Lecture 21 - Game Theory, Strategic Equilibrium, and Repeated Interactions

14.03 Spring 2003

1 Game Theory, Strategic Equilibrium and Repeated Interactions

• The study of the economics of information gives rise to several important modeling constructs that were not modeled in consumer choice:
  1. Asymmetric information
  2. Conjectures about the behaviors of others
  3. Interdependence of your actions on the conjectured choices of others
     – leads to strategic behavior.

• These attributes required us to specify the features of the environment more carefully:
  – Information/uncertainty
  – Conjectured behavior
  – Interdependence

• Naturally gives rise to game theory: a tool for analyzing strategic interactions among individuals (people, firms) in an economic setting.

• Like all previous models, we have rational actors maximizing their well being in a well specified environment.

• What is new here is small \( N \), that is a small number of actors such that the best choices for any one actor depends intimately on the choice of a small number of other actors. This is quite different from a standard market setting where everyone is a “price taker.” In this sense, the situations described by game theory resemble a multi-player game with strategies, payoffs and concealed information.

• An economic game has three elements:
1. Players
2. Strategies
3. Payoffs (utilities, states)

- There are both cooperative and non-cooperative games. In cooperative games, players can make binding commitments. In non-cooperative games (which are much more interesting), players cannot make binding commitments. These means that the only actions they can credibly ‘commit’ to are self-interested actions.

- Players
  - 2 persons/firms
  - \(N\) persons/firms
  - Can also be against nature. In that case, nature moves stochastically.

- Strategies
  - Each course of action that is open to a player is a strategy.
  - Usually this is a finite set; often limited to 2.
  - In a non-cooperative game, players cannot make binding commitments about which strategies they will play.

- Payoffs
  - Final returns to players at conclusion of game.
  - Usually measured in utility or profits.
  - Utilities not necessarily assumed to be the same across players.
  - Payoffs assumed to include all aspects of the pleasure/pain of final outcome. If for example you take extra glee in demolishing your opponent, this will be incorporated in final payoff and hence is not an additional consideration beyond that payoff.

- Two ways to write down a game:
  - Extensive form: Tree with nodes
  - Normal form: Payoff matrices

- Depending on the game, one or the other notation will typically be more useful.
1.1 Example: Dormitory game

- Player A prefers to play music quietly, but dislikes hearing loud music from the adjoining room.
- Player B is a head-banger. The more noise the better.
- This is a simultaneous move game.
- See Figure 21#1

Dormitory game:
Extensive form

Normal form
• Solution concept?
  – Does A have a dominant strategy? No. Prefers S if B chooses S, otherwise prefers L.
  – Does B have a dominant strategy? Yes. Prefers L regardless of A’s choice.
  – If A and B are both rational, then A should correctly anticipate that B will choose L.
  – Accordingly, A should choose L.
  – Equilibrium is L, L.
  – This is a ‘pure strategy’ equilibrium in that players just choose one action, do not randomize to ‘mix’ strategies.

• How do we know this is an equilibrium?
  – Player B has no incentive to deviate given A’s choices.
  – Player A has no incentive to deviate given B’s choices.
  – Hence, these strategies are complementary, or self-enforcing.

• This brings us to first core concept in game theory: Nash Equilibrium.

1.2 Nash Equilibrium

Definition 1 A pair of strategies (a∗, b∗) (for a 2-player game) is said to be a Nash Equilibrium if:
  a∗ is player A’s best strategy when B is playing b∗ and b∗ is player B’s best strategy when A is playing a∗.

• Truthful revelation: At a Nash equilibrium, one player would not benefit from knowing the strategy of the other in advance since A would never have incentive to play anything other than a∗ if B is playing b∗.

• (This is actually quite similar to the equilibrium concept in the signaling and adverse selection models we just completed. Expectations of buyers/sellers or workers/employers in those models must be complementary. In equilibrium, everyone’s expectation of quality and price, or productivity and wages are mutually consistent.)

• This concept seems very obvious. But it was not formalized until Nash wrote it down and proved that every game has at least one Nash equilibrium.

• Not all games have a pure strategy equilibrium, however.
1.3 Example: Rock, paper, scissors

- Recall the childhood game of rock paper scissors. This is a game where you hold out one of three symbols corresponding to rock, paper and scissors. The rules are:
  - Paper covers rock
  - Rock breaks scissors
  - Scissors cuts paper
  - (Any 2 identical objects are a tie; begin again)

- So, the payoff matrix looks like

\[
\begin{array}{ccc}
B & \text{Rock} & \text{Paper} & \text{Scissors} \\
\text{Rock} & 0,0 & -1,1 & 1,-1 \\
\text{Paper} & 1,-1 & 0,0 & -1,1 \\
\text{Scissors} & -1,1 & 1,-1 & 0,0 \\
\end{array}
\]

- There is no pure strategy equilibrium to this game. If player A announced a pure strategy (e.g., play rock), player B would choose a pure strategy that would defeat A, and then A would have incentive to deviate from the original strategy.

- But there is an obvious mixed strategy—that most children understand intuitively.

1.4 Another mixed strategy example

- The “Family Vacation” game. Family goes on vacation together. Parents prefer Paris, kids prefer Disney. But neither prefers to go without the other (that’s why it’s a family vacation).

- Payoffs:

\[
\begin{array}{ccc}
\text{Parents} & \text{Disney} & \text{Paris} \\
\text{Disney} & 1,2 & 0,0 \\
\text{Paris} & 0,0 & 2,1 \\
\end{array}
\]

- Clearly, this game has two pure strategy Nash equilibria: \(D, D\) and \(P, P\). If kids were certain that parents were playing \(P\), then they would play \(P\) too, and vice versa if parents were certain kids were playing \(D\).

- This game also has a mixed strategy equilibrium. What is it?

- Say parents play Paris with probability \(P\), play Disney with probability \(1-P\).

- Say kids play Paris with probability \(K\), play Disney with probability \(1-K\).
• What is the expected utility of the parents?

\[
E[U_p] = 2 \cdot PK + 0 \cdot (1 - P)K + 0 \cdot K(1 - P) + 1 \cdot (1 - P)(1 - K)
\]

\[
= 3KP - P - K + 1
\]

• Notice that \(E[U_p]\) depends on \(K\).
  - If \(K = \frac{1}{3}\), then \(E[U_p] = \frac{2}{3}\), which is independent of \(P\).
  - If \(K > \frac{1}{3}\), then \(E[U_p]\) is maximized by setting \(P = 1\), always going to Paris.
  - If \(K < \frac{1}{3}\), then \(E[U_p]\) is maximized by setting \(P = 0\), always going to Disney.

• What is the expected utility of the kids?

\[
E[U_k] = 2 \cdot (1 - P)(1 - K) + 0 \cdot (1 - P)K + 0 \cdot K(1 - P) + 1 \cdot KP
\]

\[
= 3KP - 2P - 2K + 2
\]

• So \(E[U_k]\) depends on \(P\).
  - If \(P = \frac{2}{3}\), then \(E[U_k] = \frac{2}{3}\), which is independent of \(K\).
  - If \(P > \frac{2}{3}\), then \(E[U_k]\) is maximized by setting \(K = 1\), always going to Paris.
  - If \(P < \frac{2}{3}\), then \(E[U_k]\) is maximized by setting \(K = 1\), always going to Disney.

• Hence, a mixed strategy equilibrium for this game has \((K = \frac{1}{3}, P = \frac{2}{3})\), and \(E(U_p) = E(U_k) = \frac{2}{3}\).

• Notice by the way the frequencies that result:

<table>
<thead>
<tr>
<th></th>
<th>Kids</th>
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<tbody>
<tr>
<td>Parents</td>
<td>Disney</td>
<td>Paris</td>
</tr>
<tr>
<td></td>
<td>1, 2</td>
<td>0, 0</td>
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<tr>
<td></td>
<td>(\left(\frac{2}{3}\right))</td>
<td>(\left(\frac{1}{3}\right))</td>
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<tbody>
<tr>
<td>Paris</td>
<td>0, 0</td>
</tr>
<tr>
<td></td>
<td>(\left(\frac{2}{3}\right))</td>
</tr>
<tr>
<td></td>
<td>2, 1</td>
</tr>
<tr>
<td></td>
<td>(\left(\frac{1}{3}\right))</td>
</tr>
</tbody>
</table>

• So, in \(\frac{5}{9}\) of cases, both kids and parents get 0, which is not desirable.

• The problem: this is a coordination game–where both players benefit by playing complementary strategies. By randomizing strategies, the non-coordinated outcomes occurs most often (more than \(1/2\) the time).

• Nash equilibrium should not be confused with a competitive market equilibrium which, under conditions that we are familiar with, has desirable efficiency properties.

• In game theory, as in information economics, equilibria result from strategic interactions rather than atomistic maximizing behavior. And so there is no reason to think that these outcomes will be desirable.
1.5 **Prisoner’s dilemma**

- You are already familiar with the prisoner’s dilemma. Review it briefly anyway because it underscores a central issue in non-cooperative games – the issue of credible commitment.

- The setup: Two criminals accused of a crime. The district attorney pulls each aside to say: if you help me convict (‘rat on’) the other prisoner, I’ll let you go free and the other prisoner will get 10 years. However, if you both rat on one another, you’ll each get 5 years. The prisoners understand that if neither rats on the other, they will each serve only 3 years. So, the payoff matrix:

  \[
  \begin{array}{ccc}
  & \text{Rat} & \text{Not} \\
  \text{A} & -5, -5 & 0, -10 \\
  \text{Not} & -10, 0 & -3, -3 \\
  \end{array}
  \]

- As is clear, Rat, Rat is the only Nash equilibrium of this game, even though both prisoners would be strictly better off if they could choose Not Rat, Not Rat.

- Hence, the prisoner’s dilemma underscores one of the interesting properties of game theoretic models: Outcomes that appear optimal are often not stable when subject to the Nash criterion. In fact, they are often dominated.

- This raises a set of questions:
  1. When are commitments credible?
  2. When should threats be believed?
  3. Is there anyway to ensure cooperation?

- These issues are explored a bit in the tragedy of the commons game

1.6 **Tragedy of the commons**

- Two yak herders, A and B.

- There is one common land that can only support so many yaks, who graze and produce milk. If the common is overgrazed, total milk production will fall.

- Milk sells for $1 per gallon. Let V equal milk per yak, and \( Y_A, Y_B \) equal the number of yaks brought to the common by each herder. The production function is:

  \[
  V(Y_A, Y_B) = 200 - (Y_A + Y_B)^2
  \]
Notice $\frac{\partial V}{\partial Y} < 0$, $\frac{\partial^2 V}{\partial Y^2} < 0$, meaning that each Yak does increasing marginal damage (note, the outcome variable is in gallons per yak, so obviously it will be optimal to bring some positive number of yaks).

- **Solution concept:** What does each Yak herder do taking the actions of the other herder as given?

- **Problem for herder A**

  \[
  \max_{Y_A} Y_A \cdot V = 200Y_A - Y_A(Y_A + Y_B)^2
  \]

  \[
  \frac{\partial}{\partial Y_A} = 200 - (Y_A + Y_B)^2 - 2Y_A(Y_A + Y_B)
  \]

  \[
  FOC : 200 - 3Y_A^2 - Y_B^2 - 4Y_AY_B = 0
  \]

- **Problem for herder B is symmetric:**

  \[
  FOC : 200 - 3Y_B^2 - Y_A^2 - 4Y_AY_B = 0
  \]

- **Given symmetry, it must be $Y_A = Y_B$, implying that:**

  \[
  200 - 3Y^2 - Y^2 - 4Y^2 = 0, \\
  200 = 8Y^2, \\
  Y_A = Y_B = 5.
  \]

- **Each herder brings 5 yaks to the common and earns $5 \cdot (200 - 10^2) = 500$, and total output is 1,000 gallons of milk.**
• What would be the social optimum?

\[
\max_Y Y \cdot V = 200Y - Y^3, \\
\frac{\partial}{\partial Y} = 200 - Y \cdot Y^2 = 0, \\
Y = \frac{10}{3} \sqrt[3]{6} \approx 8.16,
\]

and total output is

\[8.16 \cdot 200 - 8.16^3 = 1,089.\]

• So clearly, the common is being overgrazed. The Nash equilibrium is not a social optimum.

• But what if one herder promised to only bring 4 yaks? So an idea: players could announce their strategies in advance.

• Focus on the game in extensive form where \(A\) is trying to decide whether to graze 4 or 5 yaks. Prior to \(A\)'s move, \(B\) will announce to \(A\) the strategies he will play (to graze 4, 5 or 6 yaks) in response to \(A\)'s choice. So, the setup:

1. \(B\) announces the action he will take if \(A\) brings 4 yaks \(and\) the action he will take if \(A\) brings 5 yaks.
2. \(A\) takes his action.
3. \(B\) takes his action (hence, this is now a sequential game).

\[
\begin{array}{cccccccc}
B & 4,4 & 4,5 & 4,6 & 5,4 & 5,5 & 5,6 & 6,4 & 6,5 & 6,6 \\
4 & 544,544 & 544,544 & 544,544 & 476,595 & 476,595 & 476,595 & 400,000 & 400,000 & 400,000 \\
5 & 595,476 & 500,500 & 395,474 & 595,476 & 500,500 & 395,474 & 595,476 & 500,500 & 385,474
\end{array}
\]

• What are the pure strategy Nash equilibria of this game?

• Looking at the payoffs, \(A\) will choose his best response to each of \(B\)'s strategies. These are

\[
\begin{array}{cccccccc}
B & 4,4 & 4,5 & 4,6 & 5,4 & 5,5 & 5,6 & 6,4 & 6,5 & 6,6 \\
4 & 544,544 & 544,544 & 544,544 & 476,595 & 476,595 & 476,595 & 400,000 & 400,000 & 400,000 \\
5 & 595,476 & 500,500 & 395,474 & 595,476 & 500,500 & 395,474 & 595,476 & 500,500 & 385,474
\end{array}
\]

• If \(B\) assumes \(A\) is rational, he anticipates these choices and so the following would be Nash equilibria:

1. \(B\) threatens 6,6; \(A\) chooses 4: Payoffs are 400,600
2. B threatens 5, 5; A chooses 5: Payoffs are 500, 500
3. B threatens 6, 5; A chooses 5: Payoffs are 500, 500

<table>
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<tr>
<th></th>
<th>4, 4</th>
<th>4, 5</th>
<th>4, 6</th>
<th>5, 4</th>
<th>5, 5</th>
<th>5, 6</th>
<th>6, 4</th>
<th>6, 5</th>
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<td>5</td>
<td>400,600</td>
<td>500,500</td>
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</table>

- Hence, this game has three Nash equilibria. Question: Are any of these equilibria problematic?

- Yes, both 1 and 2 involve non-credible threats. Player B threatens to take actions contingent’s on A’s choice that it would not be rational for him to take. For example, if A brought 5 Yaks, then B should also bring 5. Player B should never bring 6 Yaks if player A brings more than 4.

- So the problem is that B is making threats that should not be believed. But in a simple Nash equilibrium, they are believed. The Nash equilibrium concept does not rule out these implausible threats. Why not?

- Because these threats don’t have to be carried out in equilibrium. If A brings 4 yaks, then B brings 6, which is rational. And if A brings 5, then B brings 5, which is also rational. So player B is never forced to carry out an irrational threat, which would in fact violate the Nash equilibrium.

- This example points to a problem with the Nash concept, which is that implausible beliefs about what would happen ‘off the equilibrium path,’ can lead to implausible results in equilibrium (e.g., Player A brings 4 yaks in response to B’s threat to bring 6).

- This motivates the ‘equilibrium refinement’ of Subgame Perfection.

**Definition 2** Subgame: Stage of the game at \( t > 1 \) moves.

**Definition 3** Subgame perfection: A Nash equilibrium is subgame perfect if the players’ strategies constitute a Nash equilibrium in every subgame.

- Translation: a Nash equilibrium is sub-game perfect if it does not require a non-credible threat to be sustained.

- B’s threat to bring 6 yaks if A brought 5 failed the subgame perfection test because if the players reached this subgame–i.e., player A did bring 5 yaks–player B would necessarily bring 5 yaks.
1.7 Subgame perfection: The Scorsese example

- In the (poor) 1996 film “Casino,” Joe Pesci plays a gangster who is sent to collect $50,000 from a banker. The banker has the money in a safe. Pesci threatens the banker with a baseball bat. The problem is that only the banker knows the code to the safe. If Pesci hits the banker with the bat, he probably won’t get the money. But he will go to jail. The banker gives him a knowing look... So, the payoffs look as follows (note— the Banker moves first):

<table>
<thead>
<tr>
<th></th>
<th>Pesci</th>
<th>Banker</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assault</td>
<td>−50,000, +50,000</td>
<td>−50,000, +50,000</td>
</tr>
<tr>
<td>Not</td>
<td>0, +Pain,Jail</td>
<td>0, 0</td>
</tr>
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</table>

- The banker recognizes that it is not subgame perfect for Pesci to assault him since Pesci only has jail time to gain.
- So, the subgame perfect equilibrium of this game is that the banker keeps the money and Pesci leaves without committing the assault.
- But this is not what happens... Pesci convinces the banker he is irrational and gets the money.
- It’s a great scene—and a nice application of game theory.
1.8 Related: Changing your payoffs

- Subgame perfection demonstrates that non-credible threats should not be believed. This works to the disadvantage of the person making the threats. There are at least two ways to change the game.

- The Scorsese example demonstrates one way: convince your opponent that you are irrational. Irrational people may carry out threats that are self-destructive.

- [North Korea is probably the master of this strategy.]

- A second strategy to make threats credible is, somewhat paradoxically, to make your payoffs worse—that is, destroy your fallback option.

1. Schelling island example.
2. Quitting drinking? If you could stop drinking for a while, you would no longer be addicted. But the craving is so intense after 24 hours, that you cannot credibly commit to stop. Take mucilage—does severe liver damage if consumed with alcohol.

1.9 The problem of cooperation

- Returning to the problem underscored by the prisoner’s dilemma, game theory seems to predict very little cooperation in the world. For example, consider the centipede game.

- A sequential move game: 1st player 1, then player 2, then player 1, etc.

Sequential move game

- Both players benefit from cooperation.
- But there is always a temptation to defect.
This game is solved by backward induction. Start at the last node, work your way back. As you can see, cooperation is clearly dominated in the final play. But then it is dominated in the 2nd to last move. And so on...

Iterated dominance leaves only one equilibrium, and it is an undesirable one.

This particular prediction does not have the ring of truth. Puts a hefty premium on the rationality of both players, and their belief in the rationality of one another. It is also not especially well supported by empirical evidence.

What would you do if the 1st player did not defect on move one? Game theory does not make a clear prediction. This is ‘off the equilibrium path.’

The problem of cooperation appears vexed.

1.10 Idea: Repeated interactions

One notion is that if players repeatedly face the same situation, they may recognize that it is in their mutual interest to cooperate. So, maximize the pie rather than simply grabbing the largest share of a small pie.

Consider a three period repeated prisoner’s dilemma. Like a standard prisoner’s dilemma except that if both players cooperate, there is a continuation game, up to a total of 3 periods. Assume also that there is a discount factor of $\delta < 1$.

Period 1

<table>
<thead>
<tr>
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<tr>
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<td>Coop</td>
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<tr>
<td>A</td>
<td>5, 5</td>
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<tr>
<td>Defect</td>
<td>10, −1</td>
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If they cooperate in period 1, then period 2 looks similar. So does period 3. Then the game ends.

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<td>A</td>
<td>5, 5</td>
</tr>
<tr>
<td>Defect</td>
<td>10, −1</td>
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</table>

So the payoff to cooperation is:

$$(5, 5) + (5\delta, 5\delta) + (5\delta^2, 5\delta^2),$$

whereas the payoff to defection is

$$(1, 1) + (\delta, \delta) + (\delta^2, \delta^2).$$
• Obviously, it is advantageous to cooperate.

• But of course in period 3, each ought to defect, since there is no further payoff to cooperation. And so by backward induction, each should defect in 2, and then in 1. Hence, no honor among thieves.

1.11 Infinitely repeated games
• What causes games to unravel? It appears to be related to backward induction. Because ‘defection’ is optimal in the last period, it becomes optimal in the second to last period, then the third to last, etc. What if there was no last period?

• Examples:
  – Are you more likely to tip at a local restaurant that you go to often or at an out-of-town restaurant where you'll likely never return?
  – Many drivers are extremely rude in city traffic. But they would not do this if those drivers were their neighbors (even annoying neighbors).
  – “No one in the history of the world has ever washed a rented car.” – Lawrence Summers, Economist, President of Harvard University.

• In an infinitely repeated game (or a game with no known end point), there cannot be backward induction. No player knows when the last period will occur. So, the unravelling that we’ve seen above may not occur.

• Consider the following strategy: Each player announces she will cooperate so long as the other player does so. But if the one player defects, the other will punish her with no further cooperation. This is called the “grim trigger strategy.” Is it a Nash equilibrium?

• Payoff to cooperation is
\[ 5 + 5\delta + 5\delta^2 + 5\delta^3 + \ldots + 5\delta^\infty = \frac{5}{1 - \delta}. \]

Payoff to defection is
\[ 10 + \delta + \delta^2 + \ldots + \delta^\infty = 10 + \frac{\delta}{1 - \delta}. \]

• Cooperation is therefore an equilibrium iff:
\[
\begin{align*}
\frac{5}{1 - \delta} &> 10 + \frac{\delta}{1 - \delta}, \\
\frac{5 - \delta}{1 - \delta} &> 10, \\
5 &> 10 - 9\delta \\
\delta &> \frac{5}{9} \approx .556
\end{align*}
\]
• So, if the future is sufficiently important, cooperation is sustainable as an equilibrium.

• Hence, there can be ‘cooperative’ outcomes in non-cooperative game theory, but seemingly only under restrictive conditions.

• Are the predictions of this models too strong?