14.03 Fall 2000 · *Lecture* # 3

- Optimization
  - single variable
  - multi-variable

- Implicit Function Theorem & comparative statistics

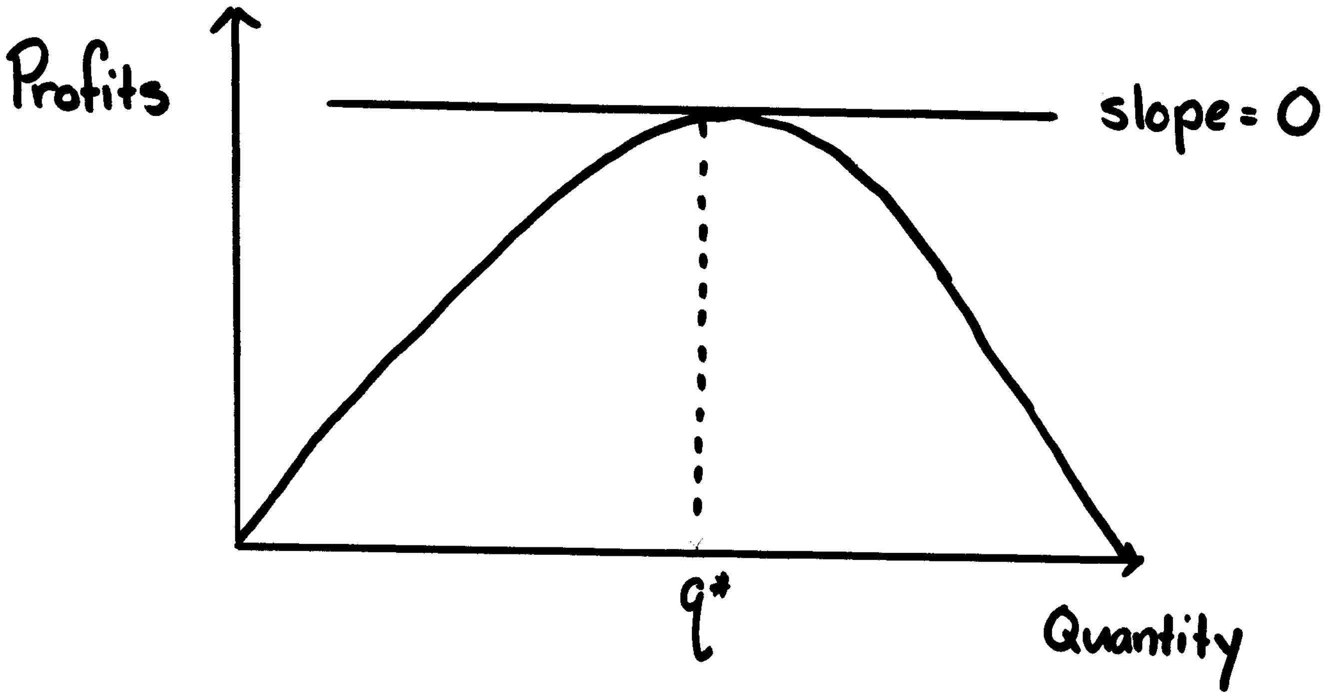
- Envelope Theorem - constrained & unconstrained

- Constrained Optimization (Langrangian Method)

- Duality

Single Variable Optimization

$\pi(q)$ is profit function
Choose q^* to maximize $\pi(q)$

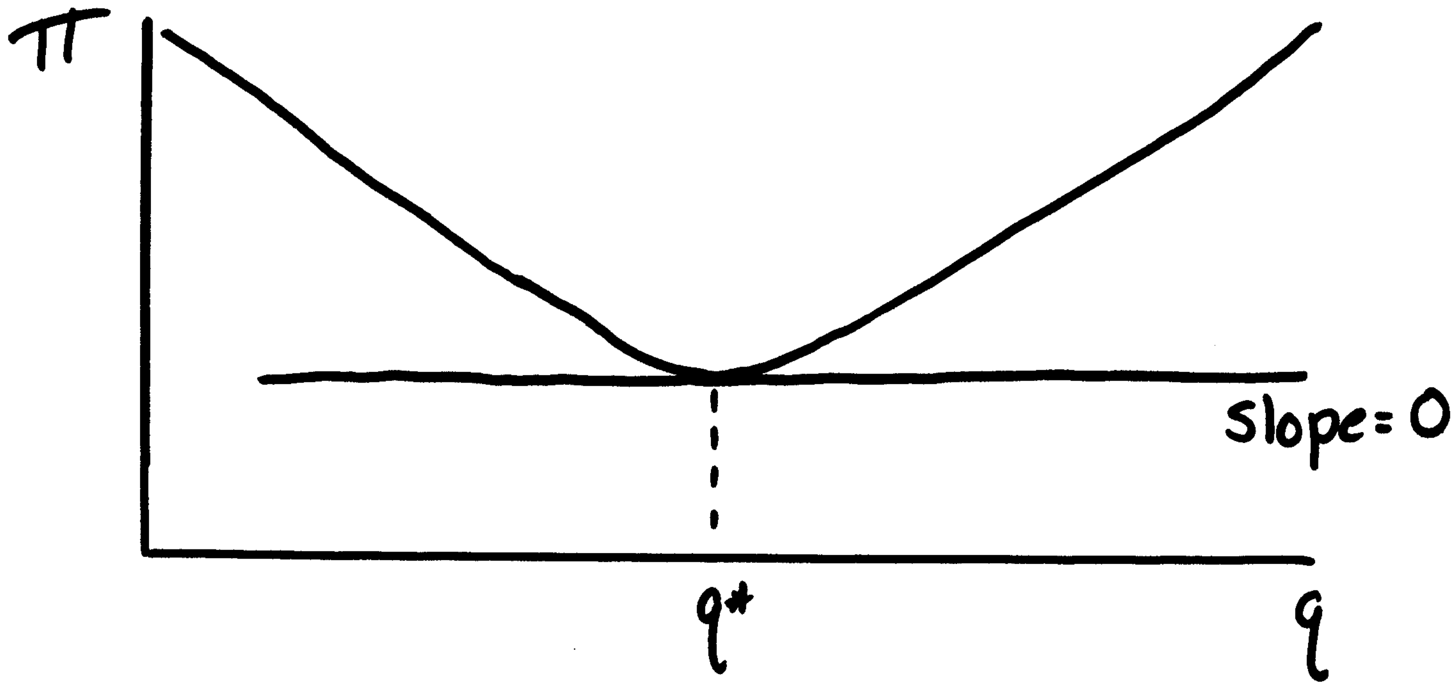


$$\frac{d\pi}{dq}\bigg|_{q^*} = 0$$ "First order condition" (FOC)

Q: Is q^* necessarily the profit max?

A: No, q^* is <u>necessary</u> but not <u>sufficient</u> for profit max.

Single Variable Optimization

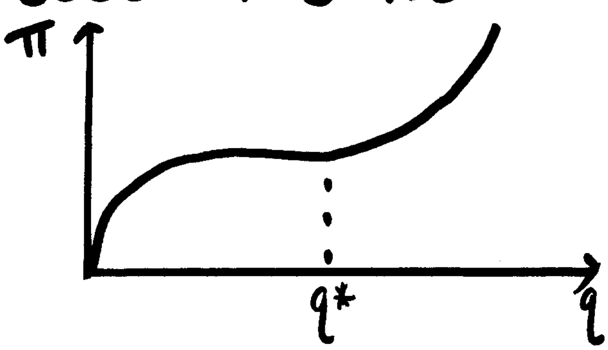


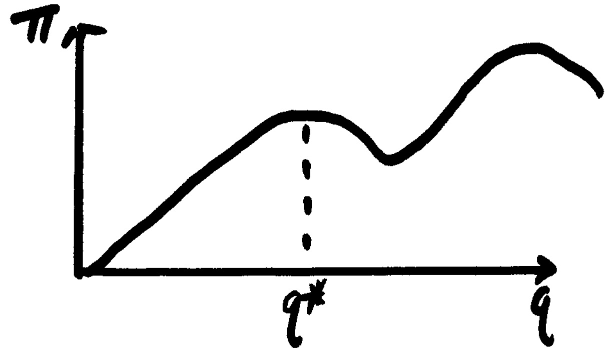
$$\frac{d\pi}{dq}\Big|_{q^*} = 0$$ but q^* is a profit minimum

Second order condition:

$$\frac{d^2\pi}{dq^2}\Big|_{q^*} < 0,$$ guarantees that q^* is a (local) maximum

Does not solve:





Optimization

We'll generally work with "well-behaved" functions: continuous, differentiable, concave. Hence, we wont focus on Second Order Conditions (read about them in Chapter 2, however).

Multi-Variate Case

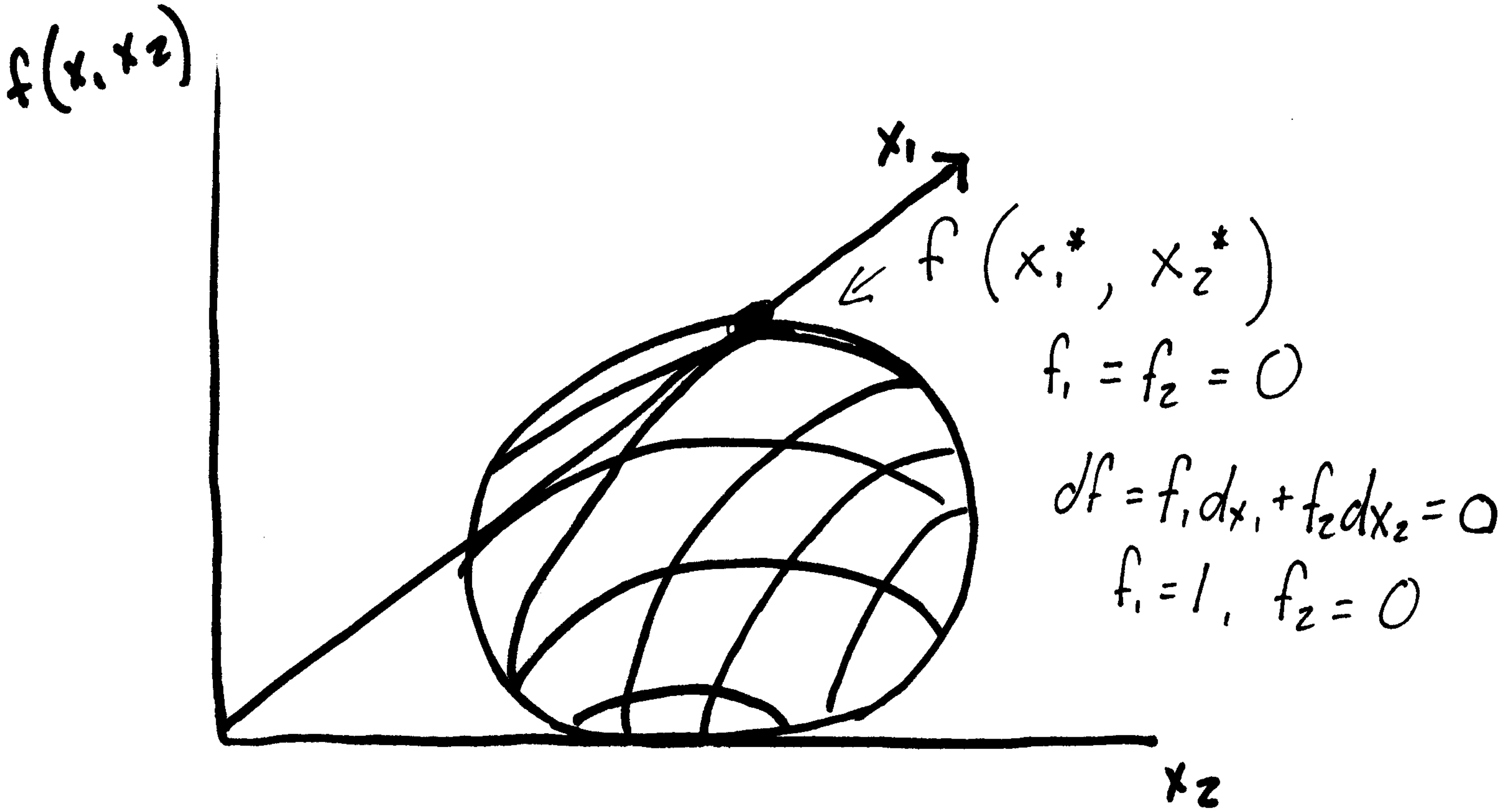
$$y = f(x_1, x_2, \ldots\ldots, x_n)$$

$$\frac{\partial f(\cdot)}{\partial x_1} \equiv f_1 \qquad \frac{\partial f(\cdot)}{\partial x_2} \equiv f_2 \qquad \ldots\ldots$$

$$\frac{\partial f(\cdot)}{\partial x_n} = f_n$$

First Order Condition (FOC) for maximum (or minimum):

$$f_1 = f_2 = \ldots\ldots f_n = 0$$



Optimization of a function with 2 variables

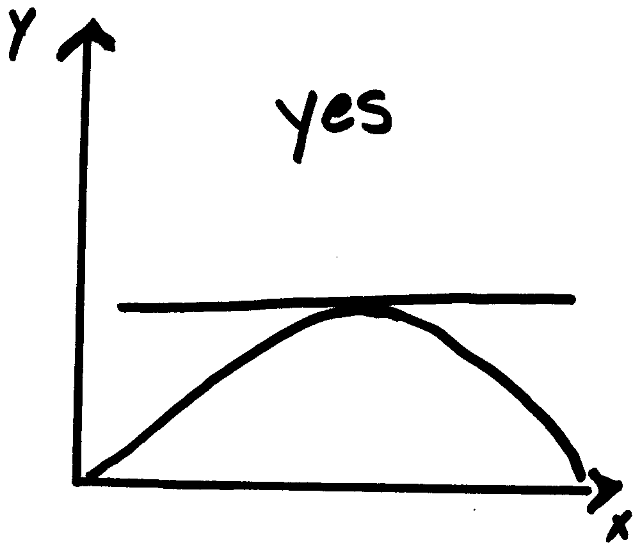
Any <u>concave</u> function will be maximized at its "flat spot" or "saddle point".

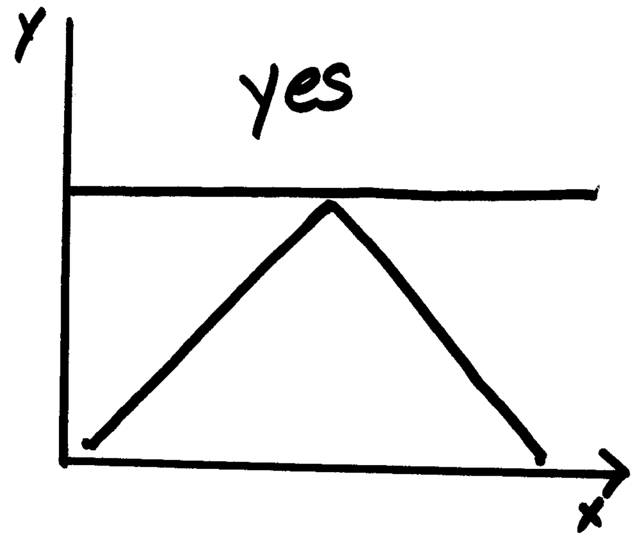
$(f_1 = f_2 = \ldots . = f_n = 0)$

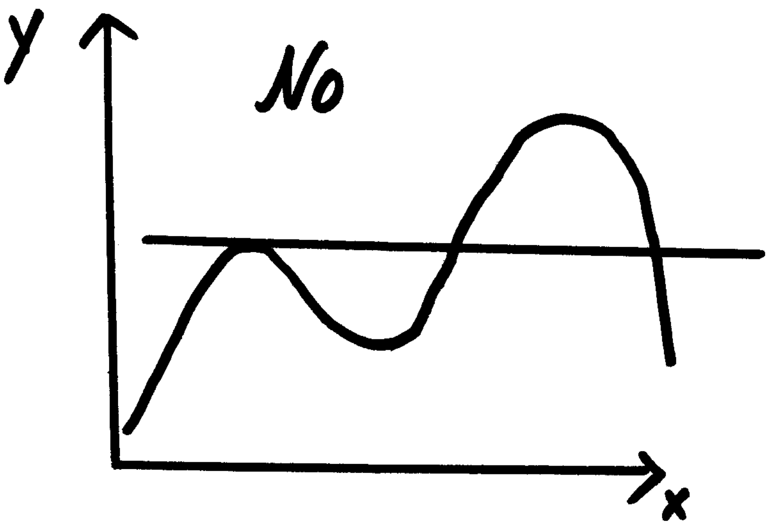
A concave function is a function that always lies below any equ-dimensional surface that is tangent to it: (1 dimension → line, 2 dimensions → plane, 3 dimensions → hyper-plane)

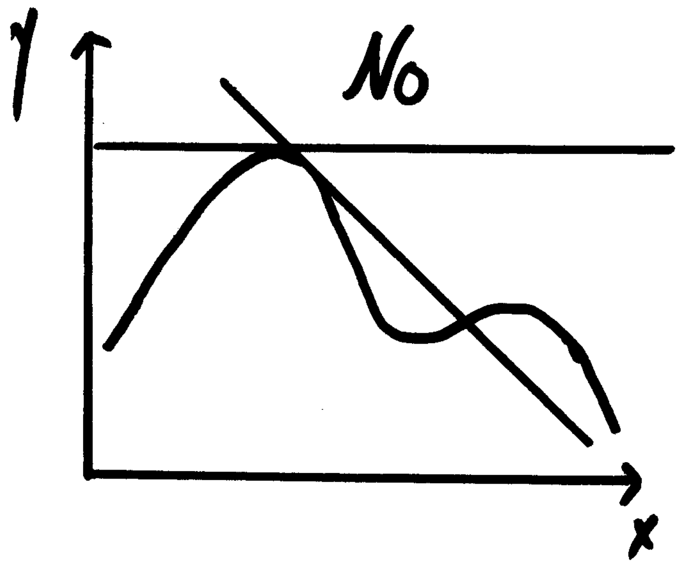
S.O.C. $\Rightarrow d^2 f = f_{11}\,(dx_1)^2 + 2 f_{12}\,dx_1\,dx_2 + f_{22}\,(dx_2)^2 < 0$

Concave Functions









Two Dimensional

- Inverted Teacup? yes
- Non-Inverted Teacup? No

Implicit Functions

endogenous exogenous

1) $y = mx + b$ Explicit
2) $y - mx - b = 0$ Implicit
3) $f(y, x; m, b) = 0$ Implicit

2 & 3 are called implicit because the relationship between the variables is implicitly present rather than explicitly shown as $y = f(x)$.

Many times in economics, we end up with implicit functions where exogenous and endogenous variables are all mixed together.

We may have no closed form expression for $y(x)$, but the derivative $\frac{dy}{dx}$ may still exist, and this is often exactly what we need.

Easy to Work With Implicit Functions

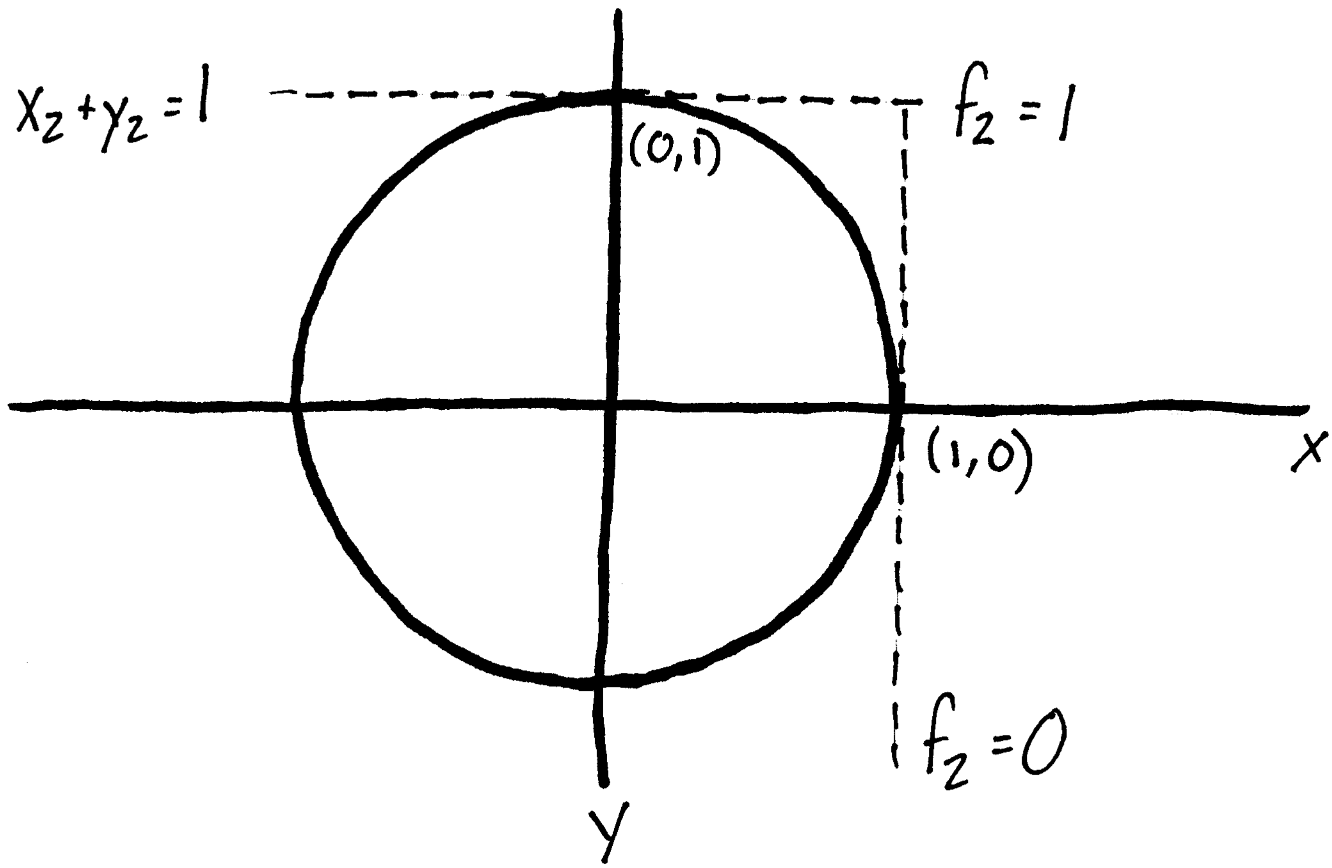
$$f(x,y)=0 \Rightarrow f(x,y(x))=0$$

$$f_x dx + f_y dy = 0$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

Caveat: $\frac{dy}{dx}$ may not exist ...

Implicit Functions



When can we write: $f(x^*, y(x^*)) = 0$?

Take implicit function:

$x^2 + y^2 - 1 = 0 \qquad 2x\,dx + 2y\,dy = 0$

$$\left.\frac{dy}{dx} = -\frac{f_1}{f_2} = -\frac{x}{y} = 0\right\} \text{ Undefined when } f_2 = y = 0$$

Q: What is the intuition for the non-existence of $f(x^*, y(x^*)) = 0$ at $(x, y) = (1, 0)$?

A: $\frac{dy}{dx}$ could be ⊕ or ⊖ here. Undefined.

Implicit Function Example

Long Way:

$$2x^2 + y^2 = 225 \quad , \quad \text{what's } \frac{dy}{dx} ?$$

$$y = \sqrt{225 - 2x^2}$$

$$\frac{dy}{dx} = \frac{1}{2}(225 - 2x^2)^{-1/2}(-4x) = \frac{-4x}{2\sqrt{225 - 2x^2}}$$

$$= -\frac{4x}{2y} = -\frac{2x}{y}$$

Implicit Function Method:

① Write: $2x^2 + y^2 - 255 = 0$

② Totally differentiate: $4x\,dx + 2y\,dy = 0$

③ Rearrange: $\frac{dy}{dx} = \frac{4x}{2y} = -\frac{2x}{y}$

You can see how this works formally:

Suppose there is a continuous solution to $y = y(x)$ for the equation $F(x, y) = C \Rightarrow F(x, y(x)) = C$

We want to know $\frac{dy}{\partial x}$ for sum $x_0, y(x_0)$

Use the chain rule to differentiate.

$$\frac{\partial F}{\partial x}(x_0, y(x_0)) = \frac{\partial F}{\partial x}(x_0, y(x_0))\frac{dx}{dx} + \frac{\partial F}{\partial y}(x_0, y(x_0)) \bullet \frac{dy}{dx}(x_0) = C$$

Simplify: $= \frac{\partial F}{\partial x}(x_0, y(x_0)) + \frac{\partial F}{\partial y}(x_0, y(x_0)) \bullet y'(x_0) = 0$

$$y'(x_0) = \frac{-\frac{\partial F}{\partial x}(x_0, y(x_0))}{\frac{\partial F}{\partial y}(x_0, y(x_0))}$$

Necessary condition for $y'(x_0)$ to exist is that $\frac{\partial F}{\partial y}(x_0, y(x_0)) \neq 0 \Rightarrow$ *Turns out this is sufficient also.*

[multivariate: $\frac{\partial F}{\partial y}(x_1^*, \ldots x_n^*, y^*) \neq 0$]

Take a more complicated example:

$$x^2 - 3xy + y^3 - 7 = 0$$

What is $\frac{dy}{dx} \Big| \begin{matrix} x=4 \\ y=3 \end{matrix}$ $\Big\rangle$ note: A Valid Solution

Use implicit function theorem:

① Totally differentiate

$$2x\,dx - 3y\,dx + 3x\,dy + 3y^2\,dy = 0$$

$$3y^2dy - 3x\,dy = 2x\,dx + 3y\,dx$$

$$\frac{dy}{dx} = -\frac{2x + 3y}{3y^2 - 3x}$$

② Plug In (Substitute)

$$\frac{dy}{dx}(4,3) = \frac{-8+9}{27-12} = \frac{1}{15}$$

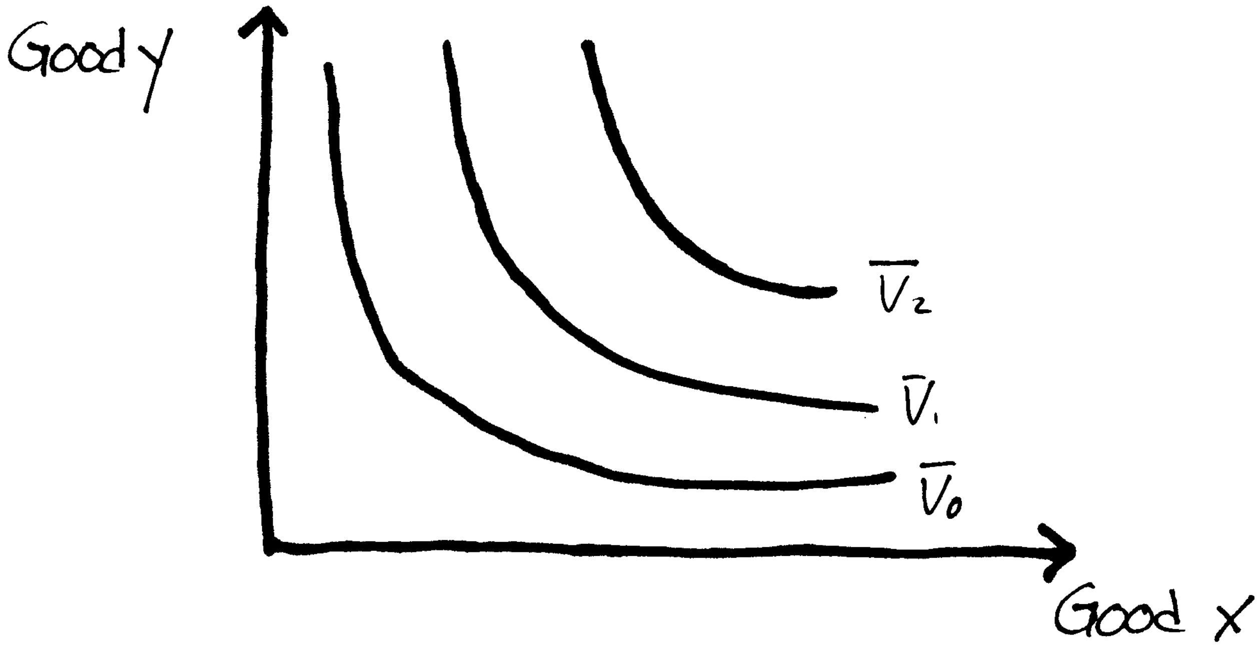
Q: What is y_1 $(x = 4.3)$? $\approx y(4) + \frac{dy}{dx}\Big|_{x=4}(.3)$

③ Take $\Delta x = .3$

$$y_1 \approx 3 + .3(1/15) = 3.02$$

To solve for y at $x = 4.3$ could only be computed numerically and equals 3.01475

Applications of Implicit Functions



- Along an indifference curve, we have $V(x, y) = \overline{V}$

- Implicit function $V(x^*, y^*(x^*)) = \overline{V}$ tells how much y we'd give up for a little more x (at the margin) while holding total utility constant.

$$V(x^*, y^*(x))^* = \overline{V} \Rightarrow \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

$$\frac{\partial y}{\partial x} = \frac{V'(x)}{V'(y)}$$

Envelope Theorems

A shortcut for taking derivatives of optimized functions with respect to their parameters.

Definition (unconstrained case). Let $f(x, a)$ be a C^1 function of $x \in \mathbb{R}^n$ and the scalar a. For each a, consider the unconstrained maximization:

$$\max f(x, a) \underbrace{\text{w.r.t.}}_{\text{"with respect to"}} x$$

Let $x^*(a)$ be a solution of this problem. Suppose that $x^*(a)$ is a C^1 function of a. Then,

$$\underbrace{\frac{d}{da} f(x^*(a), a)}_{\text{total derivative}} = \underbrace{\frac{\partial}{\partial a} f(x^*(a), a)}_{\text{partial derivative}}$$

Proof of the Envelope Theorem

$$\frac{d}{da} f(x^*(a), a) = \underbrace{\sum \frac{\partial f}{\partial x_i}(x^*(a), a) \cdot \frac{d x_i^*(a)}{da}}_{=0}$$

$$+ \frac{\partial f}{\partial a}(x^*(a), a) = \frac{\partial f}{\partial a}(x^*(a), a)$$

b/c $\frac{\partial f}{\partial x_i}(x^*(a), a) = 0 \quad \forall i$

These are the first order conditions of the maximization problem to obtain x^* [that maximizes $f(x,a)$]

(much more intuitive - <u>and</u> <u>useful</u> - than it looks.)

Envelope Theorem Example

Long Route:

① $y = -x^2 + ax$

Want to know:

$\dfrac{dy^*}{da}$ where y^* is maximized value of ①

Find x^* through single variable optimization

$$\frac{dy}{dx} = -2x + a = 0, \quad x^* = \frac{a}{2} \quad \Rightarrow$$

$$y^* = -\left(\frac{a}{2}\right)^2 + a\left(\frac{a}{2}\right) = \frac{a^2}{4}$$

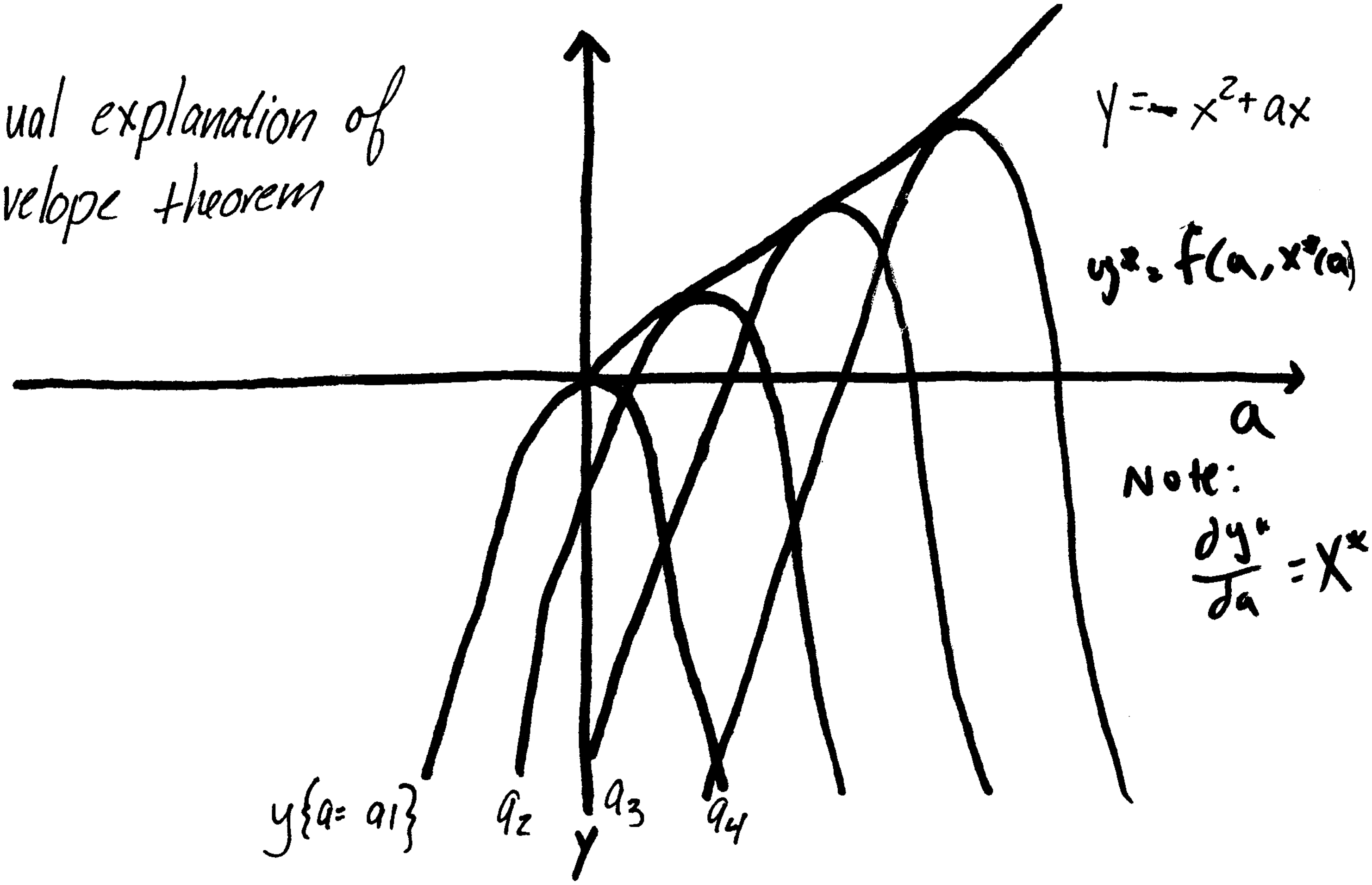
$$\frac{dy^*}{da} = \frac{a}{2} = x^*$$

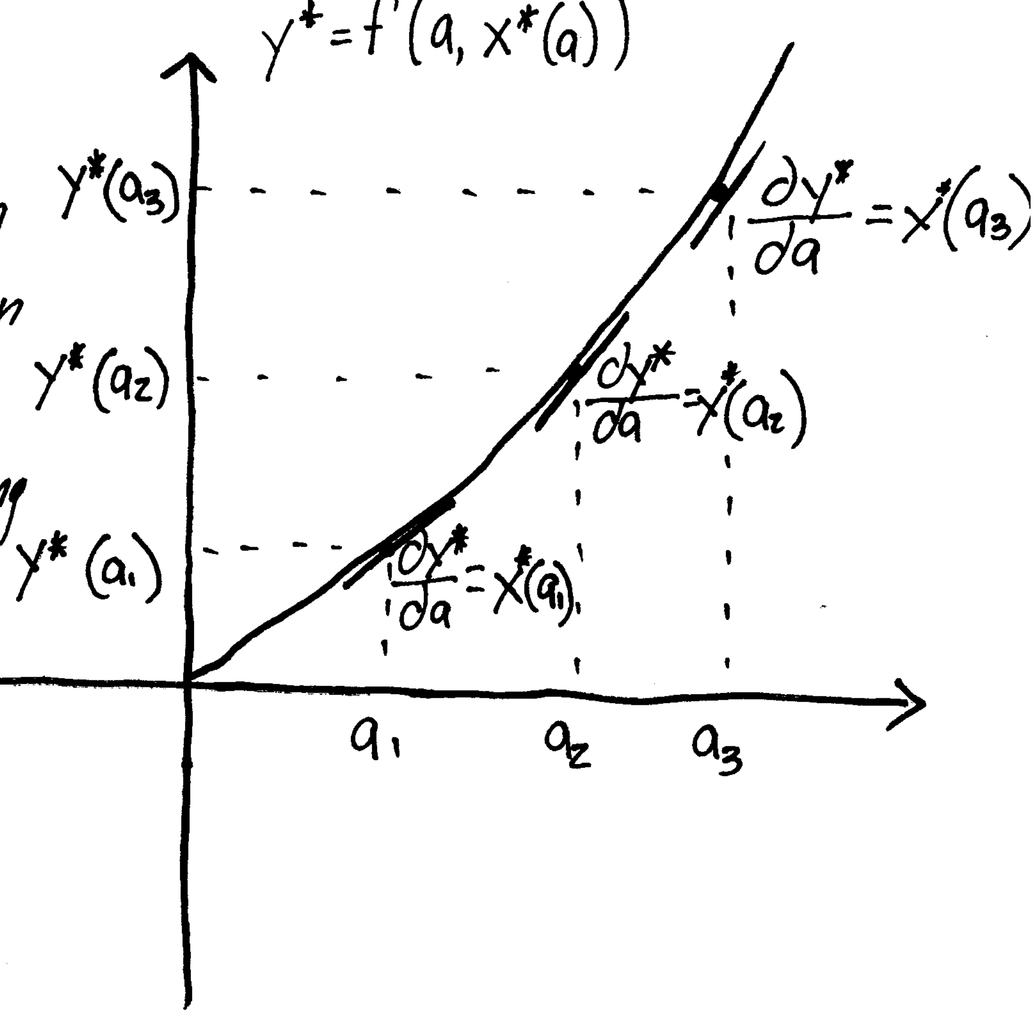
Envelope shortcut

Rewrite ①: $y^* = -(x^*)^2 + ax^*$

$$\frac{\partial y}{\partial a} = x^* = \frac{dy^*}{da} \quad \{x^* = x^*(a)\}$$

Visual explanation of envelope theorem





Note: Envelope Theorem is a linear approximation & hence only holds in an "envelope" surrounding $x^*(a)$.

Envelope Theorem is Multi-Variate

$$y^* = f\left[x_1^*(a),\ x_2^*(a),\ \ldots,\ x_n^*(a);\ a\right]$$

$$\frac{dy^*}{da} = \underbrace{\frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial a} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial a} + \frac{\partial f}{\partial a}}_{=0}$$

$$\Rightarrow \quad \frac{dy^*}{da} = \frac{\partial f}{\partial a}$$

Constrained Maximization

Most maximization problems in economics are subject to constraints:

- Maximize utility subject to budget constraint.

- Maximize social welfare subject to a resource constraint.

- Maximize profits subject to a technological constraint (e.g. can only produce so many lattes in one hour).

Tool for maximizing constrained functions:
Lagrangian Method.
A "trick" which turns out to have very useful economic content.

Lagrangian Method

Problem:

$$\max \quad y = f(x_1, x_2, \ldots x_n)$$

$$\text{s.t.} \; (\text{subject to}) \quad g(x_1, x_2, \ldots x_n) = 0$$

- Any function can be written in this *implicit* notation

$$x_1 + x_2 = 10 \iff x_1 + x_2 - 10 = 0$$

script 'L'

Setup:

$$\mathcal{L} = f(x_1, x_2, \ldots x_n) + \lambda g(x_1, x_2, \ldots x_n)$$

F.O.C.'s (First order conditions):

$$\frac{\partial \mathcal{L}}{\partial x_1} = f_1 + \lambda g_1 = 0$$

$$\vdots$$

$$\frac{\partial \mathcal{L}}{\partial x_n} = f_n + \lambda g_n = 0$$

$$* \; \frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1, x_2 \ldots x_n) = 0$$

$\Rightarrow$ Get as many equations as unknowns.
Solve simultaneously for $x_1^* \ldots x_n^*$ and λ.

Constrained Max Example

Optimal fence dimensions.

- Have fencing of perimeter length P
- Maximize fenced area

Objective:

$$\max x \cdot y$$

Constraint:

$$2x + 2y = P \quad \Rightarrow \quad P - 2x - 2y = 0$$

Lagrangian:

$$\mathcal{L} = x \cdot y + \lambda (P - 2x - 2y)$$

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = P - 2x - 2y = 0$$

$$\Rightarrow \; y/2 = x/2 = \lambda, \quad x = y = \frac{P}{4}, \quad \lambda = P/8$$

Optimal Fence (cont.)

$x = y = P/4$, $\lambda = P/8$

- Optimal fence is square $(x=y)$
- (Q: what else do we assume about shape?)
- What is the interpretation of $\lambda = P/8$?

Observe that: $\dfrac{f_1}{-g_1} = \dfrac{f_2}{-g_2} = \lambda$

Where f_1 is the marginal gain to $\mathcal{L}$ (Langrangian) from 1 more X, i.e. more area
and g_1 is the marginal cost of 1 more X in terms of tightening the constraint & hence reducing feasible y.
This ratio, λ, is called the "shadow price" of the constraint,

Interpretation of λ

$\lambda = P/8$ implies that relaxing the constraint that $x + y = P$ by 1 unit would allow us to increase the maximand (area) by $P/8$.

Check this:

$$\text{Let } P = 40 \Rightarrow x = y = 10, \quad A = 100$$

Compare to:

$$\text{Let } P = 41 \Rightarrow x = y = 10.25, \quad A = 105.06$$

$$\Delta A = 5.06 \approx 40/8$$

$\Rightarrow$ Multiplier λ is quite close to actual change in A for 1 unit change in constraint (and is exactly correct over an epsilon interval from $(x^*, y^*, P = 40)$)

- This multiplier is called the "shadow price" of the constraint.

Another Constrained Max example

max $Z = X^{1/2} Y^{1/2}$ s.t. $X + Y = 4$

$Z = X^{1/2} \cdot Y^{1/2} + \lambda(4 - X - Y)$

$$\frac{\partial Z}{\partial X} = \tfrac{1}{2} X^{-1/2} Y^{1/2} - \lambda = 0$$

$$\frac{\partial Z}{\partial Y} = \tfrac{1}{2} X^{1/2} Y^{-1/2} - \lambda = 0$$

$$\frac{\partial Z}{\partial \lambda} = 4 - X - Y = 0$$

$$\frac{1}{2} X^{-1/2} Y^{1/2} = \frac{1}{2} X^{1/2} Y^{-1/2} = \lambda \Rightarrow X = Y = 2$$

and $\lambda = 1/2$ (after some algebra)

Check multiplier's implication:

$f(x, y, 4) = 2^{1/2} \cdot 2^{1/2} = 2$

$f(x, y, 5) = 2.5^{1/2} \cdot 2.5^{1/2} = 2.5$

$\Delta Z = 1/2$ <u>exactly</u> equal to λ

Envelope Theorem for Constrained Problems

Let $x^*(a) = (x_1^*(a), \ldots x_n^*(a))$ denote a solution to max $f(x,a)$ s.t. $\underbrace{g_1(x,a)=0 \ldots g_k(x,a)=0}_{\text{constraints - could be many}}$

Let $\lambda_1 a, \ldots \lambda_n(a)$ be the Lagrange multipliers of this problem. Then:

$$\underbrace{\frac{d}{da} f(x^*(a), a)}_{\substack{\text{Total derivative of} \\ \text{original function } f(\cdot)}} = \underbrace{\frac{\partial \mathcal{Z}}{\partial a}(x^*(a), \lambda(a), a)}_{\substack{\text{Partial derivative} \\ \text{of Lagrangian}}}$$

Why is this true?

- Because at $x^*(a)$, constrained function is already maximized w.r.t. each x_i so:

$$\sum_i \frac{\partial f}{\partial x_i}(x^*(a), a) \cdot \frac{\partial x_1^*}{\partial a} = 0$$

- The only non-zero partial derivative is $\frac{\partial \mathcal{Z}}{\partial a} = \lambda(a)$.

Envelope Theorem for Constrained Problems

This result is much more obvious than it looks....

Consider our previous problem:

$$\max \quad x^{1/2} y^{1/2} \quad \text{s.t.} \quad x + y = 4$$

$$Z = x^{1/2} + y^{1/2} + \lambda(4 - x - y)$$

$$\frac{\partial Z}{\partial x} = \tfrac{1}{2} x^{-1/2} y^{1/2} - \lambda = 0$$

$$\frac{\partial Z}{\partial y} = \tfrac{1}{2} x^{1/2} y^{-1/2} - \lambda = 0$$

$$\frac{\partial Z}{\partial \lambda} = 4 - x - y = 0$$

We found: $x^* = y^* = 2, \ \lambda = \tfrac{1}{2}$

What is:

1) $\dfrac{\partial f(x^*(\overset{4}{a}), y^*(\overset{4}{a}), \overset{4}{a})}{\partial x^*}$? (0)

2) $\dfrac{\partial f(x^*(a), y^*(a), a)}{\partial y^*}$? (0)

3) $\dfrac{\partial f(x^*(a), y^*(a), a)}{\partial a}$? (λ)

Duality

Every "primal" <u>maximization</u> problem subject to a constraint has a corresponding "dual" problem that <u>minimizes</u> the constraint function subject to the original objective function being equal to its optimal value in the original problem.

Primal: $\max\ z = f(x,y)$ s.t. $x + y = \bar{K}$

yields $z^* = f(x^*, y^*)$

Dual: $\min\ K = x + y$ s.t. $f(x,y) = Z^*$

will also obtain:

$$x^*_D = x^*_P \qquad Z^*_D = Z^*_P$$

D = Dual

$$y^*_D = y^*_P$$

P = Partial

$$x^*_D + y^*_D = \bar{K}$$

Duality Example

Primal Problem (familiar):

$$\max\ x^{1/2} y^{1/2} \quad \text{s.t. } x + y = 4$$

$$z = f(x,y) = x^{1/2} y^{1/2}$$

$$g(x,y) = 4 - x - y$$

$$\mathcal{L} = x^{1/2} y^{1/2} + \lambda(4 - x - y)$$

$$x^* = 2 \qquad z^* = 2$$

$$y^* = 2 \qquad \lambda = 1/2$$

Dual problem:

$$\underline{\min}\ x + y \quad \text{s.t. } x^{1/2} y^{1/2} = 2$$

z^* from primal

$$f(x,y) = x + y$$

$$g(x,y) = 2 - x^{1/2} y^{1/2}$$

$$\mathcal{L} = x + y + \lambda_D (2 - x^{1/2} y^{1/2})$$

Yields: $x^*_D = 2$, $y^*_D = 2$, $f(x+y) = 4$

Q: Can you guess the value of λ_D?

Duality Example

Recall that:

$$\lambda p = \frac{f_1}{-g_1} = \frac{f_n}{-g_n} \qquad \text{eg.} = \frac{df/dx}{-dg/dx}$$

And for dual:

we substituted $f_D = g_P$

$g_D = f_P$

Hence:

$$\lambda_D = \frac{dg/dx}{-df/dx} = \frac{1}{\lambda p}$$

Why should we care about duality?

- Cost minimization is dual of profit maximization
- Expenditure minimization is dual of utility maximization

We'll be relying on these duality relationships all semester....