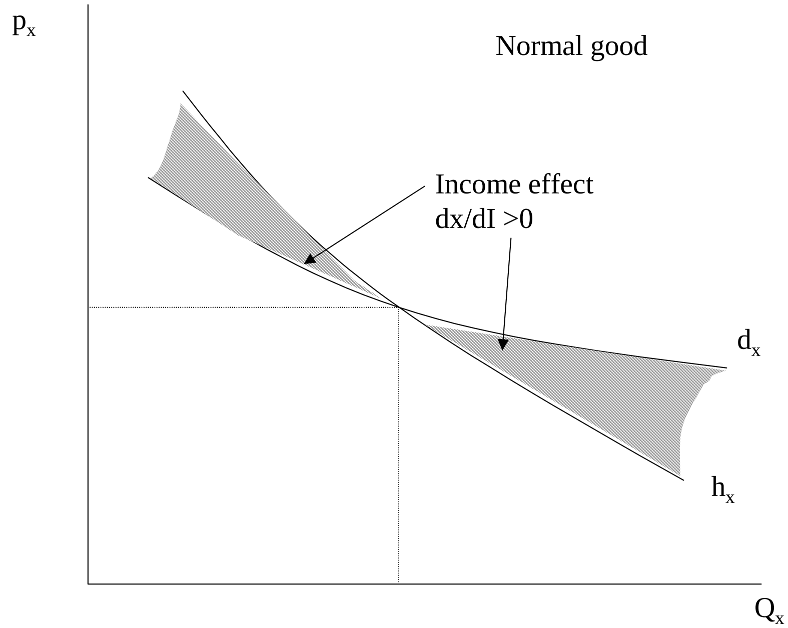
- Question: Is the X obtained from $\frac{\partial E}{\partial P_x}$ equal to h_x or d_x, i.e., compensated or uncompensated demand? Answer: h_x.

- Because the expenditure function holds utility constant, any demand function that arises from the expenditure function must also hold utility constant– and so is a compensated demand function.

- So, to reiterate: The derivative of the Expenditure function with respect to the price of a good is the Hicksian (compensated) demand function for that good.

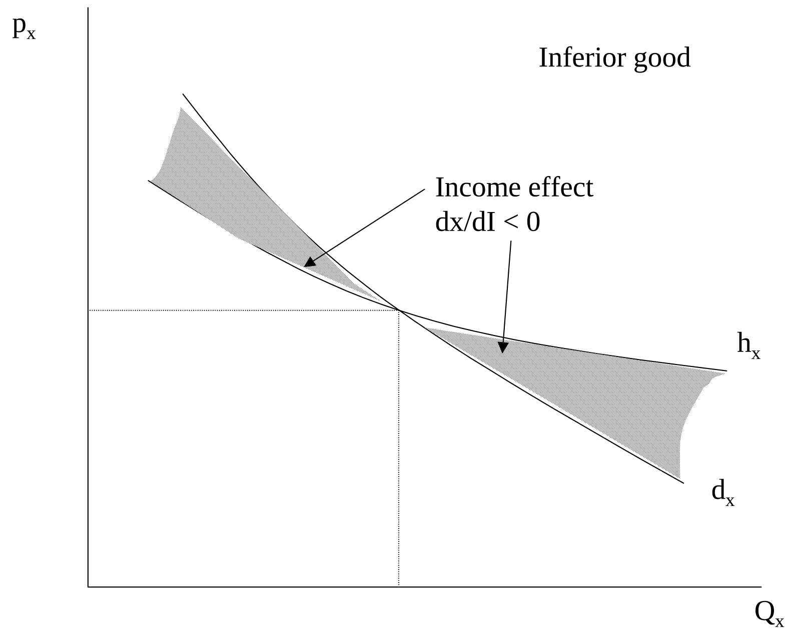
- Graphically the relationship between the two demand functions can be described as follows, according to the type of good.

6.1#7

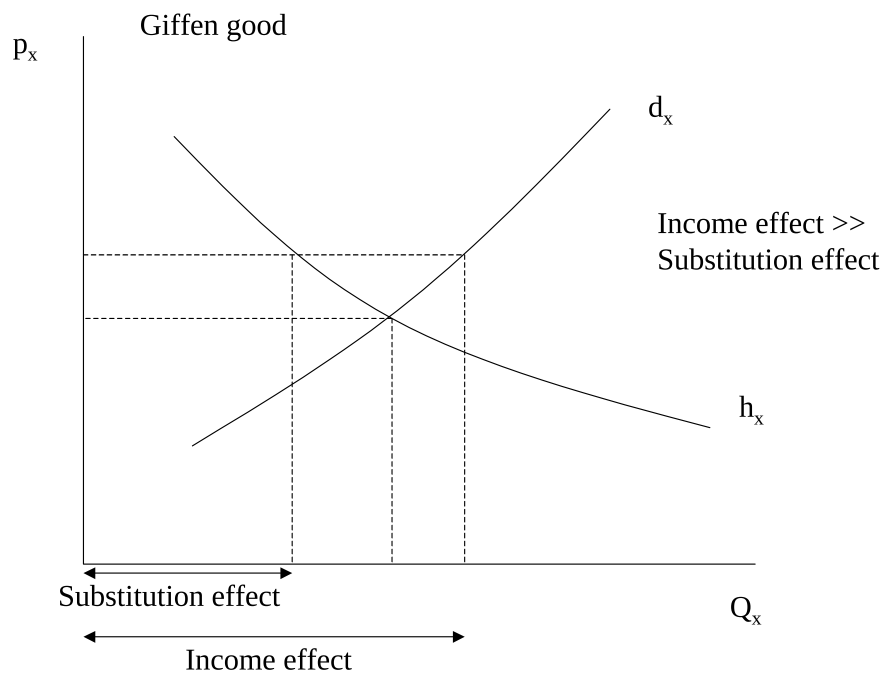


p_x
Normal good
Income effect
dx/dI >0
d_x
h_x
Q_x

6.1#8



p_x
Inferior good
Income effect
dx/dI < 0
h_x
d_x
Q_x



1.6 Applying Shephard's lemma

- Returning to equation 2, we can substitute back in using Shephard's Lemma to obtain:
$$\frac{\partial d_x}{\partial P_x} = \frac{\partial h_x}{\partial P_x} - \frac{\partial d_x}{\partial I} \cdot X.$$
- This identity is called the *Slutsky equation.*
- It says that the difference between the uncompensated demand response to a price change (the left-hand side, $\frac{\partial d_x}{\partial P_x}$) is equal to the compensated demand response $\left(\frac{\partial h_x}{\partial P_x}\right)$ minus the income effect scaled by the effective change in income due to the price change (recalling that $X = \frac{\partial E}{\partial P_x}$).
- Notice also the economic content of the final term, $\frac{\partial d_x}{\partial I} \cdot X$. The size of the income effect on total demand for good X in response to a change in P_x depends on the amount of X that the consumer is already purchasing.
- If the consumer is buying a lot of X, an increase in P_x will have a large income effect. By contrast, if the consumer is consuming zero X initially, the income effect of a change in P_x is zero.
- Applying the Slutsky equation to the three types of goods, it's easy to see that:

- For a normal good ($\frac{\partial d_x}{\partial I} > 0$), the income and substitution effects are complementary.
- For an inferior good ($\frac{\partial d_x}{\partial I} < 0$), the income and substitution effects are countervailing.
- For a Giffen good, the substitution effect dominates: $-\frac{\partial d_x}{\partial I} \cdot X > \frac{\partial h_x}{\partial P_x}$.

- **Effect of rise of P_x in two good economy (X, Y).**

	Uncompensated Demand 'Marshallian'	Compensated Demand 'Hicksian'
Consumption of X	Substitution: $-$ Income: $+/-$	Substitution: $-$ Income: 0
Consumption of Y	Substitution: $+$ Income: $+/-$	Substitution: $+$ Income: 0
Consumer Utility	$-$	0

1.7 Uncompensated demand and the indirect utility function.

- We concluded above that the compensated demand function can be derived just by differentiating the expenditure function. Is there a similar trick for deriving the uncompensated demand function? Of course!

- Recall the Lagrangian for the indirect utility function:

$$\begin{aligned} V &= \max_{x,y} U(X,Y) \ s.t. \ XP_x + YP_y \leq I, \\ \pounds &= U(X,Y) + \lambda(I - XP_x - YP_y), \\ \frac{\partial \pounds}{\partial X} &= U_x - \lambda P_x = \frac{\partial \pounds}{\partial Y} = U_y - \lambda P_y = \frac{\partial \pounds}{\partial \lambda} = I - XP_x - YP_y = 0. \end{aligned}$$

- Now, by the envelope theorem for constrained problems:

$$\frac{\partial \pounds}{\partial I} = \frac{\partial V}{\partial I} = \frac{U_y}{P_y} = \frac{U_x}{P_x} = \lambda.$$

The shadow value of additional income is equal to the marginal utility of consumption of either good divided by the cost of the good.

- And by a similar envelope theorem argument:

$$\frac{\partial V}{\partial P_x} = \frac{\partial \pounds}{\partial P_x} = -\lambda X. \tag{3}$$

- Notice the logic of this expression. The utility cost of a one unit price increase in is equal to the additional monetary cost (which is simply equal to X, the amount you are already consuming, times one) multiplied by the shadow value of additional income.

- Returning to the potato chips example, a 1 cent price rise costs you 10 cents if you were planning to buy 10 bags. And the value of 10 cents in foregone utility is simply λ times 10 cents.
- Putting together **??** and 3, we get the following expression:

$$-\frac{\partial V(P,I)/\partial P}{\partial V(P,I)/\partial I} = X(P,I), \tag{4}$$

  which is called Roy's identity.
- Roy's identity is analogous to Shephard's lemma above; both recover demand functions by differentiating solutions to the consumer's problems with respect to prices. The difference is that by differentiating the expenditure function, Shephard's lemma gives the *compensated* demand function, whereas by differentiating the indirect utility function, Roy's identity gives the *uncompensated* demand function.
- We are now ready to put these tools to work...