14.03 Problem Set 1 Solutions
Fall 2000

__Theory:__

1) $y = f(x;a) = a^2 - 2x^2 - ax$

a) FOC: $\frac{\partial f}{\partial x} = -4x^* - a = 0$

b) $\frac{\partial^2 f}{\partial x^2} = -4 < 0 \Rightarrow$ maximum

c) Solving the FOC for x^* yields $x^* = -a/4$.

d) $\frac{dx^*}{da} = -\frac{1}{4}$. $y^*(a) = f(x^*(a);a) = \frac{9}{8}a^2$, so $\frac{dy^*}{da} = \frac{9}{4}a$.

e) $-4x^*(a) - a \equiv 0$. Differentiate both sides of this identity w.r.t. a to get $-4\frac{dx^*}{da} - 1 = 0 \Rightarrow \frac{dx^*}{da} = \frac{-a}{4}$.

f) By the Envelope Theorem, $\frac{dy^*}{da} = \left.\frac{\partial f}{\partial a}\right|_{x=x^*} = 2a - x^* = 2a + a/4 = \frac{9}{4}a$.

g) $\left.\frac{\partial y}{\partial x}\right|_{x=x^*} = -4x^* - a = -4(-a/4) - a = 0$. Since $\left.\frac{\partial y}{\partial x}\right|_{x=x^*} = 0$ is the FOC for both maximization and minimization problems, the answer does not depend on whether x^* is a maximum or minimum.

2) Max $U(x,y,z) = \ln(x) + \ln(y) + \ln(2+z)$ subject to $x + 3y + 2z = 18$.

a) $L(x,y,z,\lambda) = \ln(x) + \ln(y) + \ln(2+z) + \lambda(18 - x - 3y - 2z)$

$$\frac{\partial L}{\partial x} = \frac{1}{x} - \lambda = 0$$

$$\frac{\partial L}{\partial y} = \frac{1}{y} - 3\lambda = 0$$

$$\frac{\partial L}{\partial z} = \frac{1}{2+z} - 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 18 - x - 3y - 2z = 0$$

Combine the first two equations to get $x = 3y$. Combine the first and third equations to get $x = 4 + 2z$. Substituting into the budget constraint yields the optimal choices $(x^*, y^*, z^*) = \left(\frac{22}{3}, \frac{22}{9}, \frac{5}{3}\right)$.

b) Increasing income from 18 to 19 does not affect the other first order conditions, so we can solve for the new optimal choices simply by substituting our previous relations between *x*,*y*, and *z* into the new budget constraint. This yields $(x^*, y^*, z^*) = \left(\frac{23}{3}, \frac{23}{9}, \frac{11}{6}\right)$.

c) $U(22/3, 22/9, 5/3) = \ln(5324/81) \approx 4.1855$.
$U(23/3, 23/9, 11/6) = \ln(1267/162) \approx 4.3189$.

d) Note that $\lambda^* = 1/x^*$. Hence $\lambda^* = 3/22 \approx 0.1364$ when income is 18, and $3/23 \approx 0.1304$ when income is 19. These are both close to the difference in utilities calculated in part c), which is about 0.1334, and reflects that fact that λ is the marginal utility of income.

e) Using a tangent-line approximation, $U\big|_{m=19} - U\big|_{m=18} \approx \left.\frac{dU}{dm}\right|_{m=18} (19-18)$. By the envelope theorem for constrained optimization problems we know that $\frac{dU}{dm} = \left.\frac{\partial L}{\partial m}\right|_{x^*, y^*, z^*, \lambda^*} = \lambda^*$.

Hence the approximate change in utility from increasing income by one unit is λ^*.

3) $\max z = \frac{xy}{x+y}$ subject to $x + 4y = 90$

a) $L(x, y, \lambda) = \frac{xy}{x+y} + \lambda(90 - x - 4y)$

$$\frac{\partial L}{\partial x} = \frac{(x+y)y - xy}{(x+y)^2} - \lambda = 0$$

$$\frac{\partial L}{\partial y} = \frac{(x+y)x - xy}{(x+y)^2} - 4\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 90 - x - 4y = 0$$

b) The first two equations imply that $x = 2y$. Substituting into the budget constraint yields $(x^*, y^*, \lambda^*) = (30,15,1/9)$. The optimized value of the objective function is $z^* = 10$.

c) The dual problem is $\min x + 4y$ subject to $\frac{xy}{x+y} = 10$.

d) $L^D(x, y, \lambda^D) = x + 4y + \lambda^D\left(10 - \frac{xy}{x+y}\right)$

$$\frac{\partial L^D}{\partial x} = 1 - \lambda^D \frac{y^2}{(x+y)^2} = 0$$

$$\frac{\partial L^D}{\partial y} = 4 - \lambda^D \frac{x^2}{(x+y)^2} = 0$$

$$\frac{\partial L^D}{\partial \lambda} = 10 - \frac{xy}{x+y} = 0$$

The first two FOCs again imply that $x = 2y$, and substituting into the third FOC yields $(x^*, y^*, \lambda^{D*}) = (30,15,9)$.

e) $\lambda = \frac{1}{\lambda^D}$. The economic explanation for this is that the Lagrangian from the primal problem is the marginal utility of income, or how much more utility we get from one additional unit of money. The Lagrangian from the dual problem is how much more money we need to get one additional unit of utility. At the optimum, the rate at which money is transformed into utility (λ) equals the inverse of the rate at which utility is transformed into money (λ^D).

Application: Minimum wage articles

1)

a) The policy change was the setting of a minimum wage of $9.25 a week for women with more than a year of experience in their current job, and a minimum wage of $6 a week for inexperienced women and girls aged 16 to 18.

b) The experimental groups are the groups affected by the policy change: experienced women and inexperienced women/girls working at retail stores.

c) The control group is a group of workers that is not affected by the policy change, but otherwise similar to the experimental groups. The control group in this study is composed of male workers at retail stores.

d) A control group is necessary because there may be factors other than the policy change that affect employment. In this case employment declined over the period due to a recession.

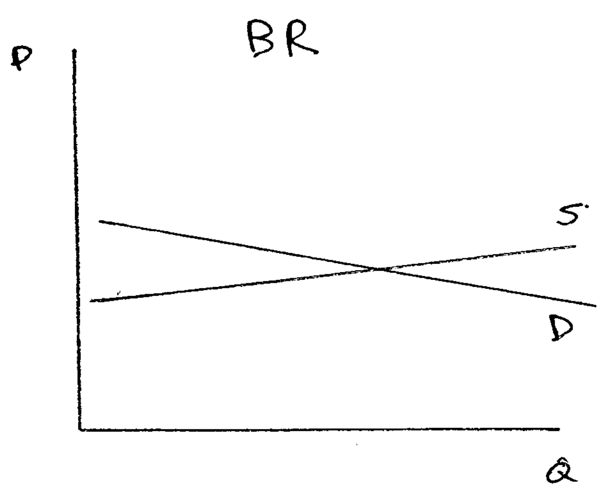
e) In order for a control group to be valid, it must be the case that whatever happens to the control group would have happened to the experimental groups in the absence of the experiment. The authors of the study note that men held jobs that were less vulnerable to the recession than women. Hence using men as a control group will tend to overstate the reduction in employment due to the minimum wage.

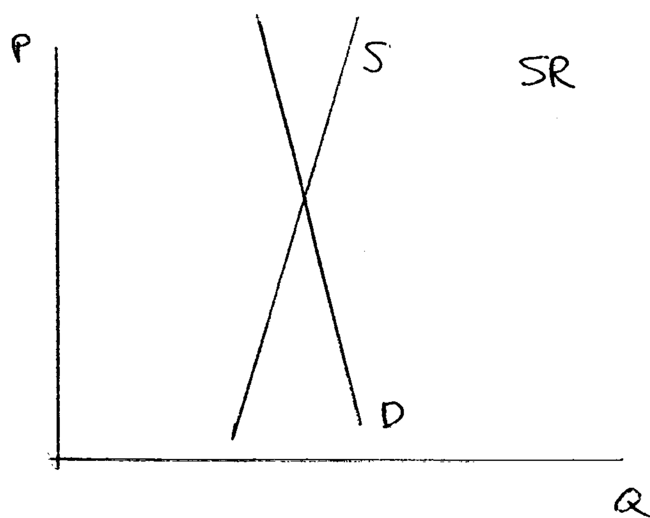
f) The study finds that the ratio of women to men employed fell by about six percent. Given the problems with the use of men as a control group, it is difficult to attribute this decline to the minimum wage. The ratio of girls to men employed rose by over twenty percent, and evidence from interviews suggests that at least some of this effect resulted from employers hiring girls rather than women.

2)

a) BRs believe that there are big responses to price changes, which means that supply and demand curves are elastic. An example of a BR labor market might be the market for gas station attendants. The demand for gas station attendants is elastic because a small increase in the wage would cause gas station owners to fire attendants and have customers pump their own gas. The supply of gas station attendants is elastic because it is an unskilled position with a large pool of potential workers.

b) SRs believe that there are small responses to price changes, which means that supply and demand curves are inelastic. An example of a SR labor market might be the market for airline pilots. The demand for airline pilots is inelastic because there are no good substitutes for pilots. The supply of airline pilots is inelastic because it is a highly skilled position that requires years of training. Hence in the short run, the supply of pilots is fixed.

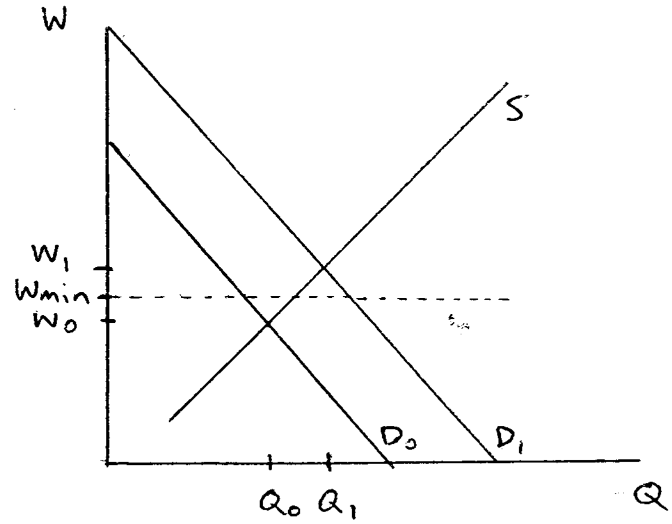




3)

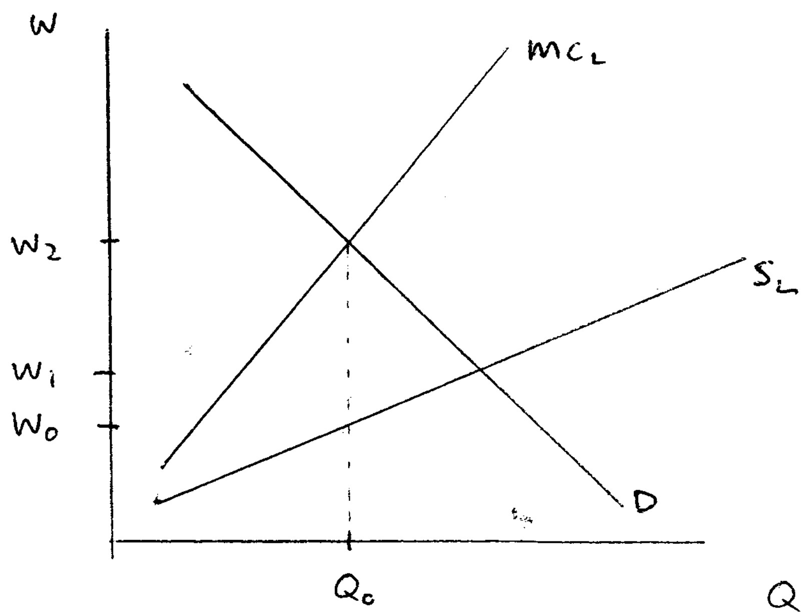
a) An economic expansion shifts out the labor demand curve, which invalidates the natural experiment if the minimum wage is less than the new equilibrium wage.

Suppose that for both New Jersey and Pennsylvania the original demand curve is D_0 and the supply curve is S. Now suppose that demand in both states shifts out to D_1 at the same time that New Jersey imposes a minimum wage of w_m. Then it looks like the minimum wage had no effect on employment, when in fact it would have decreased employment in the absence of the expansion.



b) If Pennsylvania is a valid control for New Jersey, then the decline in employment that occurred in Pennsylvania is what would have happened in New Jersey in the absence of the minimum wage increase. Hence the fact that employment in New Jersey was unchanged indicates that raising the minimum wage increased employment.

4) If the initial minimum wage increase did raise employment, then we should analyze the effects of further increases in the minimum wage using the monopsony model.



If the minimum wage is set below w_1, small increases in the minimum wage increase employment. However, in the region between w_1 and w_2 small increases in the minimum wage reduce employment. If the minimum wage rises above w_2, employment would be higher if there were no minimum wage.

5) There are several reasons why the long-run response to a minimum wage increase would be larger:

- An increase in the minimum wage may stimulate development of technologies that allow restaurant owners to substitute capital for labor, such as self-serve order terminals and automatic burger flippers.
- If it is costly for owners to fire workers, then the reduction in employment will occur gradually over time as workers quit.