Lecture 10 - Risk and Insurance

14.03 Spring 2003

1 Risk Aversion and Insurance: Introduction

- To have a passably usable model of choice, we need to be able to say something about how risk affects choice and well-being.
- What is risk? We'll define it is as:
  - "Uncertainty about possible 'states of the world,"' e.g., sick or healthy, war or peace, rain or sun, etc.
- Why do we need a theory of risk? To understand:
  - Insurance: Why people buy it. How it can even exist.
  - Investment behavior: Why do stocks pay higher interest rates than bank accounts?
  - How people choose among 'bundles' that have uncertain payoffs: Whether to fly on an airplane, whom to marry.
- More concretely, we need to understand the following:
  1. People don't want to play fair games. Fair game $E(X)$ =Cost of Entry= $P_{win} \cdot Win\$ + P_{lose} \cdot Lose\$$.
  2. Example, most people would not enter into a $1,000 dollar heads/tails fair coin flip.
  3. Another : I offer you a gamble. We'll flip a coin. If it's heads, I'll give you $10 million dollars. If it's tails, you owe me $9 million.
     - Will you take it? It's worth:
     $$\frac{1}{2} \cdot 10 - \frac{1}{2} \cdot 9 = \$0.5 \text{ million.}$$
     - What would you *pay me* to get out of this gamble (assuming you were already committed to taking it)?
  4. People won't pay large amounts of money to play gambles with huge upside potential. Example "St. Petersburg Paradox."

- Flip a coin. I'll pay you in dollars 2^n, where n is the number of tosses until you get a head:

$$X_1 = \$2, X_2 = \$4, X_3 = \$8, ...X_n = 2^n.$$

- How much *would* you be willing to pay to play this game?
- How much *should* you be willing to pay?

$$E(X) = \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + ...\frac{1}{2^n}2^n = \infty.$$

- What is the variance of this gamble? $V(X) = \infty$.
- No one would pay more than a few dollars to play this game.

- The fact that a gamble with positive expected monetary value has negative 'utility value' suggests something pervasive and important about human behavior: **As a general rule, uncertain prospects are worth less in utility terms than certain ones, even when expected tangible payoffs are the same.**

- If this observation is correct, we need a way to incorporate risk preference into our theory of choice since many (even most) economic decisions involve uncertainty about states of the world:

  - Prices change
  - Income fluctuates
  - Bad stuff happens

- We need to be able to say how people make choices when:

  - Agents value outcomes (as we have modeled all along)
  - Agents also have feelings/preferences about the riskiness of those outcomes

- John von Neumann and Oskar Morgenstern suggested a model for understanding and systematically modeling risk preference in the mid-1940s: Expected Utility Theory.

- We will begin with the Axioms of expected utility and then discuss their interpretation and applications.

- Note that the Axioms of consumer theory continue to hold for preferences over *certain* (opposite of *uncertain*) bundles of goods.

- Expected utility theory adds to this preferences over *uncertain combinations of bundles* where uncertainty means that these bundles will be available with known probabilities that are less than unity.

- Hence, EU theory is a superstructure that sits atop consumer theory.

1.1 Three Simple Statistical Notions

1. Probability distribution:

   Define states of the world $1, 2...n$ with probability of occurrence $\pi_1, \pi_2...\pi_n$. A valid probability distribution satisfies:

   $$\sum_{i=1}^{n} \pi_i = 1, \text{ or } \int_{-\infty}^{\infty} f(s)\partial x = 1 \text{ and } f(x) \geq 0 \; \forall \; x.$$

2. Expected value or "expectation."
   Say each state i has payoff x_i. Then

   $$E(x) = \sum_{i=1}^{n} \pi_i x_i \text{ or } E(x) = \int_{-\infty}^{\infty} x f(x)\partial x.$$

   Example: Expected value of a fair dice roll is $E(x) = \sum_{i=1}^{6} \pi_i i = \frac{1}{6} \cdot 21 = \frac{7}{2}$.

3. Variance (dispersion)
   Gambles with the same expected value may have different dispersion. We'll measure dispersion with variance.

   $$V(x) = \sum_{i=1}^{n} \pi_i \left(x_i - E(x)\right)^2 \text{ or } V(x) = \int_{-\infty}^{\infty} \left(x - E(x)\right)^2 f(x)\partial x.$$

   In dice example, $V(x) = \sum_{i=1}^{n} \pi_i \left(i - \frac{7}{2}\right)^2 = 2.92$.

Dispersion and risk are closely related notions. Holding constant the expectation of X, more dispersion means that the outcome is "riskier" – it has both more upside and more downside potential. Consider three gambles:

1. \$0.50 for sure. $V(L_1) = 0$.

2. Heads you receive \$1.00, tails you receive 0.
   $V(L_2) = 0.5(0 - .5)^2 + 0.5(1 - .5)^2 = 0.25$

3. 4 independent flips of a coin, you receive \$0.25 on each head.
   $V(L_3) = 4 \cdot (.5(0 - .125)^2 + .5(.25 - .125)^2) = 0.062\,5$

4. 100 independent flips of a coin, you receive \$0.01 on each head.
   $V(L_4) = 100 \cdot (.5(.0 - .005)^2 + .5(.01 - .005)^2) = 0.002\,5$

All 4 of these "lotteries" have same mean, but they have different levels of risk.

2 VNM Expected Utility Theory:

2.1 States of the world

- Think of the "States of the world" ranked from $x_0, x_1 ... x_N$ according to their desirability.
- Normalize the lowest state:

$$u(x_0) = 0. \text{ "Hell on earth."}$$

- Normalize the best state:

$$u(x_N) = 1. \text{ "Nirvana."}$$

- Now, for any state x_n ask individual what "lottery" over x_0, x_n would be equally desirable to getting x_n for sure. Define these values as π_n :

$$u(x_n) = \pi_n u(x_N) + (1 - \pi_n) u(x_0) = \pi_n 1 + (1 - \pi_n) \cdot 0 = \pi_n.$$

- Hence π_n is an index of the utility of x_n *for sure* on a $[0, 1]$ scale.

2.1.1 Axioms of expected utility

We will first lay out these axioms. I will next show that if a person behaves according to these axioms, he or she will act if she is maximizing *expected utility,* that is $E(\pi)$.

Axiom 1 *Preferences over uncertain outcomes ('states of the world') are: 1) complete; 2) reflexive and transitive*
Completeness – Can always state $X_a \succ X_b, X_b \succ X_a, X_a \tilde{} X_b$
Reflexive: $X_a \succ X_b \Leftrightarrow X_b \prec X_a$
Transitive: $X_a \succ X_b, X_b \succ X_c \Rightarrow X_a \succ X_c$.

Axiom 2 *Compound lotteries can be reduced to simple lotteries.*

This axiom says the 'frame' or order of lotteries is unimportant. So consider a two stage lottery is follows:

- Stage 1: Flip a coin heads, tails.
- Stage 2:
  If it's heads, flip again. Heads yields \$1.00, tails yields \$0.75.
  If it's tails, roll a dice with payoffs \$0.10, \$0.20, ...\$0.60 corresponding to outcomes $1 - 6$.

Now consider a single state lottery, where:

- We spin a pointer on a wheel with 8 areas, 2 areas of 90^0 representing \$1.00, and \$0.75, and 6 areas of 30^0 each, representing \$0.10, \$0.20, ...\$0.60 each.
- This single stage lottery has the same payouts at the same odds as the $2-$stage lottery.
- The 'compound lottery' exam says the consumer is indifferent between these two.

Counterexamples?

Axiom 3 *Continuity. Let $x_0 < x_i < x_N$. For each outcome x_i between x_0 and x_N, the consumer can name a probability π_i such that he is indifferent between x_i with certainty and playing a lottery where he receives x_N with probability π_i and x_0 with probability $1 - \pi_i$. Call this lottery $\widetilde{x}_i$. We say that x_i is the "certainty equivalent" of lottery $\widetilde{x}_i$ because the consumer is indifferent between x_i with certainty and the lottery $\widetilde{x}_i$.*

Axiom 4 *Substitutability. The lottery $\widetilde{x}_i$ can always be substituted for its certainty equivalent x_i in any other lottery since the consumer is indifferent between them. This is closely comparable to Axiom 2.*

Axiom 5 *Transitivity. Preferences over lotteries are transitive. (Previously we said this about preferences over goods (consumer theory) and preferences over states of the world (axiom 1)).*

Axiom 6 *Monotonicity. If two lotteries with the same alternatives differ only in probabilities, then the lottery that gives the higher probability to obtain the most preferred alternative is preferred. So*

$$\pi x_N + (1 - \pi)x_0 \succ \pi' x_N + (1 - \pi')x_0 \textit{ iff } \pi > \pi'$$

- If preferences satisfy these 6 axioms, we can assign numbers $u(x_i)$ associated with outcomes x_i such that if we compare two lotteries L and L' which offer probabilities $(\pi_1 ... \pi_n)$ and $(\pi'_1 ... \pi'_n)$ of obtaining those outcomes, then:
$$L \succ L' \text{ iff } \sum_{i=1}^{n} \pi_i u(x_i) > \sum_{i=1}^{n} \pi'_i u(x_i).$$
- Rational individuals will choose among risky alternatives *as if* they are maximizing the expected value of utility (rather than the expected value of the lottery). I will offer a simple proof of this result below.
- Note that the restrictions that this set of axioms places on preferences over lotteries. For preferences over consumption, we had said that utility was only defined up to a monotonic transformation.
- For preferences over lotteries, they are now defined up to an *affine* transformation, which is a much stronger (less palatable) assumption.
- (Affine transformation: a positive, linear transformation as in $u_2() = a + bu_1()$, where $b > 0$.)

2.2 Proof of Expected Utility property

Preamble

- As above, assume there exists a best bundle x_N and a worst bundle x_0 and normalize $u(x_0) = 0, u(x_N) = 1$.
- Define a bundle x_i such that:

$$x_i \tilde{} \left\{ \begin{array}{c} \Pr(x_N) = \pi_i \\ \Pr(x_0) = 1 - \pi_i \end{array} \right\}.$$

- We know that this π_i exists by the continuity axiom.
- As per our definition of the utility index above:

$$u(x_i) = \pi_i u(x_N) + (1 - \pi_i)u(x_0) = \pi_i 1 + (1 - \pi_i) \cdot 0 = \pi_i.$$

  So, we can freely substitute $u(x_i)$ and π_i.

Proof of Expected Utility property
Consider a lottery L of the form:

$$L = \left\{ \begin{array}{c} \Pr(x_1) = z \\ \Pr(x_2) = 1 - z \end{array} \right\}.$$

What we want to show is that

$$U(L) = z \cdot u(x_1) + (1 - z)u(x_2).$$

In words, for someone with VNM Expected Utility preferences, the utility index of this lottery is simply the *expected utility* of the lottery, that is the utility of each bundle x_1, x_2 weighted by its prior probability.

1. By the substitutability axiom, the consumer will be indifferent between L and the following compound lottery:

$$L \tilde{} \left\{ \begin{array}{c} \text{with probability } z : \left\{ \begin{array}{c} \Pr(x_N) = \pi_1 \\ \Pr(x_0) = 1 - \pi_1 \end{array} \right\} \\ \text{with probability } 1 - z : \left\{ \begin{array}{c} \Pr(x_N) = \pi_2 \\ \Pr(x_0) = 1 - \pi_2 \end{array} \right\} \end{array} \right\}, \qquad (1)$$

   where π_1, π_2 are the utility indices for x_1, x_2. Note that the simple lottery has been expanded to a 2-stage lottery, one of which occurs with probability z and the other with probability $1 - z$.

2. By the "reduction of compound lotteries" axiom, we know that the consumer is indifferent between the lottery above (1) and the following lottery:

$$L \tilde{} \left\{ \begin{array}{c} \Pr(x_N) = z \cdot \pi_1 + (1 - z) \cdot \pi_2 \\ \Pr(x_0) = z \cdot (1 - \pi_1) + (1 - z)(1 - \pi_2) \\ = 1 - z \cdot \pi_1 - (1 - z)\pi_2 \end{array} \right\}.$$

3. By the definition of $u(\cdot)$, we can substitute $\pi' s$ for $u' s$:

$$L\tilde{}\left\{ \begin{array}{c} \Pr(x_N) = z \cdot u(x_1) + (1-z) \cdot u(x_2) \\ \Pr(x_0) = 1 - z \cdot u(x_1) - (1-z) \cdot u(x_2) \end{array} \right\}. \tag{2}$$

4. Since $u(x_N) = 1, u(x_0) = 0$, and recalling from above that:

$$u(x_n) = \pi_n u(x_N) + (1-\pi_n)u(x_0) = \pi_n 1 + (1-\pi_n) \cdot 0$$

we can reduce expression 2 above to:

$$\begin{aligned} u(L) &= [z \cdot u(x_1) + (1-z) \cdot u(x_2)]u(x_N) + [1 - z \cdot u(x_1) - (1-z) \cdot u(x_2)]u(x_0) \\ &= [z \cdot u(x_1) + (1-z) \cdot u(x_2)] \cdot 1 + [1 - z \cdot u(x_1) - (1-z) \cdot u(x_2)] \cdot 0 \\ &= z \cdot u(x_1) + (1-z) \cdot u(x_2). \end{aligned}$$

In other words, the utility of facing lottery L is equal to a probability weighted combination of the utilities from receiving the two bundles corresponding to the outcome of the lottery – the 'expected utility.'

- So, the utility of facing a given lottery is the utility of each outcome weighted by its probability:

$$u(L) = \sum_{i=1}^{n} \pi_i u(x_i).$$

An expected utility maximizer would be indifferent between taking $u(L)$ for sure and the lottery on the right-hand side of this expression.

2.2.1 Summary of Expected Utility property

- We've established that a person who has VNM EU preferences over lotteries will act as if she is maximizing expected utility... a weighted average of utilities of each state, weighted by their probabilities.

- If this model is correct (and there are many reasons to think it's a useful description, even if not entirely correct), then we don't need to know exactly how people feel about risk per se to make strong predictions about how they will optimize over risky choices.

- To use this model, two ingredients needed:

  1. First, a utility function that transforms bundles into an ordinal utility ranking (now defined to an affine transformation).
  2. Second, the VNM assumptions which make strong predictions about the maximizing choices consumers will take when facing risky choices (i.e., probabilistic outcomes) over bundles, which are of course ranked by this utility function.

- Intuition check. Does this model mean that when facing a coin flip for $\$1,00$ versus $\$0.00$:

  $$u(L) = 0.5(1.00) + 0.5(0) = 0.50?$$

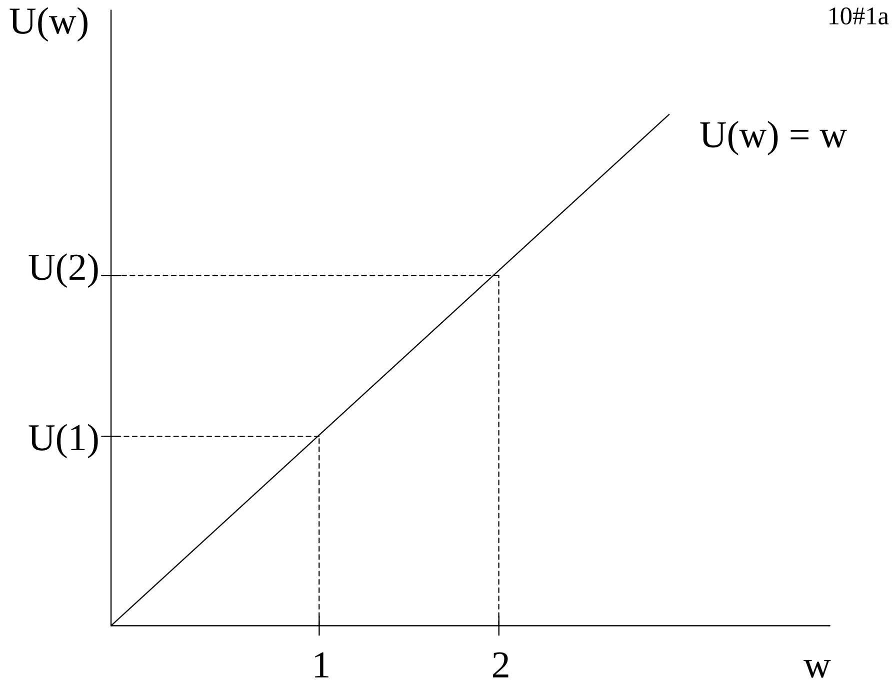
  No. It means:

  $$u(L) = 0.5u(1.00) + 0.5u(0) \gtrless u(0.50),$$

  where the sign of the inequality depends on the convexity or concavity of the utility function, as explained below.

3 Expected Utility Theory and Risk Aversion

- We started off to explain risk aversion and so far what we have done is lay out an axiomatic theory of expected utility.
- Where does risk aversion come in?
- Consider the following three utility functions:
  - $u_1(w) = w$



  - $u_2(w) = w^2$

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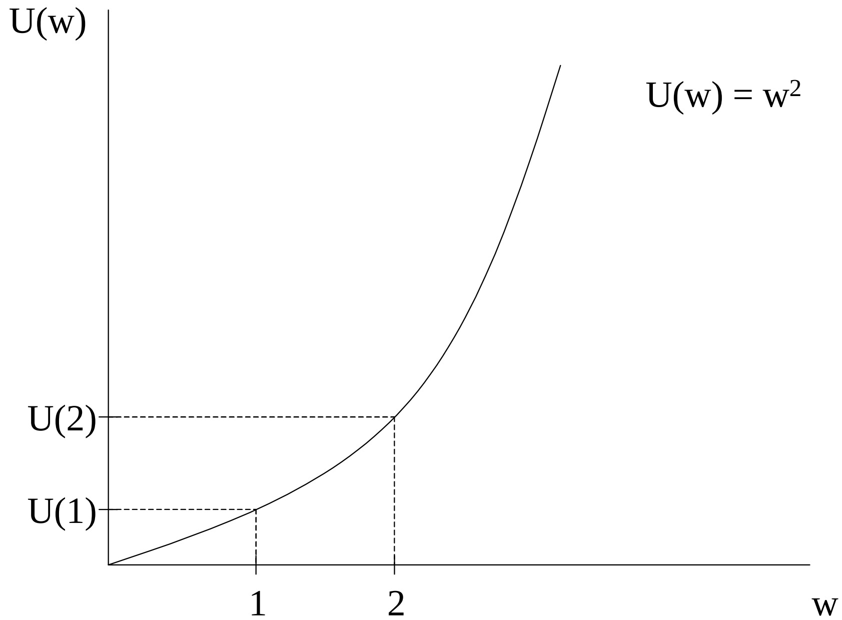

$- \; u_3(2) = w^{\frac{1}{2}}$

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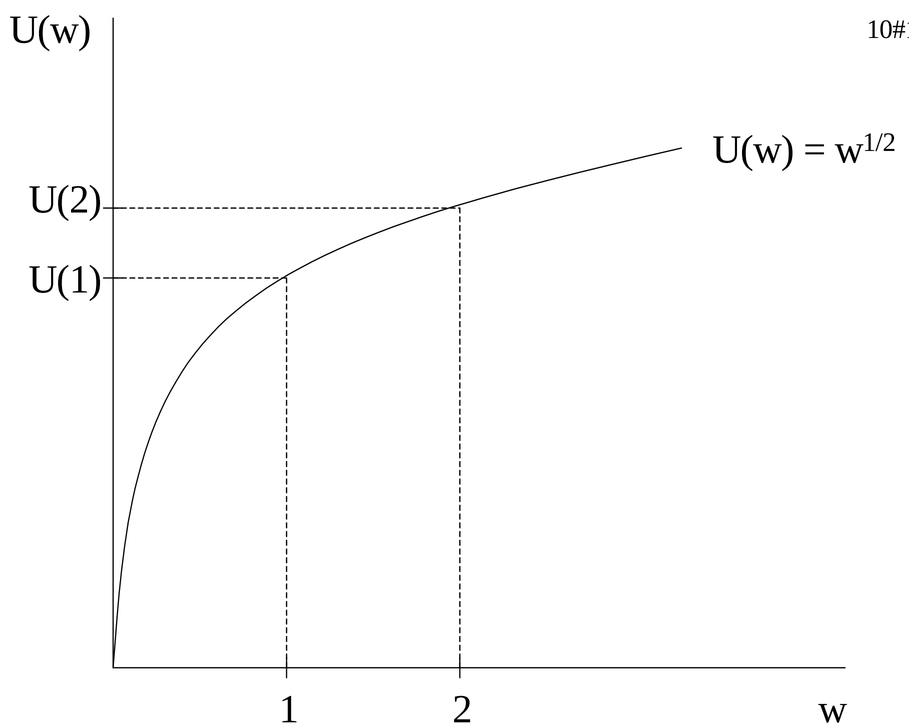

- How do they differ with respect to risk preference?
- First notice that $u_1(1) = u_2(1) = u_3(1) = 1$.
- Now consider the *Certainty Equivalent* for a lottery L that is a 50/50 gamble over \$2 versus \$0. The expected *monetary* value of this lottery is \$1.
- What is the expected utility value?
  - $u_1(L) = .5 \cdot u_1(0) + .5 \cdot u_1(2) = 0 + .5 \cdot 2 = 1$
  - $u_2(L) = .5 \cdot u_1(0) + .5 \cdot u_1(2) = 0 + .5 \cdot 2^2 = 2$
  - $u_3(L) = .5 \cdot u_1(0) + .5 \cdot u_1(2) = 0 + .5 \cdot 2^{.5} = .71$
- What is the "Certainty Equivalent" of lottery L for these three utility functions, i.e., the amount of money that the consumer be willing to accept with certainty in exchange for facing the lottery?
  1. $CE_1(L) = U_1^{-1}(1) = \$1.00$
  2. $CE_2(L) = U_2^{-1}(2) = 2^{.5} = \$1.41$
  3. $CE_3(L) = U_3^{-1}(0.71) = 0.71^2 = \$0.50$
- Hence, depending on the utility function, a person would pay $\$1, \$1.41$, or \$0.51 dollars to participate in this lottery.
- Notice that the expected value $E(Value)$ of this lottery is \$1.00.
- But these three utility functions value it differently:
  1. The person with U_1 is *risk neutral*: $CE = \$1.00 = E(Value) \Rightarrow$ Risk neutral
  2. The person with U_2 is *risk loving*: $CE = \$1.41 > E(Value) \Rightarrow$ Risk loving
  3. The person with U_3 is *risk averse*: $CE = \$0.50 < E(Value) \Rightarrow$ Risk averse
- What gives rise to these inequalities is the shape of the utility function. Risk preference comes from the concavity/convexity of the utility function:
- Expected utility of wealth: $E(U(w)) = \sum_{i=1}^{N} \pi_i U(w_i)$
- Utility of expected wealth: $U(E(w)) = U\left(\sum_{i=1}^{N} \pi_i w_i\right)$
- Jensen's inequality:
  - $E(U(w)) = U(E(w)) \Rightarrow$ Risk neutral

- $E(U(w)) > U(E(w)) \Rightarrow$ Risk loving
- $E(U(w)) < U(E(w)) \Rightarrow$ Risk averse

- So, the core insight of expected utility theory is this:
  For a risk averse agent, the expected utility of wealth is less than the utility of expected wealth (given non-zero risk).

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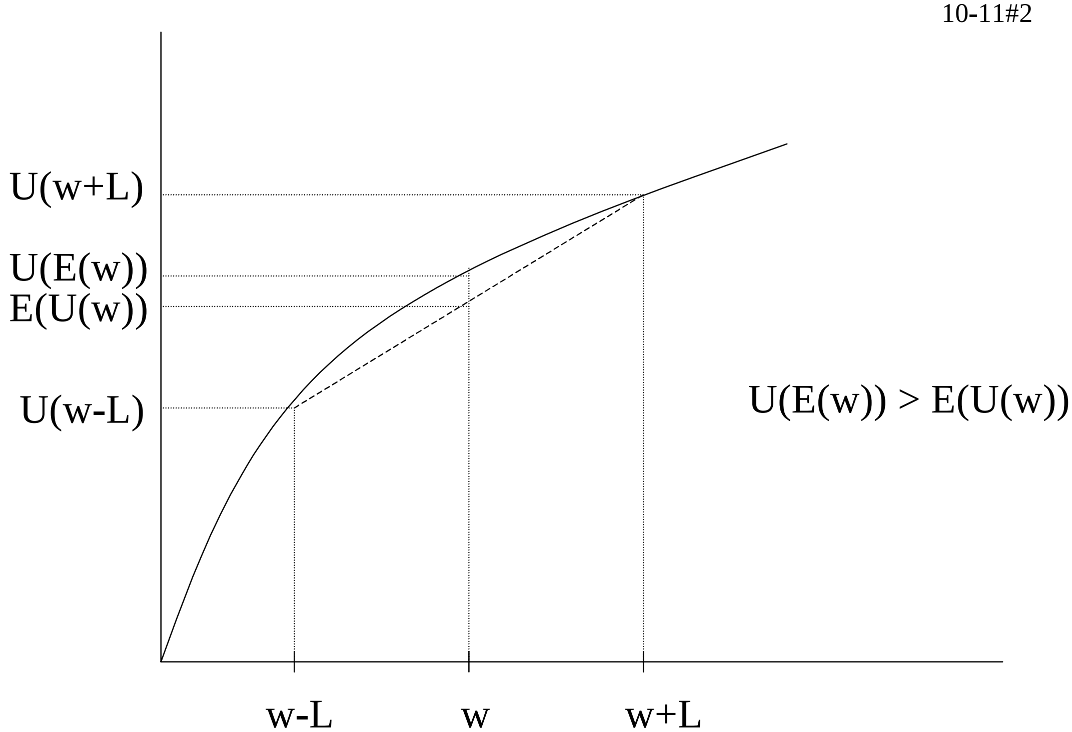


- The reason this is so:
  Wealth has diminishing marginal utility. Hence, losses cost more utility than equivalent monetary gains provide.

- A risk averse person is therefore better off with a given amount of wealth *with certainty* than the same amount of wealth *in expectation* but with variance around this quantity.

3.1 Optional: Measuring risk aversion

- Define
$$r(w) = -\frac{-u''(w)}{u'(w)} > 0.$$
- $r(w)$ is the coefficient of absolute risk aversion (ARA). The greater the curvature of $U()$, the more risk aversion is the agent.
- A person with constant ARA, i.e., $r(w) = k$, cares about absolute losses, e.g., they will always pay \$100 to avoid a \$1,000 fair bet, regardless of their level of wealth.
- Q: Should wealthy be more risk averse or less risk averse over a given (\$1,000) gamble? Most people would say *less.*
- If so,
$$\frac{\partial r(w)}{\partial w} < 0.$$
- This gives rise to the concept of relative risk aversion:
$$rr(w) = -w \cdot \frac{-u''(w)}{u'(w)}.$$
- If $rr(w) = k$, a person will pay a constant *share* of wealth to avoid a gamble over a given *proportion* of their income. Hence, as wealth rises, they will pay less and less to avoid a gamble of a given size.
- You can see this because if $rr(w) = k$, then $\frac{\partial r(w)}{\partial w} < 0$, which implies that the willingness to pay for a given absolute gamble is declining in wealth. In other words, absolute risk aversion is declining in wealth if relative risk aversion is constant.

3.2 Application: Risk aversion and insurance

- Consider insurance that is *actuarially fair,* meaning that the premium is equal to expected claims: Premium $= p \cdot A$ where p is the expected probability of a claim, and A is the amount of the claim in event of an accident.

- How much insurance will a risk averse person buy?

- Consider the initial endowment at wealth w_0, where L is the amount of the Loss from an accident:

$$\begin{aligned} \Pr(1-p) &: U = U(w_o), \\ \Pr(p) &: U = U(w_o - L) \end{aligned}$$

- If insured, the endowment is (incorporating the premium pA, the claim paid A if a claim is made,and the loss L):

$$\begin{aligned} \Pr(1-p) &: U = U(w_o - pA), \\ \Pr(p) &: U = U(w_o - pA + A - L) \end{aligned}$$

- Expected utility if uninsured is:

$$E(U|Uninsured) = (1-p)U(w_0) + pU(w_o - L).$$

- Hence, expected utility if insured is:

$$E(U|Insured) = (1-p)U(w_0 - pA) + pU(w_o - L + A - pA). \qquad (3)$$

- To solve for the optimal policy that the agent should purchase, differentiate 3 with respect to A:

$$\begin{aligned} \frac{\partial U}{\partial A} &= -p(1-p)U'(w_0 - pA) + p(1-p)U'(w_o - L + A - pA) = 0, \\ &\Rightarrow U'(w_0 - pA) = U'(w_o - L + A - pA), \\ &\Rightarrow A = L. \end{aligned}$$

- Hence, a risk averse person will optimally buy *full insurance* if the insurance is actuarially fair.

- You could also use this model to solve for *how much* a consumer would be willing to pay for a given insurance policy. Since insurance increases the consumer's welfare, s/he would be willing to pay some positive price *in excess of the actuarially fair premium* to defray risk.

- What is the intuition for this result?

- **The agent is trying to insure against changes in the marginal utility of wealth holding constant the mean wealth.**
- **Why? Because the utility of average income is greater than the average utility of income for a risk averse agent.**
- The agent therefore wants to distribute wealth evenly across states of the world, rather than concentrate wealth in one state.

- This is exactly analogous to convex indifference curves over consumption bundles.
  - Diminishing marginal rate of substitution across goods (which comes from diminishing marginal utility of consumption) causes consumer's to want to diversify across goods rather than specialize in single goods.
  - Similarly, diminishing marginal utility of wealth causes consumers to wish to diversify wealth across possible states of the world rather than concentrate it in one state.

- Q: How would answer to the insurance problem change if the consumer were *risk loving?*
- A: They would want to be at a corner solution where all risk is transferred to the least probable state of the world, again holding constant expected wealth.
- The more risk the merrier. Would buy "uninsurance."
- **OPTIONAL:**
  - For example, imagine the agent faced probability p of some event occurring that induces loss L.
  - Imagine the policy pays $A = \frac{w_0}{p}$ in the event of a loss and costs pA.

$$
\begin{aligned}
W(\text{No Loss}) &= w_0 - p\left(\frac{w_0}{p}\right) = 0, \\
W(\text{Loss}) &= w_0 - L - p\left(\frac{w_0}{p}\right) + \frac{w_0}{p} = \frac{w_0}{p} - L. \\
E(U) &= (1-p)U(0) + pU\left(\frac{w_0}{p} - L\right).
\end{aligned}
$$

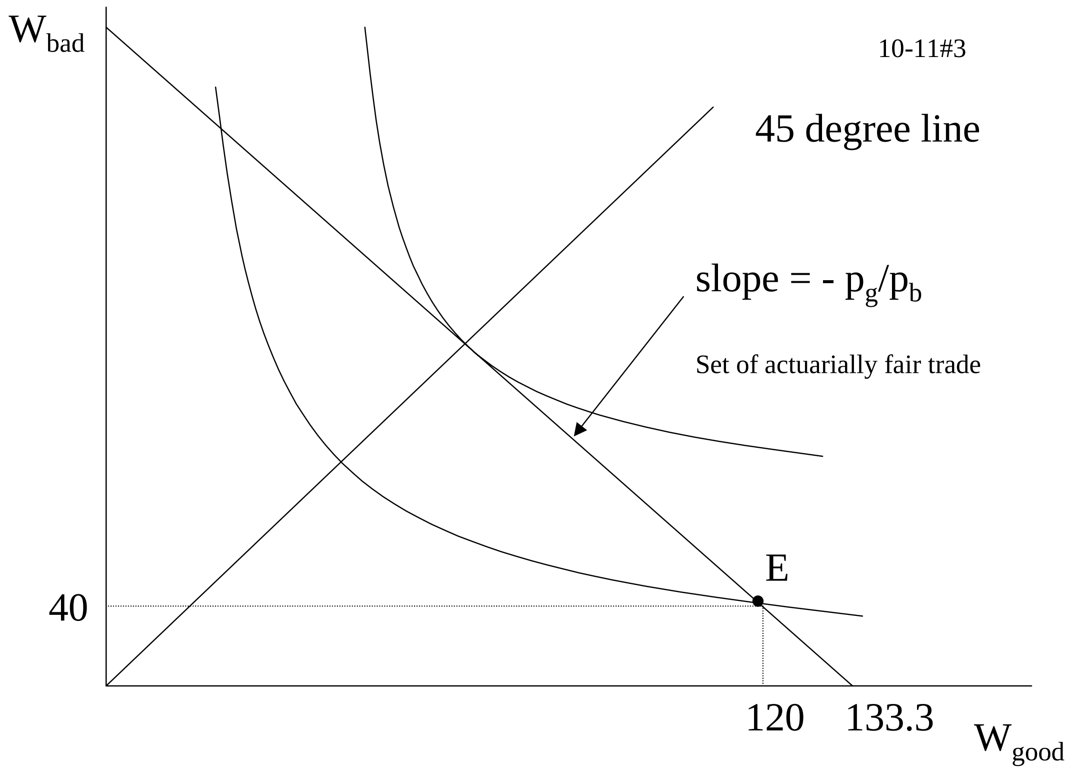
  - For a risk loving agent, putting all of their eggs into the least likely basket maximizes expected utility.

3.3 Operation of insurance: State contingent commodities

- To see how risk preference generates demand for insurance, useful to think of insurance as purchase of 'state contingent commodity,' a good that you buy but only receive if a specific state of the world arises.
- Previously, we've drawn indifference maps across goods X, Y. Now we will draw indifference maps across states of the world *good*, *bad*.
- Consumer can use their endowment (equivalent to budget set) to shift wealth across states of the world via insurance, just like budget set can be used to shift consumption across goods X, Y.
- Example: Two states of world, good and bad.

$$\begin{aligned} w_g &= 120 \\ w_b &= 40 \\ \Pr(good) &= P = 0.75 \\ \Pr(bad) &= (1-P) = 0.25 \\ E(w) &= 0.75(120) + .25(40) = 100 \\ E(u(w)) &< u(E(w)) \text{ if agent is risk averse.} \end{aligned}$$

- See FIGURE.



- Think insurance as a state contingent claim: you are buying a claim on \$1.00 that you can only make if the relevant state arises.
- This insurance is purchased before the state of the world is known.
- Let's say that this agent can buy actuarially fair insurance. What will it sell for?
- If you want \$1.00 in Good state, this will sell of \$0.75 *prior to the state being revealed.*
- If you want \$1.00 in Bad state, this will sell for \$0.25 *prior to the state being revealed.*
- So the price ratio is

$$\frac{X_g}{X_b} = \frac{P}{(1-P)} = 3.$$

- Hence, the set of fair trades among these states can be viewed as a 'budget set' and the slope of which is $-\frac{P_g}{(1-P_g)}$.
- Now we need indifference curves.

- Recall that the utility of this lottery (the endowment) is:

$$u(L) = Pu(w_g) + (1 - P)u(w_b).$$

- Along an indifference curve

$$\begin{aligned} dU &= 0 = Pu'(w_g)\partial w_g + (1 - P)u'(w_b)\partial w_b, \\ \frac{\partial w_b}{\partial w_g} &= -\frac{Pu'(w_g)}{(1 - P)u'(w_b)} < 0. \end{aligned}$$

- Provided that $u()$ concave, these indifference curves are bowed towards the origin in probability space. Can readily be proven that indifference curves are convex to origin by taking second derivatives. But intuition is straightforward.
  - Flat indifference curves would indicate risk neutrality – because for risk neutral agents, expected utility is linear in expected wealth.
  - Convex indifference curves mean that you must be compensated to bear risk.
  - i.e., if I gave you \$133.33 in good state and 0 in bad state, you are strictly worse off than getting \$100 in each state, even though your expected wealth is

    $$E(w) = 0.75 \cdot 133.33 + 0.25 \cdot 0 = 100.$$

  - So, I would need to give you more than \$133.33 in the good state to compensate for this risk.
  - Bearing risk is psychically costly – must be compensated.
- Therefore there are potential utility improvements from reducing risk.
- In the figure, $u_0 \to u_1$ is the gain from shedding risk.
- Notice from the Figure that along the 45^0 line, $w_g = w_b$.
- But if $w_g = w_b$, this implies that

$$\frac{dw_b}{dw_g} = -\frac{Pu'(w_g)}{(1 - P)u'(w_b)} = \frac{P}{(1 - P)}.$$

- Hence, the indifference curve will be tangent to the budget set at exactly the point where wealth is equated across states.
- This is a very strong restriction that is imposed by the expected utility property:
  The slope of the indifference curves in expected utility space must be tangent to the odds ratio.

4 The Market for Insurance

Now consider how the market for insurance operates. If everyone is risk averse (and it's safe to say they are), how can insurance exist at all? Who would sell it? That's what we discuss next. There are actually three distinct mechanisms by which insurance can operate: risk pooling, risk spreading and risk transfer.

4.1 Risk pooling

Risk pooling is the main mechanism underlying most *private* insurance markets. It's operation depends on the Law of Large Numbers. Relying on this mechanism, it defrays risk, which is to say that it makes it disappear.

Definition 7 *Law of large numbers: In repeated, independent trials with the same probability* p *of success in each trial, the chance that the percentage of successes differs from the probability* p *by more than a fixed positive amount* $e > 0$ *converges to zero as number of trials* n *goes to infinity for every positive* e.

- For example, for any number of tosses n of a fair coin, the expected fraction of heads H is $E(H) = \frac{0.5n}{n} = 0.5$. But the variance around this expectation (equal to $\frac{p(1-p)}{n}$) is declining in the number of tosses:

$$\begin{aligned} V(1) &= 0.25 \\ V(2) &= 0.125 \\ V(10) &= 0.025 \\ V(1,000) &= 0.00025 \end{aligned}$$

- We cannot predict the **share** of heads in one coin toss with any precision, but we can predict the **share** of heads in $10,000$ coin tosses with considerable confidence. It will be vanishingly close to 0.5.

- Therefore, by *pooling* many independent risks, insurance companies can treat uncertain outcomes as *almost known.*

- So, "risk pooling" is a mechanism for providing insurance. It *defrays* the risk across independent events by exploiting the law of large numbers – makes risk effectively disappear.

- **Example:** Each year, there is a $1/250$ chance that my house will burn down. But if it does, I lose the entire $\$250,000$ house. The expected cost of a fire in my house each year is therefore about $\$1,000$.

- Given my risk aversion, it is costly in expected utility terms for me to bear this risk (i.e., much more costly than simply reducing my wealth by $\$1,000$).

- But if $100,000$ owners of $\$250,000$ homes all kick $\$1,000$ into the pool, this pool will collect \$100 million.
- In expectation, 400 of us will have our houses burn down $\left(\frac{100,000}{250} = 400\right)$.
- The pool will therefore pay out approximately $250,000 \cdot 400 = \$100$ million and approximately break even.
- Everyone who participated in this pool was better off to be relieved of the risk.
- Obviously, there is still some variance around this 400, but the law of large numbers says this variance gets vanishingly small if the pool is large and the risks are independent.
- In particular:

$$\begin{aligned} V(FractionLost) &= \frac{P_{Loss}(1-P_{Loss})}{100,000}\frac{0.004(1-0.004)}{100000} = 3.984 \times 10^{-8} \\ SD(FractionLost) &= \sqrt{3.984 \times 10^{-8}} = 0.0002 \end{aligned}$$

- Using the fact that the binomial distribution is approximately normally distributed when n is large, this implies that:

$$\Pr[FractionLost \in 0.004 \pm 1.96 \cdot 0.0002] = 0.95$$

- So, there is a 95% chance that there will be somewhere between 361 and 439 losses, yielding a cost per policy holder in 95% of cases of \$924.50 to $\$1,075.50$.
- Most of the risk is defrayed is this pool of $100,000$ policies.
- And as $n \to \infty$, this risk is entirely vanishes.
- So, risk pooling generates a pure Pareto improvement (assuming we commit before we know whose house will burn down).

4.2 Risk spreading

- Question: when does this 'pooling' mechanism above not work? When risks are not independent:
  - Earthquake
  - Flood
  - Epidemic
- When a catastrophic even is likely to affect many people simultaneously, it's (to some extent) **non-diversifiable**.

- This is why many catastrophes such as floods, nuclear war, etc., are specifically not covered by insurance policies.
- But does this mean there is no way to insure?
- Actually, we can still 'spread' risk providing that there are some people likely to be unaffected.
- The basic idea here is that because of the concavity of the (risk averse) utility function, taking a little bit of money away from everyone incurs lower social costs than taking a lot of money from a few people.
- Many risks cannot be covered by insurance companies, but the government can intercede by transferring money among parties. Many examples:
  - Victims compensation fund for World Trade Center.
  - Medicaid and other types of catastrophic health insurance.
  - All kinds of disaster relief.
- Many of these insurance 'policies' are not even written until the disaster occurs, so there was no market. But the government can still spread the risk to increase social welfare.
- Question: is this a Pareto improvement? No, because we must take from some to give to others.

4.3 Risk transfer

- Third idea: if utility cost of risk is declining in wealth (constant absolute risk aversion for example implies declining relative risk aversion), this means that *less wealthy people could pay more wealthy people to bear this risk* and both parties would be better off.
- Example: Lloyds of London used to perform this role:
  - Took on large, idiosyncratic risks: satellite launches, oil tanker transport, the Titanic.
  - These risks are not diversifiable in any meaningful sense.
  - But companies and individuals would be willing to pay a great deal to defray them.
  - Lloyds pooled the wealth of British nobility and gentry ('names') to create a super-rich agent that in aggregate was much more risk tolerant than even the largest company.
  - For over a century, this idea generated large, steady inflows of cash for the 'names' that underwrote the Lloyds' policies.
  - Then they took on asbestos liability...

4.4 Insurance markets: Conclusion

- Insurance markets are potentially an incredibly beneficial financial/economic institution that can make people better off at low or even zero (in the case of the Law of Large Numbers) aggregate cost.
- We'll discuss in detail later this semester why insurance markets do not as perfectly in practice as they might in theory (though still work in general and create enormous social valuable).