Higgs Family Symmetry and Supersymmetry

by

Brian Lawrence Patt

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Physics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2006

Author . . . .
Department of Physics
March 24, 2006

Certified by . . . .
Frank Wilczek
Herman Feshbach Professor of Physics
Thesis Supervisor

Accepted by . . . .
Thomas J. Greytak
Associate Department Head for Education

Higgs Family Symmetry and Supersymmetry

by

Brian Lawrence Patt

Submitted to the Department of Physics
on March 24, 2006, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Physics

Abstract

In this thesis we investigate building models of family symmetry that give the Higgs fields family structure. We construct several models, starting with 2 generation models then moving onto 3 generation models. These models are described sequentially in chapters 2 through 6. All of these models are supersymmetric and they did not previously exists in the literature. In these models, quark (and lepton) masses and mixings are determined the vacuum expectation values of the family sector. These vacuum expectation values (VEV) can have a hierarchal structure because they correspond to flat directions of a superpotential. At low energies these models contain just one light pair of Higgs fields. Experimentally, the most interesting feature of these models are couplings between the low energy Higgs and moduli of the family sector. These couplings should be observable at the Large Hadron Collider.

Thesis Supervisor: Frank Wilczek
Title: Herman Feshbach Professor of Physics

Contents

Chapter 1

Introduction/background

There are many directions of research for physics beyond the standard model. These avenues are based on well known theoretical concerns, such as a desire to unify all the forces, just as the grand unified theories (GUT) and electroweak theory have unified the strong, weak, and electromagnetic force. Another concern is the hierarchy problem associated with the vast difference in scales between the electroweak scale and the Planck or GUT scale [1]. Many of these research directions lack detailed experimental data to be explained. In this respect the situation with family/flavor physics is unique in the realm of beyond Standard Model research.

In the Standard Model the masses and mixings of the quarks and leptons are determined by the Yukawa couplings of the fermions to the Higgs field. The Vacuum expectation value (VEV) of the Higgs field sets the overall scale of fermion (quarks and leptons) masses. The Standard Model treats the numerous Yukawa couplings as parameters of the theory. On philosophical grounds there are several strong reasons to look for an explanation of these parameters. First, on aesthetic grounds it's ugly for a fundamental theory to require so many parameters to be set by hand. Moreover, without some qualitative reason for the discrepancy any dimensionless parameter of a theory might be expected to be of $O(1)$, which is not true of many of these couplings.

Inspection of the couplings reveals a destinctive pattern, a pattern that doesn't look random. Most importantly, the quarks and leptons are naturally organized into three generations of particles. The three generations of particles all have Standard

Model (SM) charges. For example the electron has the same electrical charge as the muon and tau. Similarly the up, charm and top have equivalent Standard Model charges. Likewise for the down, strange and bottom. The differences between these three generations comes from their masses and mixing, with their masses increasing as we go up in generation. The masses and mixings are encoded by Yukawa couplings. In fact all the breaking of family symmetry of the Standard Model is encoded by these Yukawa couplings. We will now discuss how these Yukawa couplings appear in the Standard Model, and the experimentally observed patterns of these Yukawa Couplings which break family Symmetry.

These three generations would be identical in the Standard Model, if it were not for the Yukawa couplings, which give different masses to the three generations of particles. If it were not for the Yukawa couplings the Standard Model would have an $SU(3)_{\text{family}}$ symmetry.

The terms of the Standard Model Lagrangian which give rise to the masses and mixings of the fermions look as:

$$\mathcal{L}_{\text{quark}} = \lambda_{\mathsf{up},ij}\bar{q}_i^m \phi_m u_j + \lambda_{\mathsf{down},ij}\epsilon_{mn}\bar{q}_i^m (\phi^\dagger)^n d_j + \text{hermitian conjugate}$$

$$\mathcal{L}_{\text{lepton}} = \lambda_{e,ij}\bar{l}_i^m \phi_m e_j + \lambda_{\nu,ij}\epsilon_{mn}\bar{l}_i^m (\phi^\dagger)^n \nu_j + m_{\text{Majorana}}\nu_R^T C \nu_R + \text{ hermitian conjugate}$$

We can simplify this somewhat by implementing unitary rotations on u_j, d_j, l_j, and ν_j, which let us write the Lagrangian in terms of the quark (and lepton) mass eigenstates. The mismatch between the unitary rotation on the up quarks versus down quarks has physical significance for charged flavor changing currents mediated by $W^\pm$ exchange.

Looking below at the table of mass parameters [2], we see a hierarchal structure to the masses of the three generations of quarks and leptons. For example $\lambda_{\mathsf{up},11}$ is of $O(10^{-4})$, which is quite small for a dimensionless parameter, and thus begs for an explanation. In addition the overall hierarchical structure of the mass parameters begs for an explanation. Furthermore looking at the CKM matrix below reveals that it's close to the unit matrix, indicating the mismatch between the down-type quarks and up-type quarks is small, which is very suggestive. Moreover, the smallness of the CKM matrix elements depends is correlated with their distance off the diagonal [2].

$$\begin{array}{ccc} M_e = 0.511\,\text{MeV} & M_\mu = 106\,\text{MeV} & M_\tau = 1.78\,\text{GeV} \\ M_d = 5.0\text{–}9.0\,\text{MeV} & M_s = 105 \pm 25\,\text{MeV} & M_b = 4.3 \pm 0.3\,\text{GeV} \\ M_u = 1.5\text{–}5.0\,\text{MeV} & M_c = 1.3 \pm 0.3\,\text{GeV} & M_t = 178 \pm 4\,\text{GeV} \end{array}$$

The CKM matrix parameterizes the amount of flavor mixing due to charged flavor changing currents. If family symmetry were exact and unbroken the CKM matrix would be the unit matrix. In fact we can regard the CKM matrix as a unit matrix plus higher order term corrections. These higher order corrections are generated by family symmetry breaking, and therefore should be paramatrized in terms of the family symmetry breaking. For example these higher order corrections may be characterized by a paramater of the form $\frac{v_{\text{Fam}}}{\Lambda}$. Here v_{Fam} is VEV breaking family symmetry, and Λ is a large scale. In addition the first, second and third generation masses would be equal if family symmetry was unbroken. As a result we expect the fermion masses to depend on the family VEVs, also.

The charged flavor changing currents are parametrized by the CKM matrix, which is close to unity. The CKM matrix is an $SU(3)$ matrix that has been simplified through phase redefinitions of the quark fields. A general parameterization of the CKM matrix is:

$$\begin{pmatrix} cos(\theta_{12})cos(\theta_{13}) & sin(\theta_{12})cos(\theta_{13}) & sin(\theta_{13})e^{-i\delta} \\ -sin(\theta_{12})cos(\theta_{23}) - cos(\theta_{12})sin(\theta_{23})sin(\theta_{13})e^{i\delta} & cos(\theta_{12})cos(\theta_{23}) - sin(\theta_{12})sin(\theta_{23})e^{i\delta} & sin(\theta_{23})cos(\theta_{13}) \\ sin(\theta_{12})sin(\theta_{23}) - cos(\theta_{12})cos(\theta_{23})sin(\theta_{13})e^{i\delta} & -cos(\theta_{12})sin(\theta_{23}) - sin(\theta_{12})cos(\theta_{23})e^{i\delta} & cos(\theta_{23})cos(\theta_{13}) \end{pmatrix}$$

However, because the CKM matrix is close to unity a convenient form is the Wolfenstein parameterization [3]:

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

There are many candidate textures for the form of the Yukawa couplings of the Standard Model Lagrangian, which are consistent with the experimental data for the fermion masses and mixings. The textures are of a generic form that is parameterized by a small expansion parameter. For example, the Cabbibo angle, θ_{Cabibbo} [4], which paramatrizes the mixing between the first generation and second generation of quarks, can be used as the expansion parameter. These small parameters not only explain why the CKM matrix is close to unity, but also the hierarchical scale of the quark/lepton masses, for example $m_u = m_c(\theta_{\text{Cabibbo}})^2$, which makes the up quark much smaller than the charm quark. Measured values for the CKM matrix are [2]:

$$|V_{\text{CKM}}| = \begin{pmatrix} 0.9734 \pm 0.0008 & 0.2196 \pm 0.0020 & 0.0036 \pm 0.0007 \\ 0.224 \pm 0.016 & 0.996 \pm 0.013 & 0.0412 \pm 0.002 \\ 0.0077 \pm 0.0014 & 0.0397 \pm 0.0033 & 0.9992 \pm 0.0002 \end{pmatrix}$$

1.1 Flavor Changing Neutral Currents

An additional signature of the family symmetry is the lack of Flavor Changing Neutral Currents (FCNC's). In contrast to the CKM matrix, the matrix corresponding to FCNC's is measured to be the unit matrix, within experimental limits. In fact, when you build supersymmetric Standard Models, it is problematic to avoid these FCNC's, and one way of avoiding them is by assuming a universal mass term for squarks and sleptons. This means that the squarks and sleptons from different generations have the same masses. In other words, for supersymetric models we have experimental evidence that the mass term is family symmetric.

Inorder to understand the nature of FCNC's in a supersymmetric theory, we consider one chiral quark field. There are gluino-quark-squark coupling terms. Through these couplings there will be box and penguin diagrams which correspond to FCNC's. However these diagrams will only change flavors if the propagators of the squarks has mixing terms in the flavor basis for the quarks. The squarks mass term is universal if it is proportional to the identity as [5]:

$$(M_{\text{squark}})^i{}_j = M_{\text{universal}} \delta^i{}_j + \text{higher order corrections (nonuniversal terms)}$$

From experimental limits we know that the squark mass matrix can have a nonuniversal term no bigger than:

$$\frac{\Delta}{M_{\text{universal}}} < 10^{-2}$$

1.2 Lepton Mixing

The CKM matrix, the flavor mixing matrix for the quark sector, is known to be near unity. However, the corresponding flavor mixing matrix for the lepton sector, which tell us how the three flavors of neutrinos mix is not near unity. In fact, unlike the mass hierarchy for the quarks, the three neutrinos have a hierarchy of the form $m_{\nu_1} \approx m_{\nu_2} << m_{\nu_3}$ or $m_{\nu_3} << m_{\nu_1} \approx m_{\nu_2}$ (the experimental data is consistent with both possibilities). The mixing angles are not small, and hence the mixing matrix can not be approximated as being near unity. If we assume the CP violating phase is ± 1 then the mixing matrix can be approximated as:

$$\begin{pmatrix} cos(\theta_{12})cos(\theta_{13}) & sin(\theta_{12})cos(\theta_{13}) & \pm sin(\theta_{13}) \\ -sin(\theta_{12})cos(\theta_{23}) \mp cos(\theta_{12})sin(\theta_{23})sin(\theta_{13}) & cos(\theta_{12})cos(\theta_{23}) \mp sin(\theta_{12})sin(\theta_{23}) & sin(\theta_{23})cos(\theta_{13}) \\ sin(\theta_{12})sin(\theta_{23}) \mp cos(\theta_{12})cos(\theta_{23})sin(\theta_{13}) & -cos(\theta_{12})sin(\theta_{23}) \mp sin(\theta_{12})cos(\theta_{23}) & cos(\theta_{23})cos(\theta_{13}) \end{pmatrix}$$

The experimental data shows $\theta_{13} \approx 0$, hence the above matrix does not contain such terms [2].

$$|U_{\text{MNS}}| = \begin{pmatrix} 0.73-0.89 & 0.45-0.66 & <0.24 \\ 0.23-0.66 & 0.24-0.75 & 0.52-0.87 \\ 0.06-0.57 & 0.40-0.82 & 0.48-0.85 \end{pmatrix}$$

The leptonic sector is different than the quark sector, because it contains SM singlet fields, the right handed neutrinos. These SM singlets leads to a seesaw mechanism which results in an extremely small mass for the neutrinos of order $\frac{v_{\text{weak}}^2}{\Lambda_{\text{GUT}}} \approx 10^{-2}\,\text{eV}$.

1.3 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is a powerfull principle of physics used to solve many puzzles. Spontaneous symmetry breaking is used to show that a unified electroweak symmetry leads to an exact electromagnetic symmetry with the rest of the symmetry broken. Thus explaining that two forces both associated with symmetries can be unified and leads to a universe where we observe just one of the symmetries is exact. Furthermore, chiral symmetry which is broken spontaneously by QCD, explains the approximate masslessness of the pions. In addition, spontaneous symmetry breaking is necessary to explain Grand Unified Theories(GUTs), in which all three gauge symmetries sit nicely in a larger gauge group. The spontaneous symmetry breaking explains why the large symmetry group factors into the three symmetries we see at low energies. Given the fantastic successes of spontaneous symmetry breaking in high energy physics it make sense to examine the hypothesis, that the apparent family symmetry results from an exact family symmetry that is spontaneously broken.

This expectation that particle physics contains a broken family symmetry, has inspired large literature [6, 7, 8, 9]. The model building generally involves the introduction of fields that break the family symmetry, but are singlets under the Standard Model gauge group. When these fields receive a vacuum expectation value (VEV) they break the family symmetry, but keep the Standard Model gauge symmetry in tact. The models may specify the details of the potential for the family breaking sector. Or in other cases, the models may not build the explicit structure of the family sector, but may just specify the fields that break the symmetry. Furthermore, the form of the VEVs may be determined by the group theory of the symmetry being broken. Once the form of family breaking VEVs are known then low energy effective theory can be constructed from this knowledge alone. Two common scenarios for this sort of model building is a $U(1)$ flavor symmetry [6] or the closely related Froggatt-Nielsen mechanism [7].

In both scenarios we define a set of symmetries then introduce what are commonly called the spurion fields, which in this case break the family symmetry. Then construct the low energy effective theory by including all terms allowed by the symmetry. We assign the spurion fields appropriate VEVs, and examine the structure that results for masses and mixings.

1.4 Review of Models with Family Higgs Multiplets

Next we will discuss the operation of the $U(1)$ flavor scenario. It is assumed there exists fields S_i, which are Standard Model singlets, but have $U(1)$ charges, and these field can couple at nonrenormalizable order to fermion-Higgs Yukawa couplings. The S_i fields spontaneously break the $U(1)$ symmetries. For each abelian symmetry that is introduced will be corresponding VEV that breaks the symmetry. These VEVs will then naturally account for the mixings and the heirarchy of mass scales of the fermions.

A quick illustration of how abelian symmetries can be used to give the mass for the up quark many times smaller than the weak scale is explained below. Imagine that S has a charge of -1, and imagine the charges of $\bar{q}_1$ and u_1 add up to 2 then the nonrenomalizable term for the up quark looks like:

$$\frac{1}{\Lambda_{\mathrm{Fla}}^2}\bar{q}_1\phi u_1 S^2$$

Where, Λ_{Fla} is the family breaking scale. We see this gives a mass term of the scale $\frac{S^2}{\Lambda_{\mathrm{fla}}{}^2}v_{\mathrm{weak}}$ to the up quark, where we assume that the VEV of $S = \epsilon^2\Lambda_{\mathrm{fla}}$, where ϵ is a small parameter. In such a case we get an up quark mass proportional to $\epsilon^4 v_{\mathrm{weak}}(\propto \epsilon^4 t_{\mathrm{mass}})$. So we see this scheme can easily give a mass hierarchy between the top quark mass and the up quark mass. Thus a combination of $U(1)$ symmetry and nonrenormalizable terms naturally creates small Yukawa coupling constants. Looking at the CKM matrix we see the off-diagonal terms are also characterized by small couplings. Therefore this type of scheme can also account for the texture in the CKM matrix.

A different scheme (the Froggatt-Nielsen mechanism) assumes that the family symmetry is $SU(2)$. We now assume that the third generation particles are singlets under the $SU(2)_{\rm fam}$ symmetry. And as before there is a renormalizable term which gives mass to the heaviest generation fermions on the order of the weak scale. Although the lightest two generations transform in fundamental and antifundamental representations of the $SU(2)_{\rm fam}$ symmetry. Now in this case we introduce various spurion fields, S^{ab}, f^a, and A which transform as 3, 2, and a singlet of $SU(2)_{\rm fam}$ respectively. And the $SU(2)$ spurions are assumed to have the following VEVs:

$$S^{ab} = \begin{pmatrix} v' & v' \\ v' & v \end{pmatrix} \qquad f^a = \begin{pmatrix} v' \\ v \end{pmatrix} \qquad A = v'$$

The Lagrangian for the leptons, allowed by this symmetry is:

$$\begin{aligned}\mathcal{L} &= (\bar{q}_3 u_3 + \bar{q}_3 u_a f^a + \bar{q}_a u_3 f^a + \bar{q}_a u_b S^{ab} + \epsilon^{ab}\bar{q}_a u_b A)\phi \\ &\quad +(\bar{q}_3 d_3 + \bar{q}_3 d_a f^a + \bar{q}_a d_3 f^a + \bar{q}_a d_b S^{ab} + \epsilon^{ab}\bar{q}_a d_b A) i\sigma\phi^\dagger\end{aligned}$$

This leads to the following Yukawa coupling matrices with the following schematic form:

$$\lambda_{\rm u} = \begin{pmatrix} \frac{v'}{M} & \frac{v'}{M} & \frac{v'}{M} \\ \frac{v'}{M} & \frac{v}{M} & \frac{v}{M} \\ \frac{v'}{M} & \frac{v}{M} & 1 \end{pmatrix} \qquad \lambda_{\rm d} = \begin{pmatrix} \frac{v'}{M} & \frac{v'}{M} & \frac{v'}{M} \\ \frac{v'}{M} & \frac{v}{M} & \frac{v}{M} \\ \frac{v'}{M} & \frac{v}{M} & 1 \end{pmatrix}$$

This is one possibility for the texture of quark masses and mixings. In this scheme the lighter two generations are distinguished from the heaviest generation through family charges. Ideally, we would expect that the symmetry that governs family treats all three generations equally before symmetry breaks. This is a major aesthetic concern, that makes the $U(1)$ flavor symmetries look unattractive. However, for a nonabelian symmetry, we can imagine that there is an $SU(3)$ that breaks down to $SU(2)$, and the Lagrangian we wrote down was the effective Lagrangian after $SU(3)$ is broken, but before $SU(2)$ breaks. Furthermore, if we assume the scale of $SU(3)$ breaking is the same as the cutoff scale then we would expect the renormalizable Yukawa terms for the third generation will have natural, $O(1)$, Yukawa couplings.

1.5 Family Higgs Fields

Charactersitically, the two schemes above assume the that Higgs fields is a singlet under family symmetry. The signifigance of this thesis is to investigate the possibility that the Higgs itself has family symmmetry properties. Firstly all the currently observed non-gauge particles do in fact have family properties, and all the observed particles are built from Standard Model particles which arrange themselves into three identical generations. From that point of view, one might be a little surprised that the Higgs has no family structure. Additionally, there exists a famous model, the Peccei-Quinn model which gives the Higgs an axial $U(1)$ symmetry[11, 12]. Peccei-Quinn symmetry explains why CP symmetry is not strongly broken. The presence of the QCD θ term prevents us from rotating away a uniform phase from the quark mass matrix. However, we know that the physical effect of the quark mass matrices having a uniform phase should be easily verifiable experimentally, and the angle of this phase is mysteriously less than 10^{-11}. In effect we can consider the quark mass matrix as having an extra symmetry. However, we already know that the quark mass matrix does indeed have a somewhat systematic pattern built into the Yukawa couplings, evidenced in the hierarchy structure of the masses, and the small mixing angles. Thus, we explain one aspect of the quark mass matrix that the phase of $\mathrm{Det}M$ is real, by introducing a new symmetry that the Higgs fields transform under. From that perspective it seems natural to explain the rest of the pattern of the quark mass matrix through the introduction of another symmetry, in this case the appropriate symmetry is the family symmetry. And therefore it is appropriate to consider that the Higgs has family charges.

In the literature there are scattered examples of models which contain Higgs with family symmetry. Lets be more specific about the models we are envisioning. Consider the following Lagrangian term:

$$\bar{q}_i \phi_k u_j f^{ijk}$$

Here we are omitting Standard Model gauge indices, and i,j, and k, represent some yet to be specified family indices, and f^{ijk} represents a structure constant such that this term is invariant under the family symmetry. The intriguing feature about this term is that the VEV of the Higgs has family structure, and can potentially explain the structure of all the Yukawa couplings.

If we try to build a model of this kind without supersymmetry, we will run into several problems. Firstly, we observe this sort of model has a large number of Higgs field, but our goal is a model that reduces to the Standard Model at low energy, which has at most a pair of Higgs fields. For example, the number of low energy Higgs fields affects the running of the Standard Model (SM) gauge coupling constants, which nicely unify at GUT scales, we do not want to disturb this feature [10]. So the first question then becomes, what are the masses of the Higgs multiplets. Requiring that there is at least one weak scale Higgs mass requires that the parameter m_{H}^2 in the following Lagrangian term is no greater than the weak scale.

$$\mathcal{L} = m_{\mathrm{H}}^2 \phi_k^* \phi_k$$

The mass parameter is constrained because such a term gives mass to all the components, ϕ_k, unless there is some fine tuning.

However, there are additional terms which could give mass to this Higgs multiplet, namely couplings between the Higgs and family breaking fields. For example if the family symmetry is broken at a low scale near the weak scale, the model predicts a multiplet of low energy Higgs that should be observable at the weak scale. Alternatively if the family breaking scale is at a large scale, generically all the Higgs get masses at this larger scale. This can be understood from the permitted term $t_j^* t_j \phi_k^* \phi_k$, which give masses to all the Higgs at the scale of t_j's VEV. Here t_j is assumed to be a family breaking field. We could possibly have other terms which cancel out the mass of one Higgs component giving us one weak scale Higgs, but such a cancellation is a fine tuning.

In brief, we see there are three different choices for family models of the Higgs field without SUSY.

1) Hypermultiplet of weak scale Higgs

2) Hypermultiplet of very large scale Higgs, no weak scale Higgs

3) Hypermultiplet of very large scale Higgs, with one weak scale Higgs and fine tuning

The last of these three possibilities is the most viable choice, but still not very attractive.

The models we construct and present in this thesis are based on the idea of giving the Higgs family properties. But we aim to construct models which have family breaking at a large scale with no fine tuning required. It turns out we can solve this fine tuning problem, the same way other fine tuning problems are solved by constructing Supersymmetric models of family symmetry. The models will be designed to reproduce the Minimal Supersymmetric Standard Model (MSSM) at low energies. We recall in the MSSM that the Higgs sector contains two weak Higgs, an up Higgs and down Higgs which both give masses to the fermions. The up Higgs gives mass to the up type quarks and the neutrinos, likewise the down Higgs gives mass to the down type quarks and the leptons. The superpotential terms are:

$$W_{\text{quark}} = \lambda_u \epsilon^{ij} q_i U_j u^c + \lambda_d \epsilon^{ij} q_i D_j d^c + \lambda_e \epsilon^{ij} l_i D_j e^c + \lambda_\nu \epsilon^{ij} l_i U_j N + m_{\text{Majorana}} N N$$

Therefore we want our model of family symmetry breaking to produce a pair of weak Higgs with hypercharges $1/2$ and $-1/2$. As we discussed previously the family breaking tends to give masses to the Higgs at a large scale, but we want one pair of Higgs to remain light.

Supersymmetric models have a natural solution to this problem. The vacuum space of supersymmetric models tend to have nontrivial moduli spaces. Moduli spaces are manifolds of VEVs that all minimize the potential terms in the Lagrangian. More specifically these are flat directions, which are not Goldstone modes (associated with symmetry breaking). This is natural in supersymmetry because of the positive definite nature of the potential, as a result any field configuration which has zero energy is a global minimum. So it is common for theories with SUSY to have flat directions in the potential. Since the mass of the scalar fields is simply the quadratic term in the potential, any flat direction is massless. This provides a simple mechanism to have most of the Higgs hypermultiplet heavy, with one pair remaining light. That pair will just correspond to the flat directions.

Moduli fields can also serve as a potential solution to the mini-hierarchy problem of the quark and lepton masses. Since the moduli are completely flat, the scale of their VEV is completely arbitrary, of course effects that spoil the moduli's flatness could remove this arbitrariness of scale. Alternatively, the VEVs of the moduli field might be governed by anthropic reasoning.

Chapter 2

Overview of Models

A key feature of this framework is that all the information about family symmetry breaking is encoded in the VEV of a "generalized Higgs" field. The generalized Higgs field doublet has the same Standard Model quantum numbers as the ordinary Higgs field of the MSSM (Minimally Supersymmetric Standard Model), but additionally it's assumed to transform under a family group, G_{fam}. Hence the generalized Higgs field has tensor indices for both $SU(2)_{weak}$ and the family group, G_{fam}. Now for convenience, whenever we refer to "Standard Model (SM) fermions" and "SM fermion fields", we are actually referring to the quarks and leptons and their corresponding chiral superfields. The structure of SM fermion masses and mixings will be determined, at least to first order from the renormalizable SM fermion-Higgs couplings. Now, the generalized Higgs fields will have many components. In this framework most of the components turn out to be heavy, with masses at the scale of Λ_{Fam}, a family breaking scale. However, exactly one mode, (i.e one linear combination of generalized Higgs field components) will be massless. Constructing models which produce just one massless Higgs field without fine tuning, seems to require supersymmetry.

Supersymmetric theories require a pair of Higgs fields, the up and down Higgs field[13]. So our models actually have a pair of generalized Higgs fields. The generalized Higgs fields get their masses from trilinear terms in the superpotential of the schematic form:

$$UDT$$

Here all the tensor indices have been suppressed, and T is a field with transformation properties under the family group, but it is a singlet under the Standard Model gauge group. Furthermore U and D are the generalized up and down Higgs field respectively. The other terms in the superpotential which contain the generalized Higgs field are terms that couple the Higgs field to the SM fermions. For example, the the up Higgs field couples to the up quark in a term that schematically looks like:

$$qUu^c$$

It is meant to be understood that q and u^c are fields which contain quarks in a family symmetric manner. So, for example q is a triplet of left handed quark doublets, thus containing fields for all three generations of quarks. Likewise, u^c is a triplet of right handed fields. The VEVs of these fields will correspond to some point in the moduli space of the theory. In our models we will find a portion of the moduli space in which all the of SM Fermion chiral superfields have scalar components with VEVs of zero. In such a case the masses of the Higgs field are determined by substituting the VEVs of T into the Higgs-Higgs-T trilinear term. The models we present will have one pair of Higgs field remaining massless, while all the other components of the Higgs field will get a mass at the scale of the VEVs of T. In addition, it will turn out T contains flat directions, and the VEVs of these flat directions break family symmetry, thus they determine the family symmetry breaking scale, Λ_{Fam}. Hence the heavy components of the generalized Higgs field get masses at the scale of Λ_{Fam}. In addition, these flat directions are massless modes, and they may couple to the light Higgs field through the Higgs-Higgs-T trilinear term. This suggest interesting phenomenology of the Higgs fields interacting with "family moduli". Family symmetry breaking actually takes place in a sector of the theory which does not any contain Standard Model fields, we will call it the family breaking sector.

Now we give a more detailed discussion of the symmetries of the generalized Higgs fields. Firstly, both U and D will have tensors indices of the Standard Model gauge group and family group. In order to write down the form of generalized Higgs fields, we must know the representation of the Standard Model and family group it transforms under. In terms of the Standard Model group there is no choice, it must transform as a doublet. However we are free to choice any representation for the family group. For example, suppose the family group is $SU(3)$, and the Higgs field is in the octet of $SU(3)$. In this case the explicit form of the up Higgs field would be ${U^m}_{n,j}$. Note m and n are family indices, and j is an $SU(2)_{weak}$ index. Furthermore, this generalized Higgs field is traceless in the indices m and n, because it's in the octet representation. For this choice of representation and family group, the generalized up and down Higgs field each have 8×2 components. In other words, eight pairs of $SU(2)_{weak}$ doublets. In order that our models have the correct low energy physics, we require a single mode is massless. Although, a single mode refers to a pair of $SU(2)_{weak}$ doublets, an up Higgs field and down Higgs field, both of which will be massless. The form of the generalized Higgs fields can vary from model to model.

2.1 Family Symmetry Breaking Communicated to Standard Model

The family symmetry breaking is communicated to the Standard Model through the form of the massless Higgs field mode. After expanding about the family fields VEVs, the quadratic generalized Higgs field mass term is determined by the family VEVs of T. In fact these quadratic terms will be determined completely by the VEVs of T. For now the Standard Model gauge group indices will be suppressed. Lets consider a basis of the generalized Higgs field, U_i and D_i, chosen such the the quadratic(mass) terms are diagonal. Note in this basis the family symmetry is no longer manifest. In this basis we can write down the Higgs field mass term of the Lagrangian as:

$$\Sigma_{i=1}^{N-1}(M_i^{\text{up}})^2 {U_i}^* U_i + \Sigma_{i=1}^{N-1}(M_i^{\text{down}})^2 {D_i}^* D_i$$

In this notation we are assuming there are N pairs of Higgs field doublets, and the first pair, U_0 and D_0, are the massless pair. Notice all the massive Higgs field pairs will have VEVs of zero. In other words any non-zero VEV of the generalized Higgs field must appear only in the zero mode. More generally the VEV of any non-flat direction is zero if SUSY is unbroken. This can be understood as follows: if SUSY is unbroken the minimum of the potential is zero. However if all the fields have a VEV of zero then the potential is zero. Hence we found one point that is always part of the moduli space. Moreover, if we can vary the VEV of a field away from zero, and yet still have a potential of zero then this field is flat, by definition.

In order to see the consequences of the generalized Higgs field containing a single massless mode, lets illuminate in more detail the type of models we are imagining. It will be useful to introduce the tensor indices. The following equations will have tensor indices for the family symmetry group, but the indices for the Standard Model gauge group will still be suppressed. Imagine we are considering a model with an $SU(3)$ family group. For example, one quark superpotential term could be of the following form:

$$\lambda q_m U^m{}_n (u^c)^n$$

where λ is a constant of $\mathcal{O}(1)$, also m and n are family indices. Now write the generalized up Higgs field in terms of the up Higgs field mass diagonal basis.

$$U^m{}_n = \begin{pmatrix} C^1_1 & C^1_2 & C^1_3 \\ C^2_1 & C^2_2 & C^2_3 \\ C^3_1 & C^3_2 & C^3_3 \end{pmatrix} U_0 + \Sigma_{i=1}^{N-1} (F_i)^m{}_n U_i$$

Here $C^m{}_n$ and $(F_i)^m{}_n$ are just coefficients that define the transformation from "Higgs field mass diagonal" basis to "family symmetry manifest" basis. Now examine the VEV of U^m_n, recall only U_0 has a nonzero VEV, which implies that the VEV of U^m_n is given by the first term of the above equation. In addition, if we look at the above quark superpotential term, we see that the Yukawa coupling constants between the quarks and the light Higgs field, U_0, are in fact:

$$\lambda^m{}_n = \lambda C^m{}_n$$

Henceforth, if the form of U_0 breaks family symmetry, this breaking gets introduced into the Standard Model Yukawa couplings. Also the form of U_0 is determined by the VEVs of T. So now will we explore which factors determine the VEVs of T.

2.2 Discussion of Family Sector Potential and its Symmetry Breaking

T is the field which breaks the family symmetry. In our models we postulate a family breaking sector with a corresponding superpotential. The VEVs of T can be determined entirely by the family breaking sector's potential. This is because T appears only in the family breaking sector potential and in the T-Higgs couplings. In order to find the potential for T, we imagine setting all the other fields in the model to their VEVs, in particular the generalized Higgs field will get replaced by the VEV of the light Higgs. This means the VEV of the light Higgs field could conceivably contribute to the potential for T, however it will turn out that any component of T which couples to the light Higgs field must have a zero VEV. This can be easily understood, for if the light Higgs field couples to a moduli field which has a large VEV then the light Higgs field gets a mass from this coupling. A proof that there is no way around this problem will be presented with our second model. All the VEVs of the T components that don't couple to the light Higgs field are determined by the family breaking sector.

Now recall T has some family group structure, which we have yet to enumerate. In the following paragraph we will leave the family structure of T unspecified, and suppress family indices. So any terms we write are actually written in a schematic form. We can imagine that in addition to the family breaking field(s) T, there also exists scalar fields such as η and ξ. These scalar fields are singlets under both the

Standard Model gauge group and the family group. In our framework we are regarding these models as effective field theories of a high energy theory. The cutoff scale, Λ_{cut}, may possibly be as large as the Planck scale. Consider the following possible superpotential terms:

$$\eta T^2 + \frac{1}{\Lambda_{cut}} \xi T^3$$

Now lets explore the vacuum solutions for this superpotential. We consider the possibility of vacuum solutions which satisfy the conditions $\eta = \xi = 0$. In order to find the moduli space, we compute the F-terms of the superpotential and set every F-term equal to zero. So, for example consider computing an F-term with respect to some component of T, then this F-term will schematically be:

$$\eta T + \frac{1}{\Lambda_{cut}} \xi T^2$$

This F-term is of course zero, because we assume $\eta = \xi = 0$. Now lets consider the F-terms with respect to η and ξ, which are:

$$T^2 = 0 \qquad \text{and} \qquad T^3 = 0$$

Remembering that T has a family structure, we see that these equations imply some quadratic and cubic invariant of T is zero. So long as T has more than 2 independent invariants, we see that T is not determined uniquely, hence we get a nontrivial moduli space of configurations for T. More generally, if the number of constraints from F-terms of scalar fields is less than the number of independent invariants of T then the moduli space for T is nontrivial.

All of this previous discussion has assumed SUSY was unbroken. SUSY breaking would result in additional terms in the Lagrangian, such terms would alter the potential of the theory.

Firstly, all the SUSY breaking terms could lift the flat directions, making these directions no longer flat, hence resulting in the “moduli fields” receiving masses at a scale determined by the SUSY breaking scale. There are several assumptions we could make about SUSY breaking. For the following discussion, lets make the simple minded assumption that the SUSY breaking terms are at the electroweak scale. For example, this will result in the massless Higgs field mode having a light mass at the electroweak scale. In addition, the moduli fields would not get arbitrary VEVs after SUSY breaking, because those directions have lost their flatness. In fact, it is even possible some of the moduli fields are driven to a VEV of zero depending on the sign of the SUSY breaking terms in the potential. While constructing the models, we consider the possibility that some of the VEVs of the flat directions maybe driven to zero (possibly by SUSY breaking terms).

A discussion of the flat directions after SUSY breaking, will reveal their dependence on the sign of SUSY breaking terms. Lets examine the potential that results from SUSY breaking terms and minimize this potential. We notice that if we choose negative mass parameters in this potential, this tends to result in large VEVs on the scale of the theory’s cutoff. Likewise if the mass parameter is positive, it tends to make the VEVs zero. Consider what happens to the VEVs of the moduli fields. Since flat directions do not appear in the potential before SUSY breaking, their potential is entirely a result of SUSY breaking terms. For example it could be of the following form:

$$V(f) = {m_1}^2 f^2 + \frac{{m_2}^2}{{\Lambda_{cut}}^2} f^4$$

Here we assume m_1 and m_2 are at the electroweak scale, determined by the SUSY breaking scale. The minimum is given by $f = 0$. However, if we change the sign in the potential, so that:

$$V(f) = -{m_1}^2 f^2 + \frac{{m_2}^2}{{\Lambda_{cut}}^2} f^4$$

Then we find that $f = \frac{\Lambda_{cut} m_1}{2 m_2}$. We see that the VEVs of the moduli field will be zero or large depending critically on the signs of the SUSY breaking terms.

In the previous paragraph we found it naturally possible that the naturally are either very large or zero, but there we assumed the quartic term resulted from SUSY breaking and thus had a coupling of the form $\frac{m_2{}^2}{\Lambda_{cut}{}^2}$. However a third possibility exists if f is not truly flat before SUSY breaking, rather just massless (i.e having no quadratic term). It is possible that the quadratic term results from SUSY breaking, but the quartic term is a SUSY invariant term that comes from the superpotential, in which case if the quadratic term is negative, the VEV of f would get stabilized at the electroweak scale because the Lagrangian would be as follows:

$$V(f) = -m_1{}^2 f^2 + \lambda f^4$$

where λ is an $O(1)$ coupling.

2.3 Discussion of Family Symmetry Groups

A general picture of how these models operate has been presented. Now we will present in detail a few of the models. Notice when constructing the models, we basically have to make three decisions: What is the family symmetry group? What are the fields of the model? What $U(1)$ and R-symmetries does the model possess?

Now consider which choices exist for the family symmetry group. In the Standard Model there are six types of fermion fields, q, u, d, l, e, and ν. Likewise in the MSSM there are six chiral superfields for the fermions. It is conceivable that each set of fermions transform under a separate group. For example a three generation model could have each set of fermions transforming under separate $U(3)$'s. In this case the family group would be $U(3)\times U(3)\times U(3)\times U(3)\times U(3)\times U(3)$. For three generation models the most natural groups to consider are $SU(3)$ and $U(3)$. Furthermore, the family group can be much simpler than the product group of six $U(3)$'s. For example if each SM fermion field transforms under the same $U(3)$ then the family group would be simply $U(3)$. Furthermore, we might try to consider even simpler models which do not describe all three generations but just two generations. In such cases the most natural groups to think about are $U(2)$ and $SU(2)$.

2.4 Summary of the Models to be Presented

We present several (five) different models. The first two models we present have only two generations. The purpose of the first model is to get a more detailed feel for these models and their operation. The second model is intended to show that it is possible to get the proper Cabibbo mixing and quark masses for two generations. The purpose of the third model is to illustrate the plausibility of constructing these models with three generations of SM fermions. The purpose of the fourth model is illustrate a model that very naturally has couplings between the Higgs field and the moduli fields of the family sector. The fifth and final model presented is the physically most reasonable, in that it illustrates three generations with fairly reasonable textures for all the Yukawa couplings. Furthermore with the last model we will illustrate the following point: if we try to modify our models to accommodate an a non-zero μ term which is experimentally required, this naturally results in moduli-Higgs couplings. The moduli-Higgs couplings are very interesting because they are a low energy effect that could be observed in the near future at the LHC.

We present each model with the following organization. First, we enumerate what group has been chosen for the family symmetry. Second, we list all the fields which appear in the model. In order to choose the fields of the theory, we need to choose the form of the generalized Higgs field, and pick the fields of the family breaking sector. Then we specify our choice for the $U(1)$ and R-symmetries, and we note the charges of each field under all the symmetries. Next, we write down all permissible terms in the superpotential. The next step is solving for the moduli space of the family breaking sector. Fifthly, we choose which moduli fields if any will have VEVs driven to zero by SUSY breaking terms. Then we find the form of the massless Higgs field, and compute the resulting Yukawa couplings. Finally, we examine the moduli-Higgs couplings.

Chapter 3

Model 1: A 2 Generation Model with Massless Quarks

The first model we present is a two generation model. We consider a simple choice for the family symmetry group, that is $SU(2)$. For purposes of notation we denote the family symmetry group as $SU(2)_{\text{fam}}$. Since the family symmetry is $SU(2)$, all the irreducible representations are defined by the representations of spin. Now that we have chosen the family group, we need to decide how the quark and lepton fields transform. The quark and lepton fields are doublets, because there are two generations. Furthermore, in $SU(2)$ there is no distinction between the fundamental and antifundamental representations, hence we may assume all the SM fermion fields are in the fundamental representation. For simplicity we consider this model to only have quarks and no leptons. Also denote the quark fields as $q_{n,j}$, ${u^c}_l$, and ${d^c}_l$. Notice this is very explicit, showing all the tensor indices. In fact n is an $SU(2)_{\text{weak}}$ index, while j and l are $SU(2)_{\text{fam}}$ indices. The generalized Higgs fields will be in the spin one representation, and the generalized up Higgs fields and down Higgs fields will be respectively denoted as:

$$U_m^{jl} \qquad D_m^{jl}$$

Once again j and l are $SU(2)_{\text{fam}}$ indices, but m is an $SU(2)_{\text{weak}}$ indice. These fields are symmetric in j and l, because the generalized Higgs fields have been chosen to be spin one of $SU(2)_{\text{fam}}$. The family breaking sector has two extra fields. One field that breaks family symmetry, T_{ij}, and the other field is a family singlet, η. Both of the fields are neutral with respect to the Standard Model gauge group. The indices on T_{ij} are family indices, and T_{ij} is a rank two symmetric tensor, thus spin one.

This theory will have a global $U(1)$ symmetry, which will be called T-charge or denoted as $U(1)_T$. The T-charge of T_{ij} is 1. Now we assign the following T-charges to the rest of the fields:

	$q_{i,j}$	${u^c}_k$	${d^c}_l$	U_m^{jk}	D_m^{jl}	T_{ij}	η
$U(1)_T$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	-2

Additionally, this model will have an R-symmetry, where every field has an R-symmetry charge of $\frac{2}{3}$. As as result of this symmetry only cubic terms can appear in the superpotential. We have the following set of terms in the superpotential:

1) Terms that couple the Higgs fields to the quarks.

$$W_{\text{quarks}} = \lambda_U \epsilon^{im} q_{i,j} U_m^{jk} {u^c}_k + \lambda_D \epsilon^{im} q_{i,j} D_m^{jl} {d^c}_l$$

2) Terms that couple the Higgs fields to T_{ij}.

$$W_{\text{Higgs}} = \lambda_T \epsilon^{mn} \epsilon_{kl} U_m^{ik} D_n^{jl} T_{ij}$$

3) Terms that make a potential for T_{ij} and η.

$$W_{\text{potential}} = \lambda_\eta \eta \epsilon^{ac} \epsilon^{bd} T_{ab} T_{cd}$$

In this particular model we do not assume any of the family breaking fields are driven to zero by SUSY breaking. In order to study the moduli space of the family breaking fields, we make a digression on the $SU(2)$ properties of T_{ij}.

First of all, T_{ij} is a spin one object, so that it lives in the vector representation of $SU(2)$. However, it lives in a 6 dimensional space, because it is defined by 3 complex parameters. So, we conclude T_{ij} can be defined in terms of a 3 dimensional complex vector, that transform under $SO(3)$. Lets denote this vector as $\vec{v} = \{x, y, z\}$ where x, y, and z are complex.

$$T_{ij} = \epsilon_{im} \sigma_{a\,j}^{\;m} v^a = \begin{pmatrix} x + iy & -z \\ -z & -x + iy \end{pmatrix} \tag{3.1}$$

Now we can think of this complex vector as being defined by its real part call it $\vec{v}_1$ and its imaginary part $\vec{v}_2$.

Now we assume that no "special" relations exist between $\vec{v}_1$ and $\vec{v}_2$. Then we can simplify T_{ij} most by making rotations such that $\vec{v}_1$ is along the x-axis and $\vec{v}_2$ is in the xy-plane. This gives:

$$\begin{aligned} T_{ij} &= \begin{pmatrix} x_1 + ix_2 - y_2 & 0 \\ 0 & -x_1 - ix_2 - y_2 \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \end{aligned}$$

Where α and β are by definition the elements T_{11} and T_{22} after T_{ij} is diagonalized.

It will be useful to quickly convert between the notation of symmetric tensors, T_{ij}, to the vector notation, $\vec{v}$. Now suppose that the tensors S_{ij} has a corresponding complex vector $\vec{w}$. Likewise, R_{ij} has a corresponding complex $\vec{u}$. Then we have the following useful relations to convert between tensor notation of $SU(2)$ to vector notation of $SO(3)$:

$$\epsilon^{ik} \epsilon^{jl} T_{ij} S_{kl} = 2\vec{v} \cdot \vec{w}$$

$$\epsilon^{ik} T_{ij} S_{kl} R^{jl} = 2\vec{w}^{\star} \cdot (\vec{u} \times \vec{v})$$

Note by definition $R^{jl} = (R_{jl})^\star$ where the star denotes complex conjugation. These relations are easy to verify.

Now we return to discussing model 1, and we consider how to solve for the VEVs of T_{ij}. Solving for the moduli space of the family breaking sector requires that we find the singlet F-terms, in this case the η F-term. Next it requires setting these F-terms equal to zero. Examine the F-term of η.

$$\eta: \qquad \lambda_\eta \epsilon^{ik}\epsilon^{jl} T_{ij} T_{kl}$$

Setting this equal to zero, and using the fact that we can diagonalize T_{ij}.

$$\begin{aligned} T_{11}T_{22} - T_{12}T_{12} &= 0 \\ \alpha\beta &= 0 \end{aligned}$$

So we have a discrete choice. Lets make the following choice:

$$T_{11} = \alpha \neq 0 \qquad T_{22} = \beta = 0$$

Since, T_{11} can have an arbitrary VEV, it is in fact a flat direction(moduli field).

Next we determine the massless mode of the Higgs fields, we do this by plugging in the VEV of T_{ij} into the Higgs-Higgs-T trilinear term. After completing this substitution we get:

$$W_{\text{Higgs}} = \lambda_T(\alpha\epsilon^{mn} U^{11}_m D^{12}_n - \alpha\epsilon^{mn} U^{12}_m D^{11}_n)$$

The mass terms for the Higgs fields are given by summing the squares of the Higgs F-terms. In this simple model we enumerate all these F-terms, and by inspection will see which components of the Higgs fields do not appear in these F-terms, thus these components must correspond to zero mode(s). Now lets enumerate the Higgs fields

F-terms.

$$\begin{aligned} \epsilon^{mn}U^{11}_m &: \quad \alpha D^{12}_n \\ \epsilon^{mn}U^{12}_m &: \quad -\alpha D^{11}_n \\ \epsilon^{mn}U^{22}_m &: \quad 0 \end{aligned}$$

$$\begin{aligned} \epsilon^{mn}D^{11}_n &: \quad -\alpha U^{12}_m \\ \epsilon^{mn}D^{12}_n &: \quad \alpha U^{11}_m \\ \epsilon^{mn}D^{22}_n &: \quad 0 \end{aligned}$$

From inspection, we see the light mode of the Higgs field is given as:

$$D^{light}_n = D^{22}_n$$

$$U^{light}_m = U^{22}_m$$

Now for the purpose of illustration, we compute the moduli space and the form of the massless Higgs field again, this time using the $SO(3)$ notation. Recall for the tensor T_{ij} we can define a complex vector, $\vec{v}$. In terms of this complex vector the relevant superpotential term is:

$$2\lambda_\eta \eta \vec{v} \cdot \vec{v}$$

Therefore by setting the η F-term to zero, and then considering the real and imaginary parts separately we get:

$$\vec{v} \cdot \vec{v} = 0$$

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_1 - \vec{v}_2 \cdot \vec{v}_2 &= 0 \\ \vec{v}_1 \cdot \vec{v}_2 &= 0 \end{aligned}$$

We see the vectors $\vec{v}_1$ and $\vec{v}_2$ are perpendicular, and the vectors also have equal magnitudes. So, we can rotate into a coordinate system where $\vec{v}_1$ is along the positive x-axis, and $\vec{v}_2$ is along the negative y-axis, which gives:

$$\vec{v}_1 = \{x_1, 0, 0\} \qquad\qquad \vec{v}_2 = \{0, -x_1, 0\}$$

$$\vec{v} = x_1 \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$T_{ij} = \begin{pmatrix} 2x_1 & 0 \\ 0 & 0 \end{pmatrix}$$

Notice this is the same result as before with $\alpha = 2x_1$.

Now we examine the superpotential terms, W_{Higgs}. Notice when we discussed spin one tensors such as T_{ij} with the corresponding complex vector $\vec{v}$, the indices on T_{ij} were lower indices. However, the tensor T^{ij} also has a corresponding complex vector, $\vec{v}'$, whose components determine T^{ij} via equation 1. If we use the complex vectors corresponding to the upper indices, we can write down W_{Higgs} of this model in $SO(3)$ notation. Note we are suppressing $SU(2)_{\text{weak}}$ indices. Also $\vec{u}$ and $\vec{d}$ correspond to U^{jk}

and D^{jk} respectively.

$$\begin{aligned} W_{\text{Higgs}} &= 2\lambda_T(\vec{v}')^* \cdot (\vec{u} \times \vec{d}) \\ &= 2\lambda_T\vec{d} \cdot [(\vec{v}')^* \times \vec{u}] \\ &= 2\lambda_T\vec{u} \cdot \left[\vec{d} \times (\vec{v}')^*\right] \end{aligned}$$

First of all, this superpotential we just wrote down contains a complex conjugate of $\vec{v}'$, which is worrisome because superpotential are holomorphic functions. However, this conjugate is just a notational artifact. Notice, we wrote down three equivalent forms for the superpotential, to find the down Higgs field F-terms the second form is the convenient choice. All the down Higgs field F-terms can be written compactly as:

$$2\lambda_T(\vec{v}')^* \times \vec{u}$$

In order to find the massless directions of $\vec{u}$, we look for directions which do not effect the potential. This is accomplished by setting Higgs field F-terms equal to zero. Thus setting this cross product equal to zero implies that the two complex vectors $(\vec{v}')^*$ and $\vec{u}$ are "parallel". In this case "parallel" means the two vectors are proportional through a complex proportionality constant. Hence the light up Higgs field must be "parallel" to the VEV of $(\vec{v}')^*$. This condition gives the following form for the light up Higgs field:

$$\vec{u} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

This complex vector converts to the following spin one tensor:

$$U^{ij} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

This means U^{22} is exactly massless! This is the result we obtained earlier using $SU(2)$

notation. Likewise you can find the light down Higgs field mode too. This notation is nice for finding zero modes because cross products result in “parallel” conditions.

Next we compute the Yukawa couplings of the low energy effective theory, by setting the heavy components of the Higgs field equal to zero. In other words, we plug in the form of the light Higgs field into the superpotential of the model, and then look at the coupling between the light Higgs field and quark fields. This gives the following:

$$W_{\text{quarks}} = \lambda_U \epsilon^{im} U_m^{\text{light}} q_{i,2} {u^c}_2 + \lambda_D \epsilon^{im} D_m^{\text{light}} q_{i,2} {d^c}_2$$

These Yukawa couplings actually imply that the first generation of quarks are massless. In addition, since one generation is massless there can be no Cabibbo mixing. If we refer to the electroweak VEVs of the up and down Higgs fields as v_{up} and v_{down} then the mass matrices of the quarks are given as:

$$M_{\text{up}} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_U v_{\text{up}} \end{pmatrix} \qquad M_{\text{down}} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_D v_{\text{down}} \end{pmatrix}$$

Notice, these mass matrices only give masses to the second generation of quarks.

In this elementary model there is no coupling between moduli fields and the light Higgs field. This can be seen because the moduli field (the flat direction) is T_{11} which does not couple to D_n^{22} or U_n^{22}, the light Higgs field modes. As explained earlier it had to work out this way, because this model does not contain a moduli field that gets a VEV of zero.

This first model has a very distasteful feature. Mainly that it leaves one generation of SM fermions massless. As we know in nature all the SM fermions have masses. Although at one point in time the masslessness of the up quark was considered a possibility. However, it was never considered a possibility that a whole generation of particles could be massless. The pleasant thing about this model is it naturally has one generation of quarks being much heavier than another, a feature that is seen in nature. However in our model this results from the light generation being exactly massless. Now we will explore another two generation model, which does not have these bad features.

Chapter 4

Model 2: A 2 Generation Model with Cabibbo Mixings

This next model, which we will refer to as model 2 [14] is a two generation model, and will have the same family group, $SU(2)$. From now on we refer to the first model as "model 1". All the SM fermion fields in model 2 will have the same transformation laws as model 1. The generalized Higgs field will actually be in a reducible representation of $SU(2)$, more explicitly there will be a pair of spin one generalized Higgs fields. We denote the generalized Higgs fields as:

$$[U_1]^{jk}_m \quad [U_2]^{jk}_m \quad [D_1]^{jk}_m \quad [D_2]^{jk}_m$$

This model has a pair of generalized up Higgs fields and generalized down Higgs fields. Each generalized Higgs field transform as a spin one object, symmetric rank 2 tensor, under $SU(2)_{\text{fam}}$. Recall, the generalized Higgs field in model 1 was also spin one under $SU(2)_{\text{fam}}$. Also the T family breaking fields are more complicated. T family breaking fields will be made up of a set of five spin one fields, only one of which we actually label as T. The five T fields are:

$$X_{ij} \quad T_{ij} \quad S_{ij} \quad R_{ij} \quad P_{ij}$$

Model 2 will also have three singlet fields, η, ξ, and ζ.

We introduce four $U(1)$ global symmetries, each associated with a generalized Higgs field field. Now we list the various $U(1)$ charges on the fields in the following table:

field	$U(1)_E$	$U(1)_F$	$U(1)_G$	$U(1)_H$
U_1	1	0	0	0
U_2	0	1	0	0
D_1	0	0	1	0
D_2	0	0	0	1
$q_{i,j}$	0	0	0	0
${u^c}_k$	0	-1	0	0
${d^c}_l$	0	0	0	-1
X_{ij}	0	0	0	0
T_{ij}	-1	0	-1	0
S_{ij}	0	-1	-1	0
R_{ij}	-1	0	0	-1
P_{ij}	0	-1	0	-1
η	0	0	0	0
ξ	0	1	1	0
ζ	1	0	0	1

Notice, that each generalized Higgs field is charged under a different $U(1)$. Furthermore, we assume every field has an R-charge of $\frac{2}{3}$. (So that only cubic terms are allowed)

We have all the information necessary to construct the superpotential for Model 2. Lets write down all the permissible terms. We have the following set of terms in the superpotential:

1) Terms that couple the Higgs fields to the quarks.

$$W_{\text{quarks}} = \lambda_U \epsilon^{im} q_{i,j} {U_2}^{jk}_{m} {u^c}_k + \lambda_D \epsilon^{im} q_{ij} {D_2}^{j,l}_{m} {d^c}_l$$

2) Terms that couple the Higgs fields to the family breaking fields.

$$\begin{aligned} W_{\text{Higgs}} &= \lambda_T \epsilon^{mn} \epsilon_{kl} {U_1}^{ik}_{m} {D_1}^{jl}_{n} T_{ij} + \lambda_S \epsilon^{mn} \epsilon_{kl} {U_2}^{ik}_{m} {D_1}^{jl}_{n} S_{ij} \\ &+ \lambda_R \epsilon^{mn} \epsilon_{kl} {U_1}^{ik}_{m} {D_2}^{jl}_{n} R_{ij} + \lambda_P \epsilon^{mn} \epsilon_{kl} {U_2}^{ik}_{m} {D_2}^{jl}_{n} P_{ij} \end{aligned}$$

3) Terms that make a potential for the family breaking fields and scalars.

$$W_{\text{potential}} = \lambda_\eta \eta \epsilon^{ac}\epsilon^{bd} X_{ab} X_{cd} + \lambda_\xi \xi \epsilon^{ac}\epsilon^{bd} X_{ab} S_{cd} + \lambda_\zeta \zeta \epsilon^{ac}\epsilon^{bd} X_{ab} R_{cd}$$

Now in model 2 we assume that SUSY breaking terms will drive the VEVs of P_{ij} to zero.

Next we want to determine the moduli space of this model's family breaking sector. Once again, we have to compute the singlet F-terms, in the case for η, ξ, and ζ. We convert to the $SO(3)$ notation to find the moduli space and the form of the light Higgs field mode. For the complex vectors we use lower case letters. For example the complex vector $\vec{x}$ corresponds to X_{ij}. So we write the family breaking superpotential in $SO(3)$ notation as:

$$W_{\text{potential}} = 2\lambda_\eta \eta \vec{x}\cdot\vec{x} + 2\lambda_\xi \xi \vec{x}\cdot\vec{s} + 2\lambda_\zeta \zeta \vec{x}\cdot\vec{r}$$

Now we examine the η F-term:

$$2\lambda_\eta \eta \vec{x}\cdot\vec{x} = 0$$

The solution of this equation was found in model 1, and the solution is any vector whose real part is orthogonal to its imaginary part, and the real part has the same magnitude as the imaginary part. This vector can be simplified by rotating to a frame where the real part is in the positive x direction, and then by rotating about the x-axis such that the imaginary part is in the negative y direction. After these rotations the solutions for $\vec{x}$ is:

$$\vec{x} = \begin{pmatrix} \alpha \\ -i\alpha \\ 0 \end{pmatrix}$$

Now the ξ F-term gives the following constraint:

$$2\lambda_\xi \xi \vec{x} \cdot \vec{s} = 0$$

$$\alpha(s_1 - is_2) = 0$$

This constraint tells us the form of $\vec{s}$ is:

$$\vec{s} = \begin{pmatrix} s_1 \\ -is_1 \\ s_3 \end{pmatrix}$$

Furthermore, in exactly analogous manner the ζ F-term will give us the form for $\vec{r}$:

$$\vec{r} = \begin{pmatrix} r_1 \\ -ir_1 \\ r_3 \end{pmatrix}$$

Notice, we found no constraints on T_{ij}, this is because every component of T_{ij} is a flat direction. Therefore T_{ij} has completely arbitrary VEVs, and we shall parametrize the corresponding complex vector $\vec{t}$ simply as:

$$\vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

Lets determine which modes of the Higgs fields are massless. As with model 1 we write down the Higgs superpotential terms in $SO(3)$ notation. Following the example of model 1, for the tensor X^{ij} the corresponding complex vector is $\vec{x}'$. Likewise there are corresponding complex vectors for the tensors T^{ij},R^{ij},S^{ij}, and P^{ij}. Now we convert the Higgs superpotential to $SO(3)$ notation:

$$
\begin{aligned}
W_{\text{Higgs}} &= 2\lambda_X(\vec{t}')^* \cdot (\vec{u}_1 \times \vec{d}_1) + 2\lambda_S(\vec{s}')^* \cdot (\vec{u}_2 \times \vec{d}_1) \\
&\quad + 2\lambda_R(\vec{r}')^* \cdot (\vec{u}_1 \times \vec{d}_2) + 2\lambda_P(\vec{p}')^* \cdot (\vec{u}_2 \times \vec{d}_2)
\end{aligned}
$$

In order to find the massless up Higgs field, we first use vector identities to rewrite the superpotential as:

$$
\begin{aligned}
W_{\text{Higgs}} &= 2\lambda_T \vec{d}_1 \cdot \left[(\vec{t}')^* \times \vec{u}_1\right] + 2\lambda_S \vec{d}_1 \cdot [(\vec{s}')^* \times \vec{u}_2] \\
&\quad + 2\lambda_R \vec{d}_2 \cdot [(\vec{r}')^* \times \vec{u}_1] + 2\lambda_P \vec{d}_2 \cdot [(\vec{p}')^* \times \vec{u})]
\end{aligned}
$$

Now we can find the down Higgs field F-terms in the following compact form:

$$
2\lambda_T(\vec{t}')^* \times \vec{u}_1 + 2\lambda_S(\vec{s}')^* \times \vec{u}_2 = \vec{0} \tag{4.1}
$$

$$
2\lambda_R(\vec{r}')^* \times \vec{u}_1 + 2\lambda_P(\vec{p}')^* \times \vec{u}_2 = \vec{0} \tag{4.2}
$$

The massless up Higgs fields modes are any modes which satisfy the above constraints, though we need to substitute in the VEVs of $\vec{t}'$, $\vec{s}'$, $\vec{r}'$, and $\vec{p}'$. The last term is zero because the VEV of $\vec{p}' = \vec{0}$. Thus the second constraint implies that $\vec{u}_1$ is parallel to $(\vec{r}')^*$. So:

$$
\vec{u}_1 = C_1(\vec{r}')^*
$$

Where C_1 is an arbitrary constant. Now, we substitute this into the first constraint to derive a constraint on $\vec{u}_2$.

$$
\vec{0} = 2C_1\lambda_T(\vec{t}')^* \times (\vec{r}')^* + 2\lambda_S(\vec{s}')^* \times \vec{u}_2
$$

The only way these two cross products might cancel is if they are parallel. If they are parallel then their cross product with each other is zero. So this allows us to write the following constraint:

$$\left[(\vec{t}')^* \times (\vec{r}')^*\right] \times [(\vec{s}')^* \times \vec{u}_2] = \vec{0}$$

Now use the following vector identity, $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$. The constraint becomes:

$$\left[\vec{A} \cdot \vec{u}_2\right] (\vec{s}')^* - \left[\vec{A} \cdot (\vec{s}')^*\right] \vec{u}_2 = \vec{0}$$

Where we define $\vec{A} \equiv (\vec{t}')^* \times (\vec{r}')^*$. This equation implies that either $(\vec{s}')^*$ is parallel to $\vec{u}_2$, or the coefficients of both these vectors vanish. It is straight forward to verify that the coefficient of $\vec{u}_2$ does not vanish, hence these two vectors must be parallel. If these two vector are parallel then the second term of equation 4.1 is zero, which implies that the first term of equation 4.1 is also zero. The first term of equation 2 can only be zero if $C_1 = 0$, because $(\vec{t}')^*$ and $(\vec{r}')^*$ are not parallel. Thus we find the massless mode of the up Higgs field has the form:

$$\vec{u}_1 = 0 \qquad \vec{u}_2 = \begin{pmatrix} s_1 \\ i s_1 \\ s_3 \end{pmatrix}$$

Now convert this into the $SU(2)$ spin one tensor notation:

$$U_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 0 & -s_3 \\ -s_3 & -2s_1 \end{pmatrix}$$

For the purpose of notation, we introduce the variables β and γ such that the light up Higgs field is given by:

$$U_2 = \begin{pmatrix} 0 & -s_3 \\ -s_3 & -2s_1 \end{pmatrix} = \begin{pmatrix} 0 & -\gamma \\ -\gamma & \beta \end{pmatrix}$$

Furthermore, we can repeat all this analysis to find the massless down Higgs field:

$$D_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 0 & -r_3 \\ -r_3 & -2r_1 \end{pmatrix} = \begin{pmatrix} 0 & -\tau \\ -\tau & \omega \end{pmatrix}$$

Notice we introduce the variables τ and ω. For the next section, where the Yukawa couplings between quarks and the light Higgs field is computed, we need the light Higgs field to be canonically normalized. Here we list the light Higgs field modes with proper normalizations:

$$U_2 = \frac{1}{\sqrt{2\gamma^2 + \beta^2}} \begin{pmatrix} 0 & -\gamma \\ -\gamma & \beta \end{pmatrix} U_{\text{light}}$$

$$D_2 = \frac{1}{\sqrt{2\tau^2 + \omega^2}} \begin{pmatrix} 0 & -\tau \\ -\tau & \omega \end{pmatrix} D_{\text{light}}$$

Where U_{light} and D_{light} are the massless up and down Higgs field respectively.

It is straight forward to calculate the Yukawa couplings between the quark fields and the light Higgs field. In a manner similar to model 1, we plug in the form of the light Higgs field to quark-Higgs coupling terms of the superpotential. We get the following result for Yukawa coupling matrices:

$$\frac{\lambda_U}{\sqrt{2\gamma^2 + \beta^2}} \begin{pmatrix} 0 & -\gamma \\ -\gamma & \beta \end{pmatrix} \qquad \frac{\lambda_D}{\sqrt{2\tau^2 + \omega^2}} \begin{pmatrix} 0 & -\tau \\ -\tau & \omega \end{pmatrix}$$

If we make the assumptions that γ is small compared to β, and τ is small compared to ω then we can easily compute the Cabibbo mixing angle and quark mass matrices. The above assumptions gives a simple result, because the $SU(2)$ transformations that are applied to the Yukawa coupling matrices are near the identity, thus these $SU(2)$ transformations can be easily approximated. For example, the following transformation diagonalizes the matrix of up Yukawa couplings:

$$e^{i(\frac{\gamma}{\beta})\sigma_y}\begin{pmatrix} 0 & -\gamma \\ -\gamma & \beta \end{pmatrix} e^{-i(\frac{\gamma}{\beta})\sigma_y} \approx \begin{pmatrix} 1 & \frac{\gamma}{\beta} \\ -\frac{\gamma}{\beta} & 1 \end{pmatrix}\begin{pmatrix} 0 & -\gamma \\ -\gamma & \beta \end{pmatrix}\begin{pmatrix} 1 & -\frac{\gamma}{\beta} \\ \frac{\gamma}{\beta} & 1 \end{pmatrix}$$
$$\approx \begin{pmatrix} -\frac{\gamma^2}{\beta} & 0 \\ 0 & \beta \end{pmatrix}$$

In fact, the above Yukawa couplings will lead to the following well known result [15]:

$$\theta_{\text{Cabibbo}} \approx \frac{\tau}{\omega} - \frac{\gamma}{\beta}$$

$$M_{\text{up}} = \frac{\lambda_U v_{\text{up}}}{\sqrt{2\gamma^2+\beta^2}}\begin{pmatrix} \frac{\gamma^2}{\beta} & 0 \\ 0 & \beta \end{pmatrix} \qquad M_{\text{down}} = \frac{\lambda_D v_{\text{down}}}{\sqrt{2\tau^2+\omega^2}}\begin{pmatrix} \frac{\tau^2}{\omega} & 0 \\ 0 & \omega \end{pmatrix}$$

Recall that v_{up} and v_{down} are the electroweak VEVs for the up and down Higgs fields respectively.

In this model, we see that the mass hierarchy actually results from a hierarchy in the VEVs of the flat directions. Explicitly the VEVs γ and τ were assumed to be small compared with the VEVs β and ω. This model nicely predicts the structure of the quark mass matrices and quark mixing angle for two generations.

4.1 General Considerations of Higg-Moduli Couplings

Now we shall discuss the moduli coupling to the light Higgs field. In the Lagrangian of the low energy effective theory the following term is not prohibited by symmetry considerations:

$$\lambda_{\text{mod}} U^{\dagger} U F^{\star} F$$

Here U is the light up Higgs field and F is a moduli field. We want to explore when these terms will actually arise from our models. Schematically the superpotential term of interest is:

$$UDT$$

First we note these coupling terms contains many heavy modes and some light modes, but we are only interested in interactions which involve only light modes. In order to isolate the interactions of the light fields we integrate out the heavy modes of the theory.

First lets consider how we integrate out the heavy components in our models. The idea will be as follows: use the equations of motion for the heavy components, and then solve for the heavy components in terms of the light components. Finally, substitute these solutions back into the Lagrangian of the theory, hence obtaining the low energy effective Lagrangian. Lets explore these ideas more carefully. Suppose we have a heavy set of fields call them H_i, also a set of light fields L_j, and finally a set of flat directions F_l, recall "flat directions" are fields such that if we set all the other fields in the theory to their VEVs, the flat directions disappear from the potential. Furthermore, the potential of the theory we label as $V(H_i, L_j, F_l)$. For example suppose the VEVs of H_i and L_j are all zero, then the function $V(0, 0, F_l)$ will be independent of F_l (i.e. a constant).

In a supersymmetric theory the potential $V(H_i, L_j, F_l)$ will be positive definite, and the VEVs of the non-flat directions, H_i and L_j, consequently zero. Furthermore in supersymmetric theories the minimum of $V(H_i, L_j, F_l) = 0$, so this implies that $V(0, 0, F_l) = 0$. We are interested in determining the potential after we integrate out the heavy fields:

$$V_{\text{eff}}(L_j, F_l) = V(H_i(L_j, F_l), L_j, F_l)$$

There we have solved for H_i using their equations of motion. Now if we set, L_j to their VEVs of zero, and F_l to any arbitrary VEV then $H_i(0, F_l)$ should equal the VEVs of H_i, which are zero. Therefore we make the following conclusion:

$$V_{\text{eff}}(0, F_l) = V(H_i(0, F_l), 0, F_l) = V(0, 0, F_l) = 0$$

In order to explore the physics of this "effective potential", we Taylor expand it in terms of its fields, L_j and F_l. However it vanishes when $L_j = 0$, which implies all of its terms must contains at least one power of L_j. In other words there are no self interactions of F_l. More generally combining this sort of reasoning with the considerations of symmetry restricts the permissible terms to a small set.

Now we consider all the fields in model 2, and the renormalizable couplings that result after integrating out the heavy modes. We use the following notation H_i to refer to all the heavy fields, U and D to refer to the light up and down Higgs fields, and F_k to refer to the flat directions in X_{ij}, T_{ij}, R_{ij}, and S_{ij}. And we refer directly to P_{ij}. We make some observations, all which are made by writing down the full potential for model 2, and then inspecting it carefully. First observation, if P_{ij} gets nonzero VEVs then U and D will not be flat, and hence are driven to VEVs of zero. Next observation is if U has a nonzero VEV then P_{ij} and D are not flat, and driven to VEVs of zero. Similarly, we observe if D has a nonzero VEV then P_{ij} and U are not flat, and driven to VEVs of zero. We know after integrating out all the heavy modes and setting all the non-flat fields to their VEVs, their "effective potential", $V_{\text{eff}}(U, D, P_{ij}, F_k)$ should vanish. For example if U and D are set to zero then all the remaining fields are flat directions and hence the effective potential vanishes. We Taylor expand the effective potential, in order to isolate the renormalizable terms. Since the potential vanishes when $U = D = 0$, each term in the effective potential must contain at least one power of U or D. Likewise by analogous reasoning we also conclude each term of the effective potential contains at least one power of U or P_{ij}. Also each term must contain at least one power of U or P_{ij}. Furthermore, symmetry considerations imply any term that contains U should also contain its complex conjugate $U^\dagger$. Likewise every term with P_{ij} contains $(P_{ij})^\star$, and every term with D contains $D^\dagger$. However, integrating out the heavy fields has destroyed the family symmetry, hence we allow terms which violate family symmetry. We see the lowest order terms are fourth order terms, these are precisely the renormalizable interactions we are interested in determining. These interactions have the following generic form:

$$\sum_{ij} B_{ij} U^\dagger U (P_{ij})^\star P_{ij} + \sum_{ij} C_{ij} D^\dagger D (P_{ij})^\star P_{ij} + E U^\dagger U D^\dagger D$$

Where B_{ij}, C_{ij}, and E are real positive constants, which follows because the potential is positive definite.

Now we discuss how we actually integrate out the heavy modes, and find these renormalizable interactions. Hence we want to determine $H_i = H_i(U, D, P_{ij}, F_k)$, once again note that if U and D are set to zero then all the other directions are flat, and hence H_i equals its VEVs which are all zero. By exactly the same reasoning H_i equals zero if P_{ij} and U are set to zero. And finally H_i equals zero if P_{ij} and D are set to zero. Once again we do a Taylor expansion, in this case we expand H_j in terms of the light fields of model 2. Since H_j vanishes under the three conditions listed above we conclude that each term must contain one of the following three factors UP_{ij}, DP_{ij}, or UD.

Writing out the potential of the full theory explicitly will reveal that it's a quartic polynomial in the heavy fields. Schematically it will look like:

$$V(H_i, U, D, P_{ij}, F_k) = a(H_i)^4 + b(H_i)^3 + c(H_i)^2 + d(H_i) + e$$

Here the coefficients b, c, d, and e are actually functions of all the light fields in the theory. Furthermore, we wish to use the equations of motions of H_j to solve for H_j. Firstly, we notice that e does not appear in the equations of motion. In these equations we are thinking of H_j as small, because we are thinking of P_{ij} U, and D as small, therefore we can neglect the quartic and cubic terms in comparison to the quadratic and lower terms. H_i is heavy and the quadratic terms corresponds to the mass terms, therefore c must contain zeroth order terms (i.e. terms which do not depend on P_{ij}, U, or D). Here c actually corresponds to the mass matrix of H_i, thus the lowest order term of the solution for H_i is:

$$H_i = -\frac{1}{2}(c_0)^{-1} d_2 \tag{4.3}$$

where c_0 is a matrix and the zeroth order term of c. Likewise d_2 is a vector and the lowest order term in d is quadratic in the fields P_{ij}, U, and D. This lowest order term of H_i gives renormalizable interactions if we substitute it into the potential of the full theory. In fact we find only quartic terms. This process for model 2 does not give a clean a result, but does verify that these interactions actually exist and are non-zero.

Chapter 5

Model 3: A 3 Generation Model using Adjoint Higgs fields

We now present a three generation model. In constructing such models, we have many choices for the family breaking fields and the generalized Higgs fields. However, it is not obvious that any particular choice of fields, will result in exactly one massless Higgs field mode. In fact, it is possible to construct models with no massless Higgs field modes, or too many massless Higgs field modes. We previously showed it is possible to construct two generation models which have one massless Higgs field mode, but it is not obvious that such a construction can be generalized to include three generation models. Now we shall present a model that demonstrates, the possibility of three generation models with nontrivial Yukawa couplings. We refer to this model as "model 3" in the rest of our discussion.

The family symmetry group we choose is $G_{\rm fam} = SU(3)$. Furthermore, the SM fermions will transform under the fundamental and antifundamental representations. That is the quark doublet, q^i, will transform under the fundamental representation, and the up and down quark fields, ${u^c}_j$ and ${d^c}_k$ transform as antifundamentals. Furthermore, l_i is the lepton doublet which transforms under the fundamental representation, and the right handed leptons, ${e^c}_j$ and ${\nu^c}_k$, transform as antifundamentals.

Now we consider generalized Higgs fields that transform as $SU(3)_{\rm fam}$ octets, this suggests the following type of term for the quark fields:

$$u^c{}_i U^i{}_j q^j$$

Before we go into all the gory details of model 3, we would like to discuss a few illustrious examples that motivates our interest in octet Higgs fields.

Before we consider Model 3, we consider toy superpotential Higgs terms. Our goal is to see mechanisms which give exactly one massless Higgs field mode. We do not justify the superpotentials terms based on symmetries, rather accept the superpotential terms as given and explore their consequences. Consider the following superpotential terms:

$$W_{\text{Higgs}} = \lambda_{UD} U^i{}_j D^j{}_k T^k{}_i + \lambda_{DU} D^i{}_j U^j{}_k T^k{}_i$$

Firstly, all three fields live in the octet of $SU(3)$, that is we are assuming U, D, and T are traceless matrices. Note λ_{UD} and λ_{DU} are arbitrary dimensionless constants. Furthermore, we will assume that T satisfies a constraint such that T^{-1} is traceless. This assumption may appear mysterious at first glance.

Therefore, we would like to show that such a constraint is plausible. We know that the family breaking sector tends to give constraint equations on the family breaking T fields. So for example, a plausible constraint that might result from the family breaking sector is:

$$T^i{}_j T^j{}_i = 0$$

We can easily verify the following formula for the inverse of T:

$$\left[T^{-1}\right]^b{}_a = \frac{\epsilon_{ace}\epsilon^{bdf} T^c{}_d T^e{}_f}{2\mathsf{Det}|T|}$$

Now use the above equation to calculate the trace of T^{-1}

$$\begin{aligned}\left[T^{-1}\right]^a{}_a &= \frac{\epsilon_{ace}\epsilon^{adf}T^c{}_dT^e{}_f}{2\mathsf{Det}|T|} \\ &= \frac{(\delta^d_c\delta^f_e - \delta^d_e\delta^f_c)T^c{}_dT^e{}_f}{2\mathsf{Det}|T|} \\ &= \frac{T^c{}_cT^e{}_e - T^c{}_eT^e{}_c}{2\mathsf{Det}|T|} \\ &= \frac{-T^c{}_eT^e{}_c}{2\mathsf{Det}|T|}\end{aligned}$$

In the above manipulations we used the fact that T is traceless. From the above equation we see that T^{-1} is traceless, if it obeys the constraint $T^i{}_jT^j{}_i = 0$.

Now return to considering the Higgs superpotential, which we enumerated above. Since D and U are traceless it is useful to parameterize these matrices by 8 components using the Gell-Mann matrices, that is:

$$U = U^a\lambda_a \qquad D = D^a\lambda_a$$

We use this parameterization because each component D^a and U^a are all canonically normalized, and thus are proper components for deriving F-terms. For example, lets find the up Higgs field F-terms (i.e the U^a F-terms):

$$\mathsf{Tr}\left[\lambda_a(\lambda_{UD}DT + \lambda_{DU}TD)\right] = 0$$

Here we have set the F-terms equal to zero, because we are looking for the zero modes of the down Higgs field. This equation is of the following form:

$$\mathsf{Tr}\left[\lambda_aM\right] = 0 \tag{5.1}$$

Here M is just any arbitrary matrix, we can write any 3×3 matrix M in the following form:

$$M = \mathbf{I}M^0 + \lambda_b M^b$$

Note b runs 1 through 8. If we substitute in this form of M into equation 5.1 , we find that all M^b except M^0 must vanish, that is $M \propto \mathbf{I}$. We use this fact over and over again, when analyzing the F-terms of the Higgs field sector. Each time we use this fact, we introduce a new proportionality constant, which we are free to choose. We will denote these proportionality constants as c_i. Hence we write down the following equation:

$$\lambda_{UD}DT + \lambda_{DU}TD = c_1\mathbf{I}$$

It is easy to see that this equation is solved by $D = \frac{c_1}{\lambda_{UD}+\lambda_{DU}}T^{-1}$. In fact this is the most general solution, a fact we prove later when we discuss solving these equations in more detail. The key point is the tracelessness of T^{-1}. Hence the solution we found for D is also traceless, which is required since D lives in the octet representation, if T^{-1} had not been traceless, we would have been forced to conclude that $c_1 = 0$. This would imply that there is no massless mode contained in D, thus the tracelessness of T^{-1} was key to generating a massless mode. The down Higgs field F-terms are:

$$\mathsf{Tr}\left[\lambda_a(\lambda_{UD}TU + \lambda_{DU}UT)\right]$$

Setting these F-terms to zero, tell us we get a single massless up Higgs field also of the form $U \propto T^{-1}$. We have just presented a mechanism that appears to be a plausible method for naturally getting exactly one pair of massless Higgs fields, in 3 generation models. However, in this case the up Higgs field and down Higgs field have exactly the same form, which is very problematic in terms of mixing angles, and mass ratios of quarks.

Now we discuss another simple model, which is interesting because it suggests that we can construct 3 generation models with the massless up Higgs field and down

Higgs field of different forms. The superpotential terms we consider are:

$$\lambda_T U_1 D_1 T + \lambda_S U_1 D_2 S + \lambda_R U_2 D_1 R$$

Once again, U_1, U_2, D_1, D_2, T, S, and R are all traceless matrices. Furthermore we assume that R is constrained such that R^{-1} is traceless. Now consider the up Higgs field F-terms, and set them equal to zero:

$$\begin{aligned} \mathsf{Tr}\left[\lambda_a(\lambda_T D_1 T + \lambda_S D_2 S)\right] &= 0 \\ \mathsf{Tr}\left[\lambda_a(\lambda_R D_1 R)\right] &= 0 \end{aligned}$$

Or equivalently:

$$\begin{aligned} \lambda_T D_1 T + \lambda_S D_2 S &= c_1 \mathbf{I} \\ \lambda_R D_1 R &= c_2 \mathbf{I} \end{aligned}$$

We can easily solve the second equation for D_1, and get $D_1 = \frac{c_2}{\lambda_R} R^{-1}$, recall R^{-1} is traceless, hence this a valid solution for D_1. Now we solve for D_2 in the second equation in terms of D_1:

$$\begin{aligned} D_1 &= c_2' R^{-1} \\ D_2 &= \frac{1}{\lambda_S}\left[c_1 S^{-1} + \lambda_T c_2' R^{-1} T S^{-1}\right] \end{aligned}$$

Where ${c_2}' = \frac{c_2}{\lambda_R}$. Notice neither term in the solution for D_2 is generically traceless, however we are free to choose the parameters c_1 and c_2', such that the trace of the two terms cancel, and hence D_2 is traceless as required. Notice our solution has two parameters, c_1 and c_2', with a single constraint on them, hence we are left with one free parameter, (i.e. the solution space is one dimensional). Thus we get exactly one massless down Higgs field. Notice, in this simple model the component of the light mode appears in two generalized down Higgs fields, also the D_1 components of the massless mode are not parallel to the D_2 components. Next we do the same analysis for down Higgs field F-terms:

$$\begin{aligned} \lambda_T U_1 T + \lambda_R U_2 R &= c_3 \mathbf{I} \\ \lambda_S U_1 S &= c_4 \mathbf{I} \end{aligned}$$

Now we solve the second equation for U_1, and $U_1 = \frac{c_4}{\lambda_S} S^{-1}$, but S^{-1} is generically not traceless, hence we must choose $c_4 = 0$, as a result the first equation becomes:

$$\lambda_R U_2 R = c_3 \mathbf{I}$$

We know this is solved by $U_2 = \frac{c_3}{\lambda_r} R^{-1}$, and this is traceless. So for the up Higgs field sector we also get one massless mode, though its components are not spread over both U_1 and U_2, they are contained in only U_2. This model is notable, because we can imagine in the quark sector of the theory, that up quarks couple to U_1, while the down quarks couple to D_2, resulting in different textures for the up quarks versus the down quarks. However, if we also consider the leptons, which must couple to a generalized down Higgs field, either D_1 or D_2, the lepton's Yukawa couplings would have the same texture as either the up quarks or down quarks, which is phenomenologically unacceptable. Although this model seems quite promising, we need a slightly more complicated setup to give proper results for the leptons.

Notice in the superpotential we considered terms such as $\mathsf{Tr}\,[U_1 D_1 T]$, but not terms such as $\mathsf{Tr}\,[D_1 U_1 T]$. Presumably any model with the first term would also contain the second term. If we included both types of terms, the analysis to find the massless Higgs field mode becomes more complicated. In fact its not clear that we can write down a clean expression for the massless Higgs field mode. However, it can still be shown that generically we get exactly one massless Higgs field mode, and that its components get spread out in U_2, D_1, and D_2. In fact the form of U_2 and D_1 remain the same, it is D_1 which is no longer given by a simple expression.

We make a brief digression on solving the type of equations which we were just examining and arise when looking for massless modes of the generalized Higgs fields. For example suppose we have equations of the following form:

$$\mathsf{Tr}[\lambda_a(\lambda_X DT + c_2\lambda_Y TD)] = 0$$

Here D and T are matrices, also λ_X and λ_Y are numbers. This constraint is equivalent to the following:

$$\lambda_X DT + \lambda_Y TD \propto \mathbf{I}$$

We are imagining that the matrix T has some fixed VEV determined by the family breaking sector, and that D corresponds to the generalized down Higgs field, and is therefore traceless. Furthermore if we assume T is invertible, which is generically true, then one solution to the above equation is simply $D = c_1 T^{-1}$. However, we may wonder if this is the general solution. If we set λ_Y equal to zero, we can directly solve for D and see that $D = c_1 T^{-1}$ is the general solution. Now when we allow λ_Y to be generic, the dimensionality of solution space can not possibly increase, hence this is also the general solution for generic λ_X and λ_Y. However, recall that D is traceless, so we only want to consider solutions which are traceless, so nonzero solutions only exist when T^{-1} is traceless.

Another possible constraint equation is of the following similar form:

$$\mathsf{Tr}[\lambda_a(\lambda_X D_1 T_1 + \lambda_Y D_2 T_2 + \lambda_Z T_1 D_1 + \lambda_W T_2 D_2)] = 0$$

This implies the following equation:

$$\lambda_X D_1 T_1 + \lambda_Y D_2 T_2 + \lambda_Z T_1 D_1 + \lambda_W T_2 D_2 = c_2 \mathbf{I}$$

Now suppose that D_1, T_1, and T_2 have fixed values, and we want to solve for D_2. Once again we are assuming D_1 and D_2 are traceless. This equation can be rewritten as:

$$\lambda_Y D_2 T_2 + \lambda_W T_2 D_2 = -\lambda_X D_1 T_1 - \lambda_Z T_1 D_1 + c_2 \mathbf{I}$$

In the above equation we think of the left hand side as a linear operator acting on D_2. Lets call this operator $\mathbf{L}(\lambda_Y, \lambda_W)$. The above equation can be rewritten as:

$$\mathbf{L}(\lambda_Y, \lambda_W) \circ D_2 = -\lambda_X D_1 T_1 - \lambda_Z T_1 D_1 + c_5 \mathbf{I} \tag{5.2}$$

Note "$\circ$" denotes a linear operator acting on a matrix, (i.e. it does not denote matrix multiplication). Now we note that if we set λ_W equal to zero, this linear operator, $\mathbf{L}(\lambda_Y, 0)$ is invertible because the equation: $\lambda_Y D_2 T_2 = M$ has the unique solution $D_2 = M(\lambda_Y T_2)^{-1}$, since T_2 is generically invertible. For a linear operator to be non-invertible requires the determinate of its corresponding matrix is zero, therefore if this determinate does not equal zero when $\lambda_W = 0$, it generically does not equal zero when we allow λ_W to be generic. Hence $\mathbf{L}(\lambda_Y, \lambda_W)$ is generically invertible, which means equation 5.2 has the following unique solution:

$$D_2 = \mathbf{L}^{-1}(\lambda_Y, \lambda_W) \circ [-\lambda_X D_1 T_1 - \lambda_Z T_1 D_1 + c_5 \mathbf{I}]$$

However we demand that D_2 be traceless. We satisfy this condition by picking c_5 such that D_2 is traceless. In fact, the trace of D_2 contains the following contribution $c_5 \mathsf{Tr}\,[\mathbf{L}^{-1}(\lambda_Y, \lambda_W) \circ \mathbf{I}]$, so if this trace is not zero, we can make D_2 traceless by adjusting the value of c_5. It easy to see that this trace is not generically zero, because if $\lambda_W = 0$, then this trace is $\mathsf{Tr}[(T_2)^{-1}]$, which is generically not zero. This discussion outlines the method we use to solve the equations that we encounter when looking for massless modes in model 3. Notice we made important assumptions, regarding when the inverse of matrices are traceless or are not traceless.

For the model we will discuss, call it model 3, we will have 4 pairs of up and down octet Higgs fields:

$$[U_1]^i{}_j \quad [U_2]^i{}_j \quad [U_3]^i{}_j \quad [U_4]^i{}_j$$

$$[D_1]^i{}_j \quad [D_2]^i{}_j \quad [D_3]^i{}_j \quad [D_4]^i{}_j$$

In other words the generalized Higgs field sector has 4×8 pairs of up Higgs field and down Higgs field doublets. That is the generalized Higgs field sector has a total of $4 \times 8 \times 2 \times 2 = 128$ complex components. A global $U(1)$ symmetry will be introduced for each generalized Higgs field octet that is we will introduce 8 $U(1)$ symmetries. The family breaking sector in model 3, has 16 octets. These octets will be labeled as follows:

$$[T_{ab}]^i{}_j$$

Now the indices a and b go from 1 to 4, hence there are 16 different possible fields, also i and j are $SU(3)$ family indices. Furthermore, since each T_{ab} is in an octet of $SU(3)$ they are traceless in the indices i and j. The idea is that for each combination of generalized up and down Higgs fields, there is one flavor breaking field that couples with the pair. More explicitly, model 3 contains coupling terms of the following form:

$$[U_a]^i{}_j [D_b]^j{}_k [T_{ab}]^k{}_i$$

We see that the a and b indices on T, label which pair of generalized Higgs fields couple to T_{ab}. In addition Model 3 has two scalar fields, η and ξ, these scalar fields are used to give constraints on some of the T_{ab} fields. Model 3 has four set of symmetries to consider: Standard Model gauge group, $SU(3)_{\text{fam}}$ family symmetry, $U(1)$ global symmetries, and R-symmetries. Firstly, each field will just have an R-charge of $\frac{2}{3}$, thus permitting only cubic terms in the superpotential. Only SM fermions fields and generalized Higgs fields will transform under the Standard Model gauge group. The following table summarizes how they transform:

Field	$SU(2)_{\text{weak}}$	$U(1)_Y$
q^i	doublet	$\frac{1}{6}$
${u^c}_j$	singlet	$-\frac{2}{3}$
${d^c}_k$	singlet	$\frac{1}{3}$
l^i	doublet	$-\frac{1}{2}$
${e^c}_j$	singlet	1
${\nu^c}_k$	singlet	0

There are three representations used for $SU(3)_{\text{fam}}$: the fundamental, the antifundamental, and octet representation. We summarize the transformation properties of all the fields below:

Field	q^i	${u^c}_j$	${d^c}_j$	l^i	${e^c}_j$	${\nu^c}_j$	${[U_a]^i}_j$	${[U_b]^i}_j$	${[T_{ab}]^i}_j$	η	ξ
$SU(3)_{\text{fam}}$	3	$\bar{3}$	$\bar{3}$	3	$\bar{3}$	$\bar{3}$	8	8	8	1	1

Model 3 has eight $U(1)$ global symmetries. We have two sets of 4 $U(1)$'s, which we label as $\{A1, A2, A3, A4\}$ and $\{B1, B2, B3, B4\}$. For example U_1 is charged only under $U(1)_{A1}$ with a charge of $+1$. All the up Higgs fields are charged under one of the four A $U(1)$'s, for example U_4 is charged under $U(1)_{A4}$, with a charge of $+1$. Likewise the down Higgs fields are charged under the B $U(1)$'s. For example D_1 is charged under $U(1)_{B1}$, while D_4 is charged under $U(1)_{B4}$, in each case the charge is $+1$. The charges under the $U(1)$'s for the generalized Higgs fields and T flavor breaking fields is systematically determined. The T_{ab} fields have a charge under one of the A $U(1)$'s and one of the B $U(1)$'s. For example T_{23} is charged under $U(1)_{A2}$ and $U(1)_{B3}$, its charge under both groups is -1. However the charges of the rest of the fields in the theory are less systematic, therefore we just list their charges:

Field	$U(1)_{A1}$	$U(1)_{A2}$	$U(1)_{A3}$	$U(1)_{A4}$	$U(1)_{B1}$	$U(1)_{B2}$	$U(1)_{B3}$	$U(1)_{B4}$
Q^i	0	0	0	0	0	0	0	0
${u^c}_j$	0	0	-1	0	0	0	0	0
${d^c}_k$	0	0	0	0	0	0	-1	0
l^i	0	0	0	0	0	0	0	0
${e^c}_j$	0	0	0	0	0	0	0	-1
${\nu^c}_k$	0	0	0	0	0	0	0	0
η	0	-2	0	0	0	0	-2	0
ξ	0	0	-2	0	0	-2	0	0

We assume a certain subset of T_{ab} family breaking fields are driven to VEVs of zero via SUSY breaking terms. Hence, when we calculate the massive and massless modes of the generalized Higgs field sector, we ignore terms which contain these particular T fields. The set of T fields with zero VEVs is:

$$\{T_{42}, T_{24}, T_{33}, T_{34}, T_{43}, T_{44}\}$$

We have outlined all the fields and symmetries in model 3. Now we write down all the permitted terms in the superpotential. Summation will be implied when appropriate for the Standard Model and family indices, however for the a, b indices on the generalized Higgs fields and T fields the summation will be denoted explicitly.

1) Terms that couple the Higgs fields to the quarks.

$$W_{\text{quarks}} = \lambda_U \epsilon^{im} {u^c}_k {[U_{3m}]^k}_j [q_i]^j + \lambda_D \epsilon^{im} {d^c}_l {[D_{3m}]^l}_j [q_i]^j$$

2) Terms that couple the Higgs fields to the leptons

$$W_{\text{leptons}} = \lambda_E \epsilon^{im} e^{cK} {[D_{4m}]^j}_k [l_i]^j$$

3) Terms that couple the Higgs fields to the family breaking fields.

$$W_{\text{Higgs}} = \sum_{b=1}^{4}\sum_{a=1}^{4}\{\lambda_{UDab}\epsilon^{mn}[U_{am}]^i{}_j[D_{bn}]^j{}_k[T_{ab}]^k{}_i + \lambda_{DUab}\epsilon^{mn}[D_{bm}]^i{}_j[U_{an}]^j{}_k[T_{ab}]^k{}_i\}$$

4) Terms that make a potential for the family breaking fields and scalars.

$$W_{\text{potential}} = \lambda_\eta \eta [T_{23}]^i{}_j [T_{23}]^j{}_i + \lambda_\xi \xi [T_{32}]^i{}_j [T_{32}]^j{}_i$$

Firstly, we want to solve for the moduli space of the T_{ab} fields, notice the only fields which appear in the family breaking potential are T_{23} and T_{32}. Therefore these are the only fields that are constrained. For example, setting the η F-term equal to zero gives the following:

$$[T_{23}]^i{}_j [T_{23}]^j{}_i = 0$$

We already showed this constraint implies that $(T_{23})^{-1}$ is traceless. Similarly the constraint on T_{32} implies that $(T_{32})^{-1}$ is traceless.

Now we need to determine the massless Higgs field modes of model 3. As usual this requires considering the F-terms of the Higgs field modes and setting them equal to zero:

$$\sum_{c=1}^{4} \mathsf{Tr}\left[\lambda_a(\lambda_{UDbc}D_c T_{bc} + \lambda_{DUbc}T_{bc}D_c)\right] = 0$$

This corresponds to 4 sets of 8 equations each. Each set of 8 equations is equivalent to requiring a certain matrix be proportional to the identity. Now recall that some of the T_{ab} are assumed to have VEVs of zero. So we write down the equations that result after expanding out the above sum and setting it proportional to the identity:

$$\begin{aligned} &\lambda_{UD11}D_1T_{11} + \lambda_{UD12}D_2T_{12} + \lambda_{UD13}D_3T_{13} + \lambda_{UD14}D_4T_{14} \\ &+\lambda_{DU11}T_{11}D_1 + \lambda_{DU12}T_{12}D_2 + \lambda_{DU13}T_{13}D_3 + \lambda_{DU14}T_{14}D_4 = c_1\mathbf{I} \quad (6) \end{aligned}$$

$$
\begin{aligned}
\lambda_{UD21} D_1 T_{21} + \lambda_{UD22} D_2 T_{22} + \lambda_{UD23} D_3 T_{23} & \\
+ \lambda_{DU21} T_{21} D_1 + \lambda_{DU22} T_{22} D_2 + \lambda_{DU23} T_{23} D_3 &= c_2 \mathbf{I} \qquad (7)
\end{aligned}
$$

$$
\lambda_{UD31} D_1 T_{31} + \lambda_{UD32} D_2 T_{32} + \lambda_{DU31} T_{31} D_1 + \lambda_{DU32} T_{32} D_2 = c_3 \mathbf{I} \qquad (8)
$$

$$
\lambda_{UD41} D_1 T_{41} + \lambda_{DU41} T_{41} D_1 = c_4 \mathbf{I} \qquad (9)
$$

These equations are tricky to analyze, keep in mind that $(T_{23})^{-1}$ and $(T_{32})^{-1}$ are traceless. So we consider the last equation, and note the solution to this equation is:

$$
D_1 = \frac{c_4}{\lambda_{UD41} + \lambda_{DU41}} (T_{41})^{-1}
$$

Although this solves equation 9, T_{41} is not constrained and generically $(T_{41})^{-1}$ is not traceless, therefore the solution given for D_1 is also not traceless, except when we pick $c_4 = 0$ (i.e $D_1 = 0$). In other words, no components of the massless modes appear in D_1. Now consider equation 8, recall we just found $D_1 = 0$, thus we solve for D_2 in equation 8. As we showed before the solution to an equation of this form is:

$$
D_2 = \frac{c_3}{\lambda_{UD32} + \lambda_{DU32}} (T_{32})^{-1}
$$

However, unlike equation 9, this solution is actually good because T_{32} is constrained such that $(T_{32})^{-1}$ is traceless. The next equation we examine is tricky. Since we have already solved for D_1 and D_2 we attempt to solve for D_3 in equation 7, therefore we isolate the terms which contain D_3:

$$
\lambda_{UD23} D_3 T_{23} + \lambda_{DU23} T_{23} D_3 = c_2 \mathbf{I} - \lambda_{UD22} D_2 T_{22} - \lambda_{UD22} D_2 T_{22}
$$

Denote the linear operator which acts on D_3 on the left hand side of the equation as $\mathbf{L}_2$, then the solution for D_3 is:

$$
D_3 = (\mathbf{L}_2)^{-1} \circ (c_2 \mathbf{I} - \lambda_{UD22} D_2 T_{32} - \lambda_{DU22} T_{22} D_2)
$$

We can determine $(\mathbf{L}_2)^{-1} \circ (c_2 \mathbf{I})$, by solving the following equation:

$$\mathbf{L}_2 \circ X = \lambda_{UD23} X T_{23} + \lambda_{DU23} T_{23} X = \mathbf{I}$$

The solution is $X = \frac{1}{\lambda_{UD23}+\lambda_{DU23}} (T_{23})^{-1}$, which is traceless. We see the first term in the solution for D_3 is traceless, so in order for D_3 to be traceless the sum of the second and third term must also be traceless. However this is generically not the case, which is seen by considering $\lambda_{UD23} = 0$. In such a case $\mathbf{L}_2 = \lambda_{DU23} T_{23}$, hence the sum of the second and third term is:

$$(\mathbf{L}_2)^{-1} \circ (-\lambda_{UD22} D_2 T_{22} - \lambda_{UD22} D_2 T_{22}) \;=\; \frac{1}{\lambda_{DU23}} (T_{23})^{-1} (-\lambda_{UD22} D_2 T_{22} - \lambda_{DU22} T_{22} D_2)$$

$$= \frac{c_3}{\lambda_{DU23}(\lambda_{UD32} + \lambda_{DU32})} \left[-\lambda_{UD32} (T_{23})^{-1} (T_{32})^{-1} T_{22} - \lambda_{UD22} (T_{23})^{-1} T_{22} (T_{32})^{-1} \right]$$

There is no reason that the above matrix would be generically traceless. Although we are free to choose $c_3 = 0$, which of course makes the above matrix traceless. In other words, from equations 7 through 9 we have the following conditions on the zero mode(s) of the generalized down Higgs fields:

$$\begin{aligned} D_1 &= 0 \\ D_2 &= 0 \\ D_3 &= \frac{c_2}{\lambda_{UD23} + \lambda_{DU23}} (T_{23})^{-1} \end{aligned}$$

Examining equation 6, and setting $D_1 = D_2 = 0$ gives an equation that can be solved for D_4. Earlier we showed that equations of this form could be solved uniquely for D_4, if D_3 is known and $(T_{14})^{-1}$ is not traceless. Generically $(T_{14})^{-1}$ is not traceless, because T_{14} is not required to satisfy any constraints. Note the solution for D_4 is in terms of D_3, and the only free parameter of the solution is c_2. Also c_1 is not a free parameter because it needs to be chosen in a precise manner to insure D_4 is traceless. An important property of the massless mode is that the D_3 components are not proportional to the D_4 components. In fact one finds the up massless mode has U_3 components that are not proportional to the U_4 components, if one did the same exact analysis on the down Higgs field F-terms. In addition, no pair of D_3, D_4, U_3, and U_4 are proportional, hence our model has four possible Yukawa coupling textures. In reality we only need three textures, one each for the up quarks, down quarks, and leptons.

We briefly discuss the SM fermion sector of the model. If look at the W_{quarks} terms in the superpotential, we notice that the up quark mass matrix texture is determined by U_3, likewise the down quark mass matrix texture comes from D_3. Furthermore if we look at the W_{leptons} term, we see that the leptons mass matrix texture is determined by D_4. Thus we see that all the Yukawa matrices in model three are all distinct.

Chapter 6

Models with Interesting Couplings and Textures

6.1 A Simple $SU(3)$ model with Low Energy Couplings

Now, we construct an $SU(3)$ model of family symmetry in the same mold as previous models. This model is distinct in that it naturally has massless Higgs field modes which couple to moduli fields. In this case when we say naturally, it is not meant in any technical sense of naturalness. Rather we mean inspection of the superpotential reveals the model to be fairly simple, none of the terms seem ad hoc. More specifically we do not need to assume anything about the nature of the SUSY breaking terms to conclude that the model has moduli-Higgs couplings.

This model will be relatively simple in terms of its field content, it will have a generalized Higgs field with the up(down) in the (anti)fundamental representation of $SU(3)$. The family sector will contain only one adjoint field, $S^a{}_b$, and two scalar fields η and τ.

We make the following simple choices for $U(1)$ charges of the fields, the Higgs fields have charges of -1, the adjoint $S^a{}_b(-2)$, $\eta(8)$, $\tau(12)$. This model is different from all the others because we are considering nonrenormalizable terms. The Higgs fields are given R-charges of 1, and the scalars η and τ have R-charges of 2.

The superpotential is:

$$\begin{aligned} W_{\text{Higgs}} &= \lambda_S U_j S^j{}_k D^k \\ W_{\text{potential}} &= \lambda_1 \eta (S^a{}_b S^b{}_a)^2 + \lambda_2 \tau (\epsilon^{abc} \epsilon_{def} S^d{}_a S^e{}_b S^f{}_c)^2 + \lambda_3 \tau (S^a{}_b S^b{}_a)^3 \end{aligned}$$

Although it appears the superpotential is missing some contractions of $S^a{}_b$ to the fourth or sixth power. All other contractions contain the trace $S^a{}_a$ which is zero, or are linear combinations of the terms already listed.

Lets first examine the fourth order contractions of $S^a{}_b$, of which there are just two which do not contain the trace $S^a{}_a$

$$S^a{}_b S^b{}_a S^c{}_d S^d{}_c \qquad\qquad S^a{}_b S^b{}_c S^c{}_d S^d{}_a$$

We can use the following trick to show these terms are proportional to each other. Consider the following term:

$$\epsilon^{abcd} \epsilon_{ijkl} S^i{}_a S^j{}_b S^k{}_c S^l{}_d$$

This term appears not to be an $SU(3)$ invariant, however there is an identity that allows us to write $\epsilon^{abcd}\epsilon_{ijkl}$ in terms of $\delta^i{}_j$'s, hence this term is in fact an $SU(3)$ invariant. This identity also allows us to write this term as a sum of all the possible contractions of $S^i{}_j$'s, hence the sum of the two terms enumerated above. However, we know this term is zero because in $SU(3)$ the tensor ϵ^{abcd} is zero.

More succinctly:

$$\epsilon^{abcd} \epsilon_{ijkl} S^i{}_a S^j{}_b S^k{}_c S^l{}_d = 3 S^a{}_b S^b{}_a S^c{}_d S^d{}_c - S^a{}_b S^b{}_c S^c{}_d S^d{}_a = 0$$

Which shows that these two contractions are just proportional to each other.

Now when we consider the contractions of sixth order, there were only two we left out:

$$(S^a{}_b S^b{}_c S^c{}_d S^d{}_a)(S^e{}_f S^f{}_e) \quad [1] \qquad\qquad S^a{}_b S^b{}_c S^c{}_d S^d{}_e S^e{}_f S^f{}_a \quad [2]$$

The first contraction [1] is seen to be proportional to the second sixth order term in the superpotential, by the previous trick. The previous trick can be modified for sixth order terms, and used to show that the second contraction [2] is a linear combination of both of the sixth order terms in the superpotential.

We examine the F-terms of η and τ to see what constraints we have on the VEV of $S^a{}_b$. The η F-term gives the following equation:

$$(S^a{}_b S^b{}_a)^2 = 0$$

This implies that $S^a{}_b S^b{}_a = 0$, now consider the τ F-term which gives:

$$\lambda_2(\epsilon^{abc}\epsilon_{def} S^d{}_a S^e{}_b S^f{}_c)^2 + \lambda_3(S^a{}_b S^b{}_a)^3 = 0$$

As a result of the two previous constraints, $\epsilon^{abc}\epsilon_{def} S^d{}_a S^e{}_b S^f{}_c = 0$. This constraint is that the determinate of $S^a{}_b$ is zero, and since $S^a{}_b$ is the mass matrix for the Higgs fields sector there must be at least one massless pair of Higgs fields. In summary we have 2 constraint equations on a tensor with 8 independent components, so after the constraints are imposed the tensor still has 6 degrees of freedom remaining.

A key property of this superpotential is that all the components of $S^a{}_b$ remain massless, after $S^a{}_b$ receives its VEV. Now we demonstrate that $S^a{}_b$ is massless, consider the η F-term, for this term to give mass to $S^a{}_b$ there needs to be a quadratic term after we plug in the VEVs, but this quadratic term will contain in its coefficient at least one power of $S^a{}_b S^b{}_a$, which is zero. Likewise when we examine the F-term from τ, the determinate of $S^a{}_b$ will appear in the coefficient of the quadratic term, hence $S^a{}_b$ is completely massless.

We want to show this model produces exactly one pair of massless Higgs fields, at least in the generic case. One way to demonstrate this is to produce one solution of the constraint equations for $S^a{}_b$, and then show this solution has only one zero eigenvector. The following choice satisfies both constraint equations:

$$S^a{}_b = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is straight forward to see this solution satisfies the constraints on $S^a{}_b$. We can see by inspection that the third component corresponds to the only massless mode.

It is also simple to see this model has light Higgs-$S^a{}_b$ couplings. In order to see this, consider the above solution for the VEVs of $S^a{}_b$. In that case $S^3{}_3$ is a light mode, however we know it couples to the massless Higgs field, because giving a VEV to $S^3{}_3$ results in the determinate of $S^a{}_b$ being non-zero. In other words this VEV gives mass to the light Higgs field. Fields that give mass to the light Higgs field must couple to the light Higgs field. We remember this is easy to demonstrate by looking at the low energy effective theory.

6.2 Three Generation Model with Symmetric, Fundemental and Adjoint Higgs Fields

We can construct a simple $SU(3)$ model [16] by using a generalized Higgs fields of an adjoint, 8, a symmetric, $\bar{6}$, and fundamental field 3. In other words, we have a pair of each fields for the up Higgs fields and the down Higgs fields , such as $T_{\mathrm{u}}{}^i_j$, $T_{\mathrm{d}}{}^i_j$, $S_{\mathrm{u},ij}$, $S_{\mathrm{d},ij}$, $A_{\mathrm{u}}i$, and $A_{\mathrm{d}}i$. Next we introduce the following family breaking fields a symmetric R^{ij} and P_i. If we have a $U(1)$ symmetry and give the fields the following charges: $q^i(1)$, $u^{cj}(1)$, $d^{cj}(-3)$, $l^i(-3)$, $e^{cj}(1)$, $\nu^i(5)$, $A^k_{\mathrm{u}}(-2)$, $S_{\mathrm{u},ij}(-2)$, $A^k_{\mathrm{d}}(2)$, $S_{\mathrm{d},ij}(2)$, $R^{ij}(-5)$, $P_i(-5)$.

Then we have the following superpotential

$$W = S_{\mathrm{u},ij} T_{\mathrm{d}}{}^i_k R^{jk} + \epsilon^{jkl} S_{\mathrm{u},ij} T_{\mathrm{d}}{}^i_k P_l + \epsilon_{ijk} A_{\mathrm{u}i} T_{\mathrm{d}}{}^j_l R^{kl} + A_{\mathrm{u}i} T_{\mathrm{d}}{}^j_i P_j$$

$$+S_{\mathrm{u},ij}T_{\mathrm{d}}{}^{i}_{k}R^{jk}+\epsilon^{jkl}S_{\mathrm{d},ij}T_{\mathrm{u}}{}^{i}_{k}P_l+\epsilon_{ijk}A_{\mathrm{d}i}T_{\mathrm{u}}{}^{j}_{l}R^{kl}+A_{\mathrm{d}i}T_{\mathrm{u}}{}^{j}_{i}P_j$$

Now consider the F-terms of $T_{\mathrm{d}}{}^{i}_{k}$, which is an adjoint of $SU(3)$ and therefore is traceless, thus has 8 associated F-terms. However, $S_{\mathrm{u},ij}$ and $A_{\mathrm{u}}{}^{i}$ have a total of 9 components, and the F-terms of the adjoint field can give at most 8 fields mass, therefore at least one of the 9 components of $\bar{6}$ and 3 is massless. We write the adjoint in terms of the Gell-mann matrices as $T_{\mathrm{d}}{}^{i}_{k}=\phi_a\lambda_a$, and the 8 F-terms are of the form (a is the free index):

$$\mathbf{Tr}\left[(R^{jk}S_{\mathrm{u},ij}+\epsilon^{jkl}S_{\mathrm{u},ij}P_l+\epsilon_{lji}A_{\mathrm{u}}{}^{j}R^{kl}+A_{\mathrm{u}k}P_i)[\lambda_a]^i_k\right]$$

The massless modes are defined by modes that make the F-terms vanish. Since the F-terms have the form $\mathbf{Tr}\,[M\lambda_a]$, the only way this can vanish for all 8 values of a, is if M is proportional to the identity matrix, I. Lets consider the special case where P_i vanishes, then we have the condition:

$$R^{jk}S_{\mathrm{u},ij}+\epsilon_{lji}A_{\mathrm{u}}{}^{l}R^{kj}=I^k_i$$

For generic values of R^{jk}, it's an invertible matrix, we can easily solve for the sum, $S_{\mathrm{u},ij}+\epsilon_{jli}A_{\mathrm{u}}{}^{l}$. The sum will be the inverse of R^{jk}. Furthermore since the inverse of a symmetric matrix is symmetric, we know that $\epsilon_{jli}A_{\mathrm{u}}{}^{l}$ must be zero, since it is antisymmetric. Hence $S_{\mathrm{u},ij}$ is the inverse of R^{jk} (and $A_{\mathrm{u}}{}^{l}=0$). This is a unique solution for the massless mode, implying there is only one massless mode when $P_i=0$. However, giving P_i a non-vanishing generic VEV could potentially result in adding massive modes but could never generically add massless modes. Since, we already know that in the generic case we have at least one massless mode, having only one massless mode in the special case, $P_i=0$, implies we have only one massless mode in the generic case.

We couple the Higgs fields to the leptons and quarks, to give us a sector of the

Standard Model fermion theory. The couplings:

$$\begin{aligned} W_{\text{Yuk}} &= \lambda_{\text{u}}^{(1)}\epsilon_{ijk}q^i u^{\text{c}j}A_{\text{u}}{}^k + \lambda_{\text{d}}^{(1)}\epsilon^{ijk}q^i d^{\text{c}j}A_{\text{d}}{}^k + \lambda_{\text{e}}^{(1)}\epsilon^{ijk}l^i e^{\text{c}j}A_{\text{d}}{}^k \\ &\quad +\lambda_{\text{u}}^{(2)}q^i u^{\text{c}j}S_{\text{u},ij} + \lambda_{\text{d}}^{(2)}q^i d^{\text{c}j}S_{\text{d},ij} + \lambda_{\text{e}}^{(2)}l^i e^{\text{c}j}S_{\text{d},ij} \end{aligned}$$

We can include mass terms for the neutrinos, the ordinary mass terms:

$$W_\nu = \lambda_\nu^{(1)}\epsilon_{ijk}l^i\nu^j A_{\text{u}}{}^k + \lambda_\nu^{(2)}l^i\nu^j S_{\text{u},ij}$$

Also, we can generate Majorana mass terms for the neutrinos by considering higher order terms and the family sector(for simplicity the order unity coupling constants have been omitted):

$$W_{\text{Majorana}} = \frac{1}{M_{\text{Pl}}}\nu^i\nu^j\left[\epsilon_{ikm}\epsilon_{jln}R^{kl}R^{mn} + P_iP_j + \epsilon^{ijk}R^{kl}P_l\right]$$

At leading order the mass term for the squarks and sleptons is flavor universal, and at higher order there are nonuniversal terms of the form:

$$\mathcal{L} = \mu^2(q^i)q_j\left(\delta^j{}_i + \frac{R^{jl}R_{il}}{\Lambda^2_{\text{Planck}}}\right)$$

This show us that the correction to universal mass term is determined by $\frac{\Lambda_{\text{fam}}}{\Lambda_{\text{Planck}}}$.

We know that this model has no low energy couplings between the moduli field and the Higgs field, because there is no family sector VEV that gives mass to the light Higgs field mode. This also means our model has no μ term. In order to account for the phenomenology of a μ term [17, 18], we introduce a $U(1)_X$ neutral family fields, T^{ij} and K_l. These fields have couplings of the following form(omitting order unity couplings):

$$W_\mu = \epsilon^{ijk}\epsilon^{mno}T_{im}T_{jn}T_{ko} + K^iS_{\text{d},ij}A_{\text{u}}{}^j + K^iS_{\text{u},ij}A_{\text{d}}{}^j + \epsilon_{ijk}K^iA_{\text{u}}{}^jA_{\text{d}}{}^k$$

Previously when we found terms in the Lagrangian for $S_{\text{u},ij}$ and $A_{\text{u}i}$, we only had

8 linear independent terms because they were derived from the 8 F-terms from the adjoint fields. But now since the symmetric and fundamental Higgs fields do not couple to fields besides the adjoint, we get 9 linearly independent terms, and hence no massless modes. However if we assume the the VEVs of the extra fields T^{ij} and K_l are constrained to be at the weak scale then the new massive Higgs field mode will only have mass at the weak scale. Hence introducing these fields results in both a μ term and a Higgs-moduli coupling.

Since these fields are $U(1)_X$ neutral they don't couple to the other family sector fields(whose VEVs are large), and so they get their quadratic mass terms at the weak scale from SUSY breaking terms. However we can still have cubic terms in superpotential, made from these new auxiliary family fields, thus leading to quartic restoring terms for their potential.

If T^{ij} has a positive mass squared term and K_i has a negative mass squared term then T^{ij} will get a VEV of zero, with K_i getting a non-zero VEV. We have the following superpotential:

$$W = \lambda_T \epsilon^{ijk} \epsilon^{pqr} T_{ip} T_{jq} T_{sr} + \lambda_K K^i K^j T_{ij}$$

This superpotential results in the following quartic term for K_i:

$$\mathcal{L} = |\lambda_K|^2 (K_i K^i)^2$$

This quartic term will stabilize the VEV of K^i, if it contains no flat direction, it is evident by inspection that there are no flat directions. The VEV of K^i will be at the weak scale and will generate the μ term.

Chapter 7

Conclusion

In this thesis we explicitly construct several models, all basically of the same mold, for family symmetry breaking at a high scale which is transmitted down to the electroweak scale. The two most interesting aspects of these models are the necessity of supersymmetry, and possibility/likelihood of Higgs-moduli couplings observable at low energies. These models are unique because they give the Higgs fields family symmetry, and result in just one pair of low energy Higgs fields.

While it is possible to construct a large sets of models with different textures for the masses and mixings of the quarks and leptons, no one model stands out as a compelling candidate. The construction of these models rather than being a proposal for a specific case, is rather meant as proof of principle. That is the construction of these models is meant to demonstrate that the general framework is plausible and workable. Moreover, the various constructions demonstrate that there is a certain amount of flexibility in terms of what can be constructed. Of course, if the mechanism outlined by these models is the actual mechanism by which family symmetry breaks and gets communicated down to low energies then we would expect one model to stick out as a stellar candidate.

It is our hope that a model can be successfully constructed which incorporates gauge unification. If such a model were particularly elegant, than we would believe that is very suggestive that our approach is correct. On the experimental side, if couplings between the Higgs fields and multiplet of singlets were observed at the electroweak scale that would be a signature of these schemes. Furthermore the couplings would be expected to contain information on family symmetry breaking, though encoded in a complicated manner.

Bibliography

[1] S. Dimopoulos and H. Georgi, Nucl. Phys. B193, 150 (1981)

S. Dimopoulos and H. Georgi, Phys. Lett. B117, 287 (1982)

L. Susskind, Phys. Rept. 104, 181 (1984)

[2] Particle Data Group (S. Eidelman *et al.*), Phys. Lett. B592, 1 (2004)

C.D. Froggatt, arXiv:hep-ph/0307138

[3] L. Wolfenstein, Phys. Rev. Lett. 51, 1945 (1983)

[4] N. Cabibbo, Phys. Rev. Lett. 10, 531 (1963)

[5] E. Gabrielli, A. Masiero, and L. Silvestrini, arXiv:hep-ph/9510215

F. Gabbiani, E. Gabrielli, A. Masiero, and L. Silvestrini, arXiv:hep-ph/9604387

M. Misiak, S. Pokorski, and J. Rosiek, arXiv:hep-ph/9703442

[6] L. Ibáñez and G. G. Ross, arXiv:hep-ph/9403338

V. Jain and R. Shrock, arXiv:hep-ph/9412367

P. Binétruy and P. Ramond, arXiv:hep-ph/9412385

E. Dudas, S. Pokorski, and C. A. Savoy, arXiv:hep-ph/9504292

Y. Nir, arXiv:hep-ph/9504312

E. Dudas, C. Grojean, S. Pokorski, and C. A. Savoy, arXiv:hep-ph/9606383

[7] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B147, 277 (1979)

D.S. Shaw and R.R. Volkas, Phys. Rev. D47, 241 (1993)

M. Dine, R. Leigh, and A. Kagan, Phys. Rev., 4269, (1993)

R. Barbieri and L.J. Hall, Nuovo Cim. 110A, 1 (1997)

[8] F. Wilczek and A. Zee, Phys. Rev. Lett. 42, 421 (1979)

Y. Koide and M. Tanimoto, Z. Phys. C 72, 333 (1996)

S. L. Adler, Phys. Rev. D 59, 015012 (1999) [Erratum-ibid. D 59, 099902 (1999)]

C. I. Low, Phys. Rev. D 70, 073013 (2004)

A. Zee, arXiv:hep-ph/0508278

Y. Koide, arXiv:hep-ph/0508301

[9] K. S. Babu, T. Kobayashi and J. Kubo, Phys. Rev. D 67, 075018 (2003)

E. Ma, arXiv:hep-ph/0506036

[10] S. Dimopoulos, S. Raby and F. Wilczek, Phys. Rev. D 24, 1681 (1981)

S. Dimopoulos and H. Georgi, Nucl. Phys. B 193, 150 (1981)

[11] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. 38, 1440 (1977)

R. D. Peccei and H. R. Quinn, Phys. Rev. D 16, 1791 (1977)

S. Weinberg, Phys. Rev. Lett. 40, 223 (1978)

F. Wilczek, Phys. Rev. Lett. 40, 279 (1978)

[12] J. E. Kim, Phys. Rev. Lett. 43, 103 (1979)

M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 166, 493 (1980)

A. R. Zhitnitsky, Sov. J. Nucl. Phys. 31, 260 (1980) [Yad. Fiz. 31, 497 (1980)]

M. Dine, W. Fischler and M. Srednicki, Phys. Lett. B 104, 199 (1981)

[13] S. P. Martin, arXiv:hep-ph/9709356

S. Weinberg, The Quantum Theory of Fields: Volume III Supersymmetry (2000)

[14] B. Patt, D Tucker-Smith, and F. Wilczek, arXiv:hep-ph/0509295

[15] A. De Rújula, H. Georgi, and S. L. Glashow, Annals Phys. 109, 242 (1977)

S. Weinberg, Trans. New York Acad. Sci. 38, 185 (1977)

F. Wilczek and A. Zee, Phys. Lett. B 70, 418 (1977)

[16] E. Bilgin, B. Patt, D. Tucker-Smith, and F. Wilczek, arXiv:hep-ph/0509075

[17] S. Dimopoulos and H. Georgi, Nucl. Phys. B193, 150 (1981)

F. Gabbiani and A. Masiero, Nucl. Phys. B322, 235 (1989)

S. Hagelin, S. Kelley, and T. Tanaka, Nucl. Phys. B415 293 (1994)

E. Gabrielli, A. Masiero, and L. Silvestrini, arXiv:hep-ph/9510215

F. Gabbiani, E. Gabrielli, A. Masiero, and L. Silvestrini, Nucl. Phys. B477 321 (1996)

[18] N. Polonsky, Lect. Notes Phys. **M68**, 1 (2001) [arXiv:hep-ph/0108236].