EXTENSIONS OF THE SCALING HYPOTHESIS IN N-COMPONENT SYSTEMS

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ABSTRACT

Chapter 1 presents an introduction to aspects of critical phenomena explored in the remainder of the thesis. The emphasis is placed on the scaling hypothesis and the renormalization group.

Chapter 2 describes the application of the scaling hypothesis to asymmetric systems such as single-component fluids. When the symmetry and regularity properties characteristic of magnetic materials cannot be assumed, the scaling hypothesis predicts various asymmetries and singularities. In particular, it predicts that the curvature of the vapor pressure curve should be strongly divergent and that the specific heat has a leading order asymmetry across the coexistence surface. Even if further assumptions are made to remove these singularities, the critical isochore above the critical temperature must have a weakly singular curvature.

Chapter 3 defines a class of systems more general than scaling systems, which share many of the geometrical properties of systems satisfying a scaling hypothesis. The notion of critical points of higher order is discussed and a tentative classification system for such points is proposed.

Chapter 4 consists of calculations with the renormalization group of scaling powers and critical point exponents. The corrections to the mean-field values of these exponents are calculated for magnetic-like systems with $\mathcal{O}$ simultaneously critical phases.

Chapter 5 discusses nonlinear solutions of renormalization group equations. By solving nonlinear equations, the competition between different kinds of critical behavior. We describe the crossover from asymptotically valid critical behavior to mean-field behavior for an n-component ferromagnet. We also study a system of anisotropically interacting 2n-components spins. This system has a renormalization group solution diagram similar to phase diagrams of systems exhibiting tricritical and fourth order critical point behavior.

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Title: Associate Professor of Physics

To my wife, Michaele, who has tolerated me.

"Tprot, Scot, for thy strife,
Hang up thy hatchet, and thy knife"

15th century

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CHAPTER 1

INTRODUCTION TO SELECTED ASPECTS OF CRITICAL PHENOMENA

I. Introduction

This thesis consists of separate papers on various aspects of critical phenomena. The last decade of research in critical phenomena has been an extremely productive combination of concrete model calculations, data-analysis and phenomenological speculation and classification. The progenitor of many apparently divergent notions is the scaling hypothesis, and it is again the scaling hypothesis that is the center of this work. In Chapter 1, an overview of the body of the thesis is given to provide an introduction to each paper and to show the connections between them.

In Chapter 2, the scaling hypothesis is discussed in detail for systems with no obvious symmetries. This represents, in part, a study of the application of the scaling hypothesis to fluid systems, which lack the obvious symmetrics of simple ferromagnetic substances. However, it also serves to provide a framework for the more general systems discussed in later Chapters.

In Chapter 3, a system of axioms is introduced which describes systems which are more general than scaling systems but which share many of their geometrical features. A classification of "higher order" critical points on the basis of these "critically ordered" systems is given.

In Chapter 4, renormalization group calculations for critical point exponents at a critical point of order σ (i.e., a point at which σ phases are simultaneously critical) is given for arbitrary σ. Previously, such calculations have been made in a tortuous manner for $\sigma=2$ (ordinary critical point), $\sigma=3$ (tricritical point), and $\sigma=4$ (fourth order point). As a derivation of scaling properties this Chapter complements Chapter 2.

In Chapter 5, nonlinear calculations within the renormalization group

group are given. This nonlinear work in a sense justifies the critical point exponents of Chapter 4 ; exponent calculations in the renormalization group represent a linearization of fundamentally highly nonlinear equations. Furthermore, it is also shown that the nonlinear solutions of the renormalization group equations incorporate both the "higher order" critical points typified by the "intersection of critical subspaces", and the systems termed "critically ordered" in Chapter 3 .

In the remainder of this Chapter introductions are provided for each of the following Chapters. Section II corresponds to Chapter 2 and describes the terminology used to describe ordinary critical points. Section III (corresponding to Chapter 3) discusses the notion of higher order critical points and in the perspective of the mean field theory. The simplest example of a "critically ordered" system is discussed as preparation for the extensive discussions of Chapter 3.. In Section IV an introduction to the renormalization group as applied to critical phenomena is given. The linearized theory and its connection to the calculation of critical point exponents is discussed. The corrections to mean field exponents for a point of order θ are derived in the corresponding Chapter 4 . In Section V, the necessity of a nonlinear global approach to the renormalization group is shown as an introduction to the nonlinear calculations of Chapter 5.

II. Ordinary Critical Points

Griffiths and Wheeler[1] have shown the advantage of a geometrical viewpoint of behavior near the critical point. We consider two similar ordinary critical points: the liquid-vapor critical point of a single component fluid (cf. Fig. 1a) and the Curie point of a simple ferromagnet (cf. Fig. 1b). Below the critical temperature T_c and near the coexistence surface two different types of directions are clearly distinguishable. If we follow a path which crosses the critical surface, there are drastic changes in the order parameter (the magnetization in the ferromagnetic case; for fluids it is more complicated to define (cf. Chapter 2) but prototypically, the density). On the other hand, on a path which is always tangent to the coexistence surface, the variation in the order parameter is gradual. Griffiths and Wheeler call the former direction "strong" and the latter direction "weak". Note that only the weak direction is unique at any point of the coexistence surface, since any direction not tangent will cross the surface and be a strong direction.

In the magnetic case, the coexistence surface is defined by $H=0$, $T<T_c$. Thus a direction along the temperature axis $H=0$ is weak and, for example, a path of constant temperature is strong.

It is perhaps not obvious that this distinction between weak and strong directions persists as the critical temperature is approached or above T_c. The discontinuity in the order parameter which marks the phase boundary vanishes at the critical point itself, and is of course identically zero above T_c. However, it is extremly profitable to follow Griffiths and Wheeler and assume that the distinction holds in some neighborhood of the critical point; presumably the weak direction above T_c is at least asymptotically tangent to the phase boundary.

We now compare the critical behavior of two response functions: for the magnetic case the susceptibility and the specific heat. The magnetic susceptibility χ is given by

$$\chi = \left(\frac{\partial^2 G}{\partial H^2}\right)_T , \tag{2.1}$$

while the constant field specific heat C_H is given by

$$\frac{C_H}{T} = -\left(\frac{\partial^2 G}{\partial T^2}\right)_H , \tag{2.2}$$

where G is the Gibbs free energy.

Comparing (2.1) and (2.2), we note that the susceptibility is given by the differentiation of the Gibbs potential in a strong direction (constant temperature). On the other hand, the specific heat is generated by differentiation in a weak direction, along the line H=0. Extrapolating the notions of weak and strong from the region below T_c, Griffiths and Wheeler predict that, near the critical point the susceptibility diverges more strongly than the specific heat,

$$C_H / T << \chi . \tag{2.3}$$

This is supported by the experimental and series work which indicate that near the critical point

$$\chi \sim |T-T_c|^{-\gamma},$$

$$\frac{C_H}{T} \sim |T-T_c|^{-\alpha}, \qquad (2.4)$$

where $\gamma \sim 5/4$, $\alpha \sim 1/8$.

The situation in the single-component fluid is similar, but some care must be applied since the weak direction is neither a line of constant temperature nor a line of constant pressure: both of these directions are strong. The weak direction is asymptotically tangent to the critical isochore and hence, is formed from some combination of temperature-like and pressure-like directions (cf. Fig. 1b). The details of the fluid case are considered in detail in Chapter 2.

A more complicated geometrical picture is given by an anisotropic Ising ferromagnet.[2] A model that we will return to for many examples is shown in Fig. 2 . Ising spins are arranged in layers of planes. The in-plane interaction strength is denoted by J while the interaction between the planes is RJ. A diagram in the field space (H,T,R) is shown in Fig. 1c. For all R>0 the character of the critical point is unchanged; that is although the critical temperature depends on the value of R, the critical points exponents such as γ and α do not, and have the values of the isotropic 3 dimensional Ising model . This, of course, cannot be the case if R=0, since for that particular value of R, the system is two-dimensional. At R=0, critical point exponents assume their two-dimensional values. Similarly, for R=∞ , system behaves one-dimensionally. The special critical point at R=0 will be considered in more detail below (it represents a critical point of "higher order"). Our interest

is focused, for the moment, on the smooth line of critical points generated by the variation of R. At any one of these critical points, a direction not parallel to the coexistence surface (labelled $\hat{x}_1$ in Fig. 1c) is strong; a direction in the plane of the coexistence surface but not tangent to the line of critical points (such as $\hat{x}_2$) is weak. The direction both in the plane of the coexistence surface and tangent to the line of critical points ($\hat{x}_3$) is clearly distinct from any of the weak or strong directions. Since the critical behavior is the same (as regards exponents) along the entire line, Griffiths and Wheeler term a direction such as $\hat{x}_3$ irrelevant. Differentiation in the $\hat{x}_3$ direction should not markedly change the nature of the singularity of any theremodynamic function. At the point R=0, the variation along the line of critical points is anything but irrelevant; this will be discussed below.

Griffiths and Wheeler axiomatize these relations between strong, weak and irrelevant directions. Strong directions carry the system out of the coexistence surface; weak directions leave the system in the plane of the coexistence surface but remove it from the space of critical points or critical surface (a line in the case discussed above); and finally, irrelevant directions leave the system in the critical surface itself. We may now ask what mathematical models satisfy these geometrical-analytic axioms.

The premier example of a system which obeys the Griffiths-Wheeler axioms is a scaling system.[3-4] In the form we will employ, the scaling hypothesis assumes that the portion of the Gibbs potential which determines the behavior near the critical point is a generalized homogeneous function[5] (GHF). For the simple magnetic case, this means that

$$G(\lambda^{a_H}H, \lambda^{a_t}t) = \lambda G(H,t) \tag{2.5}$$

where we use $t \equiv T-T_c$. The constants a_H and a_t are called the scaling powers of H and t respectively. It is easy to check that if $a_H > a_t > 0$, then H is a stronger variable than T.

With the assumption of the scaling hypothesis, much more is obtained than just a system obeying the Griffiths-Wheeler postulates. Most important of the scaling results is the conversion of inequalities relating several exponents to the corresponding equalities. All the usual critical point exponents can be expressed as simple rational functions of a_H and a_t so that any inequality relating three exponents must be an equality (if not tautologically true). For example, the common exponents α, γ and β (defined by $M \sim (T_c-T)^{\beta}$ on the coexistence surface below T_c) are given by

$$\begin{aligned} -\alpha &= (1-2a_t)/a_t \\ -\gamma &= (1-2a_H)/a_t \\ \beta &= (1-a_H)/a_t \end{aligned} \tag{2.6}$$

Thus the Rutherford inequality is satisfied as an equality

$$\alpha+2\beta+\gamma = 2 \tag{2.7}$$

We have confined the discussion of scaling to the magnetic case because the fluid case to which scaling was originally applied is far more subtle and complicated. A magnetic system is endowed with a natural symmetry, $H \to -H$, $M \to -M$. This exact symmetry implies that all thermodynamic functions have simple symmetric or multisymmetric properties on the coexistence surface.

This in turn implies, as is shown in Chapter 2, that the scaling variables must be taken as exactly H and t. From Griffiths and Wheeler we know only that the variable corresponding to the larger scaling power is exactly H, since the coexistence surface is, by symmetry, on the line H=0. Until the symmetry condition in M is applied we might chose any combination of H and t for the variables corresponding to the smaller of the scaling powers. Since the scaling is in the usual variables H and t, it does not matter what thermodynamic potential we choose as the basis function for a scaling hypothesis. The property of being a GHF is preserved under Legendre transformation[5] and differentiation and integration, so that we may scale the magnetization, or the Helmholtz or Gibbs free energy without loss of generality. Furthermore, we know the exact form of the coexistence surface and the "isochore" H=0; they both form parts of the line H=0.

None of this information is available even for the simplest fluid system except by careful measurement. (i) Since the scaling variable (or scaling fields) are not any of the usual thermodynamic variables (but rather some function of them) not all thermodynamic functions can be considered as candidates for scaling equations. (ii) Until the variables are specified there is no obvious choice of an order parameter; the choices $\rho-\rho_c$ and $V-V_c$ (where ρ_c and V_c are the critical point values of the density and volume, respectively) are inequivalent, since a coexistence surface symmetric in one will be asymmetric in terms of the

other. Furthermore, neither of these is the best candidate, but rather combinations of the volume and entropy or density and entropy density (see Chapter II). (iii) Without the symmetry of the magnet either additional hypotheses have to be made, or experimental evidence assembled, to describe the form of the coexistence surface and isochore. (iv) Since fluid systems abound in asymmetries and singularities, it may be necessary to emend the scaling equation of state itself to incorporate all the observed phenomena. On the other hand, it is perhaps possible to describe the system by a sufficiently carefully chosen scaling equation with properly chosen scaling potential, scaling variables, and forms for the coexistence surface and isochore.

In Chapter II, we give a systematic discussion of the scaling hypothesis in single component fluid systems. To accomodate the difficulties in (i)-(iv) we deal with a general potential of initially arbitrary variables, $\mathcal{G}(x_1,x_2)$. By applying the scaling hypothesis, it is shown that x_1 is restricted to conform with the Griffiths-Wheeler axioms, that is the line $x_1=0$ must be tangent to the critical isochore and coexistence surface. The canonical order parameter of the system upon which symmetry requirements are imposed is chosen to be $(\partial\mathcal{G}/\partial x_1)$, the "density conjugate to the field x_1". The asymmetry in the usual thermodynamic "order parameters" such as the number density is used to establish the form of x_2, the weak variable. The form of the coexistence surface is assumed to be a scaling invariant as is natural in a scaling theory; that is, on the coexistence surface, x_1 and x_2 are related by

$$x_1 = A_- |x_2|^{a_1/a_2} \qquad (2.8)$$

where a_1 and a_2 are the scaling powers at x_1 and x_2 ($a_1 > a_2$) and A_- is a constant, possibly zero. The critical isochore, however, is shown never to be a scaling invariant (since neither the density nor the volume is proportional to the canonical order parameter $\partial g/\partial x_1$); a more complicated form than (2.8) must be chosen, but with a scaling invariant leading term, $A_+ x_2^{a_1/a_2}$.

The use of the scaling invariant form (2.8) for the coexistence surface as the most singular part of the isochore has the consequence of satisfying another exponent inequality as an equality. The exponent is defined by the behavior of the coexistence surface when expressed in terms of the variables P and T, $(\partial^2 P/\partial T^2) \sim |T-T_c|^{-\Theta}$. Griffiths proved the inequality

$$\Theta \leq \alpha+\beta \tag{2.9}$$

The form given in (2.8) satisfies the inequality as an equality, $\Theta = \alpha+\beta$, if $A_\pm \neq 0$. A second consequence of non-zero A_- is that the specific heat, C_v is not symmetric across the coexistence surface, even to leading order; the asymmetry is proportional to $A_- (T_c-T)^{-\alpha}$.

Present experimental evidence indicates that the isochore of a simple fluid is smooth when expressed in terms of the chemical potential $\mu(T)$. This in turn implies that the divergence of the vapor pressure curve must be the same as the divergence of the specific heat. That is, we must have

$$\Theta = \alpha \tag{2.10}$$

and therefore $A_- = 0$, and the appropriate scaling choice is to scale the pressure as a function of T and μ. This analyticity does not constitute a "failure" of scaling since it is included in the invariant form (2.8). However, it is somewhat disconcerting that the scaling invariant constraint does not apply in a non-trivial manner. In other phenomenological studies of scalings systems,[6] scaling invariant paths such as (2.8) have been employed to predict the geometrical properties near certain "higher order" critical points such as the tricritical point of a metamagnet (see discussion in Sec. III). Thus the failure of (2.8) to hold with non-zero A_- may indicate that the geometrical predictions of Ref. 6 may not be valid.

Even in the absence of the scaling invariant term ($A_+ = 0$), the critical isochore above the critical temperature cannot be analytic. A weak singularity of the form

$$x_1 \sim x_2^{3-2(\alpha+\beta)} \tag{2.11}$$

is predicted from the scaling hypothesis.

III. Critical points of "higher order", Generalization of the Scaling Hypothesis and Classifications of critical points.

The ordinary critical point, although in itself a rich system, is not sufficiently diverse to encompass all of critical phenomena. Various terms have been used to describe new sorts of critical points, bicritical, tricritical, tetracritical, critical points of higher order, and so forth. No consensus on a systematic classification system of more complicated points has been reached, but a few of the more common hyper-critical points have established terminology and description.

For example, in the anisotropic ferromagnet discussed in the previous section, the special point R=0 is called a critical point of order four.[2] To understand this definition, we must examine the system more carefully. If we consider the same system, (in-plane interaction J and between-plane interaction RJ) but with R negative, the system forms a metamagnet with ferromagnetic ordering in each plane and antiferromagnetic ordering on alternate planes (cf. Fig. 2). For a particular value of R, this system has an ordinary critical point at its Néel temperature T_N. If a small uniform magnetic field is applied the antiferromagnetic ordering can be disordered at a lower temperature. Thus, in the H-T plane (cf. Fig. 3) there is a line of ordinary critical points. The ferromagnetic coupling in each plane is sufficiently strong that at some magnetic field strength the transition ceases to be second order (in the sense of Gibbs) and becomes first order or discontinuous. The point at which the smooth second order transition changes to a first order transition is clearly a special critical point. For reasons to be given below, this point is termed a tricritical point. As R is decreased in absolute value, the phase diagram remains qualitatively the same (as shown in Fig. 3a and Fig. 3b) with a Néel

temperature depending on R. The two tricritical points coelesce at R=0 (cf. Fig. 3c). The resulting figure in the H-T-R space for positive and negative R is shown in Fig. 4. From the negative R side, there is a surface of ordinary critical points, which is bounded by two lines of tricritical points which intersect at the special point R=0.

The name tricritical was applied to the termination point of the line of second order transitions by Griffiths who observed that in an enlarged field space this point was formed by the intersection of three lines of ordinary critical points. Returning to the meta-magnetic model, we now apply a staggered magnetic field H' which alternates in direction on alternate planes of spins, thus favoring one or the other of the antiferromagnet orderings. Upon reaching the tricritical point, instead of merely increasing the direct field H (and proceeding onto the line of first order transitions) we may increase H and also apply the staggered field H'. The staggered field conteracts the effect of the direct field and we may continue along a line of second order transitions. In the H-H'-T space, the phase diagram appears as in Fig. 5. The half-moon coexistence surface bounded by the line of ordinary critical points in the physical plane H'=0 (cf. Figs. 3 and 4), is augmented by two pairs of "wings" formed by coexistence surfaces between one of the antiferromagnetic phases and a paramagnetic phase between the wings. The lines of critical points which border the wings intersect with the line of critical points in the physical plane H'=0 at the tricritical point.

The system when viewed in four-dimensional H-H'-R-T space is somewhat difficult to visualize but is simplified by the symmetry of the system with regard to the exchanging of H and H' while reversing the sign of R. That is, a strict symmetry of the system is given by

$$H \rightarrow H'$$

$$H' \rightarrow H$$

$$R \rightarrow -R \qquad (3.1)$$

$$s \rightarrow (-1)^p s$$

which p is even and odd on alternate planes.

The point R=0, which is the junction of four tricritical lines is called a point of order four; thus, we have the sequence of surfaces of ordinary critical points (points of order two) intersecting in lines of tricritical points (points of order three) which in turn intersect in a point of order four.

It is clear that with sufficient ingenuity, this process can be continued indefinitely, with subspaces of order θ intersecting to form critical spaces defined to be of order $\theta+1$. Such a classification of critical points has been proposed by Ref. 7, who also suggest that at a point of order θ, θ of the variables scale. That is, if there are n fields or field-like variables (such as H, H', R, and T) then, at a critical point of order θ, the important part of the Gibbs free energy (for example) is a GHF in θ of the n variables. The scaling variables are to be chosen to conform with the obvious generalization of the notions of weak and strong for ordinary critical points. For example, along a tricritical line of the system considered above, the direction corresponding to the strongest variable (largest scaling power) is out of the coexistence surface. The direction corresponding to the second strongest variable (second largest scaling power) is in the plane of the coexistence surface but not parallel to the surface of critical points of order two. The third direction corresponding to the weakest scaling variable (smallest scaling power) is in the critical surface but not

parallel to the line of tricritical points. Finally, the last direction is along the line of tricritical points and corresponds to an irrelevant or non-scaling variable. These notions of attaching augmented scaling equations to these critical points of higher order is supported by series calculations on the metamagnetic model.[2]

Almost coincidentally, the order of a critical point as defined above agrees both with the number of postulated scaling variables and the number of phases that are mutually co-critical at that point. For example, at the tricritical point (point of order three) between the wings of the metamagnet, two antiferromagnetic phases and a paramagnetic phase are simultaneously critical. At the point of order four, two antiferromagnetic phases and two ferromagnetic phases are co-critical. To see that this is indeed a coincidence we consider a Landau model which models higher order critical behavior.

If we wish to consider a system with three phases, the corresponding Landau free energy in a single variable M (this is one reason why this discussion only mimics the real situation, since in most systems two very different order parameters are competing to form the tricritical point) can be represented by a polynomial of degree six in M. By a shift in the origin of M, the coefficient of the M^5 term can be made to vanish. Therefore, the free energy can be written as

$$F(x_1, x_2, x_3, x_4, M)=$$

$$x_1 M+x_2 M^2+x_3 M^3+x_4 M^4+M^6 \qquad (3.2)$$

(The thermodynamic free energy is derived by minimizing F with respect to M.) The tricritical point is reached when $x_1=x_2=x_3=x_4=0$. We therefore must be in a four dimensional space to achieve tricriticality. The metamagnetic

system considered above bypasses this difficulty by the high degree of symmetry in the order parameter. The reversal symmetry of the Hamiltonian requires that x_3 be identically zero, and thus, only x_1, x_2 and x_4 need to be adjusted to reach tricriticality. The free energy in (3.2) provides the Landau form of scaling;

$$\lambda F(x_1, x_2, x_3, x_4, M) = F(\lambda^{5/6}x_1, \lambda^{4/6}x_2, \lambda^{3/6}x_3, \lambda^{2/6}x_4, \lambda M). \quad (3.3)$$

Thus, the scaling powers of x_1, x_2, x_3, x_4 are 5/6,4/6,3/6,2/6.

The same argument can be applied to a situation in which σ phases become simulataneously critical. The Landau free energy is a polynomial of degree 2σ with the 2σ-1 term identically zero. The number of scaling variables which must be set equal to zero to make the σ minima coelesce is 2σ-2. The scaling powers take the form $(2\sigma-i)/2\sigma$ for $i=1,..,2\sigma-2$. If special symmetry requirement are placed on the Landau free energy as in the magnetic analogue, then the number of fields necessary to generate a point of σ co-critical phases is reduced with the maximum reduction occuring when all the odd terms except the first (which corresponds to the ordering field) vanish. In this case, σ phases can be co-critical in a space of σ dimensions and σ fields will scale. This represents the magnetic limiting case discussed in Ref. 7 .

In the more general situation for multi-component fluids as described in Refs. 8-9, a line of critical points terminates in a critical end point rather than at a tricritical point (cf. Fig. 6). That is, the third phase joins the two previously co-critical phases in coexistence but is not simultaneously critical with them (cf. Fig. 6a). By changing another field or field-like variable, a line of critical end points may

be generated which eventually meets another line of critical end points at a point at which three phases are simultaneously critical (cf. Fig. 6b). Thus, an asymmetric system can, in general, only increase the number of phases which are simultaneously critical in a two stage process. First the new phase must be added in coexistence with those phases previously critical; and second, the new phase must be brought to criticality.

It is becoming customary, although there is no consensus, to define the order of a critical point as the number of phases co-critical at that point. With this definition, we see that the number of variables that can be expected to scale at a point of order σ (and therefore the minimum dimension of the space in which it must be represented) varies from σ in the fully symmetric magnetic models to $2\sigma-2$ in fully asymmetric multicomponent fluid models.

Although the single component Landau analogs do not exhaust the possibilities for critical points of higher order, they are sufficiently abundant to overwhelm the available experimental evidence. For example, the "tricritical" point in NH_4Cl, was originally thought to be a representative of a Landau-like (sometimes referred to as mean-field or, inaccurately, Gaussian) critical point of order three.[10] It has been argued that it is plausibly a Landau-like point of order four,[11] and the most recent tabulations of measured exponents are even closer to that of a critical point of order five.[12] The phenomenal number of coincedences necessary to have a critical point of order five (usually requiring the adjustment of eight fields) at an experimentally accessible point mitigates against this possibility. However, the experimental data underlines the sketchy information that is available for most realistic systems.

In the metamagnetic system discussed above, one of the fields was the unphysical staggered field H'. It was previously thought that all evidence concerning such systems would have to be gathered in the "physical

plane" H'=0. On the contrary, in materials such as DAG it appears that a distorted crystal field may produce a staggered field near the tri-critical behavior but rather the critical behavior on one of the wings. The staggered field cannot be controlled externally, however, and the induced staggered field complicates the study of DAG considerably.[13]

The situation is even more difficult in more complicated systems such as multi-component fluid mixtures or the ammonium halides. In these cases it is not even clear what field variables should be chosen and there is no detailed information available about the phase diagram in the man-dimensional field spaces in which these points must be represented. Only a narrow slice of the phase diagram can be examined; this low dimensional view could obscure the phenomenological situation.

It is, therefore, unlikely that a true test of scaling can be made at any of the higher order critical points such as in multicomponent fluid mixtures and the ammonium halides. Even in model systems, the location of tricritical and higher order points by high temperature series expansions is difficult. The exponents derived from a high temperature expansion are sensitive to the location of the singularity; and, therefore, the details of the phase diagram and possible scaling properties of even simple models systems is controversial.[14]

Since the notions of weak and strong directions at ordinary critical points (and their obvious extension to more complicated critical points) have a more immediate cogency then the notions of scaling, it might be interesting to explore a class of functions which accomodate the postulates of Griffiths and Wheeler, but which do not scale (are not generalized homogeneous functions). In the first part of Chapter 3 , we introduce and discuss a class of such functions which we call "critically ordered". To illustrate what is required for a system to be critically ordered, we

will discuss the example of the ordinary critical point.

We first consider the scaling case. We assume that the singular portion of the pressure is a generalized homogeneous function of variables x_1 and x_2 which are taken to be smooth functions of the chemical potential μ and the temperature $t \equiv T - T_c$. The slope of the isochore is given by $(\partial\mu/\partial T)_\rho$. We may express this as $\partial(\mu,\rho)/\partial(t,\rho)$ and rewrite as

$$\frac{\partial(\mu,\rho)}{\partial(t,\rho)} = \frac{\partial(\mu,\rho)/\partial(x_1,x_2)}{\partial(t,\rho)/\partial(x_1,x_2)} . \qquad (3.4)$$

Expanding the Jacobians, we obtain

$$\frac{\partial(\mu,\rho)}{\partial(t,\rho)} = \frac{\left(\frac{\partial\mu}{\partial x_2}\right)_{x_1}\left(\frac{\partial\rho}{\partial x_1}\right)_{x_2} - \left(\frac{\partial\mu}{\partial x_1}\right)_{x_2}\left(\frac{\partial\rho}{\partial x_2}\right)_{x_1}}{\left(\frac{\partial t}{\partial x_2}\right)_{x_1}\left(\frac{\partial\rho}{\partial x_1}\right)_{x_2} - \left(\frac{\partial t}{\partial x_1}\right)_{x_2}\left(\frac{\partial\rho}{\partial x_2}\right)_{x_1}} . \qquad (3.5)$$

The coordinate derivatives $(\partial\mu/\partial x_1)$, $(\partial\mu/\partial x_2)$, and so forth, are smooth and non-singular by assumption. The density ρ is $(\partial P/\partial\mu)_t$ and is therefore given by

$$\rho = \left(\frac{\partial x_1}{\partial\mu}\right)_t\left(\frac{\partial P}{\partial x_1}\right)_{x_2} + \left(\frac{\partial x_2}{\partial\mu}\right)_t\left(\frac{\partial P}{\partial x_2}\right)_{x_1} . \qquad (3.6)$$

Therefore, the density ρ is the sum of two GHFs with smooth amplitudes arising from the change of variables. The scaling power of $(\partial P/\partial x_1)$ is $1-a_1$; the scaling power of $(\partial P/\partial x_2)$ is $1-a_2$. Since $a_1 > a_2$ the second

term is vanishingly small compared to the first as the critical point is approached; the ratio vanishes like $|t|^{(a_1-a_2)/a_1}$ (cf. Chap. II). The further differentiations with respect to x_1 and x_2 indicated in (3.5) ensure that

$$\left(\frac{\partial \mathcal{P}}{\partial x_1}\right) >> \left(\frac{\partial \mathcal{P}}{\partial x_2}\right) . \tag{3.7}$$

Indeed, the ratio $(\partial\mathcal{P}/\partial x_2)/(\partial\mathcal{P}/\partial x_1)$ again vanishes like $|t|^{(a_1-a_2)/a_2}$. The quotient on the right hand side of (2.5), near the critical point reduces to

$$\left(\frac{\partial \mu}{\partial t}\right)_{\mathcal{P}} \approx - \frac{\left(\frac{\partial x_1}{\partial t}\right)_{\mu}}{\left(\frac{\partial x_1}{\partial \mu}\right)_{t}} . \tag{3.8}$$

This fixes the linear part of the transformation $x_1(\mu,t)$ in precisely the correct way so that the line $x_1=0$ is tangent to the critical isochore at the critical point.

Note that we do not need to posulate that the weak axis, i.e. the line $x_1=0$; is tangent to the isochore; the scaling hypothesis guarantees it.

In passing from (3.5) to (3.8), the necessary step is that of (3.7). The scaling hypothesis gives (3.7) and measures the precise ratio of the two derivatives, but is clearly far stronger than is necessary. An example of a system for which (3.7) holds, but which does not scale is easy to construct. If the singular part of the pressure were given by the sum of two GHFs of (for simplicity) the same variables but different scaling powers,

$$\rho = \rho(x_1, x_2) + \rho'(x_1, x_2) \tag{3.9}$$

with $a_1 > a_2$ and $a_1' > a_2'$, then (3.7) would follow, but the system would not scale.

To reach the statement of Griffiths and Wheeler expressed in (3.8) we may replace the scaling hypothesis with the weaker assumption that an ordering is associated with a particular set of variables (x_1, x_2). This ordering expresses the content of (3.7): derivatives with respect to x_1 increase the singularity of a function faster than derivatives with respect to x_2. In the scaling hypothesis discussion in (3.5)-(3.7) the density plays an inessential role. In fact, the same argument shows that the critical isentrop is also tangent to the weak axis, $x_1 = 0$. Any object Q generated by differentiation or integration of any original scaling equation will satisfy

$$\frac{\partial Q}{\partial x_1} >> \frac{\partial Q}{\partial x_2}, \tag{3.10}$$

and therefore,

$$\left(\frac{\partial \mu}{\partial t}\right)_Q \simeq - \frac{\left(\frac{\partial x_1}{\partial t}\right)_\mu}{\left(\frac{\partial x_1}{\partial \mu}\right)_t} . \tag{3.11}$$

We will assume, along with the ordering of the variables x_1 and x_2 that the set of functions for which the ordering holds is sufficiently

large to describe all the thermodynamic functions of interest. A system, endowed with a complete set of such functions and equipped with an ordered set of variables is defined as being critically ordered."

In this section we have shown that the physically cogent notions of weak and strong directions can be embodied in a system more general than that described by generalized homogeneous functions. Such "critically ordered" systems include all weak corrections to scaling in which extra terms are added to a scaling equation as discussed in Chapter 2. Less trivial examples of critically ordered systems are discussed in Chapter 5 in the context of general solutions to nonlinear renormalization group equations. These solutions are roughly of the form indicated in (3.9); the thermodynamic functions are given as a sum of generalized homogeneous functions with distinct scaling powers. In Chapter 3, the definition of a critically ordered system is extended to critical points of arbitrary order.

As noted earlier in this section, at a critical point of order σ (σ phases co-critical) the number of variables that could be expected to scale (on the basis of a Landau expansion) varied from σ for the maximally sysmmetric system to $2\sigma-2$ for a system with no symmetries at all. Classification system for critical points have been made for both limits of this range; Refs. 2-7 have discussed the symmetric limit of magnetic-like systems, while the fully unsymmetric multi-component fluid systems have been described in Refs. 8-9. In the latter part of Chapter 3, we introduce a classification system which unifies these classification systems and also treats systems with intermediate symmetry properties (i.e., neither full symmetry nor completely un-symmetric). This classification system applies both to scaling and critically ordered systems.

IV. The Renormalization Group (Linearized Theory and Scaling)

A. The Kadanoff Picture

The recent application of the renormalization group to critical phenomena has provided a firmer foundation for many of the phenomenological notions of critical behavior. First, it provides a derivation of the scaling hypothesis and a method for the calculation of scaling powers (and, hence, critical point exponents). As we will see below, the scaling hypothesis follows from the existence of "fixed-points" of the "renormalization group equations". The scaling powers are calculated by determining the eigenfuncitons and eigenvalues of the renormalization group equations when "linearized around the fixed point Hamiltonian". Second, the fact that the renormalization group equations have isolated fixed points (rather than, for example, lines or surfaces of fixed points) supports the universality hypothesis. Many Hamiltonians have their critical behavior determined by a single fixed point Hamiltonian. Third, although the calculational accuracy of renormalization group determinations of critical point exponents is limited by the perturbational nature of the renormalization analysis, the renormalization group approach can be applied to many problems where more accurate techniques such as high temperature series do not exist or give ambiguous results.

The terminology of the renormalization group approach reflects a composite of field-theoretic notions and methods from the study of systems of nonlinear first order differential equations. The connections with field-theory and differential equations will be discussed below. The underlying physical intuition is the extremely heuristic scaling theory of Kadanoff.

Consider a system of ising spins on a square lattice with lattice spacing a (cf. Fig. 7a). As the critical point is approached, the correla-

tion length is very large, and spins on distant lattice sites are strongly correlated. Over a distance L which is small with respect to the correlation length, but which may be much larger than a, we may expect the spins to be almost certainly correlated. If we consider the system to be composed of block spins containing b^2 spins (b=L/a), Kadanoff argues that the block spin system is essentially identical to the orginal site spin system. In particular, the correlation length is simply reduced by a factor of b. By a leap of faith, Kadanoff supposes that any other variables also scale with some power of b. Thus, the scaling form of the correlation length is obtained,

$$\xi(b^{a_h}h, b^{a_t}t) = b^{-1}\xi(h,t), \tag{4.1}$$

where h is the magnetic field and $t \equiv T-Tc$.

B. An Exact Approach

However, it is not necessary to proceed in this manner. Instead of simply replacing the 2^{b^2} states of the block (16 in Fig. 7) with a single block spin, we can explicitly average over all of the internal block states.

The interactions of the system can be divided into inter-block and intra-block interactions. Averaging over the internal states while holding the block spins fixed gives an effective interaction between the block spins. The new interactions between block spins will generally be more complex than the site spin interactions; for example, a site spin Hamiltonian with nearest neighbor interactions might generate next nearest neighbor interactions in the block spin Hamiltonian. If we consider a very general form for the Hamiltonian which includes all possible interactions, then we may consider the process described above as a transformation on the parameters which determine the Hamiltonian. This is an example of a renormalization transformation. In the Kadanoff case, we have only two parameters, the magnetic field h and the reduced temperature t. Calling the renormalization transformation R_b, the Kadanoff "renormalization transformations" are

$$R_b\, h = b^{a_h} h \,, \tag{4.2a}$$

$$R_b\, t = b^{a_t} t \,. \tag{4.2b}$$

In general, of course, we cannot expect the renormalization transformation equations to have the diagonal, linear form of (4.2). For an arbitrary renormalization procedure and a set of parameters $\{p_i\}$ we have

$$\mathcal{R}_b p_i = F_i(\{p_j\}), \tag{4.3a}$$

$$\mathcal{F}(\{\mathcal{R}_b p_i\}) = b^{-1}\mathcal{F}(\{p_i\}). \tag{4.3b}$$

Equations (4.3) do not bear more than a passing resemblance to the Kadanoff renormalization transformation (4.2) and scaling equation(4.1). The Kadanoff equations have following distinctive properties:

(i) The critical point $h=0, t=0$ is a fixed point of the renormalization transformation equations. That is, the Kadanoff transformations do not change the Hamiltonian parameters if the Hamiltonian is at its critical point.

(ii) The transformation equations are linear equations. The new renormalized parameters are linear combinations of the original parameters.

(iii) The linear renormalization group transformations are diagonal.

The renormalization transformations in (4.3a) will, in general have none of these properties. To obtain the simple form of the Kadanoff scaling equation, we must somehow recover these three properties.

One feature present in the Kadanoff argument, but absent in the

transformation equations is the restriction on block size mentioned above. For the argument to be plausible we must have

$$a \ll L \ll \xi \quad . \tag{4.4}$$

That is, we must include in the block spins enough spins to have an effective average, but not so many spins that the assumption of strong correlation within the blocks breaks down. Thus, we may expect the exact renormalization transformations (4.3a) to have a range of b for which the transformation equations are simple; for b too small or too large, we cannot hope to obtain the Kadanoff behavior (4.2).

Secondly, the construction in (4.1) and (4.2) by-passed entirely the determination of the critical temperature. We generally do not know the values of the critical parameters. We must determine them from the renormalization group equations themselves. In analogy with (4.2), we look for fixed points of the renormalization equations; that is, values of the parameters $\{p_i^*\}$ which have the property that

$$R_b p_i^* = p_i^* \quad . \tag{4.5}$$

It is easy to show that each such fixed point of the renormalization equations corresponds to a critical point. If (4.5)holds, then we may write (4.3b) as

$$\xi(\{p_i^*\}) = b^{-1}\,\xi(\{p_i^*\}). \tag{4.6}$$

This is only possible if $\xi = 0$ or $\xi = \infty$. The vanishing of the correlation length corresponds to a so-called "infinite temperature fixed point" (see discussion in Sec. V of this Chap. p and Chapter 5). The divergence of the correlation length is a sure sign of a critical point.

A third characteristic of the Kadanoff equations which is not immediately obvious in (4.2) is the role played by so-called irrelevant variables. For example, for the Ising system shown in Fig. 2, the introduction of lattice anisotropy shifts the critical temperature but it does not change the critical point exponents (cf. Fig. 3). If, however, we included the effect of possible anisotropy in the exact renormalization equations (4.3a) we would obtain a fixed point (4.5) for some particular value of the anisotropy, and not any other. What in the renormalization group picture corresponds to the smooth line of critical points produced in the phenomenological analysis by changing the amount of anisotropy in a system? The resolution of this difficulty lies in the renormalization use of the term "irrelevant variable". We will write the anisotropy parameter R as $1+g$, so that $g=0$ corresponds to an isotropic system. We imagine that we can augment the equations (4.2) with an equation for g

$$R_b\, g = b^{-a_g} g\,, \tag{4.7}$$

where a_g is positive. As the renormalization procedure includes larger and larger blocks of spins (corresponding to approaching the critical temperature and infinite correlation length), the anisotropy parameter g becomes smaller and smaller. If we assume that the exact correlation length depends smoothly on g, we can perhaps set g=o, its fixed point value. Thus, for sufficiely large block averages, the effect of the anisotropy disappears; the anisotropic system behaves like the isotropic system.

We may also approach this issue more formally. The solution for ξ given by the renormalization equations (4.2) and (4.7) is

$$\xi(b^{a_h}h, b^{a_t}t, b^{-a_g}g) = b^{-1}\xi(h,t,g) . \tag{4.8a}$$

Setting h=0 for convenience, this may be rewritten using the properties of generalized homogeneous functions as

$$\xi(t,g) = |t|^{-1/a_t}\xi(\mathrm{sgn}\, t,\ g|t|^{a_g/a_t}). \tag{4.8b}$$

The anisotropy parameter g enters only in the combination $g|t|^{a_g/a_t}$. This tends to zero as t→0 for all values of g and the dependence on g disappears in the asymptotically valid critical behavior.

Of course, the actual renormalization group equation g is not likely to be as simple as (4.7). However, the principle is the same. If a parameter, regardless of its initial value, tends to a particular value under the renormalization transformations, we term

the parameter "irrelevant." Comparing (4.2) and (4.7) we note that the distinction between irrelevant and relevant variables in the Kadanoff linear renormalization equations is that the irrelevant variable g has a negative scaling power, $-a_g$, while the relevant variables h and t have positive scaling powers a_h and a_t.

Thus a critical point corresponds to all the relevant parameters (that would increase under renormalization, e.g. (4.2)) being set to their fixed point values. The irrelevant parameters may have any value. The renormalization equation for the correlation length at the critical point reads

$$\xi(\{R_b q_i\}, \{p_i^*\}) = b^{-1}\xi(\{q_i\}, \{p_i^*\}), \tag{4.9a}$$

where the irrelevant parameters have been denoted as $\{q_i\}$ and the relevant parameters as $\{p_i\}$. As b grows large, the irrelevant parameters tend to their fixed point values and (4.9a) becomes

$$\xi(\{q_i^*\}\{p_i^*\}) = b^{-1}\xi(\{q_i\}, \{p_i^*\}), \tag{4.9b}$$

which, since it is true for all sufficiently large b, again implies that $\xi = \infty$.

C. Formal Renormalization Group Procedure

We can now describe the four stages of a renormalization group approach to a critical system, in close analogy to the Kadanoff approach, but presumably more rigorous.

(i) We must define a renormalization group transformation R_b. Kadanoff simply assumes them to be of the form given in (4.2). The construction of an exact transformation is more difficult.

(ii) The fixed point (or fixed points; we are not guaranteed that there is only one) of the renormalization equations must be located. These correspond to critical points of teh system for a particular choice of the irrelevant variables. Kadanoff's equations have the immediate and unique fixed point h=t=o.

(iii) Since the fixed point Hamiltonian corresponds to the critical point, we will assume that small variations in the Hamiltonian parameters from their fixed points values correspond to small variations from the critical point. We accordingly linearize the renormalization group equations around the fixed point. Kadonoff's equations are, of course, already linear.

(iv) The linearized renormalization equations are then assumed to be diagonalizable; placing them in diagonal form, we arrive at the form of the Kadanoff transformation equations (4.2) and can extract the scaling powers from the eigenvalues of the linearized, diagonal equations. Kadanoff's equations are already diagonal.

The range of b for which the renormalization equations might simplify can now be specified. We choose b sufficiently large that the irrelevant parameters are driven to their fixed point values, however, b cannot be so large that the relevant parameters are carried out of the region of validity of the linearization carried out in step (iii) of the standard renormalization group procedure.

Later in this section, we will perform steps (i)-(iv) explicitly for a specific renormalization group. At this point we will just write down a set of formal equations describeing (ii)-(iv).

First we must solve the fixed point equation

$$R_b p_i^* = p_i^* . \tag{4.10a}$$

This is often the hardest part of the solution. Just as in high temperature series analysis, the determination of critical point exponents is relatively straightforward once the critical point is located. Since the renormalization equations are highly nonlinear (cf. (4.21) below), the fixed point equation is solved in many cases by some approximate or peturbational analysis. This step is that which usually limits the accuracy of the scaling powers calculated in step (iv).

Second, we set $p_i = p_i^* + \delta p_i$ and determine the linearized equations for δp_i.

$$R_b(p_i^* + \delta p_i) - p_i^* = L_b(\{\delta p_j\}) + O(\{\delta p_k \delta p_\ell\}) \tag{4.10b}$$

where L is some linear operator which depends on R_b and $\{p_i^*\}$. We must assume of course that the linearized transformation exists. In all the cases examined to date, there appears to be a well-defined linear transformation at each fixed point.

Third, we endeavor to diagonalize the linear transformation (4.10b).

It is a further assumption that this diagonalization procedure will not introduce complex numbers. Thus, we assume that we may choose linear combinations of the $\{p_i\}$ such that

$$\mathcal{U}_b \, x_i = \Lambda_i(b) \, x_i \,, \tag{4.10c}$$

where the eigenvalues $\Lambda_i(b)$ are real. This is not a trivial assumption. A simple example of a linearized "renormalization transformation" which does not have real eigenvalues is

$$\mathcal{U}_b \, p_1 = p_1 \cos(\ell n b) - p_2 \sin(\ell n b) \,, \tag{4.11a}$$

$$\mathcal{U}_b \, p_2 = p_2 \cos(\ell n b) + p_1 \sin(\ell n b) \tag{4.11b}$$

defining $z_\pm = p_1 \pm i p_2$ gives

$$\mathcal{U}_b \, z_\pm = b^{\pm i} \, z_\pm \,. \tag{4.11c}$$

The renormalization equations (4.11a) and (4.11b) describe circular

motion of the parameters p_1 and p_2 around the fixed point $p_1=p_2=0$. The solutions of these equations, the "renormalization trajectories", never enter the critical point nor leave it. No renormalization group equation seems to have anything but real eigenvalues. This corresponds to the intuition of the Kadanoff derivation. If we average over too large a block ($b\to\infty$) we do not expect the system to resemble a critical system. In renormalization group terms, the relevant parameters will "run away" from their fixed point values. Therefore, on physical grounds we expect the eigenvalues to be real.

In all our examples the eigenvalues of the linearized renormalization group equations have been chosen to be powers of the renormalization parameter b. This is a general feature of the renormalization group. Returning to the Kadanoff picture, we can imagine performing a second block transformation, averaging over blocks of blocks to form a super-block spin. This must be equivalent to performing a single Kadanoff transformation directly from the site spins to the super-blocks. If the two separate renormalization factors are b and b' we must have

$$R_{bb'} = R_b R_{b'} . \tag{4.12}$$

This represents the semi-group property of the renormalization group. It becomes a true group only when placed in its linearized form. Using (4.12) we see that the eigenvalues in (4.10c) must be of the form

$$\Lambda_i(b) = b^{a_i} x_i , \tag{4.13a}$$

so that (4.10c) can be rewritten as

$$\amalg_b x_i = b^{a_i} x_i . \tag{4.13b}$$

These equations are well defined for all values of b. The original transformation which transformed site spins into block spins was only defined for integral b; (4.13b) is well defined for all b>0.

The correlation length can be considered as a function of the parameters $\{x_i\}$ instead of the original parameters $\{p_i\}$. Combining (4.13b) with the renormalization equation for the correlation length (4.13b) we finally obtain the Kadanoff form

$$\xi(\{b^{a_i} x_i\}) = b^{-1} \xi(\{x_i\}). \tag{4.14}$$

Formulations of renormalization groups that take precisely this form of converting site spins to block spins have been considered by Niemeijer and van Leeuwen,[16] Nelson and Fisher,[17] Kadanoff and Houghton,[18] and others. In these groups, the exact nature of the lattice as well as the discrete nature of the allowed spins values is retained. However, to date these methods have been confined to one and two dimensional systems; since many of these systems are exactly soluble, the exact solutions can be compared to the renormalization soltuions to check the accuracy and validity of the renormalization equations.

Such checks seem to indicate a high reliabilty for the renormalization calculations. The extensions of the site and block renormalization schemes to three-dimensional systems appears to more difficult.

D. Field Theoretic Analogue

An alternate approach abandons the details of the lattice structure and spin quantatization in favor of a field-theoretic model. Following Wilson,[19] we replace the set of localized site spins assuming discrete values with a spin density s(x) which may take on any value $-\infty < s(x) < \infty$. Although this might appear to be a crude approximation, high temperature series analysis indicates an insensitivity of critical point exponents to the spin quantum number.[20] The lattice structure can be retained by requiring that the Fourier transform of the spin density, s(k) has its support in the first Brillouin zone (cf. Fig. 8). As such a requirement renders the theory cumbersome, and since high temperature series analysis indicates an insensitivity of critical points exponents to the details of the lattice structure,[21] it is convenient to replace the Brillouin zone by a sphere

$$|\vec{K}| \leq \Lambda , \tag{4.15}$$

where Λ is roughly the reciprocal of the lattice spacing.

Instead of averaging over all the spins is a block of size L, we average over all the momenta between some momentum p and Λ. In the site spin case, we could only expect simple behavior if we averaged over enough sites to smooth away unimportant fluctuations (corresponding to irrelevant parameters), but not over too many sites (cf. (4.4)). The corresponding restriction in momentum space is

$$\xi^{-1} << p << \Lambda . \tag{4.16}$$

Having performed the averaging over all s(k) with k between $p \equiv \Lambda/b$ and Λ, we again choose to regard the resulting system is essentially equivalent to the original system with new interaction parameters, defining a renormalization transformation(cf Fig. 7b).

This momentum space approach to renormalization was introduced to critical phenomena by Wilson[19] and developed by many other authors. It is, of course, in this formulation that the theory is closest to its field theoretic progenitor. The renormalization transformation again reduces the length scale by a factor of b; the momentum space scale factor is correspondingly increased by a factor of b. To see this directly, recall that in the renormalized system, the un-averageed over momenta are bounded by Λ/b. To put this in the same form as (4.15) the renormalized value of the cutoff momentum $R_b\Lambda$ is $b\Lambda$. Thus, the renormalization process can be considered to be a method of gradually removing the cutoff of a field theory. The inverse of the correlation length plays the role of an effective mass. Eq. (4.16) says that we are interested in the behavior of the field theory described by the spin density for mementa much larger than the effective mass. This becomes the high energy limit of the field theory as the cutoff momentum becomes infinite. The scaling form for the correlation length and other thermodynamic functions is the asymptotical scale invariance of field theoretic literature. The study of cutoff field theories in the limit of infinite cutoff is, of course, the original provenance of the renormalization group.

The advantages of this approach lie in its approximations, which have discarded details which are unimportant. We also are free to borrow the results and techniques of many years of field theoretic perturbation theory. The disadvantages are that we have encumbered ourselves with the ultraviolet divergences of field theory (when $\Lambda \to \infty$)

and must carefully rearrange and reorder all our terms to give a finite limit as the cutoff becomes infinitely large. Of course, in field theory, the cutoff is an artifice which must be eliminated ; in the original lattice system, it represents the physical fact of spins which are more or less isolated on definite lattice sites, and is therefore real. The second disadvantage is that we are forced into perturbation analysis of a poorly controlled nature. In the Niemeijer approach,[16] for example, we must, in practise, truncate the hierarchy of of interactions contained within the renormalization scheme, including nearest neighbor, second neighbor and third neighbor interactions but not any fourth or more distant interactions. The approximation has a physical basis; we may have reasons to discard such long-range interactions. The remaining interactions are treated exactly. On the other hand, in the field theoretic approach, we must assume that all the "coupling constants" (the parameters describing the "interaction" Hamiltonian, see discussion below) are small. For example, the Wilson-Fisher expansion is a perturbation in the parameter $\varepsilon \equiv 4-d$. We are, unfortunately, interested in numerical results for real physical systems for which $d=3$ and $\varepsilon=1$. This is not precisely small; in fact, it is believed that the ε-expansion may be an asymptotic expansion.[22] Good numerical agreement is found at the $O(\varepsilon^2)$ term with results of high temperature series analysis. This is extremely fortunate since the results are only known to $O(\varepsilon^4)$ for the Wilson-Fisher model.[23]

E. Differential Generators

Although the connections with field theory are many and we will continue to borrow terminology and results from it, we will not pursue it further in this section. For the most part we will use a formulation of the renormalization group due to Wegner and Houghton.[24] In this formulation, an infinitesimal or differential generator of the renormalization group is derived. By infinitesimal we mean that the behavior of the renormalization transformation is studied for b differing only infinitesimally from 1. Formally, this infinitesimal generator can be defined as

$$\mathcal{D} = \lim_{b \to 1} \frac{\mathcal{R}_b - I}{b - 1} \quad . \tag{4.17}$$

In contrast to the averaging over a finite shell of momenta between Λ/b and Λ, Wegner and Houghton consider only those momenta in a very thin shell and take the limit as the shell becomes infinitesimal. They were able to show that in this limit certain classes of Feynman diagrams which appear in the perturbation series for general $\mathcal{R}_b$ can be neglected in this limit. They were therefore able to re-sum the pertubation series to give a closed form expression for the infinitesimal generator.

Infinitesimal generators are termed differential generators because they determine differential equations for the Hamiltonian parameters. It is customary to use ℓ as the continuous parameter of the differential generator (so that for finite renormalizations $b=\exp(\ell)$). The differential generator replaces the recursive equations (4.3a) with first order nonlinear ordinary differential equations

$$\frac{\partial P_i}{\partial \ell} = F_i(\{P_j\}). \tag{4.18}$$

The correlation length scales as $\exp(-\ell)$ so that the differential equation for ξ is

$$\frac{\partial \xi}{\partial \ell} = -\xi. \tag{4.19}$$

In Chapter 4 we introduce an approximate form of the differential generator of Wegner and Houghton. Although the quality of the approximation is not subject to rigorous a priori assessment, it is equivalent to restricting the Hamiltonian densities to be of the Landau-Ginzberg form.

$$\mathcal{H} = |\vec{\nabla}\vec{s}|^2 + H(\vec{s}). \tag{4.20}$$

The "free term" in the field theory is the gradient term; $H(\vec{s})$ is the "interaction". A similar approximation and restriction was made by Wilson[19] in his derivation of the "approximate recursion formula". The approximate differentail generator given in (4.21) below is probably the differential form of Wilson's approximate renormalization group, but this has not been shown. The advantage of the differential

approach is the multitude of techniques avalilable for the solution of differential equations, some of which are unfamiliar or lacking for finite difference equations. With this differential generator we can carry out the entire four step renormalization procedure. We will define a renormalization transformation, locate fixed points, linearize around those fixed points, and extract scaling powers by diagonalization.

(i) Definition of the Renormalization Group Transformation

Wegner and Houghton choose to keep the coefficient of the gradient term in the Hamiltonian density constant. In terms of our approximation, we expect to determine a differential equation for the function H(s). We find

$$\frac{\partial H}{\partial \ell} = dH + \left(\frac{2-d}{2}\right) \vec{S} \cdot \vec{\nabla} H + \ln \det \left[I + \hat{A} \right] \tag{4.21}$$

where $\hat{H}$ is the matrix of second partial derivatives of the function H(s)

$$\hat{H}_{ij} = \frac{\partial^2 H}{\partial S_i \partial S_j} , \tag{4.22}$$

and d is the lattice dimension.

Although the details of the derivation of the Wegner-Houghton equation and, in particular, this approximation, are beyond the scope of this section, a few explanitory remarks can be made.

The first term on the right hand side of (4.21) arises from the change in effective volume. The length scale as measured by the correlation length behaves as $\exp(-\ell)$, so that the volume changes under

renormalization as $\exp(-d\ell)$. Since H is a Hamiltonian density, a factor of $\exp(+d\ell)$ is to be expected.

The operator $\vec{s}\cdot\vec{\nabla}$ is the second term of (4.21) is a power counting operator which replaces a term of order m in the spin components with m times the same term. This term in the generator accounts for the rescaling of the spin variables themselves. The gradient term in the Hamiltonian density is to be held fixed. To do so, we must scale the spins as $\exp[\ell(2-d)/2]$ to compensate for the change of length scale. In the exact formulation of Wegner and Houghton, this rescaling factor is chosen to be $\exp(\ell\,(2-d-\eta)/2)$. The critical point exponent η is introduced to cancel contributions to the gradient term which arise from the average over the infinitesimal momentum shell. In the approximation used here, these terms have been dropped. Thus, the approximation fails if η is not small. In (4.21), we have set $\eta = 0$ for consistency.

The third term in (4.21) is the only vestige of the renormalization average taken over the infinitesimal shell of momentum. The fact that it involves only the second derivatives of the Landau energy H(s) reflects the simplification achieved by taking the infinitesimal limit. The determinant represents the change of variables made in order to perform the functional integral over the states in the shell. The logarithm is simply the connection between the partition function and the Hamiltonian.

(ii) Location of a Fixed Point

The fixed point equation

$$\frac{\partial H}{\partial \ell} = 0 \qquad (4.23)$$

has many solutions. The simplest solution (which is central to all our later perturbation studies) is the trivial or Gaussian fixed point, given by H=0. Although this fixed point is obtained by inspection and is particularly simple, we must not underestimate its importance. It is the only fixed point which is exactly known. It therefore is the anchor point to which we must refer.

(iii)-(iv) Linearization Around the Fixed Point and Determination of Scaling Powers

If we linearize (4.21) around H=0 we obtain the equation

$$\frac{\partial H}{\partial \ell} = dH + \left(\frac{2-d}{2}\right) \vec{s} \cdot \vec{\nabla} H + \vec{\nabla}^2 H \quad . \tag{4.24}$$

This equation has a familiar structure; the eigenfunctions of (4.24) are the eigenfunctions of teh harmonic oscillator (as first pointed out by Wegner[25] for the Wilson approximate recursion formula). For single component spins, these are the Hermite polynomials. The eigenvalue of the ith Hermite polynomial is given by

$$\lambda_i = \frac{i}{2}(2-d) + d \quad . \tag{4.25a}$$

These eigenvalues do not look immediately familiar. First we must recall that the free energy density scales as $\exp(d\ell)$. To convert these eigenvalues to the scaling powers commonly used in the phenomenological literature, we must divide the eigenvalues in (4.25a) by d,

$$a_i = \frac{i}{2}\frac{(2-d)}{d} + 1 \qquad (4.25b)$$

Now, we borrow a result from field theory which shows that mean field theory for an order σ critical point of Landau-Ginsberg form (cf. Eq (3.2) of Sec. iii) is valid for all dimensions $d > 2\theta/(\sigma-1)$. If we insert the value of the borderline dimension for such a point,

$$d_\theta \equiv 2\theta/(\theta-1) \,, \qquad (4.25c)$$

into (4.25b) we obtain

$$a_i = (2\theta - i)/2\theta \,, \qquad (4.25d)$$

which are precisely the mean-field values of the scaling powers derived in Sec. iii (cf. (3.3))! Note also that the eigenvalue of the 2θth Hermite polynomials is proportial to the difference between the lattice dimension and the borderline dimension for an θth order critical point,

$$\lambda_{2\theta} = (\theta-1)(d_\theta - d) \,. \qquad (4.26)$$

Thus, the 2σth Hermite polynomial corresponds to an irrelevant variable for d greater than the borderline dimension (4.25c) when mean-field holds, but to a relevant variable when d is less than the borderline dimension when mean field fails. We may understand this by considering that when mean-field holds, we may neglect all the fluctuations of the spin. In terms of the Fourier transform, this means that the support of the transform is the origin of momentum space. The possibility of such condensation into a single mementum state is lost in the field theoretic formulation which depends on some non-zero support in momentum space. When this assumption is invalidated, the renormalization group incorrectly, but understanably, treats the 2σth Hermite polynomial as irrelevant; there is no contribution to the Landau form (4.20) from fluctuations when there are no fluctuations.

Similarly, when fluctuations are important, mean field fails. The fluctuations of the spin will contribute to the Landau energy. The 2σth Hermite polynomial corresponds to a relevant term and grows in importance as renormalization proceeds. The Gaussian fixed point cannot be the correct fixed point when this is the case. The eigenvalue of the 2σth Hermite polynomial is relevant; to be at the critical point we would have to set it equal to zero. The highest order term in our Landau expression would be lost. To correctly describe the critical behavior for a critical point of order σ, we must find another fixed point.

In Chapter 4 we describe the location of the new fixed point and the determination of the new eigenvalues to first order in the difference $d_\sigma - d$. Since the discussion takes the form of a Physical Review Letter and is extremely brief we will describe the technique in some detail in this section.

The case $\sigma=2$ is the now classical computation of Wilson and Fisher[23]

for the ordinary critical point. The $\sigma=3$ tricritical case has been discussed precisely at the borderline dimension d=3 by Riedel and Wegner[26]. The extension of this work below three dimensions has been studied by Stephen and McCauley[27], and Chang, Tuthill and Stanley.[28] Ref. 28 has also considered perturbations from the $\sigma=4$ case, which has borderline dimension d=8/3. The study of successively larger σ by the methods employed in these references is rendered extremely difficult by the number of simultaneous equations which need to be solved. The number of equations necessary for the first order calculations is $2\sigma-2$, which rapidly becomes unmanageable.

The simplification and extension of the earlier results to arbitrary σ is made possible by the simple, explicit form for the renormalization group transformation given in (4.21). Its closed form, differential nature makes it easier to extract the essential features of the order σ problem.

We expect the fixed point to retain reflection symmetry so we confine our attention temporarily to the even Hermite polynomials. Since the mean-field and Gaussian results are identical at the borderline dimension, we imagine that at dimensions close to the borderline, the fixed point is "small". Postulating the existence of a fixed point close to H=0, we attempt to locate it by a perturbation expansion of (4.21). We write the fixed point Hamiltonian H=H* as

$$H^* = \epsilon_\sigma H^{(1)} + \epsilon_\sigma^2 H^{(2)} + O(\epsilon_\sigma^3) \tag{4.27}$$

where ϵ_σ is some small expansion parameter to be determined and where we have discarded term of higher than second order in ϵ_σ. Inserting

into the fixed point equation (4.23) and retaining terms only up to second order we have

$$0 = \epsilon_\sigma \mathcal{L} H^{(1)} + \epsilon_\sigma^2 \mathcal{B}(H^{(1)}, H^{(1)}) + \epsilon_\sigma^2 \mathcal{L} H^{(2)} + \cdots \tag{4.28}$$

where $\mathcal{L}$ is the linear part of the renormalization equation given in (4.24) and $\mathcal{B}$ is the quadratic part (cf. Chap. 4).

The linear operator acting on $H^{(2)}$ and the quadratic term involving $H^{(1)}$ are both of order ϵ_σ^2. There is no term to balance the (apparently) $O(\epsilon_\sigma)$ term $\mathcal{L} H^{(1)}$. Therefore, the fixed point equation (4.28) cannot be satisfied unless $\mathcal{L} H^{(1)}$ is itself $O(\epsilon_\sigma)$. This, in turn, is only possible if $H^{(1)}$ is an eigenfunction of the linear operator; that is, it must be a Hermite polynomial. The expansion parameter can be indentified with the eigenvalue of the eigenfunction chosen. Thus,

$$H^* = \epsilon_\sigma v_\sigma Q_\sigma + \cdots \tag{4.29a}$$

where Q_σ is the 2σth Hermite polynomial and

$$\epsilon_\sigma \equiv \sigma(2-d) + d \tag{4.29b}$$

The borderline dimensions for the order σ point is just given by setting $\epsilon_\sigma = 0$.

To obtain the fixed point value of v_σ we choose $H^{(2)}$ orthogonal to Q_σ and take the inner product of (4.29) with Q_0. This determines v_0. Since the fixed point is small, we expect the new eigenfunctions to differ only slightly from the Gaussian eigenfunctions. This proves to be the case, and we may calculate an $O(\epsilon_\sigma)$ shift in the eigenvalues in terms of integrals of Hermite polynomials. The result is very simple

$$\hat{\lambda}_i = \frac{i}{2}(2-d)+d - \frac{2\epsilon_\sigma \binom{i}{\sigma}}{\binom{2\sigma}{\sigma}} \quad . \tag{4.30a}$$

The renormalized values of the scaling powers quoted for mean-field in (4.25b) are again given by dividing these eigenvalues by d

$$\hat{a}_i = \frac{2\sigma - i}{2\sigma} + \frac{\epsilon_\sigma(\sigma-1)}{2\sigma}\left[\frac{i}{2\sigma} - \frac{2\binom{i}{\sigma}}{\binom{2\sigma}{\sigma}}\right] \quad . \tag{4.30b}$$

Having obtained the scaling powers, the critical point exponents are determined to $O(\epsilon_\sigma)$. However, comparison with the exact Wegner-Houghton equations shows that the critical point exponent η is zero to this order. By using field theoretic techniques[29], we have been able to calculate η to $O(\epsilon_\sigma^2)$. For Ising systems, we find

$$\eta_\sigma = \frac{4\epsilon_\sigma^2}{\binom{2\sigma}{\sigma}^3} \quad . \tag{4.30c}$$

These calculations can also be carried out for n-component spins; the

details are discussed in Chapter 4.

The calculations for critical points of higher order only differ from mean-field predictions in dimensions less than the borderline dimension for each order σ; that is, $d < 2\sigma/(\sigma-1)$. Thus, it is expected that mean-field holds in three dimensions for all such higher order point with $\sigma \geq 3$. However, (4.30) does apply to higher order critical points in two-dimensional systems.

Eq. (4.29)-(4.30) determine the properties to leading order of all the usual Landau model small fixed points. Ther, are however, many other fixed points (some of which will be discussed in Chapter V). For Ising systems, we may discuss a large class of these fixed points, those characterized in Chapter 4 as "odd-dominated" Hamiltonians. In these systems, the leading term in the expansion for the fixed point Hamiltonian is an odd Hermite polynomial in the spin s. The perturbation expansions for such systems must be carried to cubic terms and the expansion parameter is given by the square root of the corresponding hermite polynomials eigenfunction. That is, we write the expansion of the fixed point Hamiltonian H* as

$$H^* = \sqrt{\epsilon_{\sigma-\frac{1}{2}}}\, V_\sigma\, h_{2\sigma-1} + \epsilon_{\sigma-\frac{1}{2}}\, H^{(1)} + \epsilon_{\sigma-\frac{1}{2}}^{3/2}\, H^{(2)} + O(\epsilon_\sigma^2), \tag{4.31a}$$

where $h_{2\sigma-1}(s)$ is an odd Hermite polynomial and

$$\epsilon_{\sigma-\frac{1}{2}} \equiv (\sigma-\tfrac{1}{2})(2-d) + d. \tag{4.31b}$$

The fixed point equation reads

$$0 = \epsilon_{\sigma-\frac{1}{2}} \mathcal{L} H^{(1)} + \epsilon_{\sigma-\frac{1}{2}} v_0^2 \mathcal{D}(h_{2\sigma-1}, h_{2\sigma-1}) + \epsilon_{\sigma-\frac{1}{2}}^{3/2} v_0 h_{2\sigma-1} + \epsilon_{\sigma-\frac{1}{2}}^{3/2} v_0^3 \mathcal{C}(h_{2\sigma-1}, h_{2\sigma-1}, h_{2\sigma-1}) + \epsilon_{\sigma-\frac{1}{2}}^{3/2} v_0 \mathcal{D}(H^{(1)}, h_{2\sigma-1}) + \epsilon_{\sigma-\frac{1}{2}}^{3/2} H^{(3/2)} + \cdots \tag{4.32}$$

where $\mathcal{C}$ represents the cubic part of the renormalization equation (4.21).

It is now easy to see why the expansion parameter must be choosen as in (4.31). The leading term, $h_{2\sigma-1}$ is an odd Hermite polynomial, and is orthogonal to any even function of the spin s. If an expansion of the form (4.29) were employed, the inner product taken to determine the fixed point would vanish identically. In this case, we must first choose $H^{(1)}$ to balance the $O(\epsilon_\sigma)$ portion of the equation. Having determined $H^{(1)}$, we choose $H^{(3/2)}$ to be orthogonal to $h_{2\sigma-1}$ and take an inner product to determine the fixed point value of v_σ. The eigenvalues of the various Hermite polynomials can be calculated at the fixed point. They differ from their values at the H=0 fixed point by an $O(\epsilon_\sigma)$ correction (see Chap. 4).

The odd-dominated systems are different from the usual even Hamiltonian fixed points in that for $\epsilon_\sigma > 0$ (when the fixed point is "stable", see Sec. V) the fixed point value of the Hamiltonian is pure imaginary (at least for σ=2,3). For $\epsilon_\sigma < 0$, the fixed point is real but "unstable". As will be discussed in Sec. V, such a fixed point probably does not contribute to the asympototically valid values of critical point exponents, but will influence the critical behavior away from the critical point.

We also introduce in Chapter 4 an approximate renormalization group

generator based on the exact equation developed by Wilson[22] using a "partial integration technique". We show that it gives the same results as the generator (4.21) based on the Wegner-Houghton equations. Since the derivational technique is more subtle than the straightforward Wegner-Houghton approach, which mimics the Kadanoff ideas precisely, we will not discuss this new generator in this section.

V. The Renormalization Group: Global Theory and "Critical Ordering"

In this section we show how to incorporate the linearized, local approach to the renormalization group into a global, nonlinear theory. In the linearized theory, a fixed point of the renormalization group transformations is found and the renormalization equations are linearized around that point. As discussed in Sec. III, this leads directly to the scaling form for thermodynamic potentials and the determination of the scaling powers. This approximation has a double nature as expressed in the term "linearized, local" used to describe it.

First, the equation have been linearized. For example, suppose the renormalization equations concerned a single parameter p with renormalization equation

$$\frac{\partial p}{\partial \ell} = 2p(1+p^2). \tag{5.1}$$

When linearized around its fixed point p=o and combined with the equation for the correlation length (4.14) we have

$$p = e^{2\ell}, \tag{5.2a}$$

$$\xi = p^{-1/2}. \tag{5.2b}$$

However, the exact solution of (5.1) is not (5.2a). Solving (4.1) exactly we have instead of (5.2) the following

$$\frac{p}{(1+p^2)^{1/2}} \simeq e^{2\ell} \,, \tag{5.3a}$$

$$\xi = p^{-1/2}(1+p^2)^{1/4} \,. \tag{5.3b}$$

The expressions given in (5.2) and (5.3) do not differ significantly for small p but have radically different behavior for large p. Thus, the use of nonlinear equations yields "corrections to scaling" terms similar to those discussed in Chap. 2 ; that is, the use of nonlinear renormalization group equations can give the deviations from the scaling behavior derived from the linearized equations.

The example (5.1) was carefully chosen to avoid the second aspect of the approximation in the linearized, local analysis; that is, locality. Eq. (5.1) has only one real fixed point; in Sec. III we were able to establish a connection between the singularities of the correlation length and the existence of fixed points. If there is only one fixed point then there is only one sort of critical behavior. In general, there will be several fixed points of the renormalization equations each "representing" a different critical behavior (each fixed point represents a different critical behavior since the eigenvalues of the linearized equations will generally differ at the

different fixed points and hence the scaling powers and critical exponents will differ).

The existence of several fixed points is a far more serious difficulty than the simple nonlinearity introduced in (4.1). Four questions need to answered:

(i) Is there still a scaling equation?

(ii) Which fixed point determines the exponents of teh scaling equation, if it exists?

(iii) How do the other fixed points influence the behavior of the system?

(iv) Under what conditions does the asymptotic scaling behavior escape the influence of one fixed point and come under the influence of another?

Before we can discuss these questions we must introduce a somewhat technical subject. Wegner[30] has termed expressions that have purely exponential dependence on the renormalization parameter nonlinear scaling fields. Thus the expression on the left hand side of (5.3a) is the nonlinear scaling field corresponding to (5.1). The questions (i)-(iv) will be addressed most easily in terms of these nonlinear fields. Wegner explored the existence of nonlinear scaling fields in formal power series expansions around a particular fixed point (i.e. for (5.1), p=0). This approach is limiting by the fact that such series will almost never have an infinite radius of convergence. The power series expansion of (4.3a) around p=0 fails to converge at p =1 due to the pure imaginary fixed points at p=±i. However, the exact nonlinear scaling field is perfectly well behaved at p =1. This is not the case for multiple real fixed points. If (5.1) is modified slightly to introduce additional real fixed points

$$\frac{dp}{dl} = 2p(1-p^2) , \tag{5.4}$$

then the solutions for p and the correlation length are given by

$$\frac{p}{(1-p^2)^{1/2}} \sim e^{2l} , \tag{5.5a}$$

$$\xi = p^{-1/2}(1-p^2)^{1/4} . \tag{5.5b}$$

In this case, the nonconvergence of the expansion at p =1 represents a real singularity in the nonlinear scaling field and a zero of the correlation length.

Any truncated power series expression for a nonlinear scaling field or a thermodynamic quantity such as the correlation length fails to distinguish between these two possibilities and totally misrepresents the nature of the solution at any fixed point other than the original fixed point about which the expansion was made.

The above discussion shows that we may expect singularities in the nonlinear scaling fields in the vicinity of other fixed points. To successfully incorporate these expected singularities, the local approach

to the renormalization group problem (as reflected, for example, in the use of series expansions) must be superceded by a global approach which considers all the fixed points of the renormalization equations even-handedly (cf. Fig. 8). In this way the major singularities of the nonlinear scaling fields can be incorporated intially and approximaties made for any remaining smooth behavior.

If the nonlinear scaling fields for the renormalization group equations can be found, (i) can be affirmatively answered. If $\{s_i\}$ is a set of nonlinear scaling fields for the nonlinear renormalization group equations for the parameters $\{p_i\}$ with eigenvalues $\{a_i\}$ then

$$S_i \sim e^{a_i \ell} \quad , \tag{5.6a}$$

$$\mathcal{G}(\{b^{a_i} S_i\}) = b^{-1} \mathcal{G}(\{S_i\}) \, . \tag{5.6b}$$

Equations (5.6) hold everywhere in the parameter space where the nonlinear scaling fields are defined. This is not a trivial restriction as is shown by considering (5.5). The general theory of nonlinear first order partial differential equations applies in this problem since the nonlinear scaling fields satisfy equations of that form. We may generally expect the solution region for the nonlinear scaling fields to be some region of the parameter space bounded by surfaces on which one or more of the scaling fields is singular.

As will be shown in Chapter 5, the answer to (ii) is more problematical. In general, a solution fo r the Gibbs potential, for example, will not be dominated by the behavior of a single fixed point. These solutions are examples of the critically ordered systems axiomatized in Chapter 3. These solutions, however, have "extra" singularities. If these singularities are removed, then the so-called "stablest" fixed point determines the asymptotically valid behavior.

The notion of relative stability of fixed points is simple. If the $\{p_i\}$ move from a fixed point A to another fixed point B then A is relatively unstable with respect to B. Thus, in (5.1) if the parameter p is not exactly 0, it tends to ± 1 as $\ell \to \infty$. The fixed point at p=0 is unstable with respect to the fixed points at ± 1. It is clear taht a fixed point is unstable with respect to some fixed point if any of the eigenvalues of the renormalization equations linearized around that point are positive (and if limit cycles are excluded). Thus, the strictly stablest fixed point is one at which all the eigenvalues are negative. In (5.4) $p=\pm 1$ are stablest fixed points in this sense. However, these are not the stablest fixed points that we want, since at such a point the correlation length is zero (cf. (5.5b)). This apparently corresponds to "infinite temperature" fixed point behavior. Since the fixed points corresponding to critical points should yield an infinite correlation length, we must retain at least one positive eigenvalue. The remaining instability represents the temperature instability. In some cases, more than one sort of instability is retained, such as a magnetic field. While this division may seem somewhat arbitrary, in any concrete problem the resolution is clear.

As will be shown in Chapter 5, the global, nonlinear approach eventually leads to a description remarkably like that of the "higher order" critical points discussed in Ref. 7 and Chap. 3. The solution

region (cf. Fig.10) of the renormalization group equations contains a critical surface on which the critical point exponents are given by the stablest fixed point within that surface. The effects of the other fixed points (question (iii) above) disappear asymptotically close to this surface except near its borders. This surface can be compared to the surface of ordinary Θ=2 critical points in Fig. 4. On the borders of the solution regions, special symmetry condition on the renormalization equations exclude the participation of the stablest fixed point and the determination of critical point exponents passes to another fixed point (question (iv) above). This is analagous to the bordering of a surface of Θ=2 critical points by a lower-dimensional surface of Θ=3 critical points (the tricritical lines in Fig. 4). In the vicinity of the border, both fixed points compete exactly as in the phenomenological discussion of competition between types of critical point incorporating "double-power" scaling laws.[2,7,31] As in the phenomenological discussion, this process can be continued indefinitely by introducing additional fixed points for each type of critical behavior.

As mentioned above, those solutions of the renormalizations group equations for the thermodynamics functions that are not dominated by the "stablest" fixed point, exhibit additional singularities. These systems are not scaling systems but rather critically ordered (cf. Chap.3). The decision to reject the singular solutions is not a simple one. If they are retained, they provide a theoretical framework for a failure of scaling in thermodynamic systems.

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CAPTIONS

1a. Phase diagram of a single component fluid system, illustrating the vapor pressure curce terminating at the critical point. The direction tangent to the vapor pressure curve at the critical point (the weak direction) is labeled $\hat{x}_2$; $\hat{x}_1$ is any direction not tangent to the curve (a strong direction).

1b. The phase diagram of a simple ferromagnet. The weak direction $\hat{x}_2$ is along the T axis; $\hat{x}_1$ may be taken parallel to the H axis.

1c. The phase diagram of an anisotropic ferromagnet. The coexistence surface lying in the RT plane is bounded by a line of ordinary critical points. The $\hat{x}_3$, or irrelevant direction, is tangent to the line of critical points, $\hat{x}_2$ lies in the RT plane, and $\hat{x}_1$ can be chosen as any direction not lying in the RT plane.

2. Schematic representation of an anisotropic ferromagnet. In each plane, the spins are coupled with an interaction J between nearest neighbors. Nearest neighbors in different planes interact with an interaction strength RJ. For positive R (as shown) the system is an anisotropic ferromagnet. For negative R, the spins on alternate planes are aligned anti-ferromagnetically, so that the system becomes a meta-magnet.

3. The phase diagram of the fourth order system of Ref. 2 is shown in the HT plane for various values of R(R negative). The antiferromagnetic coexistence surface is bounded by a line of critical points which terminates at tricritical points labelled TCP. The line of first order transitions is shown dashed; T_N is the Néel temperature for each value of R. (a), (b) and (c) show the phase diagram for successively smaller values of $|R|$.

4. The phase diagram of the fourth order system of Ref. 2 in HTR space. A coexistence volume is capped by a two-dimensional surface of ordinary critical points. This surface is bounded by two tricritical lines. The tricritical lines intersect at the fourth order point: $H=H'=R=0$ and $T=T_2$, the two-dimensional Ising model critical temperature.

5. The phase diagram of a meta-magnet in $HH'T$ space. A half-moon coexistence surface labelled CXS is bordered by a line of anti-ferromagnetic critical points. This line terminates at two tricritical points (TCP). By examining the phase diagram for non zero staggered magnetic field, the tricritical point is seen to be the point of intersection of three lines of critical points. Wings are formed from coexistence surfaces between paramagnetic and antiferromagnetic phases.

6. Three-dimensional slices of the phase diagram of a multicomponent fluid system of three or more components. The variable t may be considered to be the temperature and u and v as suitable "fields" (combinations of the pressure and the various chemical potential differences).

6a. The coexistence surface between the phases labelled A and B terminates on the coexistence surface separating those phases from a third phase labelled C. The point P is a critical end point at which A and B are critical while simultaneously being in coexistence with C. The line L_o is a line of three phase coexistence.

6b. A slice containing a point P_t at which all three phases are simultaneously critical (the multicomponent fluid tricritical point).

6c. A slice in which the coexistence surface separating B and C terminates on the A coexistence surface. The point P' is a critical end point at which B and C are critical while in coexistence with A.

7a. The Kadanoff picture of scaling. Ising spins on a lattice with lattice spacing a are grouped into blocks of size L. If L is much less than the correlation length ξ, the blocks can be treated as if all the spins within each block are aligned.

7b. The field theoretic view of renormalization. Momenta lying between p and the cutoff momentum are integrated over; the inverse of the correlation length is an effective mass for the theory. Scaling behavior is expected for $m \ll p \ll \Lambda$.

8. Diagramatic representation of the continuous spin function s(x). If we are only interested in the values of s(x) when $x=2n\,\Pi$, the two different functions shown are equivalent. For this case of a spin on a one-dimensional lattice with lattice spacing $2\,\Pi$, we need only consider wave vectors such that $|k| \leq 1$. That is, wave vectors within the first Brillouin zone of the lattice.

9. The local linearized view of analysis is contrasted with a nonlinear global approach. The linearized renormalization equations can be treated in some region of any fixed point. Two fixed points are shown at (p,q)=(0,1) and (0,0). Local renormalization group equation solutions are indicated. See Eq. (1.2) of Chapter 5.

10. Comparison between the phase diagram of the Ising metamagnet also shown in Fig. 4 and the solution region of a nonlinear crossover problem discussed in Chapter 5.

10a. Reproduces Fig. 4. A coexistence volume is covered by a surface of ordinary critical points. This surface is bounded by two tricritical lines which intersect at a point of order four.

10b. The solution region of a nonlinear renormalization group problem. A surface is characterized by two variables which scale; the scaling exponents for these variables are the same everywhere on the surface. This surface is bounded by two lines on which three variables sacle and for which there are different scaling powers. These "tricritical lines" intersect at a point at which four variables scale with new scaling powers.

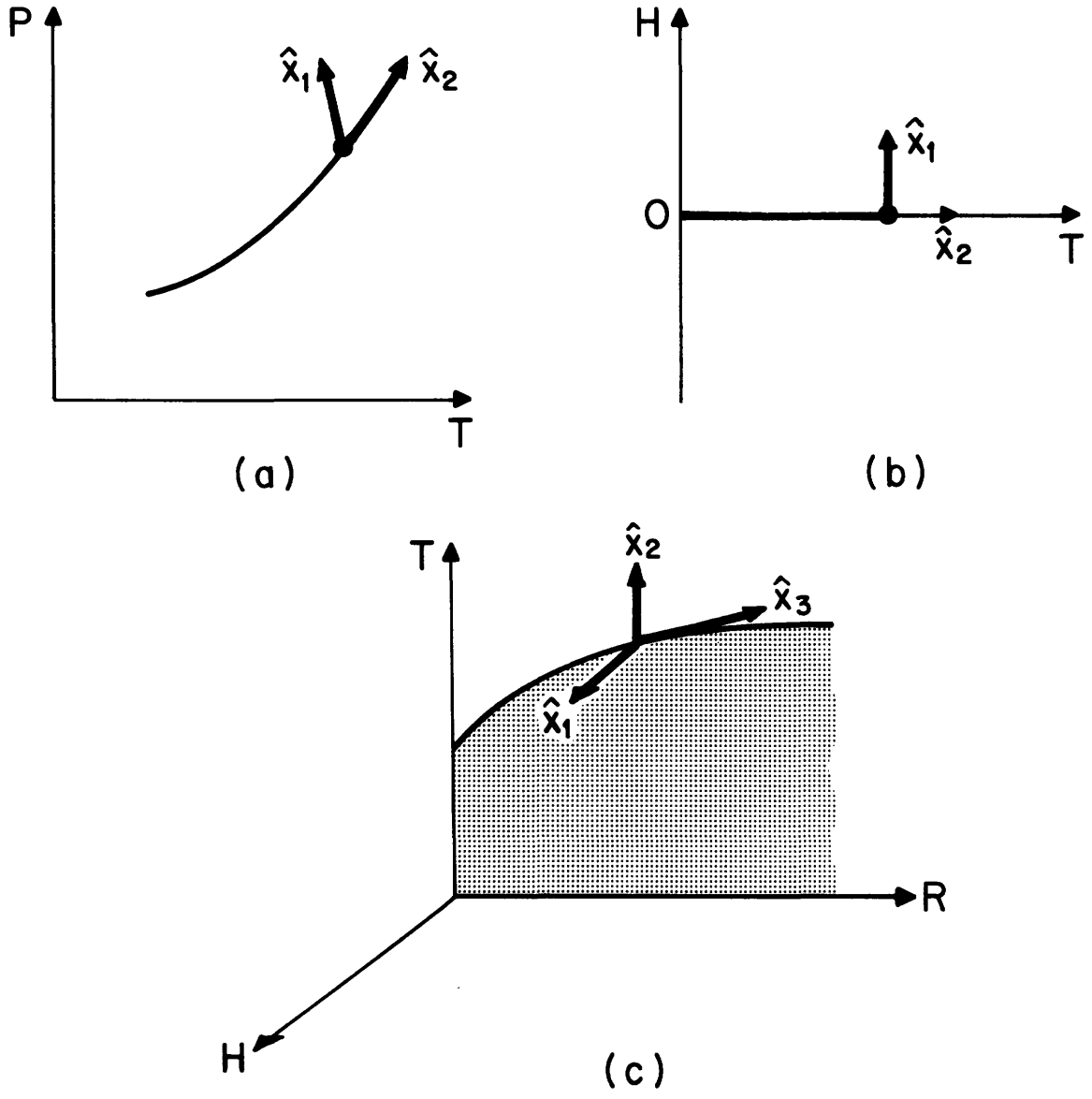


Figure 1

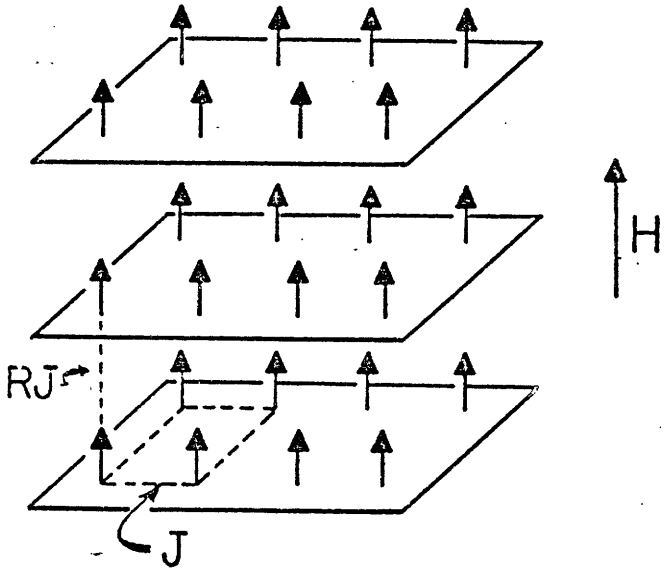


Figure 2

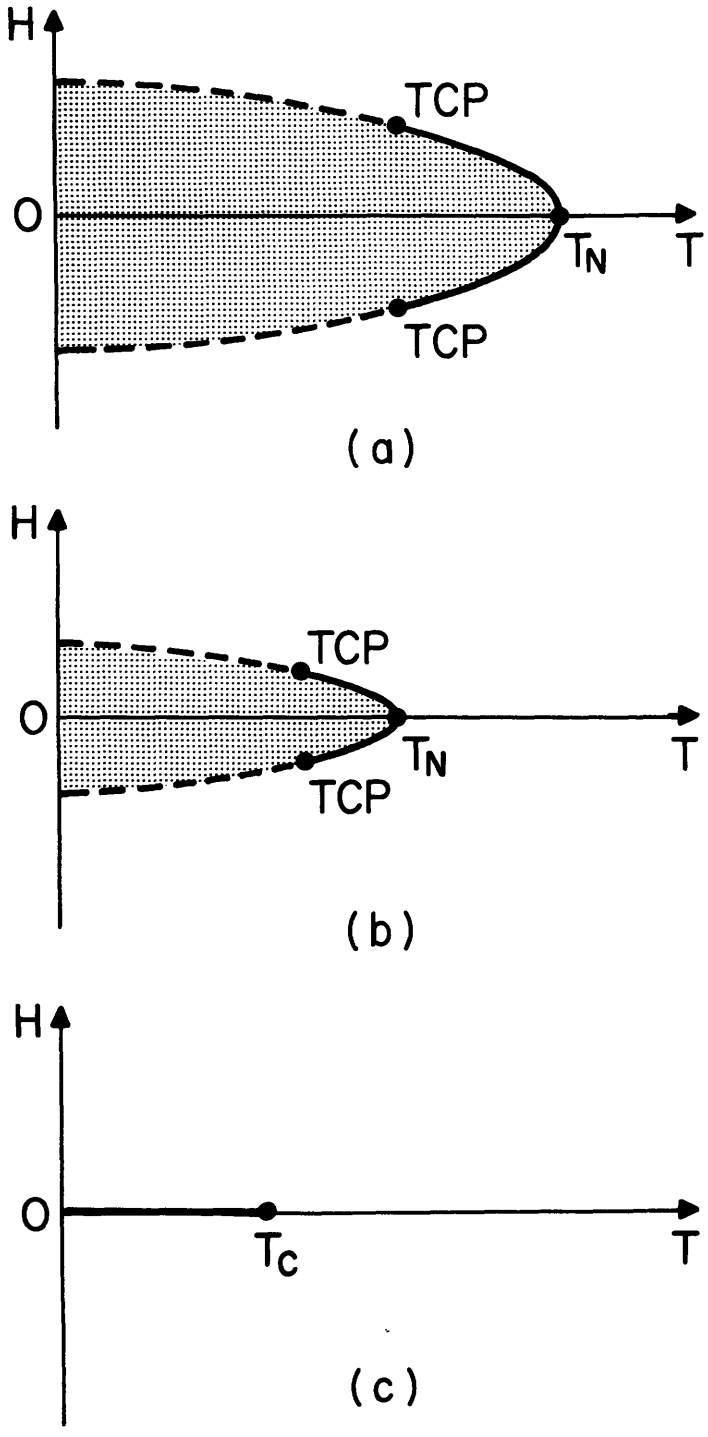


Figure 3

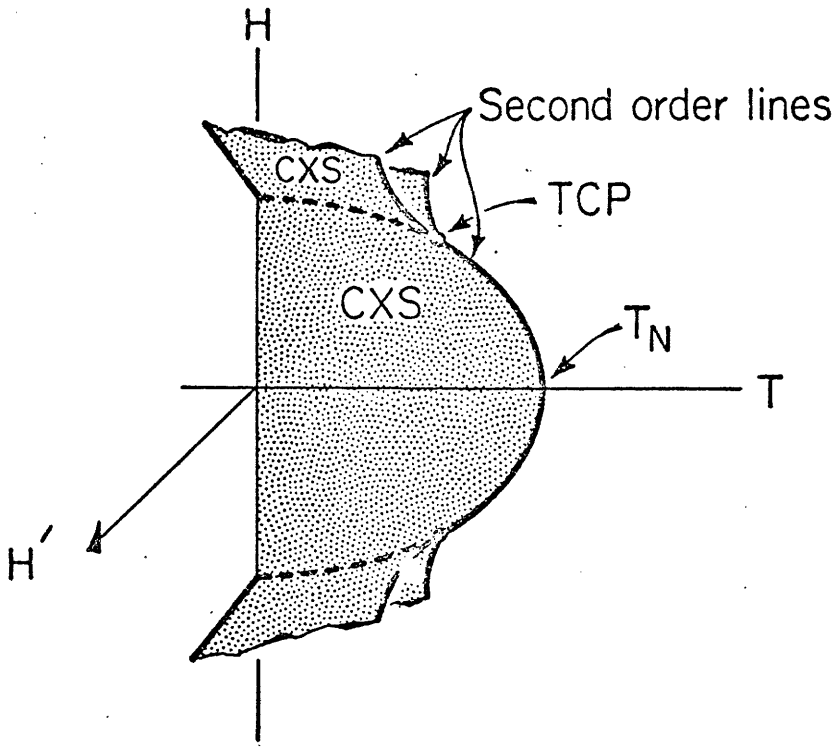


Figure 5

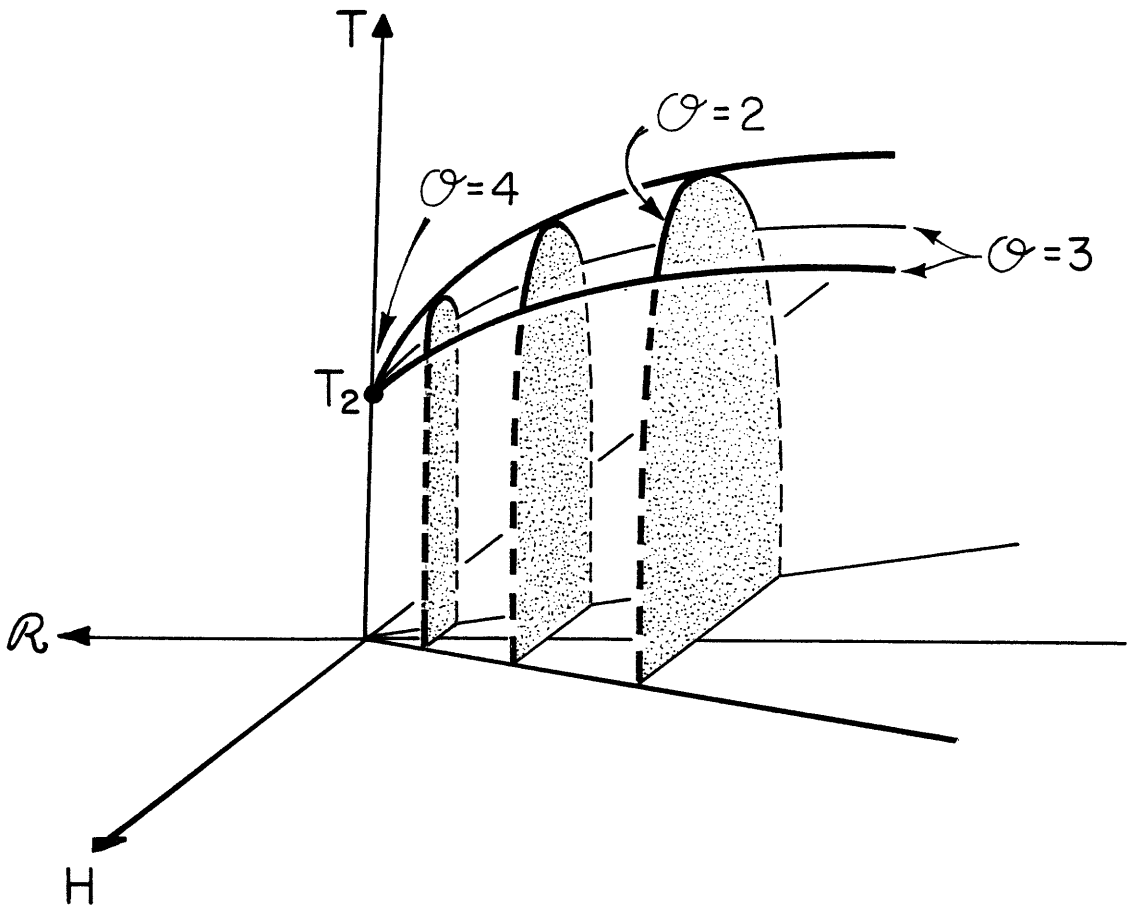


Figure 4

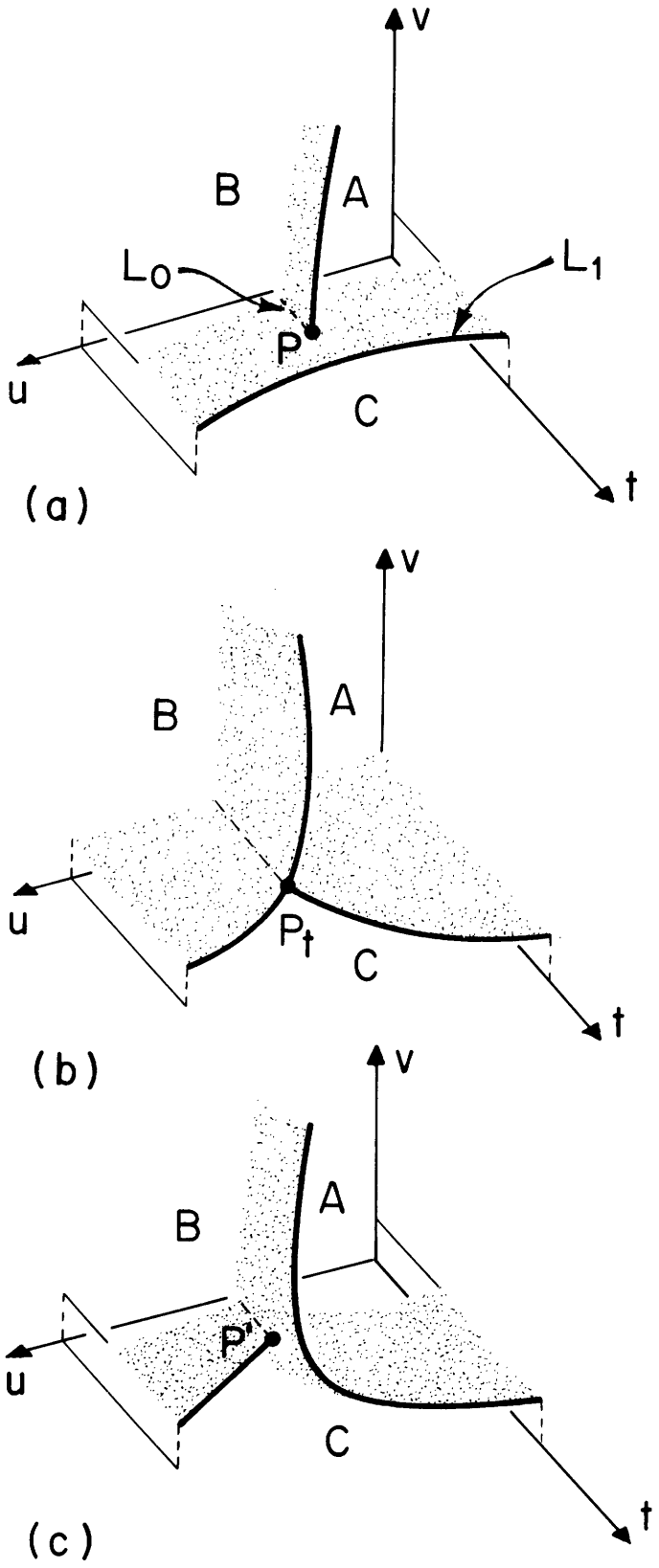


Figure 6

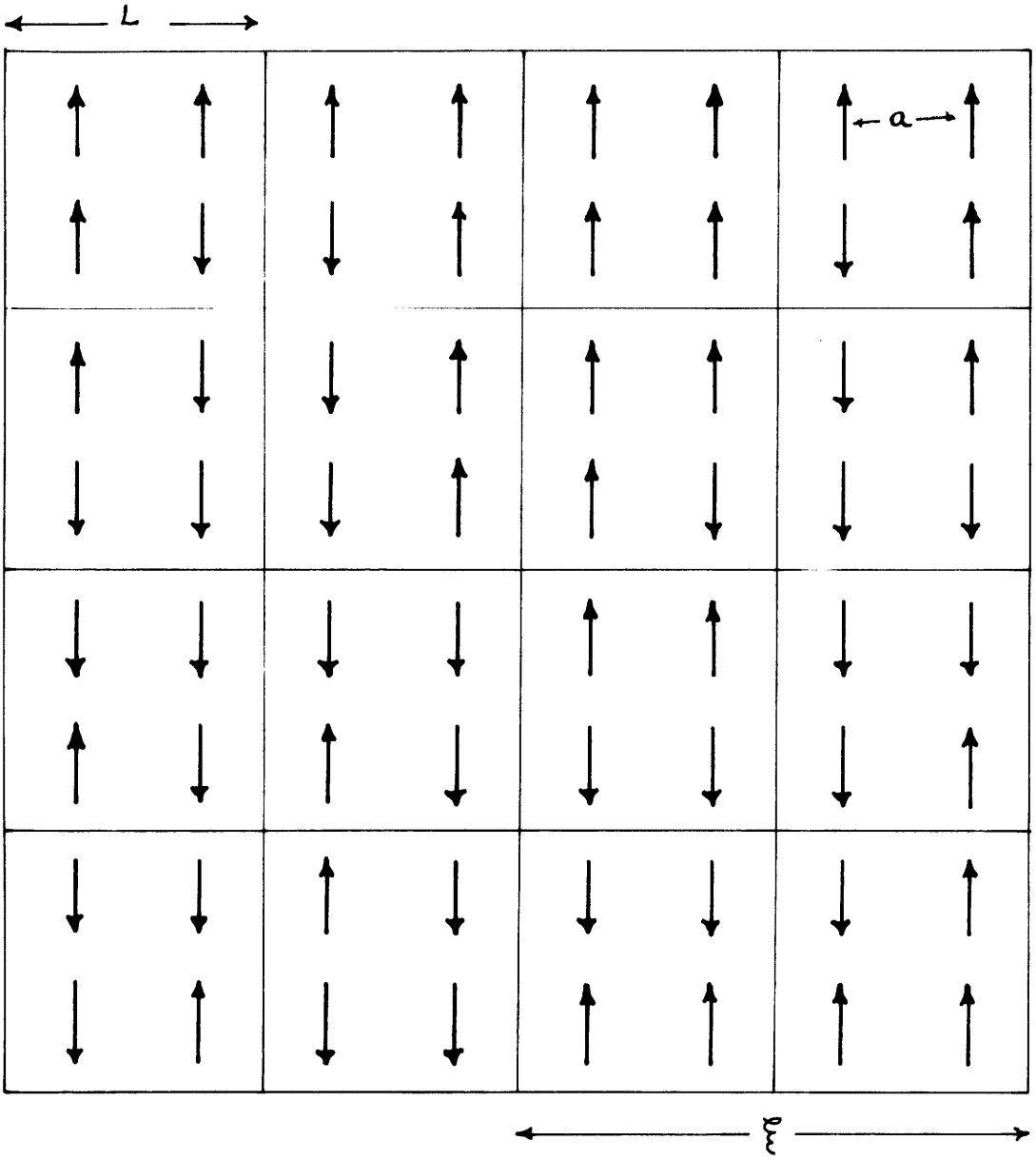


Figure 7a

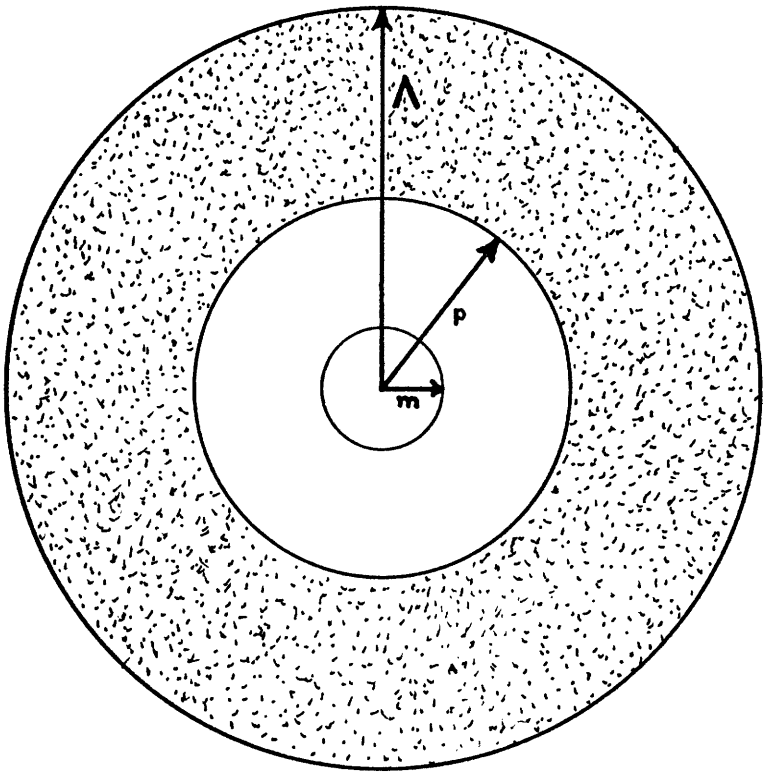


Figure 7b

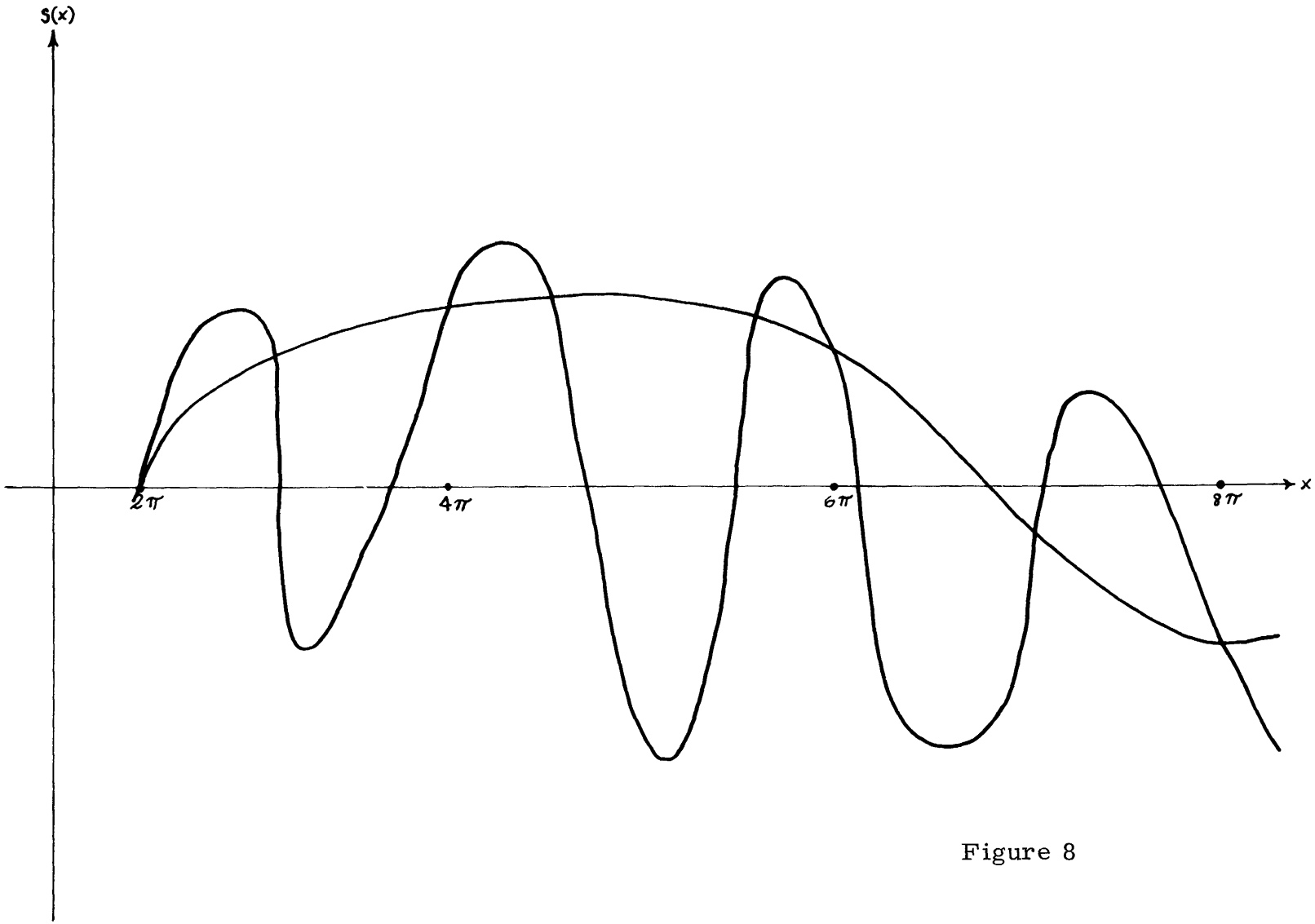


Figure 8

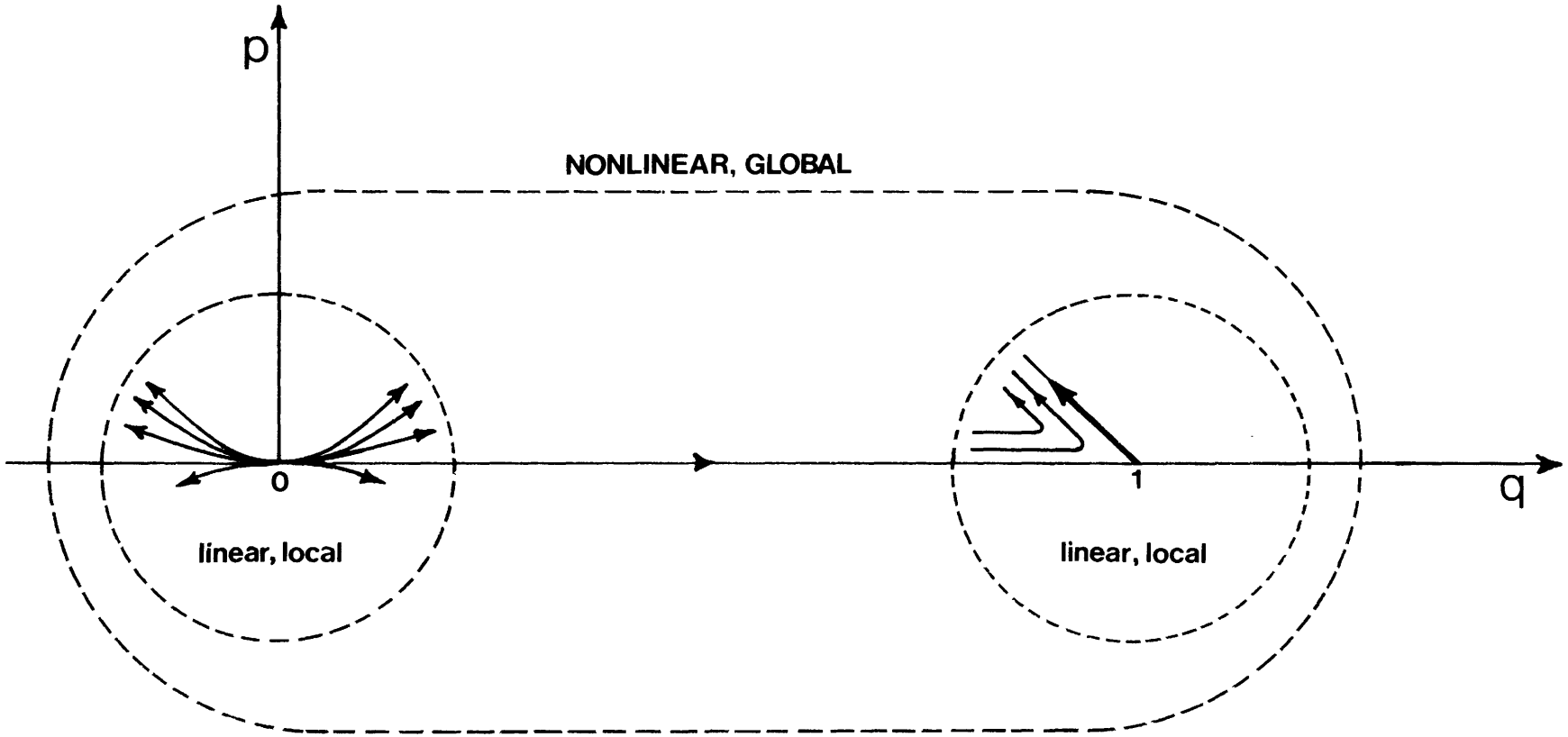
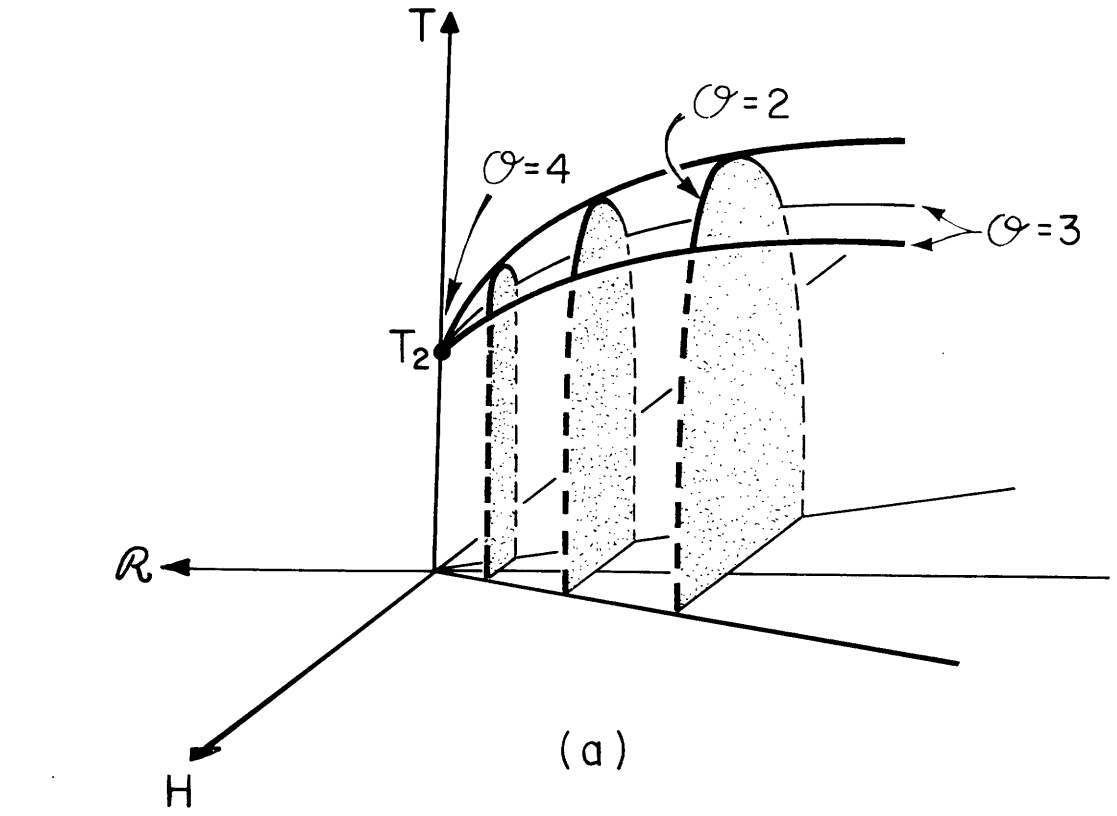


Figure 9


T
$\mathcal{O}=2$
$\mathcal{O}=4$
$\mathcal{O}=3$
T_2
R
H
(a)

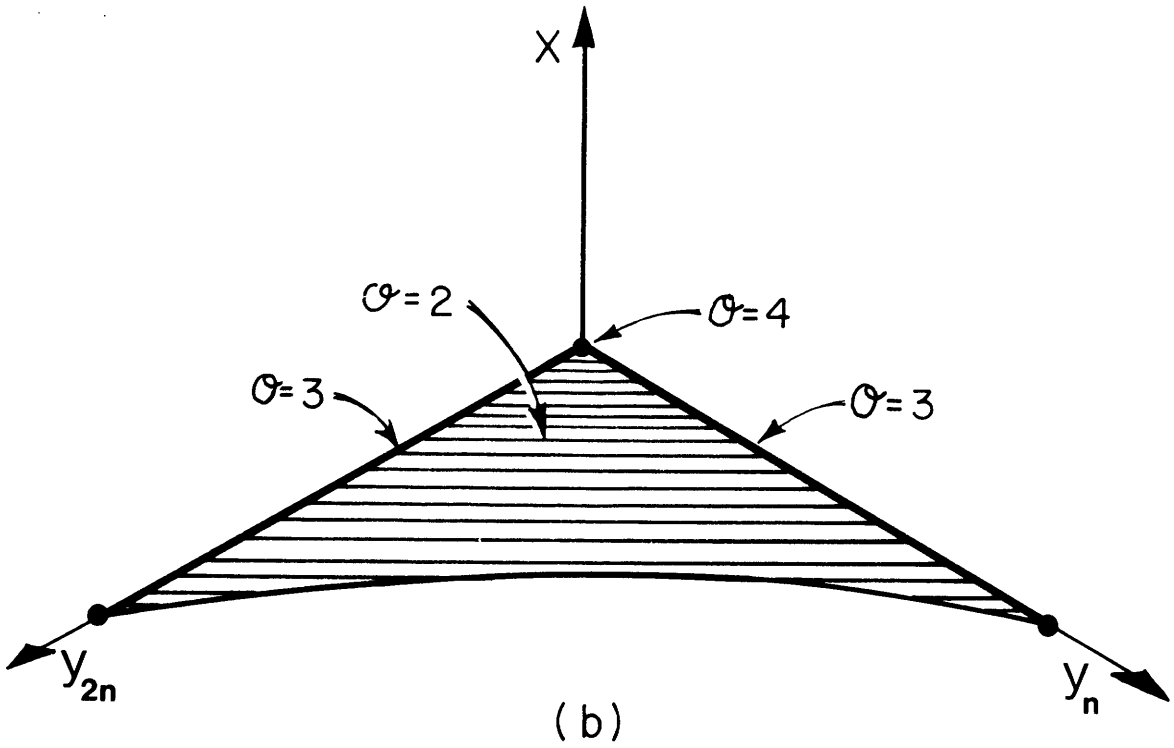

X
$\mathcal{O}=2$
$\mathcal{O}=4$
$\mathcal{O}=3$
$\mathcal{O}=3$
y_{2n}
y_n
(b)

Figure 10

Appendix: On the Application of nonlinear Renormalization Group Techniques to the Problem of "Crossover" and the Competition of Different Critical Behaviors

ON THE APPLICATION OF NONLINEAR RENORMALIZATION GROUP TECHNIQUES TO THE PROBLEM OF "CROSSOVER" AND THE COMPETITION OF DIFFERENT CRITICAL BEHAVIORS

The renormalization group, when considered in its nonlinear aspects, provides a theoretical understanding of the crossover behavior of systems which have competing critical behaviors. The result of the analysis is to confirm phenomenological description of simultaneously valid scaling hypotheses.

I. Examples of "Crossover"

The term "crossover" is used in critical phenomena in several contexts. In all cases it describes a competition between two or more distinct types of critical behaviour. The simplest example is the crossover between mean-field behaviour and true critical behavior as the critical point is approached. In Fig. 1, the phase diagram of a simple Ising ferromagnet is shown. Two phases with opposite magnetization are in coexistence surface terminates at the critical point. Semi-quantitative information about the nature of singularities of the thermodynamic functions is provided by various critical point exponents, some of which are defined below

$$\chi \equiv \frac{\partial M}{\partial h} \sim |T - T_c|^{-\gamma} \quad \text{at } h=0$$

$$C_h = \left(T \frac{\partial S}{\partial T}\right)_h \sim |T - T_c|^{-\alpha} \quad \text{at } h = 0$$

$$M \sim \pm |T - T_c|^{\beta} \quad \text{on phase boundary} \tag{1}$$

These critical point exponents are not completely independent. Rigorous thermodynamic inequalities show that $\alpha + 2\beta + \gamma \geq 2$. In fact, the equality seems to hold in most systems, both theoretically and experimentally,

$$\alpha + 2\beta + \gamma = 2 \qquad (2)$$

Mean-field or Landau theories of the critical point presume that the thermodynamic potentials are analytic in M, and $T-T_c$. This leads to the predictions that $\alpha=0$, $\beta=\frac{1}{2}$, and $\gamma=1$. At moderate distances from the critical point, these values may be ovserved.

However, nearer to the critical point, the character of the singularity appears to change. Ploting log χ versus log $(T-T_c)$ we find (as shown qualitatively in Fig. 2) that the data will fall on two separate straight lines connected by a "crossover" knee. The values of the critical point exponents as determined from the asymptotic values of the slopes also appear to satisfy (2) as an equality. Thus, we pass from one set of exponents which satisfy exponent equalities to a different set which also satisfy equalities.

A more interesting example of crossover is the competition between critical and "tricritical" singularities. As an example, we consider the phase diagram of a simple Ising metamagnet. In this sytem both antiferrmagnetic and ferromagnetic ordering are possible and compete. In the H-T plane (cf. Fig. 3) there is a coexistence surface on which two anti-ferromagnetic phases coexist. This region is bounded in part by a line of ordinary critical points. At these points, the transition is "second-order" in the sense of Gibbs. The two coexisting antiferromagnetic phases become identical along this line with a smoothly vanishing staggered magnetization. This critical line terminates, however, in two special points called tricritical points. Beyond the tricritical points the transition becomes first-order, with the staggered magnetization dropping abruptly to zero.

As pointed out by Griffiths, this phase diagram becomes clearer when examined in an augemented space which includes an unphysical staggered magnetic field, H_{st}, which couples directly to the staggered

magnetization . In this space, we see that the tricritical points are points at which three lines of critical points intersect. The addition wings (cf. Fig. 4) are coexistence surfaces separating antiferromagnetic and paramagnetic phases.

The tricritical points are obviously unique points on the phase diagram. It is not surprising that thermodynamic functions have a different singular behavior at the tricritical points than they exhibit on the line of "ordinary" critical points. It is found that on lines of constant magnetic field which do not pass through the tricritical point the divergence of the susceptibility can be described by a single critical point exponent

$$\chi \sim |T-T_c(h)|^{-\gamma} \qquad (3)$$

This "universality" of critical point exponents along the critical line is partly explained by noting that the fundamental processes near the critical line are still anti-ferromagnetic. The presence of the magnetic field h shifts the critical temperature, but does not alter the qualitative nature of the interactions. (a more mathematically precise validation of universality is obtained in the renormalization group calculations to follow).

However, on a line passing through the tricritical point, a new behavior is observed. The singularity of the susceptibility is still as power law, but with a different exponent

$$\chi \sim |T-T_{\text{tricritical}}|^{-\bar{\gamma}} \qquad (4)$$

In a neighborhood of the tricritical point (cf. Fig. 5) both the critical singularity characterized by γ and the tricritical singularity characterized by $\bar{\gamma}$ compete.

To explore this complicated situation, we first turn to a phenomenological framework which incorporates the observed facts.

II. Kadanoff Scaling

Kadanoff's heuristic scaling notions provide a basis for both the phenomenological theories of scaling and the more rigorous renormalization group techniques. Kadanoff considers a simple Ising ferromagnet near it's critical point (cf. Fig. 6). The spin-spin correlation lenght, ξ, which diverges at the critical point, is very large near the critical temperature. Many neighboring spins tend to become aligned in blocks. Kadanoff argues that we may be able to treat such blocks of spins as single spins. Thus, if the spins in a block of for example four spins are nearly always all pointed up or all pointed down, we may treat it as a unit.

This new system of block spins is similar to the original system, although it obviously is not identical to it. In particular, the internal degrees of freedom within a block have been neglected (for each block of four spins there are 16 states, not simply the two block states). However, Kadanoff argues that these differences will not affect the critical behavior of the system, and that for the purpose of studying critical behavior, the two systems can be treated as identical in nature.

The correlation length of the block spin system (as measured in, for example, lattice spacings) is less than that of the original system. For blocks of b spins on a side ($b=2$ for four-spin blocks) the correlation length is reduced by a factor of b.

Thus we have that

$$\xi_{block}\ (h_{block}, t_{block}) = \frac{1}{b}\,\xi\ (h,t) \tag{5}$$

where h_{block} and t_{block} are the effective magnetic field and reduced temperature ($t = (T-T_c)/T_c$) of the block system.

At this point Kadanoff assumes that the similarity of the two systems is sufficiently precise that the effective magnetic field and reduced temperature are themselves simply given by scale transformations.

$$h_{block} = b^{a_h} h$$

$$t_{block} = b^{a_t} t \tag{6}$$

The constants a_h and a_t are called the "scaling powers" of H and T, respectively. Combining (5) and (6) we see that the correlation length is a generalized homogenous function (GHF) of the variables h and t,

$$\xi\ (b^{a_h} h,\ b^{a_t} t) = b^{-1}\ \xi\ (h,t) \tag{7}$$

This can be rewritten to put ξ into "scaling form" as

$$\xi = |t|^{-1/a_t}\ \xi\ (\mathrm{sgn}\, t,\ h/|t|^{a_h/a_t}) \tag{8}$$

From this example of the correlation length we may extend the scaling hypothesis to other thermodynamic functions such as the Gibbs potential

$$G\ (b^{a_h} h,\ b^{a_t} t) = b^{d}\ G\ (h,t) \tag{9}$$

We have chosen the scaling power of G to be given by the dimension of the lattice d to conform with the renormalization group results to be obtained

later. From (9) we may calculate the critical point exponents α, β, and γ:

$$\gamma = \frac{2a_h - d}{a_t}; \qquad \beta = \frac{d - a_h}{a_t}; \qquad \alpha = \frac{2a_t - d}{a_t} \tag{10}$$

These values for the critical point exponents clearly satisfy (2) for any values of a_h, a_t, and d. The Kadanoff construction does not give us any way of calculating the scaling powers a_h and a_t; However, they can be calculated by the renormalization group.

III Scaling and Crossover.

The success of the scaling hypothesis in describing the critical points of ferromagnets and single-component fluid has let to its use in tricritical phenomena and crossover. To make the closest connection with the nonlinear renormalization group solution discussed below, we will consider the geometrically simple model phase diagram shown in Figure 7. A coexistence surface lied in the plane $h=0$; h represents the ordering field. The line $x=h=0$ is a line of ordinary critical points which terminates at $x=y=h=0$ at a "tricritical point." Only one of the three critical lines is shown.

At each point of the critical line, we make a two-fold scaling hypothesis. We assume that, apart from smoothly varying backround terms, the Gibbs potential is a GHF in h and x,

$$G(b^{a_h}h,\ b^{a_x}x,\ y) = b^d G(h,x,y)$$
$$\xi(b^{a_h}h,\ b^{a_x}x,\ y) = b^{-1}\xi(h,x,y) \tag{11}$$

With scaling powers a_h and a_x. The variable y only enters as a parameter. By this means the universality of critical point exponents is guaranteed. To distinguish the tricritical point, we assume that in the vicinity of the origin the variable y no longer is an unimportant parameter, but rather scales. We make a three fold scaling hypothesis

$$G_{tri}(b^{a_h}h,\ b^{a_x}x,\ b^{a_y}y) = b^{d}\, G_{tri}(h,x,y)$$

$$\mathcal{G}_{tri}(b^{a_h}h,\ b^{a_x}x,\ b^{a_y}y) = b^{-1}\, \mathcal{G}_{tri}(h,x,y) \tag{12}$$

with new scaling powers $\frac{}{a_h}$, $\frac{}{a_x}$, and $\frac{}{a_y}$.

If both (11) and (12) are to be valid near the tricritical point, then we obtain extra information in the "crossover" region where both hold. Consider, for example, the $=0$ correlation length. From (11) we have

$$\xi \sim |\mathbf{x}|^{-\nu}\ \xi\,(1, \tag{13}$$

with $\bar{\nu} \equiv 1/\bar{a}_x$. On the other hand, from (12) we have instead

$$\xi \sim |\mathbf{x}|^{-\bar{\nu}}\ \xi_{tri}(1,\ y\ /\ x^{\bar{a}_y/\bar{a}_x}) \tag{14}$$

with $\bar{\nu} \equiv 1/\bar{a}_x$. We cannot require these expressions to be identical since each scaling hypothesis represents an approximation to the critical behavior. However, we can require that they give the same asymptotic behavior when $x \to 0$ with y fixed. This is the condition for the mutual validity of both scaling hypotheses. Thus we must require that

$$\xi_{tri}(1, y / x^{\bar{a}_y/\bar{a}_x}) \sim \left| \frac{y^{\phi}}{x} \right|^{\nu - \bar{\nu}} \tag{15}$$

as $x \to 0$, $\phi \equiv \bar{a}_x/\bar{a}_y$.

One simple function that satisfies all the requirements is given by

$$\xi = |x|^{-\nu}\ |x + y^{\phi}|^{\nu - \bar{\nu}} \tag{16}$$

For $x \to 0$ with y fixed, (16) has the form expected from (13) while in the vicinity of the tricritical point it has the form (14). The ratio ϕ is called a crossover exponent since when x is the same order as y^{ϕ} that the crossover region is entered. For x much less than y^{ϕ}, critical behavior rather than tricritical behavior will be seen. We also observe that we can extract the dependence on y in the crossover region. The amplitude of the critical singularity, which may be defined as

$$\text{Amp } \xi(y) = \lim_{x \to 0} \xi(x,y) \; x^{\nu} \tag{17}$$

can be obtained from (15) or (16) as

$$\text{Amp } \xi(y) \sim y^{(\nu - \bar{\nu})\phi} \tag{18}$$

Thus the amplitude of the critical singularity scales with an amplitude exponent that is the product of the crossover exponent and the difference of the oridinary critical point and tricritical point values of the critical point exponent for the correlation length.

Although this discussion has been phased in terms of the correlation length, it is easy to see that these properties are applicable to all the thermodynamic functions. In each case, in the region in which both tricritical and critical scale might be expected to hold, we obtain constraints on the tricritical scaling functions (15), which are characteristic of "double power law" scaling forms such as (16). Furthermore, the amplitudes of critical singularities will obey scaling laws such as (18). These predictions of mutually valid multiple scaling hypotheses have been tested in various model tricritical systems by high temperature series. Within the accuracy of the series calculations they are all supported.

D. The renormalization group

To put this phenomenological discussion on firmer ground we turn to the "renormalization group". The renormalization group approach is a catch-all term describing the mathematical improvement of the block scaling notions of Kadanoff discussed earlier.

The basic physical fact utilized by Kadanoff is the divergence of the correlation length at a critical point. This implies that the system is scale invariant; that is, if the correlation length is infinite, then blocks of spins look precisely like single spins. In the immediate vicinity of a critical point we have a nearly scale invariant system. By studying how the system changes under a scale change, we hope to extract information on the behavior of thermodynamic functions.

Kadanoff used this idea purely heuristically. Wilson pointd out that a more rigourous approach is possible. We can illustrate this idea with a simple example. Consider a one-dimensional chain of Ising spins (Cf. Fig. 8a). To achieve a critical point at a non-zero temperature, we must apply a long range ferromagnetic interaction (LRI) of some kind. We will not worry about the detailed nature of this long range glue. We write the Hamiltonian as

$$\frac{\mathcal{H}}{kT} = \vec{h} \cdot \sum_i \vec{S}_i + LRI \tag{19}$$

Now we apply the Kadanoff block transformation idea to this system. We divide the spins into blocks of 3 spins (Fig. 8b). We can define an Ising block spin to each block by

$$\sigma_i = \text{sign}\left(S_{3i} + S_{3i+1} + S_{3i+2} \right) \tag{20}$$

We would like to be able to think of the block spin system in the same way as the original spin system (cf. Fig. 8c). We want to define a new magnetic field h' and a new long range interaction LRI'. We can calculate the relationship between h and h' and between LRI and LRI' instead of postulating them (as in Kadanoff) if we demand that the block system (h',LRI',s') have the same partition function as the original (h,LRI, s) system.

The partition function for the single spin system is given by

$$Z = \sum_{s_i = \pm 1} \exp\left[-\left(h \cdot \Sigma s_i + LRI\right)\right] \tag{21}$$

We must first write this in terms of the block spins σ_i. We divide the sum in (21) into two parts. First we sum over all the values of s_i consistent with a fixed set of values for the σ_j.

$$T \equiv \sum_{\substack{s_i = \pm 1 \\ \sigma_j \text{ fixed}}} \exp\left[-\left(\vec{h} \cdot \Sigma \vec{s}_i + LRI\right)\right] \tag{22}$$

Then we take a sum over the possible values of σ_j

$$Z = \sum_{\sigma_j = \pm 1} T\left(LRI, \vec{h}, \sigma_j\right) \tag{23}$$

We now must write this last expression in a form similar to (21),

$$\mathcal{Z} = \sum_{\sigma_j = \pm 1} \exp\left[-\left(h' \Sigma \sigma_i + LRI' + C\right)\right] \tag{24}$$

where C is a constant.

This can be a very complicated procedure. However, it is simple to calculate the dependence of h' on h, if LRI is even in the spin variables. If this is the case, we can calculate the relationship between h' and h as if LRI = 0 , since (as in Kadanoff) h' is a function of h only.

When we turn off the interaction, we can consider each block separate. The possible s-states for a given σ are shown in Fig. 9. There is one state for which all the spins have the same sign as σ and 3 states where 2 spins are of the same sign and one spin has the opposite sign. Therefore, we can write

$$T = \prod_j \left[\exp(-3h\sigma_j) + 3\exp(-h\sigma_j) \right] . \tag{25}$$

The pseudo partition function formed from T is

$$"\mathcal{Z}" = \left[2(\cosh 3h + 3\cosh(h)) \right]^N \tag{26a}$$

where N is the number of blocks of spins. On the other hand the partition function form using h' (with LRI'=0) is

$$"\mathcal{Z}" = \left[2 e^C \cosh(h') \right]^N \tag{26b}$$

and therefore we have

$$\cosh h' = \left[\cosh 3h + 3\cosh(h)\right] / e^{C} \tag{27}$$

We expect that if the magnetic field h = 0 in the original system that there is no magnetic field in the block spin system. This identifies $e^{C} = 4$ so that our final renormalization equation relating h and h' is

$$\cosh h' = \left[\cosh 3h + 3\cosh(h)\right] / 4 \tag{28}$$

It would be much harder to calculate the effective interaction LRI' and we will not attempt to do so. In any case, we would find

$$\text{LRI}' = \text{function (LRI)} \tag{29}$$

If we are at the critical point, $\xi = \infty$, and we expect scale invariance. That is, we expect h'=h and LRI'=LRI. This is called a fixed point of the renormalization group equations (28) - (29).

We can now study the dependence of the correlation length on the magnetic field. By the Kadanoff block construction we employed, the correlation length of the block spin system is given by

$$\xi' = \frac{1}{3}\xi \tag{30}$$

Near the fixed point (critical point) value h = 0, (26) implies that

$$(h')^2 = 3h^2 \tag{31}$$

If we stay at the critical temperature (LRI' = LRI), the correlation length ξ is a function of h alone. Therefore,

$$\xi(T=T_c) = \text{(constant)}\ /h^2 \tag{32}$$

We have derived a critical point exponent!! There was nothing magic in our choice of 3 spins per block. Any odd number leads to (30).

E. Continuum Spin Renormalization

For many applications it is more convenient for caculational purposes to replace the lattice of discrete-valued spins with an n-component spin density, $\vec{s}(\vec{x})$, which is defined at every point and which can take on all values (cf.Fig.10).

Since we are only interested in the valuesof $\vec{s}(\vec{x})$ at point of the original lattice, the Fourier transform of $\tilde{s}(\vec{x})$, which we denote $\sigma(k)$, and be taken to vanish outside the first Brillouin zone of the lattice. This restriction is illustrated in Fig. 11 for a one-dimensional Ising chain. Both the spin functions shown describe the same spin values at the lattice points. Therefore, we may discard the more rapidly varying function. Although there is an effective upper bound, there is no corresponding lower bound to the wave vectors, since the case of all spins aligned is $\vec{k}=0$. From high temperature series analysis and experimental data, it appears that the detailes of the lattice structure do not affect critical behavior greatly. In particular, critical point exponents are not sensitive to the nature of the lattice. Therefore, we are justified in replacing the Brillouin zone by a sphere (cf. Fig 12) of radius $\Lambda \sim 1/a$, where a is the lattice spacing.

We now must write our Hamiltonian in terms of the spin density s(x). Instead of the restriction on the spin values (s= ±1 for Ising systems) we introduce a spin-weighting factor such as $\exp(-s^2 - bs^4 - \ldots)$ which serves to qualitatively approximate the restriction on spin values. This sort of approximation is not expected to distrub critical point exponents. Series work indicates, for example, that exponents do not depend on spin quantum number (cf. Fig. 13).

We write the partition functions as

$$Z = \int \mathcal{D}s(x)\, e^{-\mathcal{H}(\vec{s}(\vec{x})) - W(s(x))} \tag{33}$$

where $\mathcal{H}$ is the Hamiltonian functional divided by kT and W is a weight factor (e.g. $W = s^2 + bs^4$) The notation $\int \mathcal{D}\vec{s}\,(\vec{x})$ indicates that we must integrate over all functions $\vec{s}(\vec{x})$.

We may absorb W into $\mathcal{H}$ and write the sum as Landau-Ginzberg-Wilson form

$$\mathcal{H} = \int d^d x \left[|\nabla s|^2 + \vec{h}\cdot\vec{s}(x) + r\, s^2(x) + u\, \vec{s}^4(x) + \cdots \right] \tag{34}$$

where d is the dimension of the lattice. The gradient term represents a short range interaction. The remaining terms could represent true interactions or be partly from the weighting functions.

Near the critical point of the system, the correlation length is very large. The characteristic size of fluctuations is therefore very large and we do not expect that small scale fluctuations are important.

Wilson suggested that the analogy to Kadanoff's replacement of site spins by block spins is the integration out of the large k components of $\vec{s}(\vec{x})$. The procedure is as follows:

In the expression for the partition function, perform the average over all $\sigma(k)$ ($\sigma(k)$ is the Fourier transform of $s(x)$) with $\Lambda/b \leq k \leq \Lambda$. The resulting expression will only involve $\sigma(k)$'s with $k < \Lambda/b$. Then a scale change $k \to k' = bk$ is made to restore the original form of the cutoff (cf. Fig. 14). After this is completed, we will find that the remaining evaluation of the partition function can be cast into the original form with new parameters h',r',u'.

The equations which relate the new parameters to the old parameters are in general extremely complicated. In fact, we have to allow for a form more general than (34) to encompass all the possible changes in the parameters. There are several ways by which we may simplify the situation.

As mentioned previously, the precise size of the Kadanoff block does not matter; when exponents are determined, the block size drops out of the final result. In the continuum case considered here, this means that the final results will not depend on the value of b. However, the renormalization equations will involve b at every intermediate stage. We may remove this extraneous depend from the problem by examining the renormalization proceedure in the limit $b \rightarrow 1$.

If we set $b=\exp(\delta\ell)$ with $\delta\ell \ll 1$, we are only averaging over a very thin momentum shell (cf. Fig. 15). We expect that the renormalized values of the parameters will differ only slightly from the unrenormalized values

$$\begin{aligned} h' &= h + O(\delta\ell) \\ r' &= r + O(\delta\ell) \\ u' &= u + O(\delta\ell) \end{aligned} \tag{35}$$

We can therefore study the differential change in the parameters as $\delta\ell \rightarrow 0$

$$\lim_{\delta\ell \rightarrow 0} (r'-r)/\delta\ell \equiv dr/d\ell \tag{36}$$

Such an exact differential generator has been given by Wegner and Houghton.

Even these equations are still too complicated in general. We now make the assumption that we are in dimensions close to 4. That is, we define an expansion parameter $\epsilon = 4-d$ and make a perturbative analysis of the renormalization equations. To lowest non-trival order in ϵ we obtain the following equations

$$\frac{d\xi}{d\ell} = -\xi \tag{37a}$$

$$\frac{dh}{d\ell} = \left(1+\frac{d}{\xi}\right)h \tag{37b}$$

$$\frac{dx}{d\ell} = 2x\left[1-x-\epsilon\, y(n+2)/2(n+8)\right] \tag{37c}$$

$$\frac{dy}{d\ell} = y\left[\epsilon\,(1-y)-4x\right] \tag{37d}$$

where x and y are related to r and u by

$$x = r/(1+r) + \frac{u\, d(n+2)}{(1+r)^2\ 2n\ (d-2)} \tag{38a}$$

$$\epsilon\, y = \frac{u}{(1+r)^2} \qquad \frac{d(n+8)}{2n} \tag{38b}$$

It is the study of these equations (37) that will return us to the phenomenological discussions of crossover.

F. Return to Phenomenology: Examination of the Renormalization Equations

Before we examine the more complicated equations, it is useful to examine the simple equations (37a) and (37b) for the correlation length and the magnetic field. The first equation just represents the scale change involved in the renormalization transformation analogous to $\xi' = 1/3\,\xi$ in our earlier example. The fact that the magnetic field equation involves only h is due to the absence of any odd terms in the Wilson Hamiltonian. If all the other parameters were fixed at their scale invariance values (fixed point or critical values). The correlation length would again be a function of h alone

$$\xi = \frac{\text{constant}}{h^{1+d/2}} \tag{39}$$

It is only the dependence on x and y that requires any further analysis. In most of the following discussion we will set h=0 and consider only the x and y equations.

The particular definitions of x and y employed in (38) place the fixed points of the equations at "canonical" locations. The fixed points with positive y (needed for thermodynamic stability) are (cf. Fig. 16)

x=y=0	the finite Gaussian fixed point
x=0,y=1	the Wilson-Fisher fixed point
x=1, y=o	the infinite Gaussian fixed point

Let us examine the x and y equations in the vicinity of each fixed point.

Near the infinite Gaussian fixed point we linearize the equations around x=1, y=0 and obtain

$$\frac{dz}{de} = -2z$$

$$\frac{dy}{de} = -dy \tag{40}$$

where $z=(x-1) + \text{const}\ y$. These equations have simple solutions.

$$z = z_0 \exp[-2e]$$

$$y = y_0 \exp[-de] \tag{41}$$

Combining these with the equation for the correlation length, we see that is a scaling function of z and y,

$$\xi(\lambda^2 z, \lambda^d y) = \lambda \xi(z, y)$$

or

$$\xi = z^{1/2} \xi(y/z^{d/2}) \tag{42}$$

As y and z tend to zero (with $y/z^{d/2}$ fixed), $\xi \to 0$. The infinite Gaussian fixed point corresponds to a point of zero correlation. It is scale invariant but is not the sort of scale invariance that we want: it corresponds to infinite temperature (note that x=1 implies r = ∞).

If we turn now to the Wilson Fisher point, we have the following linearized equations

$$\frac{dx}{de} = \left(2 - \epsilon(n+2)/(n+8)\right) x$$

$$\frac{dw}{de} = -\epsilon w \qquad (43)$$

where w= (1-y) + const x. This leads to a scaling form in terms of the variables x and w.

$$\xi\left(\lambda^{2-\epsilon(n+2)/(n+8)} x, \lambda^{-\epsilon} w\right) = \lambda^{-1} \xi(x, w),$$

or,

$$\xi = \left[\frac{1}{x}\right]^{2-\epsilon(n+2)/(n+8)} \xi\left(w\, x^{\epsilon/(2-\epsilon(n+2)/(n+8))}\right) . \qquad (44)$$

The correlation length diverges as $x \to 0$ regardless of the value of w. We identify x with $T-T_c$ and extract the critical point exponent ν for the Wilson - Fisher point

$$\nu_{WF} = \frac{1}{2 - \epsilon(n+2)/(n+8)} \qquad (45)$$

It is important to note that the variable w scales in equation (44) but in such a way that its presence is not important as $x \to 0$. At least in the vicinity of the Wilson-Fisher point the line x=0 is a line of critical points, each with the same critical point exponent (45). However, how real this behavior is and how far this putative line of critical points extends cannot be determined from this linearization approach alone.

As we examine the final fixed point, the finite Gaussian fixed point, the phenomenological picture comes into even clear focus. The linearized equations are

$$\frac{dx}{de} = 2x \quad .$$

$$\frac{dy}{de} = \epsilon y \quad . \tag{46}$$

and the corresponding scaling form for the correlation length is

$$\xi(\lambda^2 x, \lambda^{\epsilon} y) = \lambda^{-1} \xi(x, y) \quad ,$$

or,

$$\xi = x^{-\frac{1}{2}} \xi(y/x^{\epsilon/2}) \quad . \tag{47}$$

If we compare the renormalization group solution (44) and (47) with our phenomenological equations (12) and (13), the resemblance is quite striking (cf. Table 1). Somewhere near the finite Gaussian point the scaling power of x seems to change from $2-\epsilon(n+2)/(n+8)$ to 2. Thus the correlation length has a different singularity, a different critical point exponent, at the finite Gaussian point than it does near the Wilson-Fisher point. Moreover, the variable y, which has no effect near the Wilson-Fisher point (as $x \to 0$) is a crucial scaling variable at the finite Gaussian point.

The analogy is so close that we immediately feel that the finite Gaussian point represents a "tricritical" sort of point at which the scaling behavior changes suddenly and drastically. The Wilson-Fisher behavior(44) or rather the Wilson-Fisher value of the critical point exponent, can be expected to be the correct expression of the critical behavior for all $y > 0$ (at least as $x \to 0$). To show this we must clearly go beyond the linearized solutions of eqs. (40) - (47).

G. Solution of the Renormalization Group Equations

In the linearized analysis we obtained solutions valid in the immediate vicinities of the three fixed points. To produce a global valid solution which stiches together the solutions already obtained from the linear analysis, we must solve the renormalization equations without linearization.

The solutions of nonlinear equations is often a matter of chance and circumstance. Therefore, the method used to solve the particular equations at hadn is less interesting than the motiviations which lie behind it. It is therefore instructive to examine what we need to know to solve the nonlinear problem.

Each of the linearized equations produced a scaling function in terms of the original variables x and y or some linear combinations of x and y. Taking the finite Gaussian point as an example, we found that the variables x and y were scaling variables near x=y=0. However, near the Wilson Fisher point we again got scaling but in terms of x and w, and with different scaling powers. We call x and y (near the Gaussian point) and x and w (near the Wilson Fisher point) <u>linear scaling fields</u> . They are <u>linear</u> in x and y, they appear in <u>scaling</u> equations, and we borrow Griffiths terminology of <u>fields</u> to describe those things which appear as arguments in thermodynamic functions. They would be adequate to describe the system if we had only one fixed point or knew that we were very near a particular fixed point.

Wegner suggested that we consider nonlinear scaling fields. That is, nonlinear functions of the variables (in this case x and y) that would appear in globally valid scaling equations. We could define such scaling fields as

functions which satisfy particularly simple renormalization equations. For example,

$$\frac{d S_{Gau}}{d \ell} = 2 S_{Gau} \quad \Rightarrow \quad S_{Gau} = e^{2\ell}$$

$$\frac{d S_{WF}}{d \ell} = \left(2 - \epsilon \frac{n+2}{n+8}\right) S_{WF} \Rightarrow S_{WF} = e^{\left(2 - \epsilon \frac{n+2}{n+8}\right)\ell} \tag{48}$$

These functions have the trivial exponential dependence on ℓ and a very complicated dependence on x and y. From the form of the equations, however, we might guess that near x=y=0, S_{Gau} is essentially x (since they satisfy the same equation) while near x=0, y =1, S_{WF} is x. If we had such functions, we could write the correlation length scaling equation as.

$$\xi\left(\lambda^2 S_{Gau}, \lambda^{2-\epsilon\frac{n+2}{n+8}} S_{WF}, \lambda^{\left(1+\frac{d}{2}\right)} h\right) = \lambda^{-1} \xi\left(S_{Gau}, S_{WF}, h\right) \tag{49}$$

where we have reintroduced the magnetic field h for completeness. The scaling equation for the Gibbs free energy (which satisfied dG/dℓ =dG) would be

$$G\left(\lambda^2 S_{Gau}, \lambda^{2-\epsilon\frac{n+2}{n+8}} S_{WF}, \lambda^{1+\frac{d}{2}} h\right) = \lambda^d G\left(S_{Gau}, S_{WF}, h\right) \tag{50}$$

These nonlinear scaling functions shoudl be valid everywhere, not just near the Gaussian or Wilson-Fisher fixed points. They, therefore, should contain all the information contained in our earlier phenomenological analysis. We should be able to show that the line x =0 represents a critical line on which the critical point exponents are those given by the Wilson-Fisher fixed point; the special point x=y=0 should appear as a "tricritical" point at which we see crossover effects.

Before we can proceed further, we must write down the scaling fields S_{Gau} and S_{WF}. They are given by

$$S_{Gau} = x\, C^{-(n+2)/(n+8)} \quad (1+r)$$

$$S_{WF} = x\, y^{-(n+2)/(n+8)} \quad (1+r)^{(4-n)/(n+8)} \tag{51}$$

where the function C is given by

$$C = (1-y/\phi(x)) \quad \exp\left[((n+2)/(n+8)) \in xy/\phi \right] \tag{52}$$

where $y = \phi(x)$ is the equation of the line connecting the Wilson Fisher point to the infinite Gaussian point. In the language of nonlinear differential equation $y = \phi(x)$ is the <u>separatrix</u> connecting the two fixed points. The factors of (1+r) are present in the solutions (51) to match things up at the infinite Gaussian fixed point and are not important for our later discussion. The interesting thing to notice is that the scaling field corresponding to Gaussian fixed point behavior is infinite along the $y = \phi(x)$ separatrix, while the Wilson Fisher scaling field is infinite along the line y = 0.

We can understand this behavior most easily in terms of the line y=0. If y = 0 at some value of ℓ it stays equal to zero. The remaining equation simply states that the variable r has a purely exponential dependence $r=\exp(2\ell)$ and, hence, $\xi = r^{-\frac{1}{2}}$. The line y = 0 represents pure Gaussian behavior. In a similar way, the $y = \phi$ separatrix represents pure Wilson-Fisher behavior. Since both sorts of scaling behavior are contained in the scaling forms (49) -(50), the scaling fields must take on special values to avoid appearing in the final result.. If we return to Fig. 16 we can see that these lines are also singled out geometrically. The flow lines indicate the paths that the variables x and y take as functions of ℓ . The separatrix $y = \phi(x)$ is the only trajectory which leaves the Wilson Fisher point; the line y=0 is the only trajectory leaving the Gaussian fixed point which is not tangent to the y-axis.

Armed with the nonlinear scaling fields, we can now investigate what forms the scaling functions may have. This is entirely parallel to our earlier discussion which placed restrictions on the tricritical scaling function in order that it also describe critical behavior. However, the present discussion is more surely grounded than the purely phenomenological discussion since we know that the single form given in (49)-(50) must incorporate all the behavior.

In general, $G(S_{Gau},S_{WF},h)$ will generate critical point exponents that do not satisfy exponent inequalities as equalities. This is to be expected since G depends on three distinct scaling field with three distinct scaling powers. The usual scaling equalities which relate three exponents are satisfied because there are only two independent scaling powers . An example of a Gibbs potential which is a "nonscaling" global solution of the form given in (50) is

$$G = G_{Gau}(h,S_{Gau}) + G_{WF}(h,S_{WF}) \tag{53}$$

where G_{Gau} and G_{WF} are both scaling functions. Each of these will generate its own singularities with "exponents" that satisfy equalities. Since these exponents are not equal, the measured exponents will be the exponents representing the larger singularities. Therefore, we must have $\alpha + 2\beta + \gamma > 2$!

However, (51) is not a very good solution. Since S_{Gau} diverges on the separatrix, and S_{WF} at $y = 0$, the Gibbs energy given above is infinite on these two lines. This is certainly unacceptable especially since these two lines were to represent pure Wilson-Fisher and pure Gaussian behavior. To examine this more closely, we consider the Gibbs potential at $h = 0$. We may write it in two ways

$$G = S_{Gau}^{\,2-\alpha_{Gau}} f_{Gau}(I) \tag{54a}$$

or

$$G = S_{WF}^{\,2-\alpha_{WF}} f_{WF}(I) \tag{54b}$$

where I is the renormalization group or scaling invariant

$$I = \frac{x^{\epsilon} C^{2-\epsilon(n+2)/n+8}}{y^2 (1+r)^d} \tag{55}$$

$$= \left[\frac{S_{WF}}{S_{Gau}^{[2-\epsilon(n+2)](n+8)]/2}} \right]^{(n+8)/(n+2)}$$

We have written d/2 as $2 - \alpha_{Gau}$ and $d/(2 - \epsilon(n+2)/(n+8)$ as $2 - \alpha_{WF}$ to explicitly display the critical point exponent. If the asymptotically valid value of α were α_{Gau}, then it follows that $f_{Gau}(I=0)$ is some finite constant. However, the invariant is zero on the $y = \phi$ separatrix as well as at x=0. Therefore, the function f_{Gau} cannot help us avoid the singularity in S_{Gau} along the separatrix. On the other hand, if the asymptotically valid value of α were α_{WF}, then $f_{WF}(I=0)$ is a finite constant. This implies that the separatrix should also have Wilson-Fisher behavior, which is what we expect. Near the line y = 0 both S_{WF} and the invariant I are singular. By a proper choice of the function f_{WF}, the two singularities will cancel. An example of a choice that works is

$$G = (S_{WF})^{2-\alpha_{WF}} [I^{1/\epsilon} + 1]^{\alpha_{WF} - \alpha_{Gau}} \tag{55}$$

which near x=y=0 can be written as

$$G = x^{2-\alpha_{WF}} [x + y^{2/\epsilon}]^{\alpha_{WF} - \alpha_{Gau}} \tag{56}$$

This is precisely the sort of expression that in our phenomenological discussion served as an example of a suitable tricritical scaling function.

Thus, we have returned to our starting place. We have shown how to discuss crossover phenomena in terms of a single nonlinear group transformation (the renormalization group) instead of two linear groups (the regular critical and tricritical scaling transformations). Not only does the

renormalization group naturally give rise to the universality of critical point exponents along the critical line, (and the attendant crossover to "tricritical behavior") but it also provides a method for the calculation of scaling powers. The crossover behavior resulted not from a desire to have simultaneous validity of critical and tricritical scaling notions, but rather from simple regularity conditions on the Gibbs potential.

PHENOMENOLOGY

Near the critical line

$$f \approx |x|^{-\nu} \varphi(y)$$

Th e variable y does not change qualit ative behavior or exponents. The line x=0 is a line of critical singularities.

Near the tricritical point

$$f = |x|^{-\bar{\nu}} \varphi\left(y/x^{\bar{a}_y/\bar{a}_x}\right)$$

The variable y scales and there is a change of exponent.

RENORMALIZATION GROUP

Near the Wilson-Fisher fixed point

$$f = |x|^{-\nu_{WF}} \varphi\left(x^{\epsilon \nu_{WF}} w\right)$$

The variable w does not change behavior or e xponents. A line of singularities at least fo r $w \ll 1$ (close to Wilson -Fisher point).

$$1/\nu_{WF} = 2 - \epsilon(n+2)/(n+8)$$

Near the Gaussian fixed point

$$f = |x|^{-\nu_G} \varphi\left(y/x^{\epsilon/2}\right)$$

The variable y scales and there is a change of exponent.

$$1/\nu_G = 2$$

Table 1.

CAPTIONS

1. The phase diagram of a simple ferromagnet. The scaling directions x_1 and x_2 in this case coincide with the magnetic field H and the temperature $T-T_c$.

2. Qualitative view of crossover for the susceptibility. A simple power law dependence of would imply a straight line relationship between log and log $T-T_c$. The observed relationship shows two straight-line regions connected by a crossover "knee".

3. Phase diagram of a simple Ising metamagnet. The antiferromagnetic co-existence surface is bounded by a line of critical points which terminates at tricritical points labelled TCP. The line of first order transitions is shown dashed; T_N is the Neel temperature.

4. The phase diagram of a meta-magnet in HH' T space. A half-moon coexistence surface labelled CXS is bordered by a line of anti-ferromagnetic critical points. This line terminates at two tricritical points (TCP). By examining the phase diagram for non zero staggered magnetic field, the tricritical point is seen to be the point of intersection of three lines of critical points. Wings are formed from coexistence surfaces between paramagnetic and antiferromagnetic phases.

5. In the vicinity of the tricritical point, three different types of behavior can be expected. Meanfield behavior will hold far from the tricritical point and critical line. Near the critical line, "true critical" behavior will be observed. Finally, in the immediate neighborhood of the tricritical point, the tricritical singularity

will be observed. In the overlapping regions and borders, crossover behavior will be observed.

6. The Kadanoff picture of scaling. Ising spins on a lattice with lattice spacing a are grouped into blocks of size L. If L is much less than the correlation length ξ , the blocks can be treated as if all the spins within each block are aligned.

7. Geometrically simple tricritical system. A coexistence surface lies in the zero-ordering field plane h = 0 . In that plane a line of critical singularities (x = 0) terminates at a tricritical point at the origin. The crossover core, within which critical rather than tricritical behavior, is obtained is of the form $x \sim y^{\phi}$, where ϕ is the crossover exponent.

8. The renormalization group principle as applied to a one dimensional Ising chain.
   a. The system consists of a chain of Ising spins coupled to a magnetic field h and interacting via a long range interaction LRI.
   b. The spins may be grouped into blocks at spins. The interactions can be divided into inter-block and intra-block interactions.
   c. Performing an average over the "internal degrees of freedom", we obtain a new Ising system, with an effective magnetic field h' and an effective interaction LRI'.

9. Contributions of internal states of block spins to free energy. There is one state in which all the spins have the same sign as the block spin and three states in which one of the spins has the opposite sign.

10. A system of Ising spins on a lattice is replaced by a continuum-valued spin density defined everywhere.

11. Diagramatic representation of the continuous spin function s(x). If we are only interested in the values of s(x) when $x=2n\pi$, the two different functions shown are equivalent. For this case of a spin on a one-dimensional lattice with lattice spacing 2π, we need only consider wave vectors such that $k \leq 1$, that is, wave vectors within the first Brillouin zone of the lattice.

12. A spherical Brillouin zone approximates the true Brillouin zone approximates the true Brillouin zone of the lattice. Details in lattice structure are not expected to change critical behavior.

13. Approximation of quantized spin values by a weight function: The value of the spin quantum number ($S = \frac{1}{2}, 1$, and $^3/_2$ are shown) does not change the critical behavior. A continuous weight function serves as an average over all spin values.

14. Renormalization procedure for a contiuum spin system. The wave-numbers of the spin fluctuations are divided into two classes. The more rapid fluctuations are integrated over and the remaining wave-vectors rescaled to restore the original form.

15. A differential generator is calculated by performing the renormalization procedure for an infinitesimal change of scale. A thin shell of wave-vectors of thickness $\Lambda\, \delta\ell$ is integrated over. The limit $\delta\ell \rightarrow 0$ is then taken.

16. Solution region of the renormalization group equations. The region includes three fixed points, denoted as the finite Gaussian, infinite Gaussian, and Wilson-Fisher fixed points. The separatrix connecting the Wilson-Fisher and infinite Gaussian fixed points is labeled $y = \phi(x)$. The line $x = 0$ corresponds to the surface of critical Hamiltonian. The origin corresponds to the tricritical point.

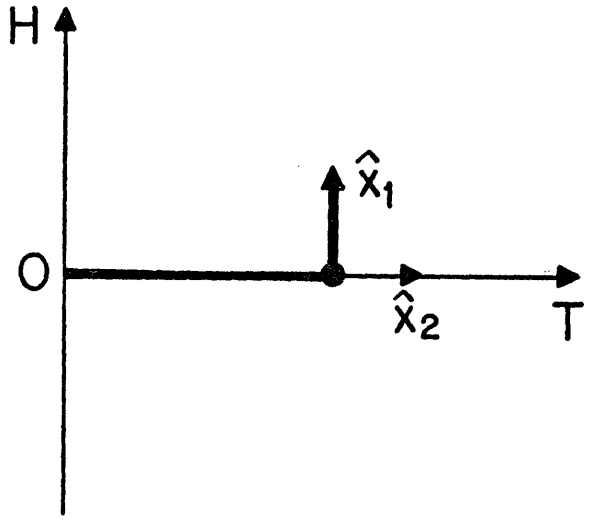
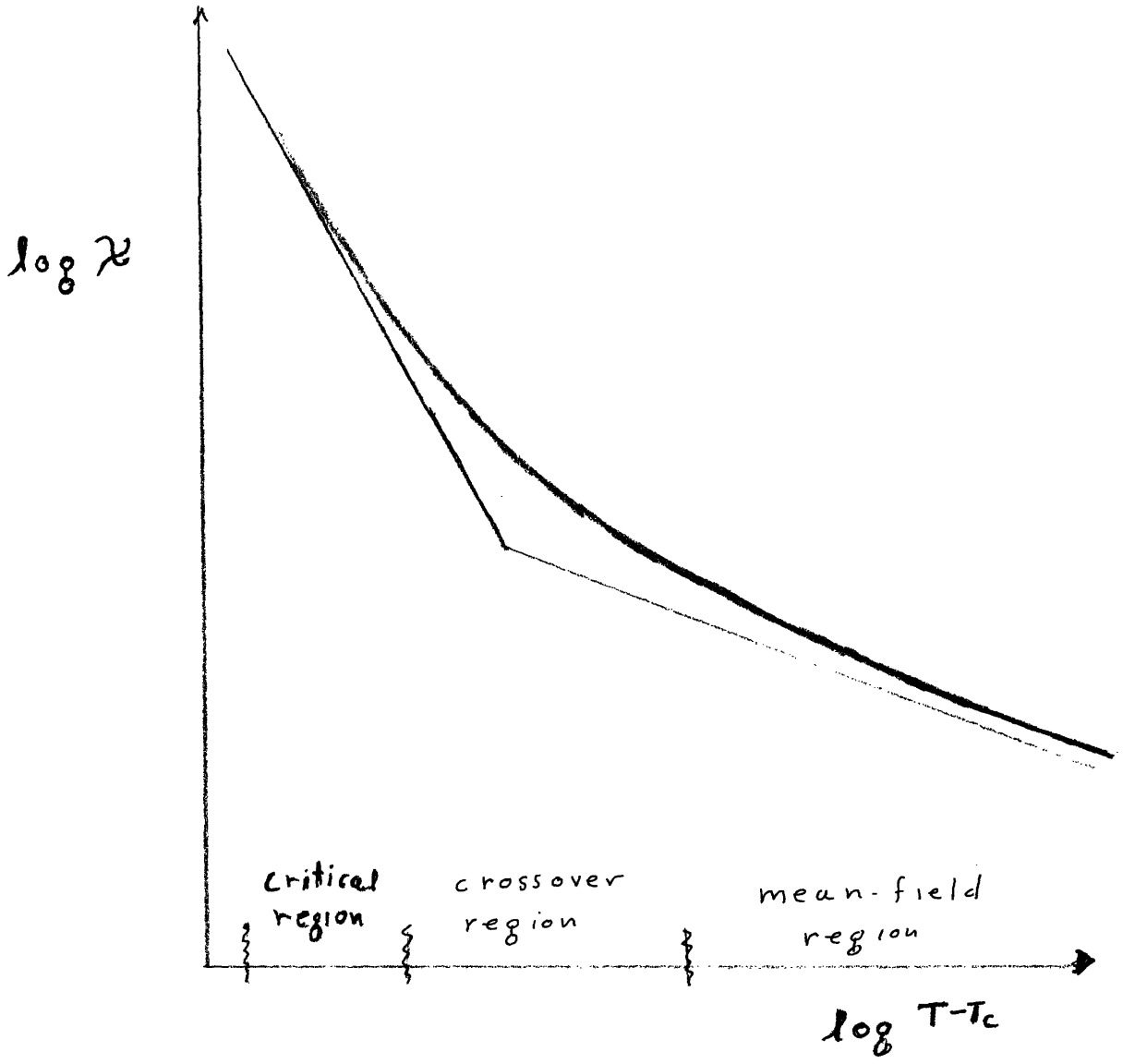


Figure 1.


log χ
critical region
crossover region
mean-field region
log T-Tc

F igure 2.

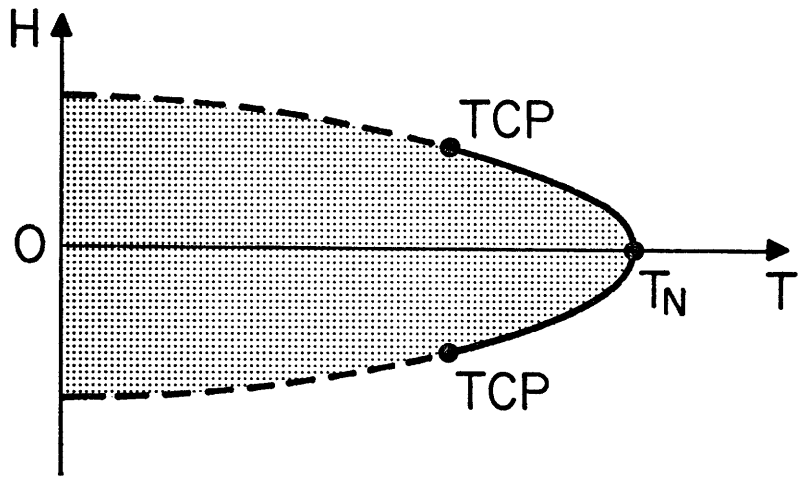


Figure 3.

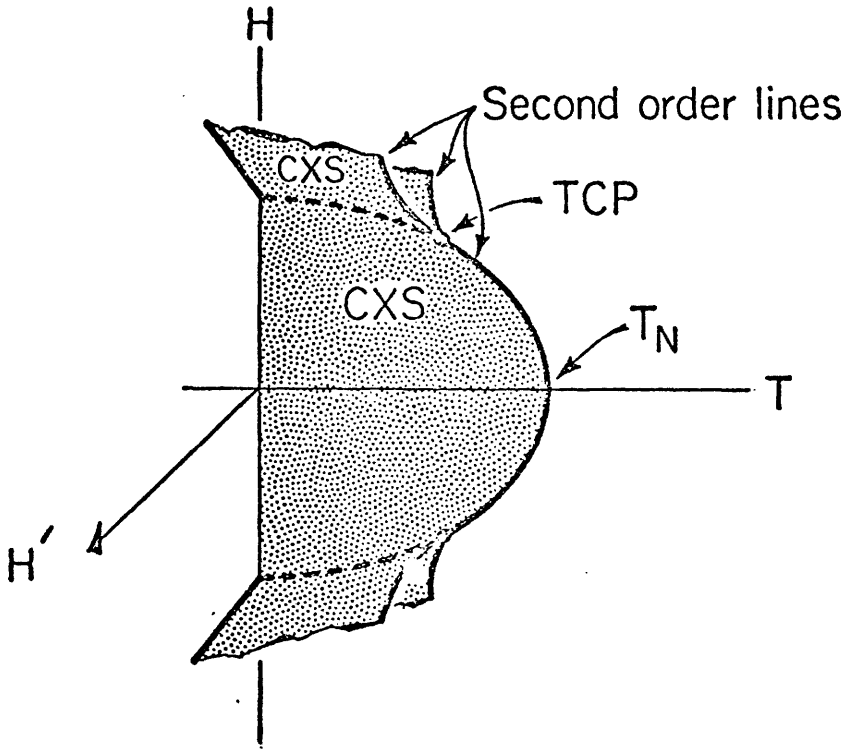


Figure 4.

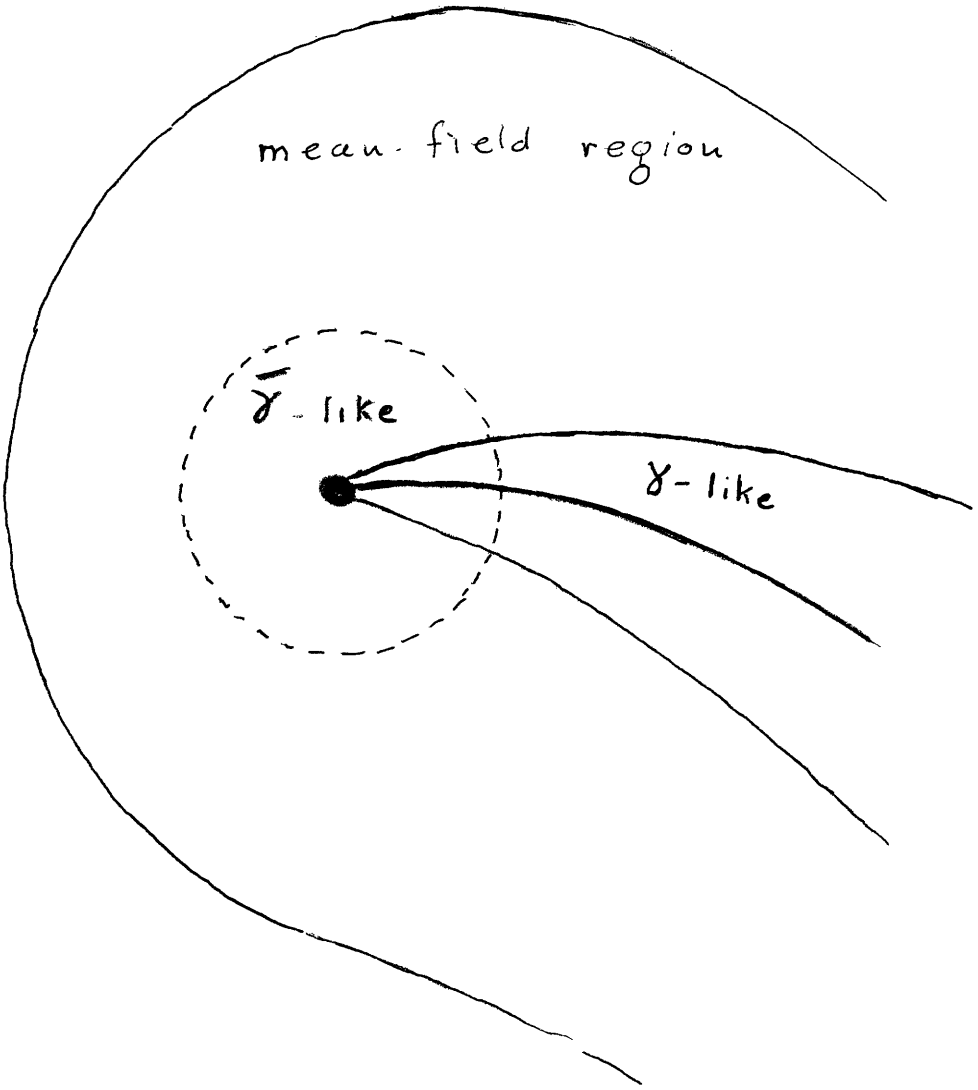
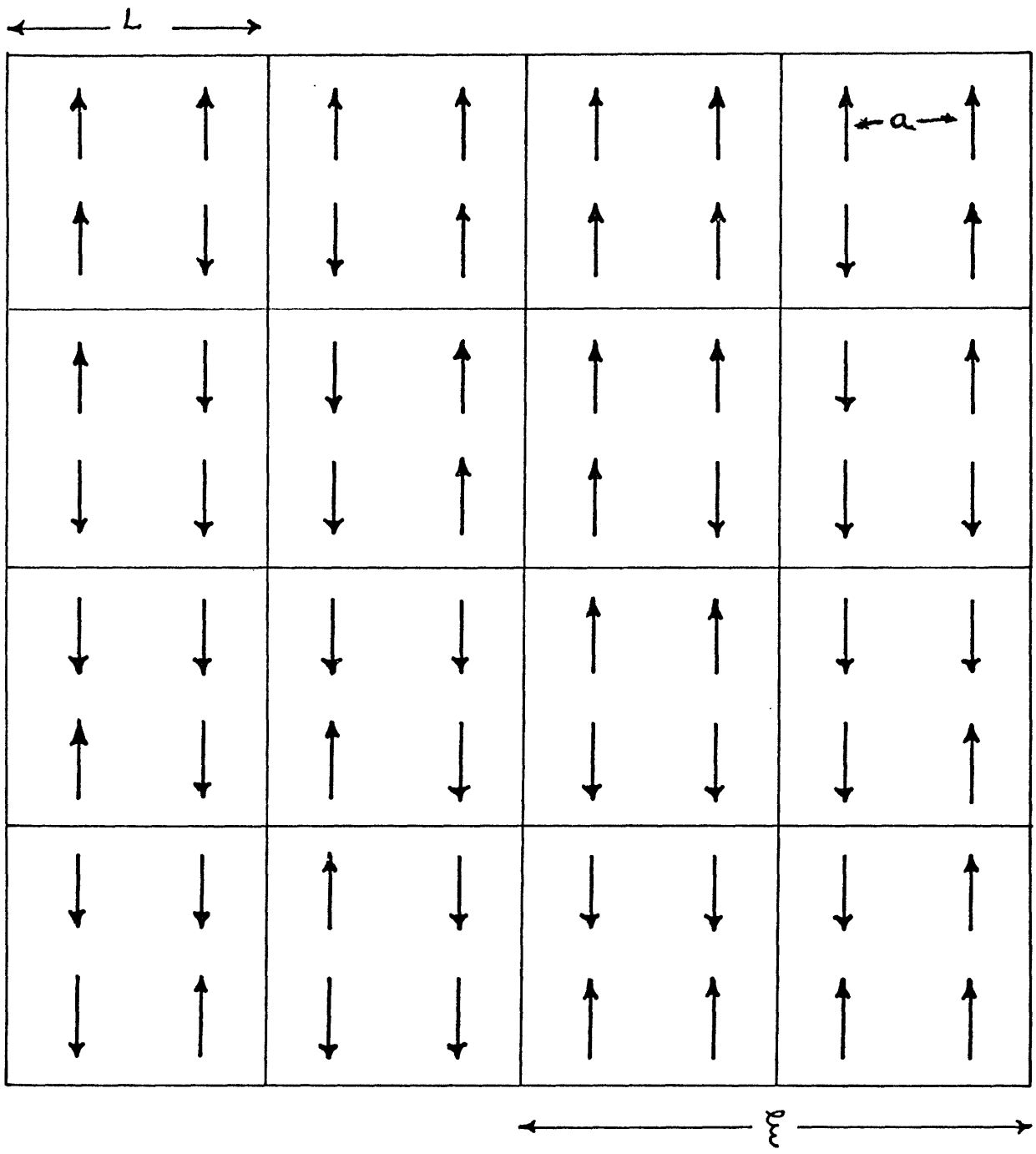


Figure 5.



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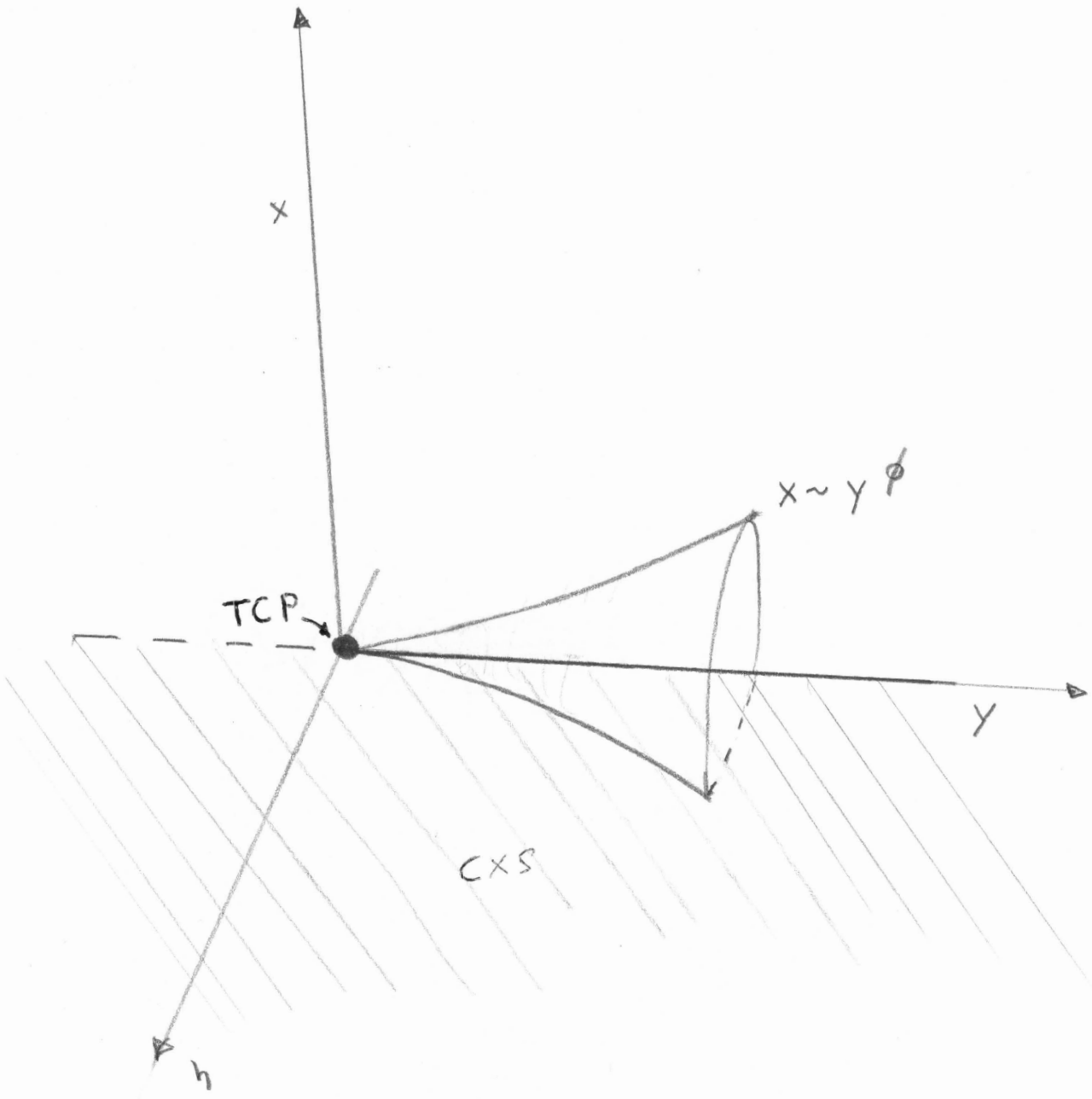


Figure 7.

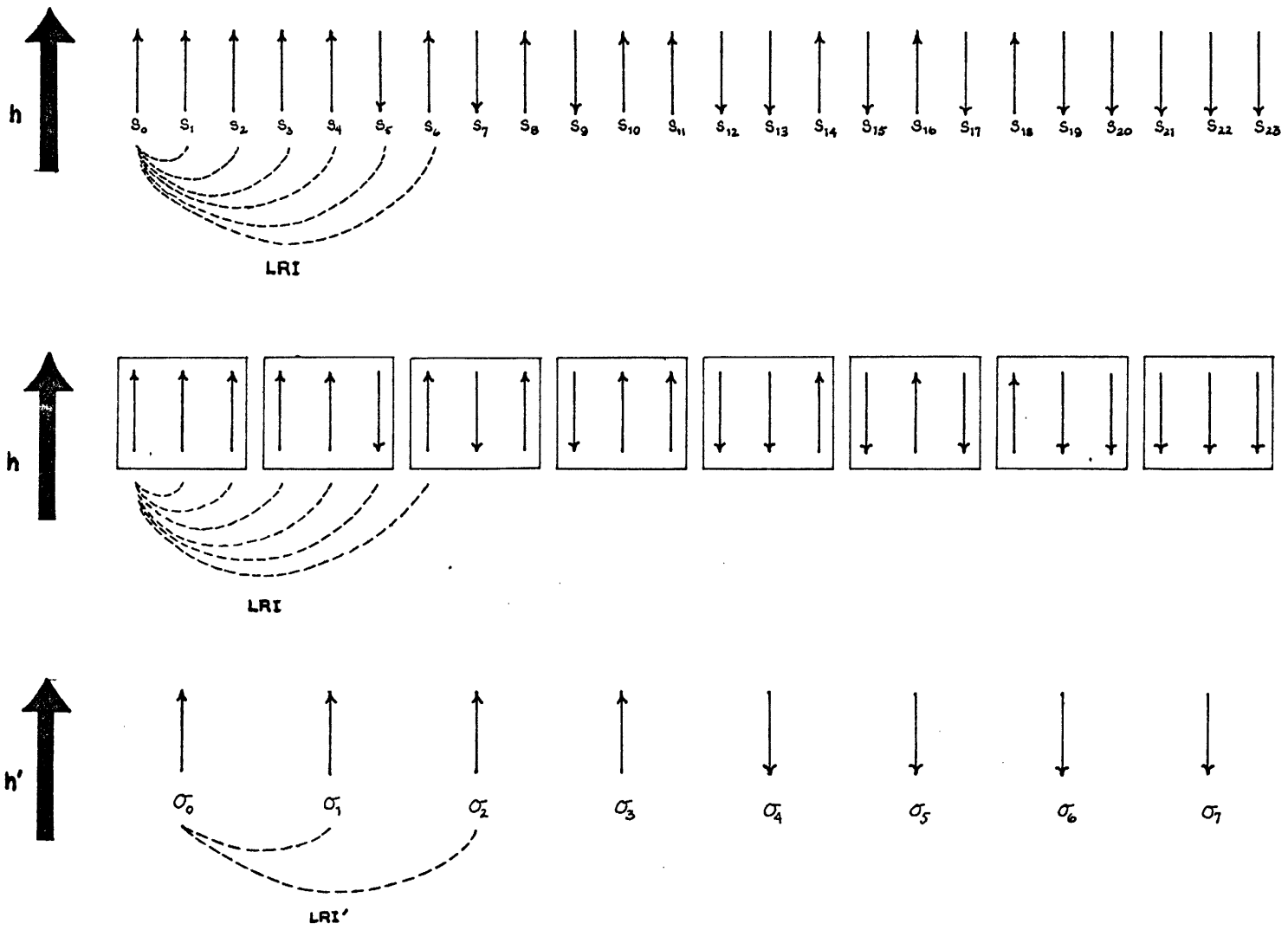

h
S_0 S_1 S_2 S_3 S_4 S_5 S_6 S_7 S_8 S_9 S_{10} S_{11} S_{12} S_{13} S_{14} S_{15} S_{16} S_{17} S_{18} S_{19} S_{20} S_{21} S_{22} S_{23}
LRI
h
LRI
h'
σ_0 σ_1 σ_2 σ_3 σ_4 σ_5 σ_6 σ_7
LRI'

Figure 8.

$\sigma = +1$

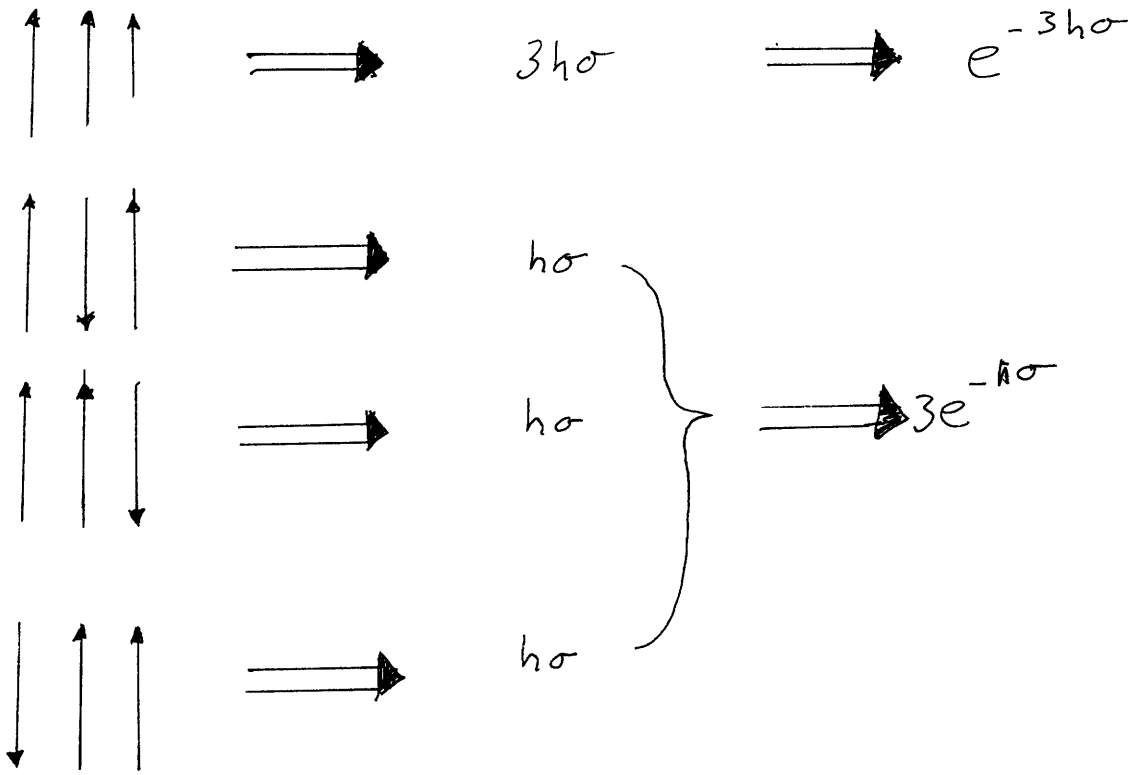


Figure 9.

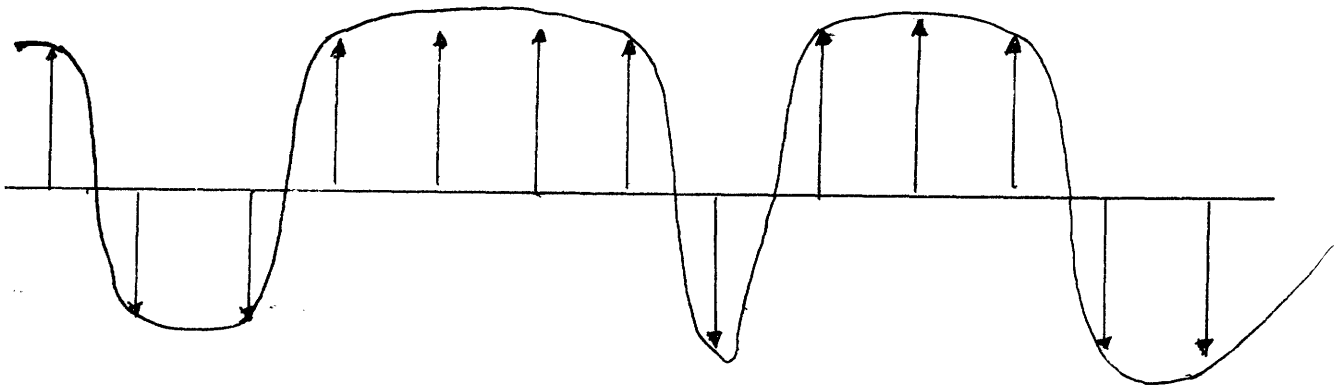


Figure 10

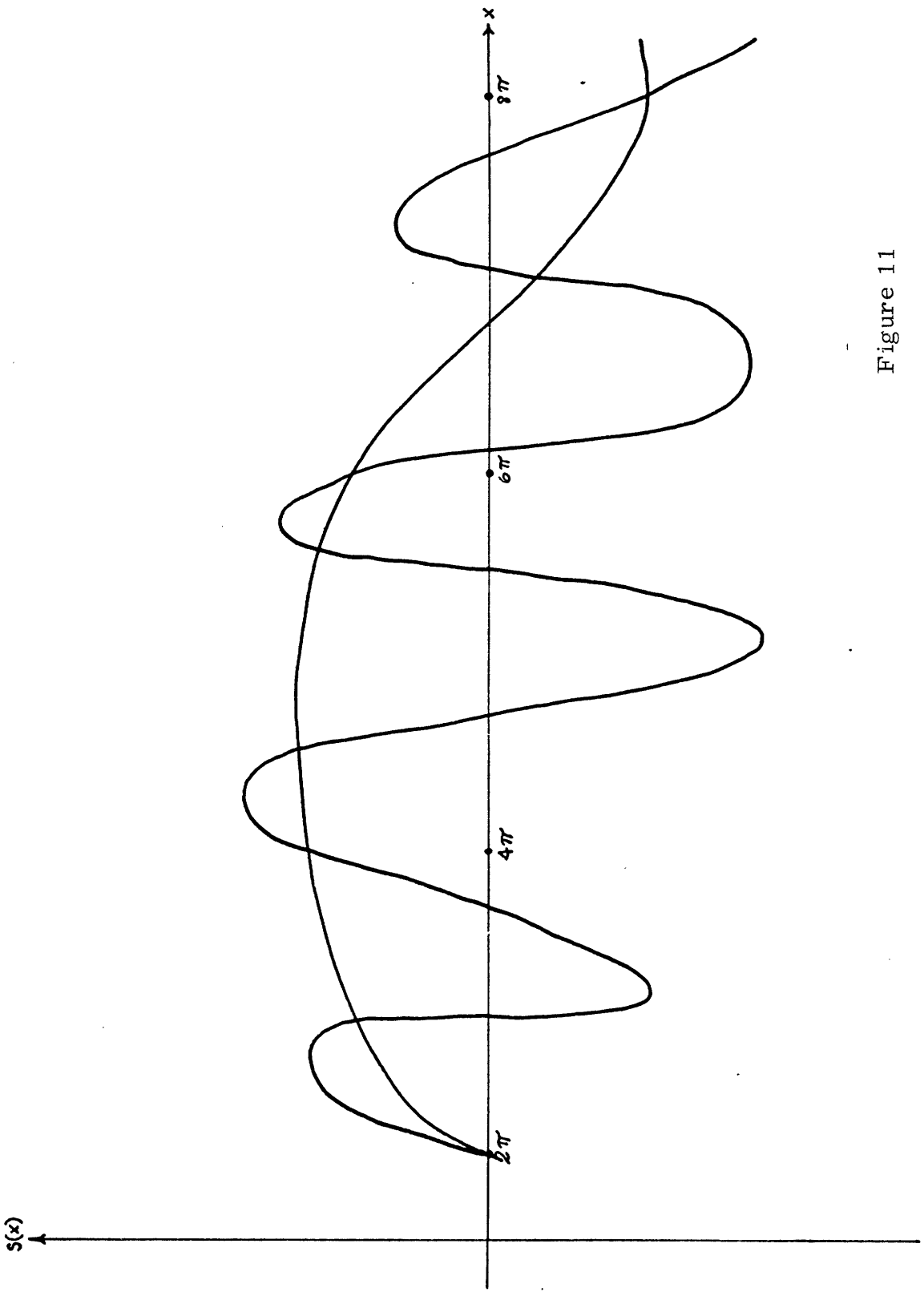
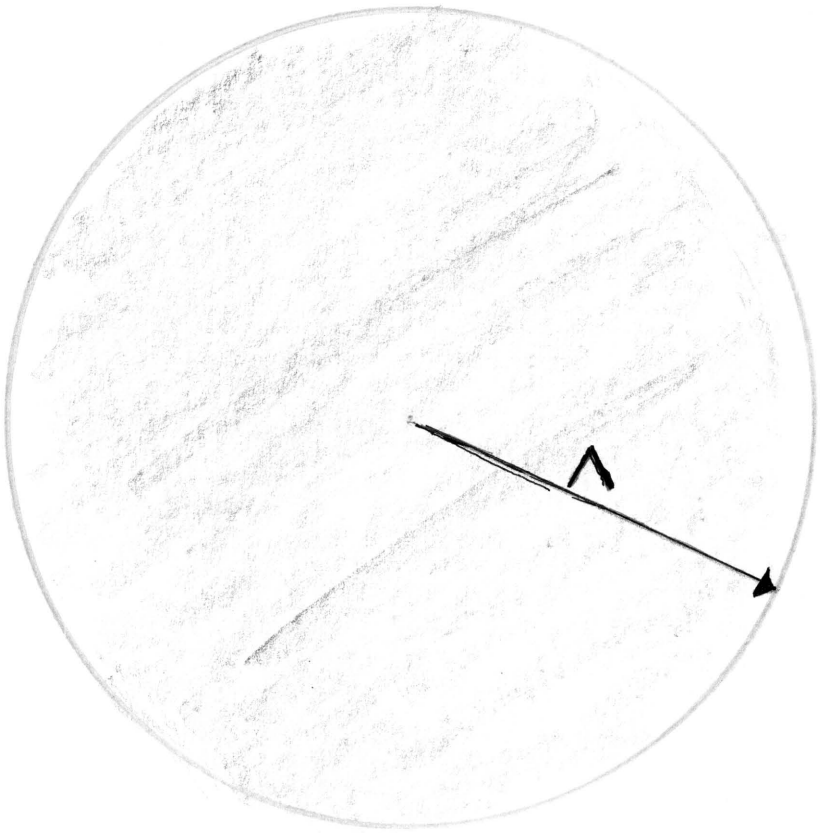


Figure 11



F igure 12

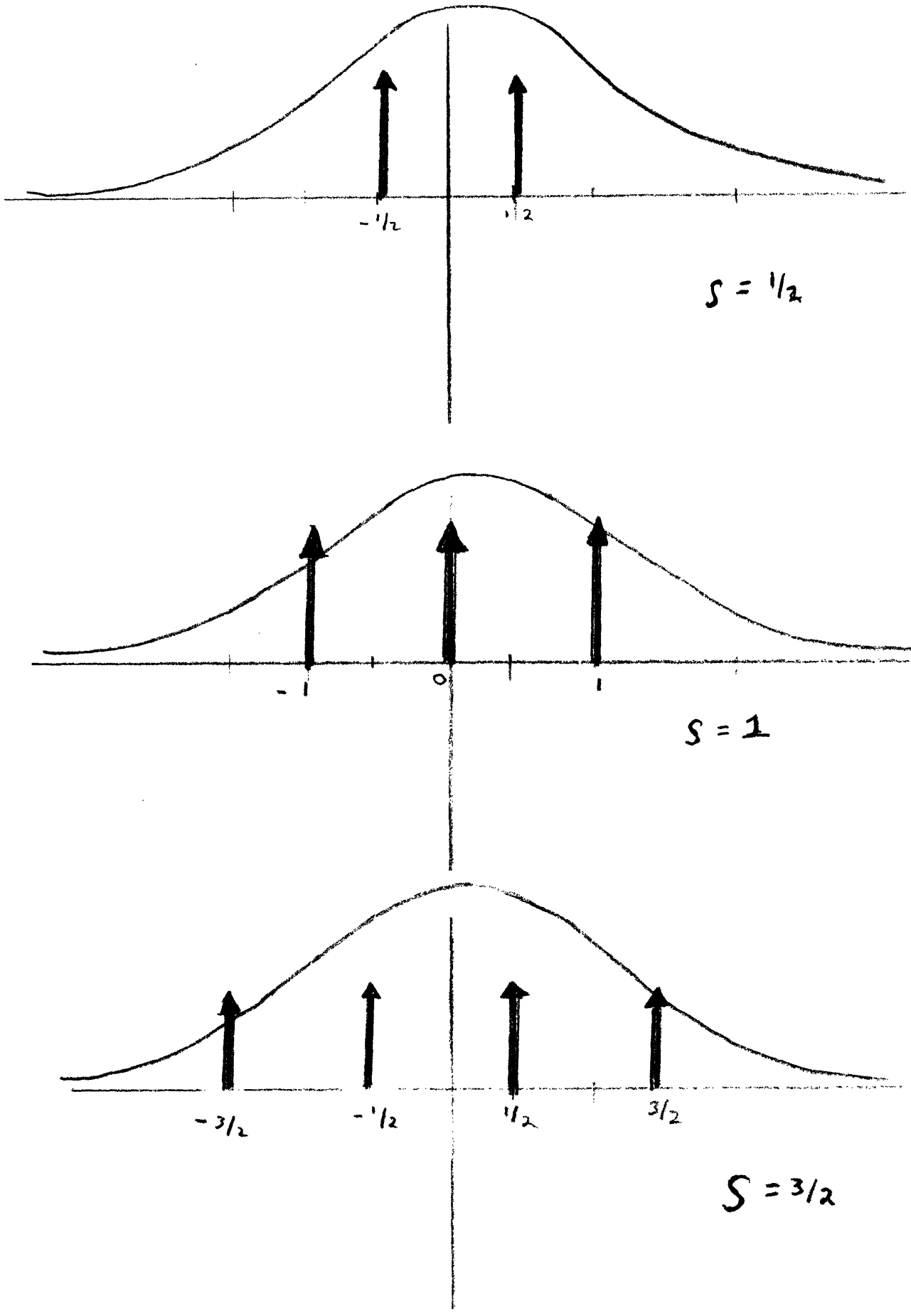


Figure 13

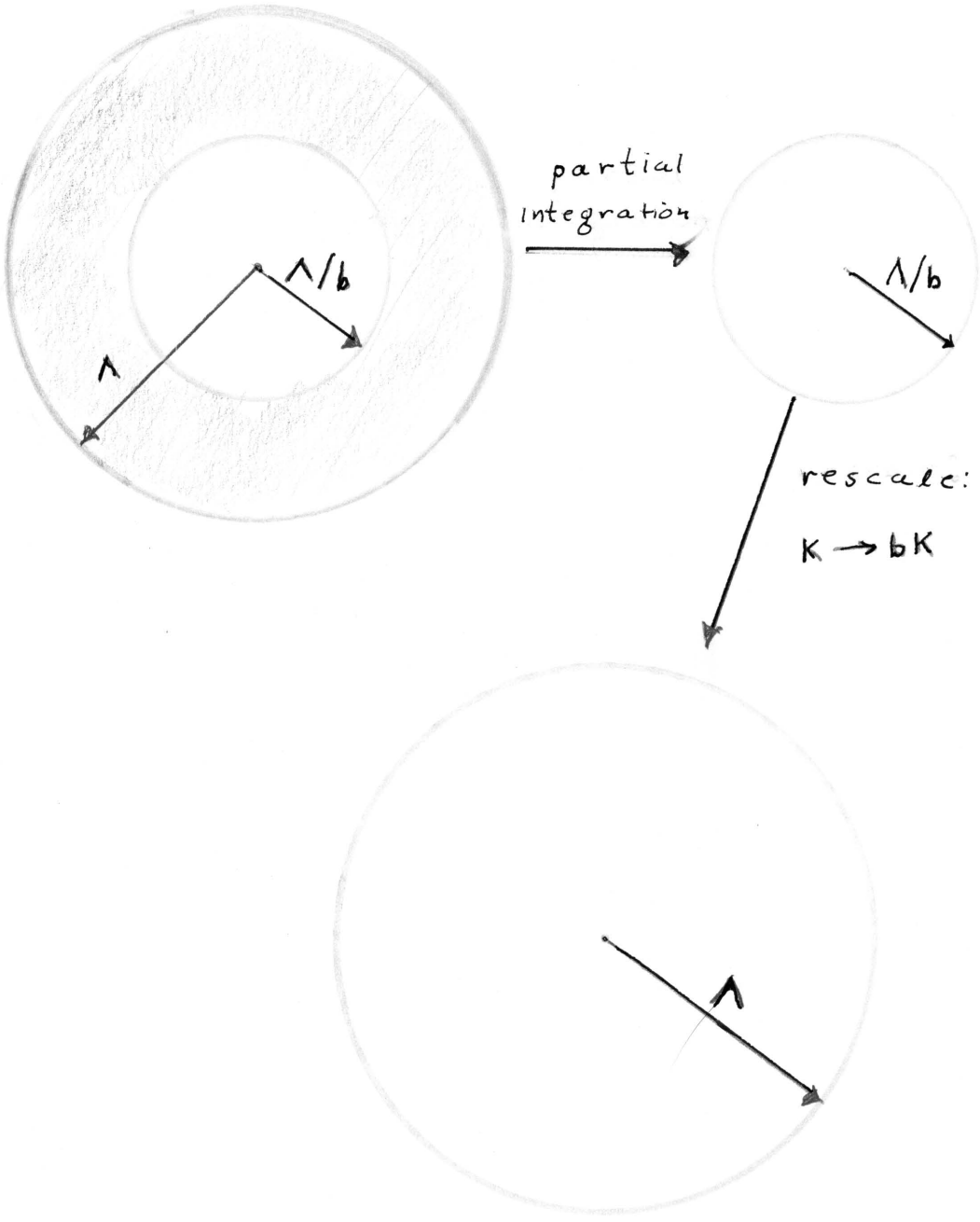


Figure 14

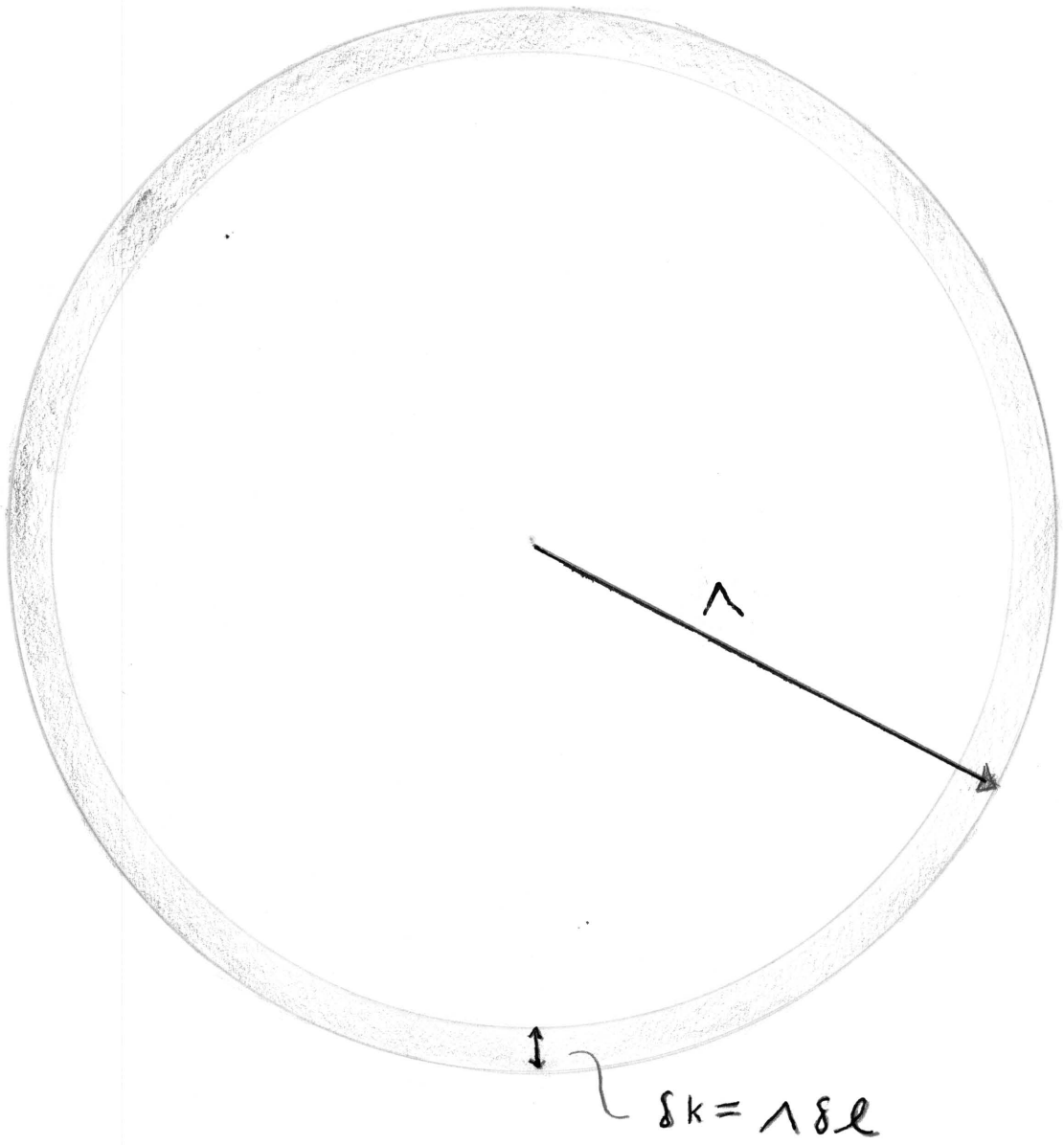


Figure 15

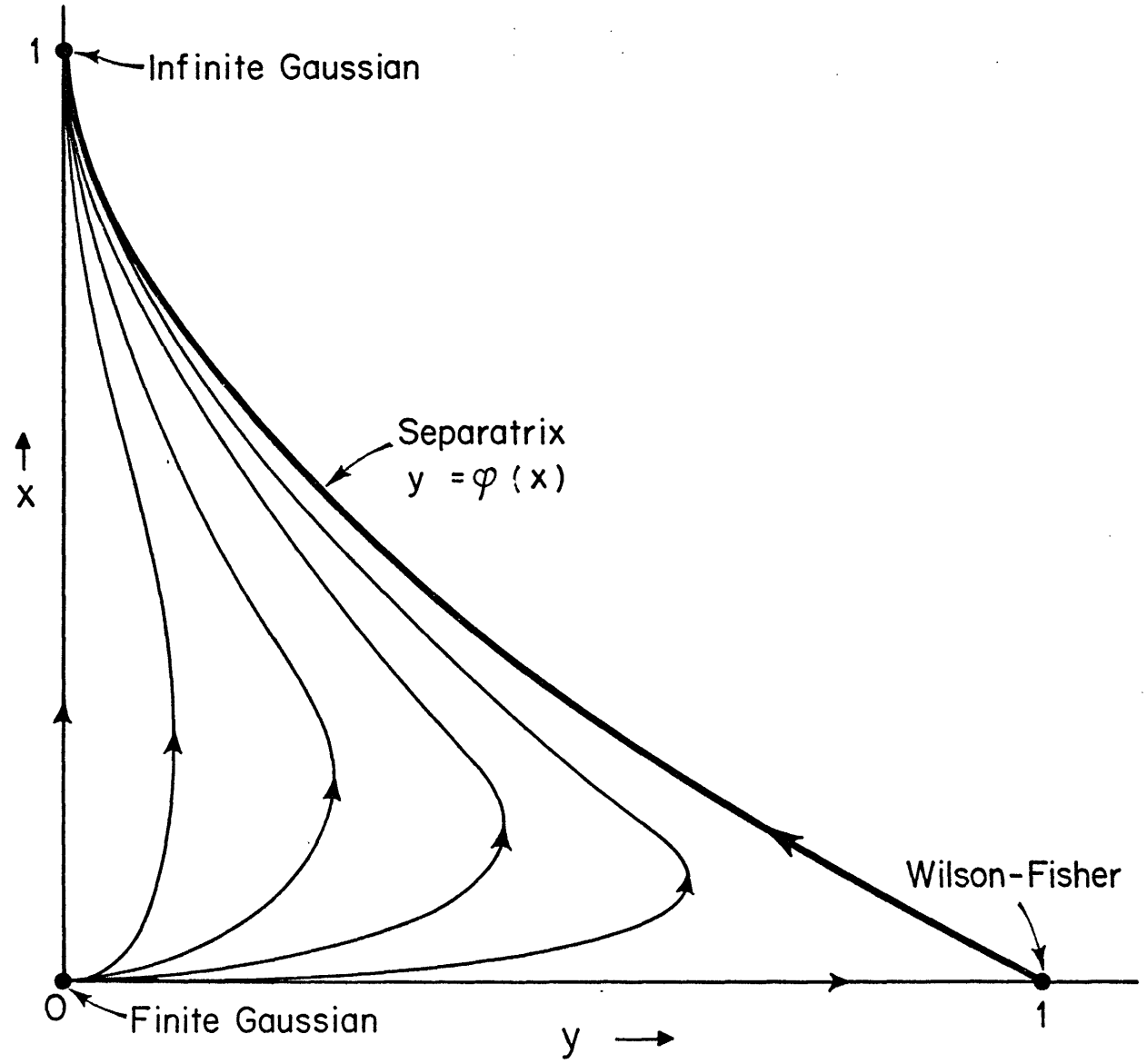

1
Infinite Gaussian
Separatrix
y = φ (x)
x
Wilson-Fisher
0
Finite Gaussian
y
1

Figure 16

CHAPTER 2

SCALING LAWS FOR FLUID SYSTEMS USING GENERALIZED HOMOGENEOUS FUNCTIONS OF STRONG AND WEAK VARIABLES*

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Scaling laws for fluid systems using generalized homogeneous functions of strong and weak variables*

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(Received 28 December 1973)

We present a systematic approach to scaling at ordinary critical points with special emphasis on the critical point of a single-component fluid. Recent work on scaling in fluids has avoided the possibility of a singular coexistence surface. In particular, the consequences of satisfying the inequality $\theta \leq \alpha + \beta$ as an equality have not been explored. We show that $\theta = \alpha + \beta$ is a prediction of scaling, and that, if $\theta = \alpha + \beta$, the specific heat at constant volume has a leading-order (α-divergent) asymmetry across the coexistence surface. We further show that the asymmetric nature of the fluid critical point precludes the analyticity of the critical isochore above the critical temperature, whether the critical isochore is expressed in terms of $\mu(T)$ or $P(T)$. A weak singularity of the form $|T - T_c|^{\beta-2(\alpha+\beta)}$ is predicted for the isochore, which may be dominated by stronger singularities.

I. INTRODUCTION

The original scaling hypothesis was made by Widom[1] (and, independently, by others[2]) to describe behavior near the critical point of a fluid. More recently, it has been realized that this form of the scaling hypothesis may only be adequate to describe the *leading-order* behavior of models and real fluid systems. The interest in extending the domain of validity of the scaling hypothesis has been stimulated by the discovery of a singular diameter[3] in certain models[4,5] and by recent renormalization-group calculations[6]; the singularity behaves like $|T - T_c|^{1-\alpha}$, where α ($\alpha > 0$) is the exponent characterizing the divergence of the specific heat, C_V. Mermin and Rehr[7] have suggested that this $(1 - \alpha)$ singularity may be expected generally in fluids.

To incorporate the diameter singularity, Cook and Green[8] have suggested a very general equation of state. It contains as its leading term the scaling equation of state and many less singular terms. These corrections to scaling yield the diameter singularity and other weakly singular corrections to leading-order scaling behavior. On the other hand, Rehr and Mermin[9] have shown that the singular diameter can be derived from a simple modification of the original scaling equation.

In 1965, Griffiths[10] derived the rigorous inequality $\theta \leq \alpha + \beta$, where θ is the exponent characterizing the divergence of the curvature of the vapor pressure curve, $(\partial^2 P/\partial T^2)_V$. In Refs. 8 and 9 this Griffiths inequality is not satisfied as an equality since assumptions are made about the smoothness of the chemical potential (Ref. 9) or the degree of symmetry about the liquid-vapor coexistence surface. This suggests that the approaches of Refs. 8 and 9 do not explore the consequences of the scaling hypothesis for fluids in the most general way.

In this work, we show how to formulate a scaling hypothesis which can satisfy the Griffiths inequality $\theta \leq \alpha + \beta$ as an equality. To separate physical assumptions from mathematical assumptions, we consider the general problem of making a scaling hypothesis at an ordinary critical point with special attention paid to the liquid-vapor critical point. To make a scaling hypothesis for a fluid system the following four decisions must be made: (i) which thermodynamic variable to select as the dependent variable of the scaling equation; (ii) what independent variables to choose in the scaling equation; (iii) what curves in the thermodynamic space to describe and how to express them in the variables chosen; (iv) whether to augment the scaling equation with correction terms. We will illustrate these four decisions by examining the assumptions implicit in the original scaling hypothesis proposed by Widom[1]:

(i) First, a particular thermodynamic potential must be selected as a candidate for a scaling equation. For fluids, Widom chooses the pressure, P. Each choice of a potential carries with it a natural set of variables (here μ and T, where μ is the chemical potential and T the temperature) *and* a natural set of associated thermodynamic quantities given by the partial derivatives of the potential with respect to these natural variables. For example, the number density ρ is given by $\rho = (\partial P/\partial \mu)_T$. On the other hand, if one chooses the Gibbs potential, $G(P, T)$, then the volume V is given by $V = (\partial G/\partial P)_T$. [For a magnetic system, $-M = (\partial G/\partial H)_T$, where M is the magnetization and H is the magnetic field.] Symmetries or asymmetries in quantities such as ρ or V, will differ depending on the choice of variables and potential (cf. Fig. 1).

(ii) Second, one must choose variables in which to state the scaling equation. Widom chooses $\mu - \mu(T)$ and T, where $\mu(T)$ describes the coexistence surface for $T < T_c$, and the critical isochore for $T > T_c$. The choice of T as the second variable instead of some combination of μ and T, coupled with Widom's use of a single scaling function, has the consequence that the density diameter is rectilinear (cf. Sec. III). The $|T - T_c|^{1-\alpha}$ dependence of the diameter can be obtained in two ways (a) by allowing the second variable to be a function of μ and T, and (b) by adding correction terms to the scaling equation. As we will see in Sec. III, the amplitude of the $|T - T_c|^{1-\alpha}$ term in the diameter is explicitly related to the amplitudes of leading order singularities in case (a), while in case (b) the amplitudes need have no relationship. Thus, the use of a different "second variable" (instead of T), which Rehr and Mermin[9] call "revised scaling," gives the expected form of the diameter singularity but may not correctly give the associated amplitude, as they have pointed out.

(iii) Third, one must decide which surfaces in the thermodynamic space to describe and how to describe them in terms of the scaling variables. By his choice of $\mu - \mu(T)$ as his first variable, Widom singles out the liquid-gas coexistence surface (i.e., the vapor pressure curve) and the critical isochore and describes them both by the homogeneous[11] equation $\mu - \mu(T) = 0$. The coexistence surface is a natural choice in that it is the real phase boundary. The critical isochore is not a natural choice in this sense, but both the coexistence surface critical isochore correspond to paths utilized in experimental measurements, making it highly desirable to describe these paths. The use, however, of a homogeneous[11] description of these paths by an equation $x_1 = 0$, where x_1 is some appropriate variable, limits the scaling approach to systems described by smooth[12] surfaces. To see this statement, suppose that $\mu(T)$ were singular on the coexistence surface or critical isochore. Then the variable $x_1 = \mu - \mu(T)$ would have singularities at $T = T_c$ even away from the critical point. This behavior is undesirable although perhaps tolerable if the singularity were sufficiently weak.[13] Since the singularity in the vapor pressure $P(T)$ [and its critical isochore continuation for $T > T_c$] is expected to be strong, having a divergent curvature, $(\partial^2 P/\partial T^2)_V \to \infty$ as $T \to T_c$, there has been a nearly universal avoidance of $p - t$ scaling of G. (Here we utilize the reduced variables, $p \equiv P - P_c$ and $t \equiv T - T_c$.) In this work we weaken Widom's assumption to allow inhomogeneous[11] descriptions of the critical isochore and coexistence surface in order to encompass the more general situation. This permits us to consider p-t scaling of the Gibbs potential G as well as μ-t scaling of the pressure P.

(iv) Fourth, one must decide whether to describe the system with a single scaling function or to augment the scaling equation with correction terms. Widom's choice of a single function accounts for the leading-order singular behavior and, as extended in Ref. 9, places strong restrictions on the forms and amplitudes of asymmetries and other weakly divergent corrections to the leading-order behavior. On the other hand, multiple correction terms give considerable freedom to the equation of state. Therefore, it is possible that the revised scaling approach of Rehr and Mermin[9] gives only the qualitative nature of the diameter, but cannot correctly predict the amplitude.

In Sec. II A we introduce a general potential Ψ which could be taken to be (within a linear term subtracted off) either $P(\mu, T)$ or $G(P, T)$ for fluids [or, for the sake of comparison, $G(H, T)$ for a magnetic system]. We initially choose to describe the system with a single scaling function to simplify the exposition and to explore the limitations of this approach.

In Sec. II B we discuss the restrictions that can be placed on the forms of the scaling variables used to describe the system. We show that the preferred ("weak") direction of Griffiths and Wheeler,[14] as reflected in the scaling variables, is an automatic consequence of the scaling hypothesis. That is, one of the scaling variables, x_1, must be chosen such that the line $x_1 = 0$, the x_2 axis, is tangent to the coexistence surface at the critical point. We further show that a change in the second variable, x_2, generates a series of correction terms to the scaling equation of a form suggested by a restriction of the formalism of Cook and Green.[8]

In Sec. II C we form a hypothesis for the description of the coexistence surface and critical isochore in terms of the scaling variables x_1 and x_2. The scaling-invariant form $x_1 = A x_2^{\beta\delta}$ is suggested by the scaling hypothesis. We show that this choice in a fluid system corresponds to a vapor-pressure curve with a divergent curvature, $(\partial^2 P/\partial T^2)_V$, characterized by an exponent $\theta = \alpha + \beta$, and an asymmetry in the amplitudes of the specific-heat divergence across the coexistence surface (cf. Sec. III).

In Sec. III we derive the usual critical-point exponents and the relationships between the amplitudes of the leading-term singularities and those of the asymmetries, such as the diameter, utilizing the scaling-invariant parametric form $x_1 = A x_2^{\beta\delta}$ for the coexistence surface and critical

isochore. We then show that if a single scaling function is used, this scaling-invariant form cannot suffice on the critical isochore, but it must be modified by the addition of a specific correction term.

II. A MODIFIED SCALING HYPOTHESIS AT AN ORDINARY CRITICAL POINT

A. Choice of potential and scaling equation

We consider a system that can adequately be described near its critical point by a potential Ψ which can be expected to scale. By keeping Ψ general, we can discuss p-t and μ-t scaling in fluids and H-t scaling in a simple magnetic system, simultaneously. For example, in a magnetic system, we can choose $\Psi = G(H, T) - G_c + S_c(T - T_c)$,[15] where S is the entropy (the subscript c denotes the value at the critical point).

For simplicity we choose to describe Ψ with one scaling function and we employ initially aribtrary scaling variables. In the simple case of a single function, we write

$$\Psi = \mathcal{S} + \mathcal{B}, \tag{2.1}$$

where the scaling function $\mathcal{S}(x_1, x_2)$ is a generalized homogeneous function (GHF)

$$\lambda \mathcal{S}(x_1, x_2) = \mathcal{S}(\lambda^{a_1} x_1, \lambda^{a_2} x_2), \tag{2.2}$$

and $\mathcal{B}(x_1, x_2)$ is a C^∞ background term which vanishes and whose first partial derivatives vanish at the critical point (0, 0). We choose x_1 and x_2 to be smooth invertible functions of the usual thermodynamic variables, y_1 and y_2. For example, in the simplest Ising ferromagnet, $x_1 = y_1 = H$ and $x_2 = y_2 = t$. The convention $a_1 > a_2$ labels x_1 and x_2 as the strong and weak variables of Griffiths and Wheeler.[14, 16] By restricting the transformation to be smooth and invertible, we exclude those cases in which the geometry of the transformation may be as important or more important than that of the scaling function $\mathcal{S}$. In particular, the parametric representations of Schofield[17] embody the singularities directly into the transformed variables used in the parametrization. The transformation is singular and noninvertible at the critical point.

Of the large number of thermodynamic quantities, we will discuss in detail three: C, which is an "order parameter" for the system; D_s, a typical strongly divergent quantity; and D_w, a typical weakly divergent quantity.[14] In terms of the potential Ψ, these are given by

$$C \equiv \Psi^1, \tag{2.3}$$

$$D_s \equiv \Psi^{11}, \tag{2.4}$$

$$D_w \equiv (\Psi^{11}\Psi^{22} - \Psi^{12}\Psi^{12})/\Psi^{11}; \tag{2.5}$$

we introduce the notation

$$F^i \equiv \frac{\partial F}{\partial y_i}; \quad F_i \equiv \frac{\partial F}{\partial x_i}. \tag{2.6}$$

Table I lists the specific symbols for Ψ, C, D_s, y_1, y_2, and D_w, for the three cases considered in this work.

B. Restrictions on the forms of the variables x_1 and x_2

It is straightforward to show[18] that the scaling hypothesis (2.2) implies that near the critical point

$$\frac{b_2^1}{b_1^1} = -\left(\frac{\partial y_1}{\partial y_2}\right)_C, \tag{2.7}$$

where we use the notation

TABLE I. Values of symbols used in the text in three cases. Symbols not defined in the text are $s \equiv S/V$ (entropy density) and K_T, the isothermal compressibility.

	$\mu - t$ scaling	$p - t$ scaling	Magnetic system
Ψ	$P - P_c - \rho_c(\mu - \mu_c) - (T - T_c)s_c$	$G - G_c + (T - T_c)S_c - (P - P_c)V_c$	$G - G_c + (T - T_c)S_c$
y_1	$\mu - \mu_c$	$p \equiv P - P_c$	H
y_2	$t \equiv T - T_c$	$t \equiv T - T_c$	$t \equiv T - T_c$
C	$\rho - \rho_c$	$v \equiv V - V_c$	$-M$
D_s	$\rho^2 K_T$	$\left(\frac{\partial V}{\partial P}\right)_T$	$-\chi_T = -\left(\frac{\partial M}{\partial H}\right)_T$
D_w	$\rho C_V/T$	$-C_V/T$	$-C_M/T$
$\left(\frac{\partial y_1}{\partial y_2}\right)_C$	$\left(\frac{\partial \mu}{\partial T}\right)_\rho$	$\left(\frac{\partial P}{\partial T}\right)_V$	$\left(\frac{\partial H}{\partial T}\right)_M$

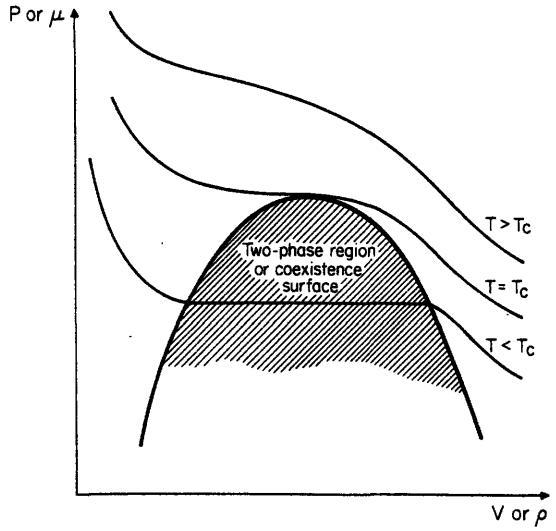

FIG. 1. View of the coexistence surface in $P-V$ or $\mu-\rho$ plane for the case of an asymptotically symmetric top.

$$b_j^i \equiv \frac{\partial x_i}{\partial y_j} . \tag{2.8}$$

Equation (2.7) is proved under less restrictive hypotheses (than the scaling hypothesis) in Ref. 18. Equation (2.7) shows that the line $x_1 = 0$, which is the x_2 axis, must be tangent to the critical isochore at the critical point, as postulated in Ref. 14 (cf. Fig. 2). We assume that the derivative (2.7) is continuous from above T_c to below T_c so that the x_2 axis is also tangent to the coexistence surface at the critical point. Equation (2.7) determines x_1 to linear order. In general, x_1 will have higher-order dependence on y_1 and y_2; however, this dependence cannot be extracted from leading-term or even first-nonleading-term behavior of any thermodynamic quantity. The linear dependence of x_2 must be determined from nonleading-order terms, so that the higher-order dependence of x_2 is even more difficult to extract. For this reason, we will discard any term which does not dominate these weak singularities coming from the nonlinear portions of the transformation between (x_1, x_2) and (y_1, y_2).

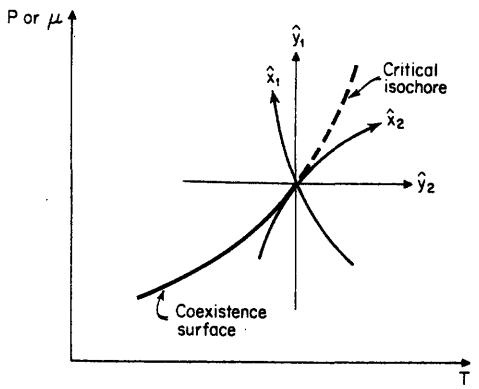

FIG. 2. Relationship between the x_2 axis and the coexistence surface. The dashed line denotes the critical isochore for $T > T_c$.

The original postulate of Griffiths and Wheeler[14] (that the strong direction, the x_1 axis, is arbitrary), however, is correct if and only if we consider solely leading terms in the expressions for thermodynamic quantities. Our scaling hypothesis (2.1) is stated in terms of a GHF, $\mathcal{S}$; for a function which is a GHF to remain a GHF after a change of variables, the change of variables is severely limited. The transformation cannot be linear in both variables unless either it is the unit transformation or the scaling powers are equal. The proof of these statements is given in Appendix A.

If we consider a linear transformation in one variable, setting $X_2 = x_2 - (\text{const})x_1$, our GHF, $\mathcal{S}(x_1, x_2)$, can be expanded as a sum of GHF's:

$$\mathcal{S}(x_1, x_2) = \mathcal{S}(x_1, X_2) + (\text{const})x_1 \mathcal{S}_2(x_1, X_2) + [(\text{const})^2/2!]\, x_1^2 \mathcal{S}_{22}(x_1, X_2) + \cdots . \tag{2.9}$$

Equation (2.9) is a series of correction terms of the form suggested[8] to account for certain of the asymmetries in a fluid. For example, one can easily show that the second term on the right-hand side of (2.9) can be written

$$|X_2|^{(1-a_2+a_1)/a_2} Q\left(\frac{x_1}{|X_2|^{a_1/a_2}}\right) = |X_2|^{2-\alpha+\beta\delta-1} Q\left(\frac{x_1}{|X_2|^{\beta\delta}}\right) . \tag{2.10}$$

For the convenience of the reader, the right-hand side of (2.10) utilizes the expressions in terms of a_1 and a_2 of the critical-point exponents to be obtained in Sec. III.

If we try a more general smooth change of variables, we generate a more general series of corrections. However, the most singular of these corrections will still be given by the linear part of the change of variables. *It is important to observe that a linear change in the strong variable x_1 generates a series of correction terms, each of which is more singular than the previous terms.* Therefore, even if we did not have (2.7) we would not be free to choose x_1 arbitrarily. This statement is a special case of the situation for n variables discussed in Appendix A.

The correction terms of (2.9) are explicitly related to the original GHF, $\mathcal{S}$. They are not, of course, the most general correction terms of the same form. However, if for some choice of x_2 the corrections of the form (2.10) vanished identical-

ly, then revised scaling[9] alone would be sufficient to account for all the asymmetries discussed (at least to leading order in those asymmetries). On the other hand, one may be forced to include correction terms from the beginning. In this case, a change of x_2 simply changes the exact form of the correction terms without changing their qualitative nature; x_2 can be chosen "arbitrarily." The use of revised scaling relates the amplitudes of the weaker singularities to those of the stronger singularities, and is therefore capable of sharper testing when compared to models or experiment, than the corrections-to-scaling approach[8] which leaves the weak amplitudes independent of the leading-term amplitudes. If revised scaling[9] does correctly give the weak amplitudes, then the x_1 axis forms a second preferred direction in the sense that it defines the most appropriate variables in which to state the scaling equation.

The x_2 variable, although unspecified, has a simple form on many paths. It is essentially equal to y_2. To see this, observe that on any path Γ,

$$\left(\frac{dx_2}{dy_2}\right)_\Gamma = \frac{J}{b_1^1 - b_1^2(dx_1/dx_2)_\Gamma}, \tag{2.11}$$

where J denotes the Jacobian of the transformation between (x_1, x_2) and (y_1, y_2),

$$J = b_1^1 b_2^2 - b_2^1 b_1^2. \tag{2.12}$$

We assume that J is nearly constant near the critical point; this is consistent with our assumption of a C^∞ transformation. For convenience we will normalize our variables so that at the critical point $J=1$ and $b_1^1 = 1$.

In this case, an approximate integral of (2.12) is

$$x_2 = y_2 + b_1^2 x_1(x_2). \tag{2.13}$$

We will see that on the coexistence surface and critical isochore $dx_1/dx_2 \to 0$ at the critical point, so that the approximation in (2.13) is a good one.

C. Forms of the coexistence surface and critical isochore

We choose to describe the critical isochore as well as the coexistence surface, since experiments are performed along both paths; as we will show in Sec. III, the critical-isochore path is slightly inconvenient theoretically.

On all paths passing through the critical point, the singularities of Ψ in (2.1) are assumed to come from terms involving $\mathcal{S}$. We know from (2.7) that $x_1 \approx 0$ on both the critical isochore and coexistence surface. Using the properties of GHF's we write for $\mathcal{S}_i$ and $\mathcal{S}_{ij}$:

$$\mathcal{S}_i(x_1, x_2) = |x_2|^{(1-a_i)/a_2} \mathcal{S}_i(x_1/|x_2|^{a_1/a_2}, \pm 1), \tag{2.14a}$$

$$\mathcal{S}_{ij}(x_1, x_2) = |x_2|^{(1-a_i-a_j)/a_2} \mathcal{S}_{ij}(x_1/|x_2|^{a_1/a_2}, \pm 1). \tag{2.14b}$$

The upper sign is used for x_2 positive and the lower sign for x_2 negative.

Equations (2.14) suggest the possible validity of the following scaling-invariant form for the relationship between the scaling variables on paths of interest:

$$x_1 = A_\pm |x_2|^{a_1/a_2} = A_\pm |x_2|^{\beta\delta}. \tag{2.15}$$

In (2.15), $A_\pm$ is a constant, possibly zero, which may differ above and below T_c (the subscript denotes the sign of $T - T_c$).

The limiting case, $A_\pm = 0$, reduces to the homogeneous relation $x_1 = 0$, corresponding to a smooth form for the phase boundary and critical isochore. The case of $A_\pm \neq 0$ gives a power-law singularity. We observe that on any path Γ,

$$\left(\frac{dy_1}{dy_2}\right)_\Gamma = -b_2^1 + \left(\frac{dx_1}{dx_2}\right)_\Gamma \left(\frac{dx_2}{dy_2}\right)_\Gamma. \tag{2.16}$$

The b_j^i are smooth by assumption and by (2.13), $(dx_2/dy_2)_\Gamma$ is nearly constant. We therefore expect that the curvature of the path d^2y_1/dy_2^2 on a scaling-invariant path (2.15) will be given approximately by

$$\left(\frac{d^2y_1}{dy_2^2}\right)_\Gamma \sim \left(\frac{d^2x_1}{dx_2^2}\right)_\Gamma \sim |y_2|^{(a_1-2a_2)/a_2} \sim |y_2|^{-\theta}, \tag{2.17}$$

where the exponent is given by

$$\theta = \alpha + \beta. \tag{2.18}$$

In both p-t and μ-t scaling, (2.18) satisfies the inequality[10] $\theta \leq \alpha + \beta$ as an equality.[19(a)] Physical necessity (as well as convenience) suggests the relationship (2.15) for the coexistence surface. In the two-phase region below T_c, there are two branches of $\mathcal{S}$ corresponding to the liquid and gas phases. The coexistence surface is defined by the equality of $\mathcal{S}$ on these two branches; that is, on the coexistence surface,

$$\mathcal{S}^{\text{liquid}}(x_1, x_2) = \mathcal{S}^{\text{gas}}(x_1, x_2). \tag{2.19}$$

Only the form given in (2.15) allows (2.19) to be satisfied exactly,[19(b)] if a single scaling function is used. On the other hand, we will show in Sec. III that if a single scaling function is used, (2.15) cannot be used on the critical isochore and must

be modified by the addition of corrections.

Before we present the detailed results of our modified scaling hypothesis, a discussion of the use of any inhomogeneous description is in order.[11] In the p-t scaling case, for example, we can write the form of the coexistence surface as

$$p-(\text{const})t-(\text{const}')t^2+\cdots=A_-|t|^{2-\theta}+\cdots \tag{2.20}$$

The left-hand side of (2.20) is an acceptable choice for x_1; that is, it satisfies (2.7). Using (2.13) we could rewrite (2.20) to lowest order as

$$x_1=A_-|x_2|^{2-\theta}. \tag{2.21}$$

(A similar situation might exist in μ-t scaling with the possibility that $\theta<0$.) We assume, therefore, that any nonanalyticity on the coexistence surface or critical isochore can be expressed in an inhomogeneous way such as (2.15) or (2.21). This is equivalent to defining x_1 to be some or all of the smooth part of the coexistence surface and critical isochore. If both of these are smooth, we have the case treated by Widom. If one is smooth, but the other is not, x_1 is the analytic continuation of the smooth surface.

III. RESULTS OF REVISED SCALING

In this section we develop the results of a revised-scaling hypothesis using a single scaling function with no correction terms [cf. Eq. (2.1)] and utilizing the scaling invariant form (2.15) to describe the coexistence surface and critical isochore. We derive the usual critical-point exponents and relate the amplitudes of the weaker singularities to the amplitudes of the dominant singularities in C, D_s, and D_w. We show that the use of (2.15) with $A_\pm$ nonzero changes both the leading and nonleading amplitudes and gives a leading-term (α-divergent) liquid-gas asymmetry in D_w across the coexistence surface. Finally, we show that the use of a single scaling function and the scaling-invariant form are incompatible on the critical isochore; to retain a single scaling function, Eq. (2.15) must be modified. This modification has the consequence that a weak singularity is predicted for the pressure of a fluid system on the isochore similar to one found in the corrections-to-scaling approach of Cook and Green.[8]

Employing Eq. (2.15), and utilizing the properties of GHF's, we can express the quantities Ψ, C, D_s, and D_w as follows[20,21]

$$\Psi=Q_1|t|^{2-\alpha}+Q_2|t|^{3-2\alpha-\beta}, \tag{3.1a}$$

$$C=C_1|t|^{\beta}+C_2|t|^{1-\alpha}, \tag{3.1b}$$

$$D_s=K_1|t|^{-\gamma}+K_2|t|^{\beta-1}, \tag{3.1c}$$

$$D_w=D_0+D_1|t|^{-\alpha}+D_2|t|^{1-2\alpha-\beta}. \tag{3.1d}$$

We have replaced y_2 with $t\,(=T-T_c)$ for the sake of clarity since this substitution is appropriate in the three cases we are considering. The constants in Eqs. (3.1) are given in Table II. The constant D_0 is not derived from the scaling function $\mathcal{S}$, but comes instead from the background term $\mathcal{B}$ of Eq. (2.1). Background terms have been dropped from the other expressions. The divergence in D_w tends to be weak ($\alpha\simeq 0.1$), and hence the terms coming from the background may be important and measurable. The constant Q_2 and the final terms in C_2, K_2, and D_2 come from the expansion of x_2 given in (2.13) with the upper and lower signs applying to the critical isochore and coexistence surface, respectively. We note that the independent parameters in Table II are b_1^2, Q_1, C_1, K_1, and $A_\pm$.[19(a)] The critical-point exponents are obtained in the usual way,[15]

$$\beta=(1-a_1)/a_2, \tag{3.2a}$$

$$-\gamma=(1-2a_1)/a_2, \tag{3.2b}$$

$$-\alpha=(1-2a_2)/a_2. \tag{3.2c}$$

On the critical isotherm, $x_2\propto x_1$, so that by using GHF properties we have (dropping all but the leading term)

$$C=|x_1|^{(1-a_1)/a_1}\mathcal{S}_1(\pm 1,0). \tag{3.3}$$

Here the upper sign corresponds to x_1 positive and the lower sign to x_1 negative, and

$$\delta=a_1/(1-a_1). \tag{3.4}$$

Combining (3.2) and (3.4) we observe that the usual exponent inequalities involving α, β, γ, and δ are satisfied as equalities; for example,

$$\alpha+2\beta+\gamma=2, \tag{3.5a}$$

$$\gamma=\beta(\delta-1). \tag{3.5b}$$

TABLE II. Values of constants in Eq. (3.1).

$Q_1=\mathcal{S}(A_\pm,\pm 1)$
$Q_2=\pm(2-\alpha)b_1^2A_\pm Q_1$
$C_1=\mathcal{S}_1(A_\pm,\pm 1)$
$C_2=b_1^2[(2-\alpha)Q_1+C_1A_\pm(-\beta\delta\pm\beta)]$
$K_1=\mathcal{S}_{11}(A_\pm,\pm 1)$
$K_2=2b_1^2\beta C_1-K_1b_1^2A_\pm(2\beta\delta\pm\gamma)$
$D_1=(2-\alpha)(1-\alpha)Q_1-\beta^2C_1^2/K_1-C_1\beta\delta A_\pm(1-\alpha-\beta)$
$D_2=-b_1^2D_1[2\beta C_1/K_1+A_\pm(-2\beta\delta\pm\alpha)]$

Below the critical temperature, Eq. (2.19) provides some information which relates the amplitudes of the scaling function $\mathcal{S}$ on the liquid and gas sides of the phase boundary. If we also require that the coexistence surface have an asymptotically symmetric top, then

$$\mathcal{S}_1^{\text{liquid}}(x_1, x_2) = -\mathcal{S}_1^{\text{gas}}(x_1, x_2), \tag{3.6}$$

at least to lowest order, on the coexistence curve. Similarly, if the strongly divergent quantities are to have the same leading-term behavior in the two phases, we must require

$$\mathcal{S}_{11}^{\text{liquid}}(x_1, x_2) = \mathcal{S}_{11}^{\text{gas}}(x_1, x_2), \tag{3.7}$$

to leading order.[22(a)] Note that (2.15) is the only relationship between x_1 and x_2 which allows (3.6) and (3.7) to hold exactly.

Using (2.19), (3.6), and (3.7), we can evaluate the asymmetries across the phase boundary:

$$C^{\text{liquid}} + C^{\text{gas}} = 2b_1^2 Q_1 (2-\alpha)|t|^{1-\alpha}, \tag{3.8a}$$

$$D_s^{\text{liquid}} - D_s^{\text{gas}} = 4b_1^2 \beta C_1^{\text{liquid}} |t|^{\beta-1}, \tag{3.8b}$$

$$\begin{aligned} D_w^{\text{liquid}} - D_w^{\text{gas}} = -2C_1^{\text{liquid}} &(A_- \beta\delta(1-\alpha-\beta)|t|^{-\alpha} + b_1^2 |t|^{1-2\alpha-\beta} \\ &\times \{\beta\delta(1-\alpha-\beta)(2\beta\delta+\alpha)A_-^2 \\ &+ 2\beta[(2-\alpha)(1-\alpha)Q_1 - \beta^2 C_1^2/K_1]/K_1\}). \end{aligned} \tag{3.8c}$$

Observe that the asymmetries in (3.8a) and (3.8b) are proportional to b_1^2. If $b_1^2 = 0$, then the implied symmetry leads to a rectilinear diameter as is the case in the original Widom formulation. This is also true of the *weaker* asymmetry in (3.8c); however, the *leading* asymmetry depends only on A_-. If A_- is nonzero, the *amplitudes of the weak divergence differ in the liquid and gas phases.*[22(b)] Thus, although we can maintain symmetry in the order parameter and the strong divergence, $\theta = \alpha + \beta$ breaks the symmetry of the weak divergence, typically, the specific heat.

The expressions given in Table II have one unfortunate consequence. On the critical isochore, setting $C_1 = C_2 = 0$ implies $D_1 = 0$. That is, there is no α singularity of the specific heat. The difficulty arises because we have implicitly assumed that the isochore is a natural path in the same sense that the coexistence surface is a natural path. This is not the case in an asymmetric system. The most natural order parameter ϕ is not simply the density or entropy, but some function of ρ and s given by

$$\phi = \frac{\partial \Psi}{\partial x_1}. \tag{3.9}$$

Lines of constant ϕ might be expected to be scaling invariant. If, however, we wish to describe the isochore, we cannot expect that path to be scaling invariant as well. We could abandon revised scaling and add correction terms to the equation of state (2.1). If, however, we retain revised scaling, we find on the isochore

$$-b_1^2 x_2^{1-\alpha-\beta} = \frac{\mathcal{S}_1(x_1/x_2^{\beta\delta}, 1)}{\mathcal{S}_2(x_1/x_2^{\beta\delta}, 1)}. \tag{3.10}$$

Inverting this equation we find that

$$x_1/x_2^{\beta\delta} = f(x_2^{1-\alpha-\beta}). \tag{3.11}$$

Since we have neglected background terms and excluded corrections to scaling, we are only justified in expanding the right-hand side of Eq. (3.11) to first order in $x_2^{1-\alpha-\beta}$. Using the explicit forms given in (3.10) we discover that on the isochore,

$$\mathcal{S}_1(A_+, 1) = 0, \tag{3.12a}$$

$$x_1 = A_+ x_2^{\beta\delta} - \frac{b_1^2(2-\alpha)\mathcal{S}(A_+, 1) x_2^{3-2(\alpha+\beta)}}{\mathcal{S}_{11}(A_+, 1)}. \tag{3.12b}$$

Since (3.12b) is not a scaling-invariant relationship for $b_1^2 \neq 0$, the arguments of $\mathcal{S}_i$ and $\mathcal{S}_{ij}$ will no longer be constant; the expansion of these amplitudes gives corrections of the same order as the corrections due to revised scaling. Employing (3.12b), we write for the critical isochore only

$$\Psi = Q_1' |t|^{2-\alpha} + Q_2' |t|^{3-2\alpha-\beta}, \tag{3.13a}$$

$$D_s = K_1' |t|^{-\gamma} + K_2' |t|^{\beta-1}, \tag{3.13b}$$

$$D_w = D_0 + D_1' |t|^{-\alpha} + D_2' |t|^{1-2\alpha-\beta}. \tag{3.13c}$$

The constants are given in Table III. All the formulas given in this section are special cases of the equations developed in Appendix C (cf. Tables IV–VI).

TABLE III. Values of constants in Eq. (3.13).

$Q_1' = \mathcal{S}(A_+, 1)$

$Q_2' = b_1^2(2-\alpha)A_+ Q_1'$

$K_1' = \mathcal{S}_{11}(A_+, 1)$

$K_2' = -b_1^2[A_+ K_1'(2\beta\delta+\gamma) + (2-\alpha)\mathcal{S}_{111}(A_+, 1)Q_1'/K_1']$

$D_1' = (2-\alpha)(1-\alpha)Q_1'$

$D_2' = b_1^2 D_1' A_+ [2\beta\delta - \alpha + \beta\delta(1-\alpha-\beta)/(1-\alpha)]$

In the case of $\mu - t$ scaling of the pressure (3.12) gives the following expression for the pressure on the critical isochore ($T > T_c$):

$$\begin{aligned} p = (\text{const})t + (\text{const}')t^2 + Q_1 t^{2-\alpha} \\ + b_1^2 Q_1 A_+ (2-\alpha) t^{3-2\alpha-\beta} + \rho_c A_+ t^{2-\alpha-\beta} \\ - \rho_c b_1^2 \left(\frac{Q_1(2-\alpha)}{K_1} - A_+^2(2-\alpha-\beta) \right) t^{3-2(\alpha+\beta)}. \end{aligned} \tag{3.14}$$

An expression of this form with $A_+ = 0$ and the coefficient of the last term unrelated to δ follows from a correction to the scaling approach as well.[23]

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APPENDIX A: PROPERTIES OF GENERALIZED HOMOGENEOUS FUNCTIONS

In this appendix we will develop some properties of generalized homogeneous functions (GHF's) necessary to the body of the work. We will always be discussing a GHF, $F(x_1, \ldots, x_n)$,

$$\lambda^{a_f} F(x_1, \ldots, x_n) = F(\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n) . \tag{A1}$$

Since thermodynamic functions are often evaluated along particular paths (such as the isochores and isotherms), the behavior of GHF's along different paths is crucial. The simplest path is one along a variable axis, that is, a path on which all the variables except one, x_j, are zero. On such a path, F is given by

$$F(x_1, \ldots, x_n) = |x_j|^{a_f/a_j} F(0, 0, \ldots, \mathrm{sgn} x_j, 0, \ldots) . \tag{A2}$$

This homogeneous description[11] gives a pure power-law behavior. However, there are many other paths that give a pure power-law dependence. If we write F as

$$F(x_1, \ldots, x_n) = |x_j|^{a_f/a_j} F(x_1/|x_j|^{a_1/a_j}, \ldots, \mathrm{sgn} x_j, \ldots, x_n/|x_j|^{a_n/a_j}) , \tag{A3}$$

we recognize a class of paths which we term *scaling-invariant paths*, which are characterized by

$$x_k = A_k |x_j|^{a_k/a_j} , \quad k \neq j . \tag{A4}$$

On such a path we have the simple power-law dependence of the homogeneous paths

$$F(x_1, \ldots, x_n) = |x_j|^{a_f/a_j} F(A_1, \ldots, \mathrm{sgn} x_j, \ldots, A_n) . \tag{A5}$$

However, there is a still larger class of paths which give essentially the same behavior as (A4). If on a path we can write

$$x_k = A_k |x_j|^{a_k/a_j} + B_k |x_j|^{a_k/a_j + q_k} , \tag{A6}$$

with $q_k > 0$, and if F is sufficiently nice (analytic) near $(A_1, A_2, \ldots, \mathrm{sgn} x_j, \ldots, A_n)$ we would expect (A5) to hold approximately. We would write

$$F(x_1, \ldots, x_n) = x_j^{a_f/a_j} \times \left(F(A_1, \ldots, A_n) + \sum_{k \neq j} \frac{\partial F(A_1 \cdots A_n)}{\partial x_k} B_k |x_j|^{q_k} \right) . \tag{A7}$$

Equation (A7) has (A5) as its leading term. We will term paths such as given in (A6) as *asymptotically tangent to the x_j axis*. Note that in the case of scaling-invariant paths (A4), the path is asymptotically tangent to all the axes with nonzero A_k.

An example of particular interest is the "straight line," for which the path is described by a linear parametrization,

$$x_k = c_k x_J . \tag{A8}$$

Then if $a_J > a_k$, for all k such that $c_k \neq 0$, the path is asymptotically tangent to the x_J axis since $1 = a_k/a_J + 1 - a_k/a_J$ and, by assumption, $1 - a_k/a_J > 0$.

The variables employed in (A1) and in the body of this work may seem arbitrary and unconnected to the thermodynamic variables one is accustomed to. If we make a change of variables from $(x_1, x_2, \ldots, x_n)$ to $(y, x_2, \ldots, x_n)$ with $y = y(x_1, x_2, \ldots, x_n)$ we cannot expect the GHF F to remain a GHF. Writing $\bar{F}$ for F as a function of $(y, x_2, \ldots, x_n)$, the statement that $\bar{F}$ is still a GHF, i.e.,

$$\lambda^{a_f} \bar{F}(y, x_2, \ldots, x_n) = \bar{F}(\lambda^{a_y} y, \lambda^{a_2} x_2, \ldots, \lambda^{a_n} x_n) , \tag{A9}$$

is equivalent to the following differential equation for $y(x_1, x_2, \ldots, x_n)$:

$$(\Delta y + \tilde{\Delta} y) \Big/ \frac{\partial y}{\partial x_1} = \tilde{\Delta} F \Big/ \frac{\partial F}{\partial x_1} \tag{A10}$$

where the differential operators Δ and $\tilde{\Delta}$ are defined by

$$\Delta = a_y - \sum_{j=1}^{n} a_j x_j \frac{\partial}{\partial x_j} , \tag{A11}$$

$$\tilde{\Delta} = \sum_{j=2}^{n} (a_j - \bar{a}_j) x_j \frac{\partial}{\partial x_j} . \tag{A12}$$

From (A10)–(A12) we see that if the transformation leaves the basic scaling exponents unchanged along paths asymptotically tangent to the x_j axes $(j \geq 2)$, that is, $a_j = \bar{a}_j$, then y satisfies the homogeneous equation[11]

$$\Delta y = 0 . \tag{A13}$$

The solutions of (A13) are GHF's,

$$\lambda^{a_y} y(x_1, \ldots, x_n) = y(\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n). \tag{A14}$$

The converse is not true. If we rewrite (A10) as

$$\frac{\partial F}{\partial y} \Delta y = \hat{\Delta} F, \tag{A15}$$

where the new differential operator, $\hat{\Delta}$, is given by

$$\hat{\Delta} = \sum_{j=2}^{n} (a_j - \bar{a}_j) x_j \left(\frac{\partial}{\partial x_j}\right)_y, \tag{A16}$$

we see that y, a GHF, implies that $\hat{\Delta} F = 0$. This always has the trivial solution $y = F$.

In general, the right hand side of (A10) is a GHF W such that

$$\lambda^{a_1} W(x_1, \ldots, x_n) = W(\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n). \tag{A17}$$

Now if y is taken to be an analytic function of $(x_1, \ldots, x_n)$, then W must also be analytic. The right-hand side of (A10) is analytic only if

$$n_j a_j = a_1, \tag{A18}$$

for some integers n_j. Furthermore, if the transformation is linear, then the left hand side of (A10) is linear and we must have $n_j = 1$, or $a_j = a_1$ for all j.

If we consider a linear transformation

$$y = x_1 + \sum_{j=2}^{n} c_j x_j,$$

we can expand F to yield

$$F = F(y, x_2, \ldots, x_n) - \sum_{j=2}^{n} c_j x_j \frac{\partial F}{\partial x_1}(y, x_2, \ldots, x_n). \tag{A19}$$

From (A19) we see that if $a_1 < a_j$ for all j such that $c_j \neq 0$, then the correction terms generated are weaker than the leading term, and that, conversely, if $a_1 > a_k$ for some k such that $c_k \neq 0$, then the correction term is stronger than the original term. Therefore, linear (and, in general, smooth transformations with a nonvanishing linear part) variable changes can only involve variables stronger than the variable undergoing transformation. The strongest variable cannot be changed at all, the second strongest can have mixtures of the strongest added to it, the third strongest, mixtures of the first and second, and so forth, down to the weakest variable, which can be considered as arbitrary.

APPENDIX B: EXACT FORMS OF C, D_w, AND D_s

From the definitions it is straightforward to obtain

$$C = b_1^1 \Psi_1 + b_1^2 \Psi_2, \tag{B1}$$

$$D_s = b_1^1 b_1^1 \Psi_{11} + 2 b_1^1 b_1^2 \Psi_{12} + b_1^2 b_1^2 \Psi_{22} + b_{11}^1 \Psi_1 + b_{11}^2 \Psi_2, \tag{B2}$$

$$D_w = [J^2 (\Psi_{11} \Psi_{22} - \Psi_{12} \Psi_{12}) + W_1 + W_2] / D_s, \tag{B3}$$

where

$$b_{jk}^i \equiv \frac{\partial^2 x_i}{\partial y_j \partial y_k}.$$

The quantities W_1 and W_2 are given by the lengthy expressions

$$W_1 = \Psi_1 \Psi_{11} (b_{11}^1 b_2^1 b_2^1 + b_{22}^1 b_1^1 b_1^1 - 2 b_{12}^1 b_1^1 b_2^1) + 2 \Psi_1 \Psi_{12} [b_{11}^1 b_2^1 b_2^2 + b_{22}^1 b_1^1 b_2^2 - b_{12}^1 (b_1^1 b_2^2 + b_1^2 b_2^1)]$$

$$+ \Psi_1 \Psi_{22} (b_{11}^1 b_2^2 b_2^2 + b_{22}^1 b_1^2 b_1^2 - 2 b_{12}^1 b_1^2 b_2^2) + \Psi_1 \Psi_1 (b_{11}^1 b_{22}^1 - b_{12}^1 b_{12}^1) + \tfrac{1}{2} \Psi_1 \Psi_2 (b_{11}^2 b_{22}^1 + b_{11}^1 b_{22}^2 - 2 b_{12}^1 b_{12}^2), \tag{B4a}$$

$$W_2 = W_1, \; 1 \rightleftarrows 2. \tag{B4b}$$

The b_j^i and b_{jk}^i must be understood to be smooth functions of y_1 and y_2 so that the singularity structure of D_s, for example, is given by

$$D_s = k_0 + k_1 |t|^{-\gamma} + k_2 |t|^{\beta - 1} + k_3 |t|^{-\alpha} + k_4 |t|^{\beta} + k_5 |t|^{1-\alpha} + k_6 |t|^{1-\gamma}. \tag{B5}$$

The constant k_6 comes entirely from the t dependence of $b_1^1 b_1^1$; the constants k_4 and k_5 are a mixture of b_{11}^1 and b_{11}^2, on the one hand, and the t dependence of $b_1^1 b_1^2$ and $b_1^2 b_1^2$, on the other. Terms like the $1 - \gamma$ divergent term k_6 in (B5) could also arise by replacing the scaling function in (2.1) with the product of a smooth function of y_1 and y_2 and the same scaling function. This is related to the idea employed by Domb[24] to generate corrections to scaling for the Ising ferromagnet.

The detailed dependence of D_w as expressed in W_1 and W_2 in (B4) is very complicated and generally unenlightening. Each term is proportional to second derivatives of the transformation between (y_1, y_2) and (x_1, x_2) which, unlike the first derivatives, we have no method of estimating. A change of variables which changed the second derivatives (but left the first derivatives unchanged) leads to the generation of a series of corrections to the

scaling equation that are always weaker than the original scaling function. Therefore, if we are willing to carry correction terms of that type, we may choose the second derivatives in any way we wish.

We observe that if $\alpha+\beta<\frac{1}{2}$, the leading singular term in W_1 is in fact larger than the $|t|^{1-2\alpha+\beta}$ term discussed in Sec. III and Appendix C. We write D_w as

$$D_w = D_0 + D_1|t|^{-\alpha} + D_2|t|^{1-2\alpha-\beta} + D_3|t|^{\alpha-\alpha} + D_4|t|^{\beta}. \tag{B6}$$

The new constant D_4 is given by

$$D_4 = C_1(b_{11}^1 b_2^1 b_2^1 + b_{22}^1 - 2b_{12}^1 b_2^1). \tag{B7}$$

Since it is proportional to C_1, the $|t|^{\beta}$ term vanishes on the critical isochore ($T>T_c$).

APPENDIX C: EXTENSIONS OF REVISED SCALING

The symmetry requirements on the coexistence surface [cf. Eqs. (2.19), (3.6), and (3.7)] preclude any modification of the scaling-invariant form (2.15) without a corresponding modification of the fundamental equation (2.1). That is, corrections to scaling must be added. To illustrate what can be done in a corrections-to-scaling approach, we replace (2.1) with

$$\Psi = \mathcal{S} + \tilde{\mathcal{S}} + \mathcal{B}, \tag{C1}$$

where $\tilde{\mathcal{S}}$ is a correction-to-scaling term.[8] We restrict ourselves to a single correction term for simplicity; the further generalization is only an algebraic complication. We have no *a priori* reason to restrict $\tilde{\mathcal{S}}$ in any way. However, for simplicity, we assume that $\tilde{\mathcal{S}}$ is a GHF in the same scaling variables as $\mathcal{S}$,

$$\lambda^{1+qa_2}\tilde{\mathcal{S}}(x_1, x_2) = \tilde{\mathcal{S}}(\lambda^{a_1}x_1, \lambda^{a_2}x_2). \tag{C2}$$

Note that the exponent of λ on the left-hand side of (4.2) is not 1 but $1+qa_2$ with $q>0$. By assumption, the dominant behavior of any thermodynamic quantity is given by $\mathcal{S}$. The correction-to-scaling term $\tilde{\mathcal{S}}$ cannot affect leading-order scaling. If we assume that $\tilde{\mathcal{S}}$ is a GHF, we know by the discussion in Sec. II B that the weaker variable can be chosen freely; there is no loss in assuming it to be simply x_2. The stronger variable has a natural definition as the smooth part of the coexistence surface and critical isochore, and we assume that this preferred variable can be carried over to the correction term. The scaling powers a_1 and a_2 have been chosen equal to those of the scaling function $\mathcal{S}$ for further simplicity. One of the scaling powers can always be so chosen (cf. Ref. 15), and if the coexistence surface or critical isochore is singular with $A_{\pm}\neq 0$, the usefulness of the scaling-invariant path suggests that both of the scaling powers are equal to the corresponding scaling powers of the leading-term GHF.

Correction terms of the same order as those coming from $\mathcal{S}$ can be generated by modifying (2.15). We write for the coexistence surface

$$x_1 = A_-|x_2|^{\beta\delta} + B_-|x_2|^{\beta\delta+q}. \tag{C3}$$

Anticipating the difficulties on the critical isochore, we write for the isochore:

$$x_1 = A_+ x_2^{\beta\delta} + B_+ x_2^{\beta\delta+q} - b_1^2(2-\alpha) \times \frac{\mathcal{S}(A_+, 1)}{\mathcal{S}_{11}(A_+, 1)} x_2^{3-2(\alpha+\beta)}. \tag{C4}$$

TABLE IV. Values of constants in Eq. (C5).

$Q_1 = \mathcal{S}(A_-, -1)$

$Q_2 = -b_1^2 A_-(2-\alpha)Q_1$

$Q_3 = B_- C_1 + \tilde{\mathcal{S}}(A_-, -1)$

$C_1 = \mathcal{S}_1(A_-, -1)$

$C_2 = b_1^2(2-\alpha)[Q_1 - A_- C_1]$

$C_3 = B_- K_1 + \tilde{\mathcal{S}}_1(A_-, -1)$

$K_1 = \mathcal{S}_{11}(A_-, -1)$

$K_2 = b_1^2[2\beta C_1 - K_1(2\beta\delta - \gamma)]$

$K_3 = B_- \mathcal{S}_{111}(A_-, -1) + \tilde{\mathcal{S}}_{11}(A_-, -1)$

$D_1 = (2-\alpha)(1-\alpha)Q_1 - \beta^2 C_1^2/K_1 - C_1 A_-\beta\delta(1-\alpha-\beta)$

$D_2 = b_1^2 D_1[A_-(2\beta\delta+\alpha) - 2\beta C_1/K_1]$

$D_3 = B_- C_1[\beta(3-2\alpha-\beta+2q) - (1-\alpha+q)(2-\alpha+q)] + \beta^2 C_1^2 K_3/K_1^2 - 2\beta \times (\beta+q)C_3 C_1/K_1 + (1-\alpha+q)(2-\alpha+q)Q_3 + C_3\beta\delta A_-(\alpha+\beta-1)$

TABLE V. Value of the constants in Eq. (C9).

$$\bar{C}_2 = 2b_1^2(2-\alpha)Q_1$$
$$\bar{C}_3 = 2B_-K_1 + \tilde{\$}_1^{\text{liquid}}(A_-,-1) + \tilde{\$}_1^{\text{gas}}(A_-,-1)$$
$$\bar{K}_2 = 4b_1^2\beta C_1^{\text{liquid}}$$
$$\bar{K}_3 = B_-[\$_{111}^{\text{liquid}}(A_-,-1) - \$_{111}^{\text{gas}}(A_-,-1)] + \tilde{\$}_{11}^{\text{liquid}}(A_-,-1) - \tilde{\$}_{11}^{\text{gas}}(A_-,-1)$$
$$\bar{D}_1 = 2C_1^{\text{liquid}}A_-\beta_-(1-\alpha-\beta)$$
$$\bar{D}_2 = \bar{K}_2[(2-\alpha)(1-\alpha)Q_1 - \beta^2C_1^2/K_1]/K_1 + b_1^2A_-(2\beta\delta+\alpha)\bar{D}_1$$
$$\bar{D}_3 = 2B_-C_1^{\text{liquid}}[\beta(3-2\alpha-\beta+2q) - (1-\alpha+q)(2-\alpha+q)] + \beta^2C_1^2\bar{K}_3/K_1^2 - 2\beta(\beta+q)C_1^{\text{liquid}}\bar{C}_3/K_1 + (C_3^{\text{liquid}} - C_3^{\text{gas}})\beta\delta(\beta+\alpha-1)A_-$$

TABLE VI. Values of constants in Eq. (C10).

$$Q_1' = \$(A_+,1)$$
$$Q_2' = b_1^2(2-\alpha)A_+Q_1'$$
$$Q_3' = \tilde{\$}(A_+,1)$$
$$K_1' = \$_{11}(A_+,1)$$
$$K_2' = -b_1^2[A_+K_1'(2\beta\delta+\gamma) + (2-\alpha)\$_{111}(A_+,1)Q_1'/K_1']$$
$$K_3' = B_+\$_{111}(A_+,1) + \tilde{\$}_{11}(A_+,1)$$
$$D_1' = (2-\alpha)(1-\alpha)Q_1'$$
$$D_2' = b_1^2D_1'A_+[2\beta\delta - \alpha + \beta\delta(1-\alpha-\beta)/(1-\alpha)]$$
$$D_3' = (1-\alpha+q)(2-\alpha+q)Q_3'$$

From our discussion of changes of variable in Sec. II B, we note that if we employ a nonzero $\tilde{\$}$ with $q = 1-\alpha-\beta$, we may set $b_1^2 = 0$.

Evaluating the consequences of (4.1) and (4.3) we find on the coexistence surface:

$$\Psi = Q_1|t|^{2-\alpha} + Q_2|t|^{3-2\alpha-\beta} + Q_3|t|^{2-\alpha+q}, \quad \text{(C5a)}$$
$$C = C_1|t|^{\beta} + C_2|t|^{1-\alpha} + C_3|t|^{\beta+q}, \quad \text{(C5b)}$$
$$D_s = K_1|t|^{-\gamma} + K_2|t|^{\beta-1} + K_3|t|^{q-\gamma}, \quad \text{(C5c)}$$
$$D_w = D_0 + D_1|t|^{-\alpha} + D_2|t|^{1-2\alpha-\beta} + D_3|t|^{q-\alpha}. \quad \text{(C5d)}$$

The constants in (C5) are given in Table IV. The independent parameters have increased to Q_1, C_1, K_1, Q_3, C_3, K_3, $A_\pm$, and $B_\pm$.

In the simple revised-scaling scheme of Sec. III, $p-t$ and $\mu-t$ scaling, although handled in similar ways, cannot be compatible due to the intrinsic differences in symmetry. Even a pure $|t|^{\beta}$ dependence in $\rho(v)$ leads to a $|t|^{2\beta}$ diameter in $v(\rho)$. However, if we set $q=\beta$, we produce $|t|^{2\beta}$ terms in C (and $|t|^{\beta-\gamma}$ terms in D_s). If we assumed that $\mu-t$ scaling were valid with a $|t|^{1-\alpha}$ density diameter, then $p-t$ scaling would also be valid if in the $p-t$ scaling formalism, the following relations between the constants of (C5) held:

$$C_3 = C_1C_1/V_c, \quad \text{(C6a)}$$
$$K_3 = 3K_1C_1/V_c, \quad \text{(C6b)}$$
$$D_3 = D_1C_1/V_c. \quad \text{(C6c)}$$

These relations are particularly interesting in the case $A_- = 0$ but $B_- \neq 0$, since (C3) would then imply that the divergence of the curvature of the coexistence surface, $(\partial^2p/\partial t^2)_v$,[19(a)] would be characterized by an exponent θ with

$$\theta = \alpha. \quad \text{(C7)}$$

Equation (C7) gives the same value of θ as that expected in the case of a smooth coexistence surface $\mu(T)$.

The generation of new singular terms in (C5) changes the asymmetries across the coexistence surface. To evaluate the asymmetries we first observe that in addition to (2.19), (3.6), and (3.7),

$$Q_3^{\text{liquid}} = Q_3^{\text{gas}}. \quad \text{(C8)}$$

We find for the asymmetries

$$C^{\text{liquid}} + C^{\text{gas}} = \bar{C}_2|t|^{1-\alpha} + \bar{C}_3|t|^{\beta+q}, \quad \text{(C9a)}$$
$$D_s^{\text{liquid}} - D_s^{\text{gas}} = \bar{K}_2|t|^{\beta-1} + \bar{K}_3|t|^{q-\gamma}, \quad \text{(C9b)}$$
$$D_w^{\text{liquid}} - D_w^{\text{gas}} = \bar{D}_1|t|^{-\alpha} + \bar{D}_2|t|^{1-2\alpha-\beta} + \bar{D}_3|t|^{q-\alpha}. \quad \text{(C9c)}$$

The constants in (C9) are given in Table V. We observe that the amplitudes of the asymmetries arising from the correction term cannot be evaluated in terms of the amplitudes of the leading-order singularities. The amplitude $\bar{D}_3$ can be expressed in terms of leading-term amplitudes and $\bar{C}_3$ and $\bar{K}_3$. $\bar{C}_3$ and $\bar{K}_3$, on the other hand, involve derivatives of $\$$ and $\tilde{\$}$ which do not play a part in any of the other amplitudes. Therefore, only the difficult measurement of C_3, K_3, and $\bar{D}_3$ can give a test of this extended revised-scaling approach.

On the critical isochore, Eqs. (C1) and (C4) combine to give

$$\Psi = Q_1't^{2-\alpha} + Q_2't^{3-2\alpha+\beta} + Q_3't^{2-\alpha+q}, \quad \text{(C10a)}$$
$$D_s = K_1't^{-\gamma} + K_2't^{\beta-1} + K_3't^{2-\gamma}, \quad \text{(C10b)}$$
$$D_w = D_0 + D_1't^{-\alpha} + D_2't^{1-2\alpha-\beta} + D_3't^{q-\alpha}. \quad \text{(C10c)}$$

The constants in Eq. (C10) are given in Table VI.

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[9]J. J. Rehr and N. D. Mermin, Phys. Rev. A 8, 472 (1973).

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[11]We use homogeneous in the sense of the vanishing of a constriant. That is, $x = 0$ is a homogeneous constraint on x; $x = F(y)$ is *inhomogeneous*.

[12]By smooth we mean having a sufficient number of continuous derivatives to avoid any difficulties.

[13]Near the critical point a weak singularity might not be noticeable; as we move away from the critical point the spurious ridge of singularities coming from the variable change would become relatively more important. We might accept this unphysical behavior away from the critical region in order to simply describe the critical region.

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[16]This terminology corresponds to the fact that if we take n successive derivatives of (2.2) with respect to x_1, the exponent of λ appearing on the left-hand side of (2.2) (i.e., the scaling power of the function $\partial^n \mathcal{S}/\partial x_1^n$) changes from 1 to $1 - na_1$. Then the critical-point exponent for this function is $(1 - na_1)/a_1$ or $(1 - na_1)/a_2$, if the path is asymptotically tangent to the x_1 or x_2 axis, respectively (see Appendix A). These exponents are algebraically smaller than the corresponding exponents obtained by differentiating $\mathcal{S}$ with respect to x_2, and, hence, we say that the x_1 derivatives diverge more strongly.

[17]P. Schofield, Phys. Rev. Lett. 22, 608 (1969).

[18]J. F. Nicoll and H. E. Stanley (report of work prior to publication). Equation (2.7) is derived by expressing the derivative on the right-hand side by a ratio of Jacobians,

$$\left(\frac{\partial y_1}{\partial y_2}\right)_C = \frac{\partial(y_1, C)/\partial(x_1, x_2)}{\partial(y_2, C)/\partial(x_1, x_2)}.$$

Writing C as $b_1^1 \mathcal{S}_1 + b_1^2 \mathcal{S}_2$ and expanding the resulting Jacobians, we obtain (2.7) directly from the scaling hypothesis (since $a_1 > a_2$).

[19](a) Experimental evidence seems to indicate that $(\partial^2 \mu/\partial T^2)\rho$ is not singular, leading to the suggestion that $\theta = \alpha$. For description of relevant experimental work, see, for example, M. Vicentini-Missoni and J. M. H. Levelt-Sengers, J. Res. Natl. Bur. Stds. A 73, 563 (1969); and J. M. H. Levelt-Sengers and S. C. Greer, Int. J. Heat Mass Transfer 15, 1865 (1972), J. M. H. Levelt-Sengers and W. T. Chen, J. Chem. Phys. 56, 595 (1972), and references contained therein. If the possibility of $\theta = \alpha + \beta$ is excluded, then we expect that there is a hidden symmetry (similar to the $H \to -H$ symmetry in magnetic systems) in the fundamental interactions. In our formalism, this is expressed as a lowest-order symmetry by $A_+ = 0$. (b) We may write (2.19) as $\mathcal{S}^{\text{liquid}}(x_1/x_2^{\beta\delta}, -1) = \mathcal{S}^{\text{gas}}(x_1/x_2^{\beta\delta}, -1)$. If (2.15) holds, the arguments of the functions are constants; if (2.15) does not hold, then the above equation would hold for a range of values of the arguments. This is clearly impossible since, for example, we show in Sec. III that $\mathcal{S}_1^{\text{liquid}} = -\mathcal{S}_1^{\text{gas}}$ on the coexistence surface.

[20]A discussion of some of the weaker singularities is given in Appendix B.

[21]The value for D_1 given in Ref. 9 differs from that given here because we have evaluated the weak divergence along the coexistence surface rather than on the critical isochore ($T < T_c$) as in Ref. 9. The isochore value is $D_1^{\text{iso}} = (1 - \alpha)(2 - \alpha)Q$, independent of A_-.

[22](a) In Appendix C we show that if (2.15) is not used, we can only require that the leading-order behavior correspond to the symmetries given in (3.6) and (3.7). (b) If the amplitudes of the weak divergence do not differ in the liquid and gas phases, then the implied symmetry requires $A_- = 0$, which in turn indicates $\theta < \alpha + \beta$.

[23]See Ref. 8, Table II.

[24]C. Domb, in *Critical Phenomena*, edited by M. S. Green (Academic, London, 1971).

CHAPTER 3

AN AXIOMATIC APPROACH TO GEOMETRICAL ASPECTS OF CRITICAL PHENOMENA IN MULTICOMPONENT SYSTEMS*

*(submitted for publication)

An Axiomatic Approach to Geometric Aspects of Critical Phenomena in Multicomponent Systems*

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Griffiths and Wheeler (GW) have shown the utility of considering the coexistence and critical surfaces of simple and complex systems. In particular, GW have proposed postulates which connect the geometry of the coexistence and critical surfaces to the relative strengths of singularities at the critical surface. The GW postulates combine axioms of an analytic nature with strictly geometrical ideas. GW assume that derivatives taken in various directions have different well-defined strengths and that these directions are associated with the coexistence surface and critical surface in a certain way. In this work, we show that the first assumption (that different directions can be assigned relative strengths) is sufficient to imply the geometrical association postulated by GW. Using the reduced set of axioms, which we call critical ordering, we discuss the application of GW to critical points of higher order. We show that in the absence of critical ordering, competing coexistence and critical surfaces may not merge smoothly at their intersection. Several systems exhibiting a critical point of order four that have this singular geometry are discussed. Application of the analytic axioms is extended to multicomponent fluids and a classification system unifying previous systems for fluids and magnetc materials is introduced. The formulation of scaling hypotheses is treated and a discussion of corrections to scaling is given.

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In 1970, Griffiths and Wheeler[1] (GW) proposed that the qualitative behavior of systems near their critical points could be understood geometrically. They argues that the coexistence and critical surfaces [2] give directions which are singled out thermodynamically as well as geometrically. In this paper we discuss analytic postulates which imply the geometrical considerations of GW.

I. POSTULATES OF GRIFFITHS AND WHEELER

For a single component fluid (cf. Fig. 1a), the direction tangent to the coexistence surface at the critical point (labeled $\hat{x}_2$) is the only direction determined by the geometry. GW point out that motion along the coexistence surface produces gradual changes in the character of the fluid while, on the other hand, motion across the coexistence surfaces produces dramatic changes. For this reason, GW call $\hat{x}_2$ the weak direction; any direction oblique to $\hat{x}_2$, such as $\hat{x}_1$ in Fig. 1a, is called strong.

In a simple ferromagnet (cf. Fig. 1b) the coexistence line and hence $\hat{x}_2$ s precisely along the temperature axis, T, and the strong direction $\hat{x}_1$ can be taken to be parallel to the magnetic field, H. Since T is weak and H is strong, GW predict that the constant-field specific heat,

$$-C_H = \left(\frac{\partial^2 G}{\partial^2 T}\right)_H , \tag{1.1}$$

diverges weakly at the critical point, and that the isothermal susceptability

$$\chi_T = \left(\frac{\partial^2 G}{\partial^2 H}\right)_T , \tag{1.2}$$

diverges strongly. Here G = G (H,T) is the Gibbs potential, H the magnetic field, and T the temperature.

It is found that the divergences of C_H and χ can be represented by power laws,

$$C_H \sim |T-T_c|^{-\alpha} \tag{1.3a}$$

and

$$\chi \sim |T-T_c|^{-\gamma} \tag{1.3b}$$

with $\alpha \sim 0.1$ and $\gamma \sim 1$.

In an anisotropic ferromagnet with interplanar interations J_Z given by RJ_{xy}, changing the lattice anisotropy parameter R strings out the ferromagnetic critical point (for fixed R) into a line of critical points bounding a two-dimensional coexistence surface (cf. Fig. 1c). A direction tangent to the line of critical points ($\hat{x}_3$) is called irrelevant.

In a general system described by n field variables, GW postulate that if an (n-1)- dimensional coexistence surface is bounded by an (n-2)- dimensional surface of ordinary critical points, then directions out of the coexistence surface are <u>strong</u>, directions in the coexistence surface but not in the critical surface are weak; and, finally, that the remaining variables are <u>irrelevant</u>.

It is convenient for our purposes to restate these postulates in terms of variables, instead of directions. In terms of variables, the GW postulates are given as follows. At any point of the critical surface, we can choose variables, $(x_1, x_2, x_3 \ldots x_n)$ such that

$$x_1 = 0 \tag{1.4a}$$

is the tangent surface to the coexistence surface at the critical point in question and such that

$$x_1 = x_2 = 0 \qquad (1.4b)$$

is the surface tangent to the critical surface at the critical point. The remaining variables $(x_3 \ldots x_n)$ parametrize the critical surface near the critical point. Any variables chosen in this way have the further property that derivatives of a thermodynamic potential with respect to x_1 are, in general, strongly singular, derivatives with respect to x_2 are, in general, weakly singular, and, finally, derivatives with respect to $(x_3 \ldots x_n)$ are, in general, finite.

The GW postulates specifically apply to determinants of matrices of second partial derivatives of the thermodynamic potential (e.g., the Gibbs potential)

$$D(A) \quad || \phi_{ij} || \qquad x_i, x_j \in A \qquad (1.5)$$

and we use the notation $\phi_{ij} \equiv \partial^2\phi / \partial x_i \, \partial x_j$.

where A is a set of the variables x_k . The simplest case is that of a set containing a single variable $A = \{x_k\}$. In this case, the determinant in (1.5) is just the susceptibility associated with the variable x_k,

$$D(\{x_k\}) = \phi_{kk} \qquad (1.6)$$

As in (1.3), we expect that $D(\{x_1\})$ diverges strongly and $D(\{x_2\})$ diverges weakly. The susceptibilities associated with $x_3, \ldots x_n$, (that is, the susceptibilities associated with irrelevant variables) are expected to be finite.

If the set A contains two variable, $A = \{x_k, x_l\}$, then D(A) involves the ordinary and "cross" susceptibilities,

$$D(x_k, x_l) = \phi_{kk}\,\phi_{ll} - \phi_{k\bar{e}}\,\phi_{ke} \qquad (1.7)$$

GW propose that the character of this combination be determined by the ordinary susceptibility terms. That is, they suppose that the second term in (1.7) is not more singular than the first term, and that, in general, no cancellations occur that would reduce the divergence of the sum. For example, $D(\{ x_1, x_3\})$ diverges strongly-times-weakly, $D(\{x_1, x_3\})$ diverges strongly, $D(\{x_2, x_3\})$ diverges weakly, and $D(\{x_3, x_4\})$ is finite.

This lack of cancellation is a very important feature of the GW postulates. Each term in the determinant

$$J = \begin{Vmatrix} \frac{\partial^2 G}{\partial P^2} & \frac{\partial^2 G}{\partial P\, \partial T} \\ \frac{\partial G}{\partial P^2 T} & \frac{\partial G}{\partial T^2} \end{Vmatrix} \tag{1.8}$$

is strongly divergent. However, by a unimodular transformation, this determinant is seen to be exactly $D(\{x_1, x_2\})$. Thus, the determinant in (1.8) is strongly-times-weakly divergent, not strongly-times-strongly divergent. Cancellation of the most singular parts of the determinant reduces the strength of the singularity. However, there is no cancellation in the determinants (1.7) which are written in terms of the geometrically distinguished variables.

The properties of D(A) when A has three or more members are determined in the same way. In each case, the strength of the singularity can be read off from the product at the diagonal terms. Thus, the postulates are:

(a) If A contains x_1 but not x_2, D(A) diverges strongly;
(b) If A contains x_2 but not x_1, D(A) diverges weakly;
(c) If A contains both x_1 and x_2, D(A) diverges strongly-times-weakly;
(d) If A contains neither x_1 nor x_2, D(A) is finite.

Recently 3-8, there has been great interest in higher order critical points. Prototypically, in a metamagnet[3], three lines of critical points meet at a special point termed the tricritcal point (cf. Fig. 2a). The directions tangent to the critical lines, which were irrelevant everywhere along the critical line, certainly is not irrelevant at the tricritcal point. We may expect that the critical behavior may be different at the tricritcal point. In Ref. 3, a hierarchy of critical points is defined to incorporate the tricritical and even more complicated critical points. An ordinary critical point is defined to be a point of order two. A d-dimensional surface of such point of order 2 is denoted ${}^{2}R_{d}$. It is expected that in a space of n field variables d=n-2. In a space of three variables, there are <u>lines</u> of ordinary critical points which may intersect at a critical point of order three (a tricritical point). Similarly, in a four dimensional field space, <u>surfaces</u> of ordinary critical points may intersect in <u>lines</u> of tricritical points (points of order three), which in turn intersect at a point or order four. Reference 3 proposes that several ${}^{P}R_{n-p}$ intersect at a ${}^{p+1}R_{n-p-1}$. While such simple systems do not exhause the possible complexities of higher order critical points (it does not, for example, include the possible termination of a critical line in a critical end point rather than a tricritical point; the critical end point is characteristic of multicomponent fluid systems 4-5, cf. Sec. III), they do suggest the appropriate extention of GW to higher order critical points.

We temporarily adopt the geometrical picture of Ref. 3 and further suppose that on an ${}^{P}R_{n-p}$ there are p distinct degrees of divergence which can be classified as follows. We can choose a set of variables at each point of the ${}^{P}R_{n-p}$ such that

$$x_1 = 0 \tag{1.9a}$$

is the surface tangent to the coexistence surface,

$$x_1 = x_2 = 0 \tag{1.9b}$$

is the surface tangent to the surface of ordinary critical points; and, in general

$$x_1 = x_2 = \ldots x_j = 0 \tag{1.9c}$$

is tangent to the surface of critical point of order j (the ${}^{j}R_{n-j}$) for all j, $1 \leq j \leq p$. (for (1.9c) to hold for j=1, we also define the coexistence surface to be $a^{1}R_{n-1}$).

We now suppose that the susceptibilities ϕ_{jj} are divergent for $1 \leq j \leq p$ and that, as the obvious extension of GW.

$$\phi_{11} >> \phi_{22} >> \cdots >> \phi_{pp} \tag{1.10}$$

The susceptibilities corresponding to the remaining n-p variables are assumed to be finite. Just as for the p=2 ordinary critical point case treated by GW, we assume that the singularity of the determinants such as (1.5) can be estimated from the diagonal terms which only involve the ordinary susceptibilities. Thus, the cross susceptibility terms are again supposed to produce no singularities stronger than the product of the diagonal terms and they do not combine to reduce the strength of the singularity given by that product.

Since we now have p different degrees of divergence, we must abandon giving each divergence a name such as strong and weak; we may say instead that x_1 is stronger than x_2 which is stronger than x_3 and so forth. Thus, the set of variables $(x_1 \ldots x_n)$ is partially ordered by the degree of singularity of the susceptibilities corresponding to the irrelevant variables $(x_{p+1} \ldots x_n)$ are all assumed to be finite. However, the ordering is complete among the variables $(x_1 \ldots x_p)$ which we will term relevant variables to distinguish them from the irrelevant variables.

We see that the GW postulates fall into two parts, one analytical and the other geometrical. The analytical assumptions are (i) at any point of a pth order critical surface we may divide the n field variables into two classes. There are p relevant variables whose susceptibilities are infinite at the critical point. The remaining n-p irrelevant variables have finite susceptibilities. (ii) The relevant variables can be ordered by the strength of the divergence of their susceptibilities (1.10). (iii) Determinants which contain both direct and cross susceptibilities have the divergence properties of the products of the direct susceptibilities (cf (1.7)). The geometrical portion of the postulates connect the relevant variables to the geometry of the critical surfaces near the critical point. That is (as in (1.9)) the surface determined by

$$x_1 = x_2 \cdots = x_j = 0 \tag{1.11}$$

is tangent to the surface of order j critical points, the ${}^{j}R_{n-j}$, for all $j \leq p$.

In Section II, we show that the anlytical assumptions (i) - (iii) are sufficient to imply the geometrical association (1.11). We also discuss the situation in which the relevant variables are not completely ordered (several variables have the same "strength"). We point out that the analytical axioms are obeyed by more general systems than the scaling systems discussed in Ref. 3.

In Section III we discuss multicomponent fluid systems for which the geometrical picutre of Ref. 3 fails. We extend our GW formalism to these systems and introduce a classification system for surfaces on which some non-critical phases are in coexistence with other critical phases.

In appendix A, we discuss the application of this work to the formulation or scaling hypotheses; a brief discussion of corrections to scaling is given.

II CRITICAL ORDERING AND SIMPLE CRITICAL SYSTEMS

In part A of this section we show that a system obeying the axioms (i) - (iii) of the previous section automatically have the geometrical properties summerized in (1.11). We call a system that obeys the analytical portion of the GW postulates "critically ordered", where the ordering is, of course, that of the susceptibilities (1.10). In part B, we discuss various scaling and non-scaling critically ordered systems and their relation to the renormalization group. In part C, we discuss systems with "partial" critical ordering . The geometry of such systems is more complicated and is illustrated by two fourth order" examples.

A. Geometrical Properties of Critically Ordered Systems

We will use two sets of variables in this discussion. The first $(y_1 \ldots y_n)$ are field like variables which are not associated with the geometry of any of the critical surfaces. In asymmetric systems, they may be chosen to be the usual thermodynamic fields. For example, in a single component fluid, we could choose the pressure and temperature. In symmetric systems such as a simple Ising ferromagnet, the usual variables of temperature and magnets field are associated with the geometry; the coexistence surface is contained in the line H=0. However, we can imaging choosing slightly skewed fields. This restriction is just a technical one to allow certain simplifications of the discussions to follow (cf 2.3).

The second set of variables is the ordered set $(x_1 \ldots x_p; x_{p+1} \ldots x_n)$. We assume that ϕ_{kk} is divergent for k=p, and finite otherwise. We assume that determinants of second partial derivatives of the potential ϕ of the form (1.5) have the same singular behavior as the product of the diagonal terms and that

$$\phi_{11} >> \phi_{22} >> \phi \ldots >> \phi_{pp} \qquad (2.1)$$

The assumption of complete ordering of $(x_1 \ldots x_p)$ will be relaxed in part C of this section.

The basic geometrical connection is provided by the following relationship between the densities $\phi_k \equiv \partial\phi/\partial x_k$ and the variables $(x_1 \ldots x_p)$ at the critical point.

$$\left(\frac{\partial y_\kappa}{\partial y_\ell}\right)_{\phi_1 \ldots \phi_j \{y\}} = \left(\frac{\partial y_\kappa}{\partial y_\ell}\right)_{x_1 \ldots x_j \{y\}} \tag{2.2}$$

for all $j \leq p$, and all k, ℓ and where $\{y\}$ denotes any set of $n-j$ y_i's. The proof is as follows.

The left hand side of (2.2) can be written as the quotient of two Jacobians[8]

$$\left(\frac{\partial y_k}{\partial y_\ell}\right)_{\phi_1 \ldots \phi_j \{y\}} = \frac{\partial(\phi_1 \cdots \phi_j \{y\}\, y_\kappa)}{\partial(x_1 \ldots x_j; x_{j+1} \ldots x_n)} \Bigg/ \frac{\partial(\phi_1 \ldots \phi_j \{y\}\, y_\ell)}{\partial(x_1 \ldots x_j; x_{j+1} \ldots x_n)} \tag{2.3}$$

Each Jacobian on the right hand side (2.3) can be written as a sum of determinants involving the susceptibilities multiplied by factors involving the derivatives of the $(y_1 \ldots y_n)$ with respect to the $(x_1 \ldots x_n)$. These coordinate derivatives may be assumed to be non-vanishing since we have not specified the $(y_1 \ldots y_n)$. The largest of the susceptibility determinants is $\partial(\phi_1 \ldots \phi_j)/\partial(x_1 \ldots x_j)$. This is of the form (1.5) and therefore, this leading divergence does

not have singularity reducing cancellations (we do not care if there are cancellations in the other determinants of susceptibilities).

Therefore, as $\phi_1, \ldots, \phi_j$ take on their critical values, we may write the right hand side of (2.3) as

$$\frac{\partial(\phi_1 \ \ \phi_j)}{\partial(x_1 \ \ x_j)} \ \frac{\partial(\{y\} \ x_k)}{\partial(x_{j+1} \ x_n)} \Big/ \frac{\partial(\phi_1 \ \ \phi_j)}{\partial(x_1 \ \ \phi_j)} \ \frac{\partial(\{y\} \ y_\ell)}{\partial(x_{j+1} \ \ x_n)}$$

The leading divergent term is identical in both numberator and denominator. Cancelling this term we have

$$\left(\frac{\partial y_k}{\partial y_\ell}\right)_{\phi_1 \ \ \phi_j \{y\}} \longrightarrow \left(\frac{\partial y_k}{\partial y_\ell}\right)_{x_1 \ \ x_j \{y\}} \tag{2.4}$$

as the critical point is approached. Equation (2.2) is proven for the case p=2 (ordinary critical point) in Ref. 1.

It may be instructive to work out the details of this proof for a particular case. Consider the Jacobian

$$J = \frac{\partial(\phi_1, \phi_2, y)}{\partial(x_1, x_2, x_3)} \tag{2.5a}$$

It may be expanded to give

$$J = \frac{\partial y}{\partial x_3} \begin{Vmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & \phi_{22} \end{Vmatrix} - \left(\frac{\partial y}{\partial x_2}\right) \begin{Vmatrix} \phi_{11} & \phi_{12} \\ \phi_{31} & \phi_{32} \end{Vmatrix}$$

$$- \left(\frac{\partial y}{\partial x_1}\right) \begin{Vmatrix} \phi_{21} & \phi_{22} \\ \phi_{31} & \phi_{32} \end{Vmatrix} \tag{2.5b}$$

By considering various determinants of the form (1.5) we can estimate the divergence of the cross-susceptibilities

$$|\phi_{12}| \lesssim (\phi_{11}\phi_{22})^{1/2}, \tag{2.6a}$$

$$|\phi_{13}| \lesssim (\phi_{11}\phi_{33})^{1/2}, \tag{2.6b}$$

$$|\phi_{23}| \lesssim (\phi_{22}\phi_{33})^{1/2}. \tag{2.6c}$$

Thus the second determinant in (2.5b) has at most the divergence of $\phi_{11}(\phi_{22}\phi_{33})^{\frac{1}{2}}$ Since $\phi_{22} >> \phi_{33}$, this is much smaller than the first determinant which is characterized by the divergence of $\phi_{11}\phi_{22}$. Similarly, the third determinant has at most the divergence of $\phi_{22}(\phi_{11}\phi_{33})^{\frac{1}{2}}$ and therefore may be neglected in comparison with the first determinant.

It is easy to see that we may replace the ϕ_k in (2.2) by any functions f_k where

$$f_k = \sum_{i=0}^{n-k} a_i \phi_{k+i} \tag{2.7}$$

where the a_i are any smooth functions of $(x_1 \ldots x_n)$, $a_0 \neq 0$. For example, if a and b are constants,

$$\frac{\partial(\phi_1 + a\phi_2 + b\phi_3, \phi_2, y)}{\partial(x_1, x_2, x_3)} = \frac{\partial(\phi_1, \phi_2, y)}{\partial(x_1, x_2, x_3)} + b\,\frac{\partial(\phi_3, \phi_2, y)}{\partial(x_1, x_2, x_3)}.$$

The largest possible divergence of the second Jacobian is negligible in comparison with the most divergent term of the first Jacobian. Furthermore, the Jacobians, (and, hence, (2.2)) are invariant under

any unimodular transformation of the $(f_1 \ldots f_j)$.

To apply (2.2), consider the single component fluid. For j=1, Eq. (2.2) implies that

$$\left(\frac{\partial P}{\partial T}\right)_V \longrightarrow \left(\frac{\partial P}{\partial T}\right)_{x_1}$$

$$\left(\frac{\partial P}{\partial T}\right)_S \longrightarrow \left(\frac{\partial P}{\partial T}\right)_{x_1} \tag{2.9}$$

as the critical point is approached in the single phase region. Thus, whether we choose to regard the volume or entropy as the order parameter, (2.9) shows that the coexistence surface (which is presumed to have the small tangent surface at the critical point as the iso-order surface) is tangent to the line $x_1 = 0$.

The same argument shows that (2.2) implies the tangency of the surface $x_1=0$ to the coexistence surface in the general case of n fields. This identification corresponds to the physical notion of GW that the strongest fluctuations should be exhibited by the order parameter.

The remaining connections between (2.2) and (1.11) may be proved by induction on j. The proof depends upon a weak version of the smoothness hypothesis[9] which is implicit in any discussion of surfaces of higher order critical points. For our classification by order to be useful, the critical properties of the critical points on a particular surface of order j critical points should not vary drastically from point to point. For our classification by order to be useful, the critical properties of the critical points on a particular surface of order j critical points should not vary drastically from point to point. We need not assume that, for example, critical point exponents are the same everywhere on the surface; a gradual variation of exponents is tolerable. We need only exclude variations in critical behavior which are so extreme as to isolate a critical point from its neighbors. This criterion applies rigorously only in the interior of the surface. The boundary of the

surface is isolated geometrically and we cannot impose our smoothness requirement there. This is, of course, to be expected; it is precisely at the boundaries of surfaces of order j that Ref. 3 suggests the existence of order j+1 critical points. However, we do not expect the ordering on the ${}^{j+1}R_{n-j-1}$ to be incompatible with that of the boundary region of the ${}^{j}R_{n-j}$. If a region of mutual validity of the two orderings exists, no difficulty is encountered.

Consider the case j=2. From (2.2) we have that the surface $x_1=0$ is tangent to the coexistence surface and to the ϕ_1= constant surface. We have also that the $x_1 = x_2 = 0$ surface is tangent to the ϕ_1 = constant, ϕ_2= constant surface. We wish to show that this latter surface must be the surface of ordinary (j=2) critical points. Suppose that it is not. Then, if we take a derivative of ϕ_2 in any direction contained within the critical surface, the resulting susceptibility will contain a ϕ_{22} term which cannot be cancelled by any other term (we are already in the $x_1 = 0$ surface) Thus, the density ϕ_2 varys in the surface of j=2 critical points strongly in such a way as to isolate the critical point in question. This contradicts our smoothness requirement. Therefore, the surface $x_1 = x_2 =0$ must be tangent to the surface of critical points of order 2.

The proof for general j proceeds in the same way to show that the surface $x_1=x_2=\ldots=x_j=0$ is tangent to the surface of order j critical points. We may write this symbolically as

$$x_1=\ldots=x_j=0 \sim {}^{j}R_{n-j} \tag{2.10}$$

where $\sim$ denotes tangency between the indicated surfaces.

B. Examples of Critically Ordered Systems

The most usual and useful critically ordered systems are scaling systems. Ref. 3 proposes that on each ${}^{j}R_{n-1}$ the"singular part" of the Gibbs potential is a generalized homogeneous function [10](GHF) of j of the n fields

$$G\left(\lambda^{a_1}x_1,\ldots,\lambda^{a_j}x_j;x_{j+1},\ldots x_n\right) = \lambda G(x_1\ldots x_n) \qquad (2.11)$$

If for each individual scaling hypothesis we have $a_1 > a_2 \ldots a_j > 0$, the scaling equation (2.11) represents a critically ordered system. For such systems, the analytical portions of the GW postulates can be checked in more detail. For example, in E9 (1.5), each term in the determinant expansion is of the same degree of singularity. Thus in the determinant $D(\{x_1, x_2\}) = G_{11}G_{22} - G_{12}G_{12}$. the cross - susceptibility term $G_{12}G_{12}$ can never be larger than the product of the direct susceptibilities $G_{11}G_{22}$. The noncancellation of the two terms is not, however, guaranteed by the scaling hypotheses (2.11)

Of course, we may add to the Gibbs potential any regular or weakly singular terms which do not disturb the leading order scaling behavior; the system will still be critically ordered. These "corrections to scaling" are or interest in single-component fluid systems which lack the simplifying symmetries of magnetic systems and yet are sufficiently simple that verification of theoretical predictions may be possible.

The scaling hypotheses (plus corrections) has the theoretical support oif the renormalization group[14] which also gives a method for the calculation of the scaling powers ($a_1 \ldots a_j$) by linearization of the renormalization group equations 6-7.[14] However, even more support for the phenomenological scaling theories can be obtained from nonlinear renormalization group equations.[15-18] In Ref. 3, it is assumed that near the boundary of an ${}^{j}R_{n-j}$ (the boundary is itself an ${}^{j+1}R_{n-j-1}$) there is a region in which both the order j scaling hypothesis and the order j+1 scaling hypotheses have validity. This assumption leads to the prediction of various "double power" scaling laws and relationships.[19] In a region in which three separate scaling hypotheses can be made (e.g. near the intersection of tricritical lines at a point of order four) " triple-power" laws are predicted. The nonlinear renormalization group solutions of Ref. 18 have precisely these properties. The competiion between various fixed points 14 expressed in these nonlinear solutions bears a striking resemblance to the phenomenological description of Ref. 3. In the "Hamiltonian space" surface of ordinary critical points is bounded by "tricritical lines" which in turn intersect at a point of "order four". Double-power scaling behaviour is found near the tricritical lines and triple-power scaling near the point of order four. The geometry of these renormalization group systems will be considered in more detail in Part C of this section.

There are also critically ordered systems which do not scale. Consider for example, a Gibbs potential given by the sum of two GHFs $G=G^a(x_1, x_2) + G^b(x_1, x_2)$, with scaling powers (a_1^a, a_2^a) and (a_1^b, a_2^b) respectively. Since G^a and G^b are both GHFs, each generates a set of "critical point exponents" which satisfy

$$\alpha^a + 2\beta^a + \gamma^a = 2 \qquad \alpha^b + 2\beta^b + \gamma^b = 2$$

2-12

We can choose the scaling powers such that $\beta^a = \beta^b$ and $\alpha^a > \alpha^b$ (and therefore, $\gamma^b > \gamma^a$); for example, we might have $a_1^a = 13/15$, $a_2^a = 8/15$, $a_1^b = 6/7$ and $a_2^b = 4/7$. The measured exponents would be α^a, β^b, and γ^b. In this case, only some critical point exponent inequalitites will be satisfied as equalities; e.g. $\alpha^a + 2\beta^a + \gamma^b > 2$. Thermodynamic potentials of this form arise naturally as singular solutions of nonlinear renormalization group equations[18]. Calculations of the free energy by a method of Wilson's [20] seem to indicate that these singular solutions will not represent the true free energy in most systems [18-21].

C. Partial Critical Ordering and Singular Geometries

We have assumed that there is a complete ordering of the divergent susceptibilities, $\phi_{11} >> \phi_{22} >> \cdots >> \phi_{jj}$, on each $^{j}R_{n-1}$. This guarantees the smooth merging of intersecting critical surfaces. When several $^{p}R_{n-p}$ intersect to form a $^{p+1}R_{n-p-1}$, the complete ordering on the $^{p+1}R_{n-p-1}$ gives unique tangent surface to all the $^{j}R_{n-p-1}$, the complete ordering on the $^{p+1}R_{n-p-1}$ gives unique tangent surface to all the $^{j}R_{n-j}$. This merging of coexistence and critical surfaces was described in terms of scaling systems in Ref. 22.

However, there are many systems which do not have this complete ordering. In the scaling case, complete ordering corresponding to distinct scaling powers. If some of the scaling powers are equal, then the ordering is only partial. In general we may have only that

$$(\phi_{11},\cdots,\phi_{j_1 j_1}) >> (\phi_{j_1+1,j_1+1}\cdots\phi_{j_2 j_2}) >> \cdots >> (\phi_{j_{r-1}+1,j_{r-1}+1}\cdots\phi_{j_r j_r}),$$

corresponding to the following relationship among the scaling powers

$$a_1 = \cdots = a_j > a_{j_1+1} = \cdots = a_{j_2} > \cdots > a_{j_{r-1}} = \cdots a_{j_r} \tag{2.14}$$

In this case, the fundamental geometrical relation (211) holds only for $j=j_1, j_2,\cdots j_r$. Thus, we cannot show the geometrical relationships of GW for all j.; previously, the obtained results apply only to $j=j_t$, $1 \leq t \leq r$.

In particular, the smooth mergining of intersecting coexistence and critical surface cannot be guaranteed. In Fig. 2b, we show schematically what is expected when $(\phi_{11},\phi_{22}) \gg \phi_{33}$, but the relative sizes of ϕ_{11} and ϕ_{22} cannot be distinguished. Two coexistence surface intersect at a finite angle but the critical lines bordering those coexistence surfaces merge smoothly. This sort of behaviour at a tricritical point was also discussed in Ref. 22 (corresponding to $a_1 = a_2 > a_3$).

The first example of such a system that we will discuss has even more singular geometry. Not only are the two strongest variables of the same strength there is also no unique tangent surface to the surface of ordinary (order two) critical points at the critical point of order four. Thus the left hand side of (2.1) is not well defined. The system is the meta magnetic system of Harbus et al[3]. The Hamiltonian is given by

$$\mathcal{H} = -[\sum_{\langle ij \rangle}^{xy} s_i s_j + R \sum_{\langle ij \rangle}^{z} s_i s_j] - H \sum_i s_i - H' \sum_i (-1)^{\eta} s_i \quad . \tag{2.15}$$

The first sum is over nearest neighbor pairs in each xy-plane; the second sum is over nearest neighbor pairs coupled in the z-direction; H is a direct field, and H' is a staggered field which changes sign on alternate xy planes (the parameter η is zero on even-numbered planes and $\eta = 1$ on odd-numbered planes). For $R > 0$, the system is a metamagnet, for $R < 0$, an anisotropic ferromagnet, and for $R = 0$, the system reduces to a set of uncoupled two-dimensional Ising planes. The Hamiltonian is invariant under the transformation

$$R \rightarrow -R$$
$$H \rightarrow H'$$
$$H' \rightarrow H$$
$$s_i \rightarrow (-1)^{i} s_i \qquad (2.16)$$

Therefore the Gibbs potential obeys the symmetry relation

$$G(H, H', R, T) = G(H', H, -R, T). \qquad (2.17)$$

For various values of R ($R < 0$), the phase diagrams in the H-T plane are shown in Fig. 3. The coexistence surface is bounded by a line of critical points, which, in turn, terminates at tricritical points (labeled TCP).

T_N is the Neel temperature at each particular value of R. In the three dimensional space H-T-R, Fig. 4 shows a coexistence volume capped by a surface of critical points or order two, which is, in turn, bounded by two tricritical lines. These tricritical lines interesect at R=0 at a point of order four. The symmetry exhibited in (2.17), combines with exact scaling results for the two-dimensional Ising model, gives rigorously the scaling powers at the point of order four:

$$a_1 = a_H = 15/16$$
$$a_2 = a_{H'} = 15/16$$
$$a_3 = a_R = 7/8 \qquad (2.18)$$
$$a_4 = a_T = 1/2$$

Two of the scaling powers are equal; the system has only partial critical ordering at the point of order four. For R 0 the full coexistence surface lies in the H=0 space; on the other hand, for R 0, the coexistence surface is entirely in the H'=0 space.

At the fourth order point, the two strongest variables are H and H' We would expect the critical surface, therefore, to be tangent to the R-T place at the fourth order point. However, the critical surface lacks a unique tangent plane at that point. For example, a place placed tangent to the critical surface at the top of the coexistence volume (at the Neel temperature) has as its limit plane the H-T plane if R 0; while a similarly placed plane for R 0 has as its limit the H' - T plane. On the other hand, if we approach the fourth order point along any of the lines of tricritical points, the tangent planes do have as their limit the R - T plane [23].

It is suggestive that for this example the "proper" limiting behaviour at the fourth order point is achieved along the line of third order points. However, we have not been able to establish conditions for this to hold generally.

As a second example taken from nonlinear renormalization group calculations,[18] we consider two internal isotropically interacting n-component spin systems, $\vec{s}_1$ and $\vec{s}_2$ which interact through a biquadratic term. The Hamiltonian density is given by

$$\mathcal{H} = |\nabla \vec{S}_1|^2 + |\nabla \vec{S}_2|^2 + r[S_1^2 + S_2^2] + Y_n[(S_1^2)^2 + (S_2^2)^2] + Y_{2n}[S_1^2 + S_2^2]^2 + \vec{h}_0[\vec{S}_1 + \vec{S}_2] \tag{2.18}$$

The properties of this system can be studied via a renormalization group perturbation expansion in the parameter $\epsilon = 4 - d$, where d is the lattice dimension. The properties of the solution depend on the number of spin components n. For $2 \leq n \leq 4$, the Hamiltonian space may be depicted as in Fig. 5. The variable x is related to r and is proportional to $T-T_c$, we have set the ordering field h=0.

The shaded portion of the x=0 plane is a surface of order two critical points. The scaling power of x is given by $da_x = 2 - \epsilon 3n/(n^2+8)$. Near the "tricritical line" $y_{2n}=0$, y_{2n} is also a scaling variable with scaling power $da_{y_{2n}} = \epsilon(4-n)/(n^2+8)$. The tricritical scaling power of x is $da_x = 2 - \epsilon(n+2)/(n+8)$, this is the usual n-component Wilson-Fisher fixed point value .[24] Near the second "tricritical line" $y_n=0$, the variable y_n scales with scaling power $da_{y_n} = \epsilon(n-2)/(n+4)$. The tricritcal scaling power of x is in this case, given by $da_x = 2 - \epsilon(n+1)/(n+4)$; this is the Wilson-Fisher value for a 2n-component system. Finally, at the point of order four, $y_n = y_{2n} = 0$, both y_n and y_{2n} are scaling variables with the same scaling power $da_y = \epsilon$. The fourth order scaling power for x is $da_x = 2$; the Gaussian or mean-field value. Since the scaling powers for y_n and y_{2n} are identical, we are not surprised to find that the tricritical lines intersect at a finite angle.

III MULTICOMPONENT FLUID SYSTEMS

In systems of high symmetry (prototypically, magnetic systems) the simple picture of Ref. 3 applies. In other words, each ${}^{j}R_{n-j}$ is bounded by a ${}^{j+1}R_{n-j-1}$. This situation need not always hold, and it is possible that a critical surface may terminate at a surface of critical end points rather than at a critical surface of higher order.[4] In Fig. 6a, a phase diagram for a multicomponent fluid system is shown. A coexistence surface between two phases labeled A and B intersects a second coexistence surface separating A and B from a third phase C. The critical line bounding the AB coexistence surface terminates at the point P, a critical end point. By changing a fourth field, under certain conditions the phase diagram may be brought to that of Fig. 6b. The three coexistence surfaced, bounded by three critical lines, meet at the tricritical point P_t. A different value of the fourth field produces the phase diagram shown in Fig. 6v, where the line of B-C criticality ends at the critical end point P'.

It is possible that under cerain conditions all the critical lines form a single connected surface in the four-dimensional field space.[5] This critical surface is bounded by two lines of critical end points (swept out by P and P') which intersect at the tricritical point, P_t. This critical surface is drawn schematically in Fig. 7. The point labeled P_o is a point on the critical surface but not on either of the two critical end lines. Universality would argue that at every point such as P_o, the critical phenomena are essentially the same; in particular, the critical-point exponents are the same all over the critical surface. The points on the lines of critical end points are certainly distinguished both thermodynamically and geometrically. They bound the critical surface, and

are points where three phases are in equilibrium (two of the phases being critical). However, the presence of the third coexisting phase may not disturb the critical phenomena connected with the two critical phases. We will assume that on the lines of critical end points there are still only relevant variables.

At the tricritical point, however, three phases become critical simultaneously. The considerations of a Landau model[25] lead us to expect that at the tricritical point all four of the variables scale. We will assume that this is the case, although experimental verification is lacking. In order to describe this more general situation, we define a new notation for critical coexistence surfaces. A surface of dimension d, on which Θ phases are simultaneously critical, while in coexistence with x additional phases are simultaneously critical, while in coexistence with x additional phases, and on which s of the variables are relevant variables, will be denoted by ${}^{\Theta}_{s}R^{x}_{d}$. We will call the order of the critical surface in harmony with Griffiths. The critical surfaces of Ref. 3 are, in this notation, ${}^{s}_{s}R^{0}_{d}$, the number of scaling variables being equal to the order of the critical surface. With this notation, the surface of ordinary critical points in a ternary mixture is a ${}^{2}_{2}R^{0}_{2}$. The line of critical end points bounding the critical surface is a ${}^{2}_{2}R^{1}_{1}$; the tricritical point is a ${}^{3}_{4}R^{0}_{0}$. Note that for these cases s+x+d=n, where n is the number of field or field-like variables used to describe the system.

Multicomponent fluid systems lack the extreme symmetry of magnetic systems. The failure of the line of critical end points to be a tricritical line (as it was in the magnetic case) exhibits this lack of symmetry[26]. If we write the constraint that one of the critical phases is in coexistence with the third phase as $F(\{x\},\beta)=0$, then

We can retain our identification of variables with geometry on the critical end line, by noting that since x_3 and x_4 are irrelevant along the critical end line, we can choose them in any way we wish. If we choose x_4 (locally) to measure distance along the end line and x_3 to measure distance from the end line at the critical point in question, then by definition

$$\left(\frac{\partial F}{\partial x_4} \right) = 0 \tag{3.1}$$

at each point along the critical end line. Therefore, at each point of the critical end line,

$$\left(\frac{\partial y_k}{\partial y_\ell} \right)_{\phi_1 \phi_2 F} = \left(\frac{\partial y_k}{\partial y_\ell} \right)_{x_1 x_2 x_3} \tag{3.2}$$

The variables (x_1, x_2) are pulled out because they are critical; x_3 is pulled out by Eq. (3.1).

At the tricritical point (if it is a ${}^{3}_{4}R^{0}_{0}$), Eq. (3.2) may hold by critical ordering. For example, if we made a scaling hypothesis at the tricritical point,

$$\lambda G(x_1, x_2, x_3, x_4) = G(\lambda^{a_1} x_1, \lambda^{a_2} x_2, \lambda^{a_3} x_3, \lambda^{a_4} x_4) \tag{3.3}$$

we might expect that the line of critical end points is described by a scaling invariant constraint function; that is, the critical end line is described by F=0, where

$$F = F\left(\frac{x_1^{a_4}}{x_4^{a_1}}, \frac{x_2^{a_4}}{x_4^{a_2}}, \frac{x_3^{a_4}}{x_4^{a_3}} \right) . \tag{3.4}$$

If F is a scaling invariant constraint, then the properties of GHFs imply that

$$\frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial x_4}} \rightarrow \infty , \tag{3.5}$$

on scaling invariant paths, i= 1,2,3, where

$$x_i = A_i x_4^{a_i / a_4} \tag{3.6}$$

In the more general situation of critical ordering, we will assume that Eq. (3.5) holds as the tricritical point is approached. In this case, Eq. (3.2) holds by critical ordering at the tricritical point. We will call F a quasi-critical constraint, since it constrains the geometry in a non-critical way except at the higher order point.

If we raised the dimension of the system by one, the tricritical point is strung out into a line of third order points. This line could terminate at a critical end point and the process given above repeated. In general, the relationship between successive surfaces might be given by

$${}^{2}_{2}R^{0}_{n-2} \rightarrow {}^{2}_{2}R^{1}_{n-3} \rightarrow {}^{3}_{4}R^{0}_{n-4} \rightarrow {}^{3}_{4}R^{1}_{n-5} \rightarrow \cdots {}^{m}_{n}R^{0}_{0} , \tag{3.7}$$

where n=2m-2. At each step in which a coexisting phase is added to the previously critical phases, a quasi-critical constraint is added. Systems described by Eq. (3.7) have been considered in Refs. 4. For these systems we can replace (2.10) by

$$(x_1 = \ldots x_{2m} = 0) \sim {}^{m+1}_{2m}R^{0}_{n-2m} \tag{2.8a}$$

$$(x_1 = \ldots = x_{2m+1} = 0) \sim {}^{m+1}_{2m}R^{1}_{n-2m-1}, \tag{3.8b}$$

where in Eq. (3.8b) we have defined the coexistence surface as a surface of order $\theta'=1$ and written as ${}^{1}_{0}R^{1}_{n-1}$ so that the convention s+x+d =n is satisfied

The magnetic systems of Ref. 3 are described by (2.10) and the succession surfaces.

$${}^{1}_{0}R^{1}_{n-1} \rightarrow {}^{2}_{2}R^{0}_{n-2} \rightarrow {}^{3}_{3}R^{0}_{n-3} \rightarrow \cdots \rightarrow {}^{n}_{n}R^{0}_{0}. \tag{3.9}$$

Eq. (3.8) and Eq. (3.9) are governed by a set of rules which can be generalized to cover different situations such as

$${}^{\theta'}_{s}R^{x}_{d} \rightarrow {}^{\theta'+\theta}_{s+\theta'+1}R^{x-\theta'}_{d-1} \tag{3.10a}$$

$${}^{\theta'}_{s}R^{x}_{d} \rightarrow {}^{\theta+\theta'}_{s+\theta'}R^{x-\theta'+1}_{d-1} \tag{3.10b}$$

The magnetic systems, after one application of (3.10a), use (3.10b) exclusively; on the other hand, the multicomponent fluids use (3.10a) and (3.10b) alternately (with $\theta'=1$ and $\theta' = 0$, successively). The rules are more generally applicable. For example, the three-phase line of a metamagnet terminates at the tricritical point,

$$ {}^{1}_{0}R^{2}_{1} \rightarrow {}^{3}_{3}R^{0}_{0} \qquad (3.11) $$

Eq. (3.11) employes (3.10a) with $\mathcal{O}' = 2$.

To explore the uses of Eqs. (3.10), consider a multicomponent system with at least four phases. We consider the appropriate generalization of Fig. 6a but sliced in such a way as to suppress the coexistence surface between phases A and B. In Fig. 8 we illustrate a three-dimensional slice of this hyperdimensional phase diagram. The line L_0 is a ${}^{1}_{0}R^{2}_{n-2}$, where three phases (A,B,C) are in coexistence. The point P is a ${}^{2}_{2}R^{1}_{n-3}$ where A and B become critical while remaining in coexistence with C. The line L_1 is a ${}^{2}_{2}R^{0}_{n-2}$, where C is critical with one other phase. L_1 terminates on a coexistence surface with the phase labeled D, which is bounded by L_2, where D is critical with the phase previously in coexistence with it. As the various undisplayed fields are changed, the point P moves on the C coexistence surface and the lines L_0, L_1, and L_2 move relative to one another. From this diagram we see that the ${}^{2}_{2}R^{1}_{n-3}$ can be bounded in the following ways:

(a) The point P hits the line L_1 (Fig. 9a). This is the usual multicomponent fluid boundary: ${}^{2}_{2}R^{1}_{n-3} \rightarrow {}^{3}_{4}R^{0}_{n-4}$

(b) The point P hits the D coexistence plane but not L_1 (Fig. 9b):

$$ {}^{2}_{2}R^{1}_{n-3} \rightarrow {}^{2}_{a}R^{2}_{n-4} $$

(c) The point P hits the line L_1 at the termination of L_1 (Fig. 9c):

$$ {}^{2}_{2}R^{1}_{n-3} \rightarrow {}^{3}_{3}R^{1}_{n-4} $$

(d) The point P hits the line L_1 at the D coexistence point precisely when that point hits the D critical line: ${}^{2}_{2}R^{1}_{n-3} \rightarrow {}^{4}_{4}R^{0}_{n-4}$ (Fig. 9d)

In case (a) when point P moves down the line L_1 to the D coexistence plance, there are two possibilities. If phase D is not critical, then ${}^{3}_{4}R^{0}_{n-4} \rightarrow {}^{3}_{4}R^{1}_{n-5}$; while if D is critical, ${}^{3}_{4}R^{0}_{n-4} \rightarrow {}^{4}_{5}R^{0}_{n-5}$.

Exhausting all the possibilities inherent in Figs. 7-8, we can make a table of successive boundaries (cf. Fig. 10). We have used the equation

$$s+x+d=n \tag{3.12}$$

to predict the number of scaling variables or critical variables on each surface. The ${}^{2}_{2}R^{1}_{n-3}$ might be bounded on one side by a ${}^{2}_{2}R^{2}_{n-4}$ and by a ${}^{3}_{4}R^{0}_{n-4}$ on another which intersect to form a ${}^{3}_{4}R^{1}_{n-5}$. The other possibilities given in Fig. 10 all exhibit greater symmetry and reach surfaces of order four more quickly.

V. CONCLUSIONS

We have shown that the analytic-geometrical postualtes of Griffiths and Wheeler[1] can be reduced to analytic statements, from which the geometrical associations follow. Our assumption of critical ordering is weaker than that of scaling, and yet the major results of scaling hypotheses, such as the smooth merging or intersecting of competing coexistence and critical surfaces, depend only on critical ordering.

The system of classification of coexistence-critical surfaces introduced here unifies multicomponent fluid systems with the highly symmetrical magnetic systems. We may think of Eq. (3.12) as a phase rule which encompasses both the fluid systems of Ref. 4 (cf (3.7)) and the magnetic systems of Ref. 3 (cf.(3.9)). The rules given in (3.10) might also govern systems of mixed symmetry; that is, systems more symmetric than fluids, but lacking the full symmetry of magnetic systems (cf. Fig. 6). The phase rule given in (3.12) and the boundary rules (3.10) are partly based on a Landau model, which may not be correct in every detail. However, the rules apply as stated to those systems for which we have a global understanding of the geometry of the critical surfaces. Eq. (3.10) are the only rules that preserve the phase rule $s+x+d=n$ and that also have the property that the number of relevant variables (i.e. those have divergent susceptibilities) always increases at least as fast as the order of the critical surface. The latter property precludes the possibility that the order could ever be greater than the number of relevant variables. This is a plausible condition on the basis of the systems presently known.

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APPENDIX A. FORMULATION OF SCALING HYPOTHESIS

The present expression of the Griffiths-Wheeler ideas in terms of variables is especially suited for the discussion of the formulation of scaling hypotheses at the various critical points, since, in the final accounting, functions are written in terms of variables. Eq.(2.10) can be regarded as a set of equations for the relevant variables $(x_1,\ldots x_p)$. Since the relationship is one of tangency at the critical point under consideration, the relevant variables can only be determined to linear terms in the fields $(y_1,\ldots y_n)$. Furthermore, if $(x_1,\ldots x_p)$ is a solution of Eq. (2.12), then $(X_1,\ldots X_p)$ is also a solution, where $(X_1,\ldots X_p)$ is given by

$$X_i = \sum_j A_{ij} x_j \quad . \tag{A 1}$$

n Eq. (A.1), A_{ij} is an arbitrary lower triangular matrix, with non-zerio diagonal entries[2]. A trivial change of variables reduces A to a matrix with unit diagonal entries. By the construction of (2.12), the transformation of (A 1) cannot affect the leading order behavior of any thermodynamic quantity. However, the non-leading terms are affected. To illustrate this effect, consider a scaling hypothesis stated in terms of GHF in p critical variables,

$$\lambda\, G(x_1\ldots x_p\,;x_{p+1}\ldots x_n) = G(\lambda^{a_1} x_1,\ldots \lambda^{a_p} x_p;x_{p+1}\ldots x_n) \quad . \tag{A 2}$$

We define the induced action of A on the function G by

$$A\colon G((x_1\ldots x_p)\ldots) = G(A^{-1}(x_1\ldots x_p),\ldots) \quad . \tag{A 3}$$

Writing $A^{-1} = I + B$ and expanding the right hand side of (A.3) in a Taylor series, we have

$$A\colon G = G(x_1\ldots x_p,\ldots) + \sum_{ij} B_{ij} x_j \frac{\partial}{\partial x_i} G(x_1\ldots x_p) + \ldots \quad . \tag{A 4}$$

We define C_{ij} by

$$C_{ij} = B_{ij} x_j \frac{\partial}{\partial x_i} \quad G \tag{A.5}$$

Then the properties of GHFs imply that

$$\lambda^{1+a_j-a_i} C_{ij} = C_{ij}(\lambda^{a_1} x_1 \cdots \lambda^{a_p} x_p) \tag{A.6}$$

Since B is lower diagonal, $a_j - a_i > 0$, and the C_{ij} are all weaker than G (that is, the singularities of the C_{ij} are all weaker than the corresponding singularities of G). If we only wish to examine leading behavior in a critical system, any solution of (2.10) can be used in a scaling hypothesis.

In cases of special symmetry, the solutions of (2.10) are more restricted. For example, in the case of a simple ferromagnet, Eq. (2.10) has the solutions

$$\begin{aligned} x_1 &= H \\ x_2 &= T - T_c + cH, \end{aligned} \tag{A.7}$$

where c is any constant. If a scaling hypothesis were made in terms of these variables, the magnetization, M, would be given by

$$-M = \left(\frac{\partial G}{\partial x_1}\right) + \left(\frac{\partial G}{\partial x_2}\right) \tag{A.8}$$

On the coexistence surface

$$-M = \pm \left| T - T_c \right|^{\beta} \frac{\partial G_1}{\partial x_1} (0,1) + c(2-\alpha) \left| T - T_c \right|^{1-\alpha} G(0,1) \tag{A.9}$$

The term proportional to c is an unphysical asymmetry. We therefore must set c=0. In a fluid system, on the other hand, the lack of such a symmetry leads to the expectation 11-13,18, that a diameter (defined as the sum of the order parameters on opposite sides of the coexistence surface) behaving like $|T-T_c|^{1-\alpha}$ should be found in a fluid.

In general, we expect that the knowledge of particular symmetries of the system, or the lack of such symmetries, will restrict the transformations allowed in Eq. (A.1). Ref. 1 has suggested as a possible equation of state for a fluid an equation that is form invariant under (A,1) and its non-linear extentions. This equation of state includes all the corrections of the form given in (A,6) and (A,5) and many more. However, the corrections of Ref. 11 are not connected to an original GHF, but are independent functions. Refs. 12-13 have suggested that a particular choice of critical variables can be used with a single GHF to describe fluid systems. The resolution of these two approaches has not been settled experimentally.

CAPTIONS

1a. Phase diagram of a single component fluid system, illustrating the vapor pressure curve terminating at the critical point. The direction tangent to the vapor pressure curve at the critical point is labelled x_2; x_1 is any direction not tangent to the curve.

b. The phase diagram of a simple ferromagnet. The x_2 direction is along the T axis; x_1 is parallel to the H axis.

c. Phase diagram of an inisotropic ferromagnet. The coexistence surface lying in the RT plane is bounded by a line of critical points. The x_3 direction is tangent to the line of critical points, x_2 lies in the RT plane and x_1 is out of the R=0 plane.

2a. Phase diagram of a metamagnet. The antiferromagnetic critical line terminates at the tricritical point where three critical lines intersect.

b. Schematic phase diagram for a system with incomplete critical ordering. Two coexistence surfaces are shown intersecting along the x_3 axis.

3. The fourth order system of Ref. 4 is shown in the HT plane. for various values of R(R negative). The antiferromagnetic coexistence surface is bounded by a critical line, which terminates at tricritical points labelled TCP. The line of first order transitions is shown dashed; T_N is the Neel temperature for each value of R. (a) gives the phase diagram for the magnitude of R moderately large; (b) for a smaller value of R ; and (c) gives the phase diagram for R=0.

4. The phase diagram of the system of Ref. 4 in HTR space. A coexistence volume is capped by a two-dimensional critical surface, which is bounded by two tricritical lines. The tricritical lines intersect at the fourth order point at H=H'=R=0 and $T=T_2$, the two-dimensional Ising model critical temperature.

5. Hamiltonian space for the nonlinear renormalization group solution of Ref. 18. The surface x=0 is a surface of ordinary critical point of order two. It is bounded by two "tricritical lines" $y_n = 0$ and $y_{2n} = 0$ (and by a separatrix connecting the fixed points. cf. Ref. 14,18). The point $x=y_n=y_{2n}=0$ is the point of order four.

6. Three-dimensional slices of a mulcomponent fluid system of three or more components. The variable t may be thought of as the temperature and u and v as suitable field variables (combinations of the pressure and various chemical potential differences).

   a. The coexistence surface between the phases labelled A and B terminates on the coexistence surface separating those phases from the phase labelled C. The point P is a critical end point at which A and B are critical while in coexistence with C.

   b. A slice containing the point P_t at which all three phases are simultaneously critical.

   c. A slice in which the coexistence surface separating B and C terminates on the A coexistence surface. The Point P' is a critical end point at which B and C are critical while coexisting with A.

7. The two dimensional critical surface in a ternary mixture is shown schematically. The points P and P' are on critical end lines which bound the critical surface and intersect at the tricritical point P_t. The point P_0 is any other point on the critical surface but not on either critical end line.

8. Phase diagram of a multicomponent fluid mixture (of at least four phases). The line L_0 (cf. Fig. 6a) is a n-2 dimensional surface where three phases A,B,C, are mutually coexisting. At the point P, phases A and B become critical. The line L_1 (cf. Fig. 6a) is a n-2 dimensional surface where C is critical with one other phase. The C coexistence surface (bounded by L_1) terminates on a coexistence surface separating the phase labelled D from the other phases. The line L_2 is a n-2 dimensional surface where D is critical with one other phase.

9. Phase diagram of a multicomponent fluid system. Various possibilities for the boundary of the surface of critical end points are shown.

a. ${}_{2}R^{1}_{n-3} \rightarrow {}_{4}R^{0}_{n-4}$. The third phase C becomes critical with the previously critical phases A and B.

b. ${}_{2}R^{1}_{n-3} \rightarrow {}_{2}R^{2}_{n-4}$. The phase D becomes coexistent with A and B (critical) and C.

c. ${}_{2}R^{1}_{n-3} \rightarrow {}_{3}R^{1}_{n-4}$. The third phase C becomes critical and simultaneously the phase D becomes coexistent with A,B, and C.

d. ${}_{2}R^{1}_{n-3} \rightarrow {}_{4}R^{0}_{n-4}$. The phases C and D. Simultaneously become critical with A and B, previously critical.

10. The illustrative example of Eq. (3.10) treated in the text and associated Figs. 8-9. All the critical chains begin from a ${}_{2}^{2}R_{n-3}^{1}$ and terminate at a point of order four (four phases simultaneously critical), a ${}_{s}^{4}R_{d}^{0}$.

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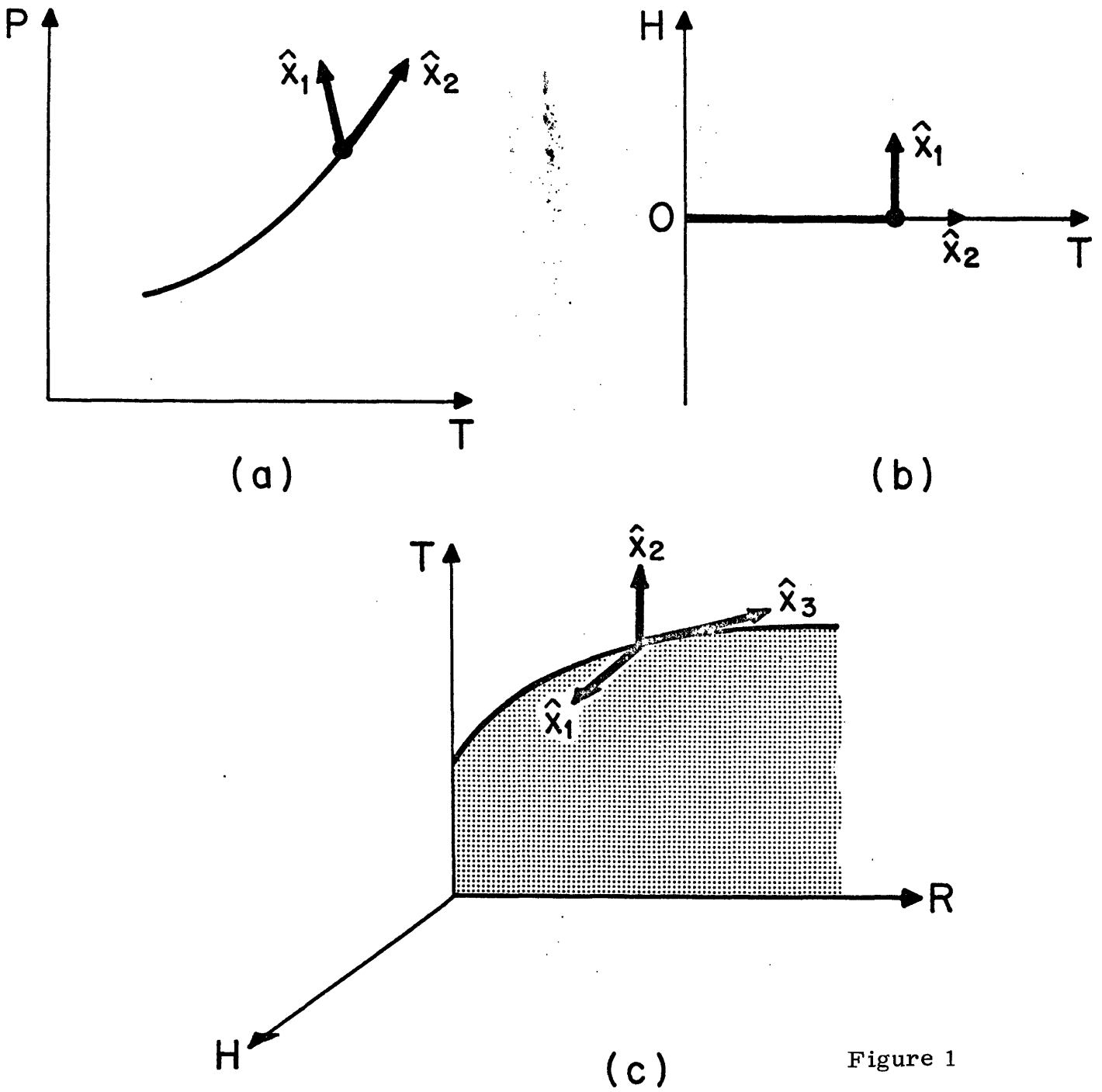
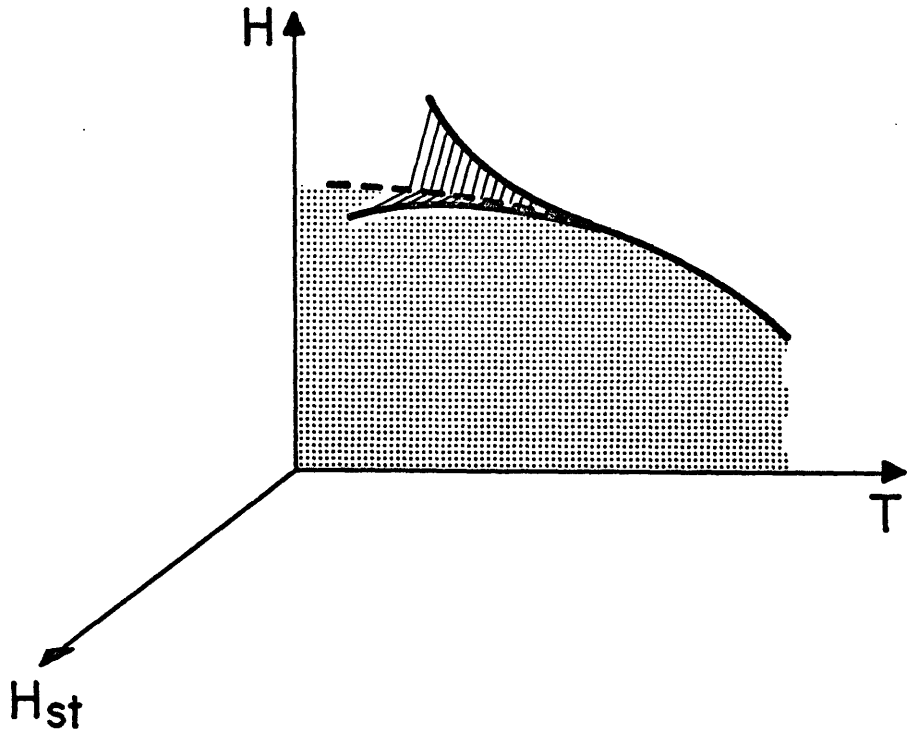


Figure 1



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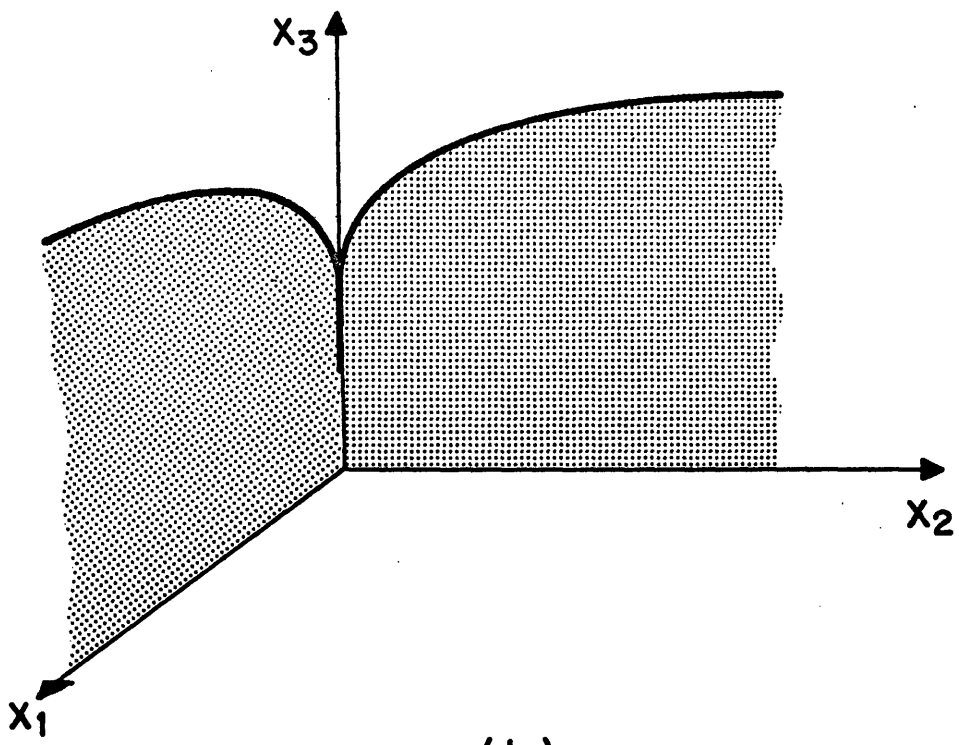

(b)

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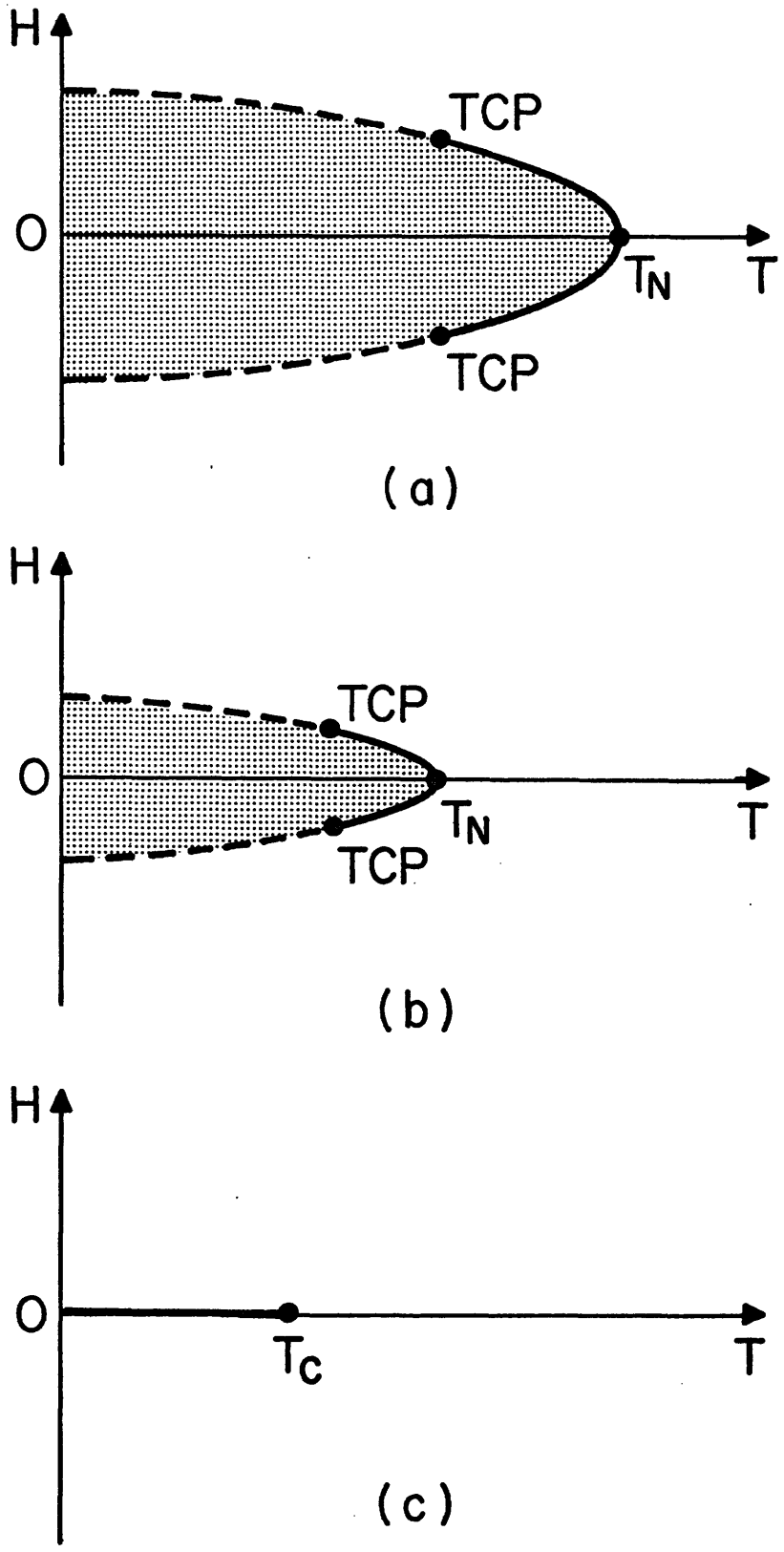

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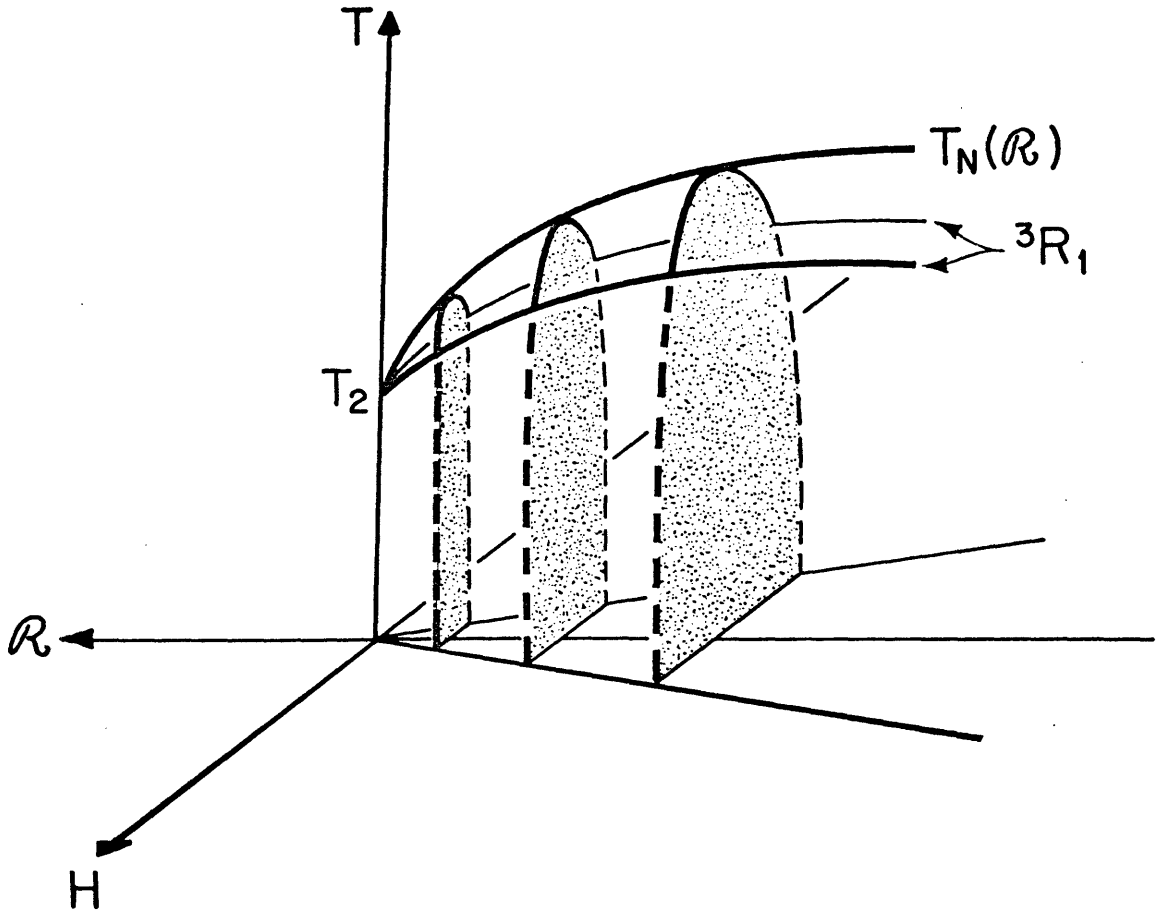

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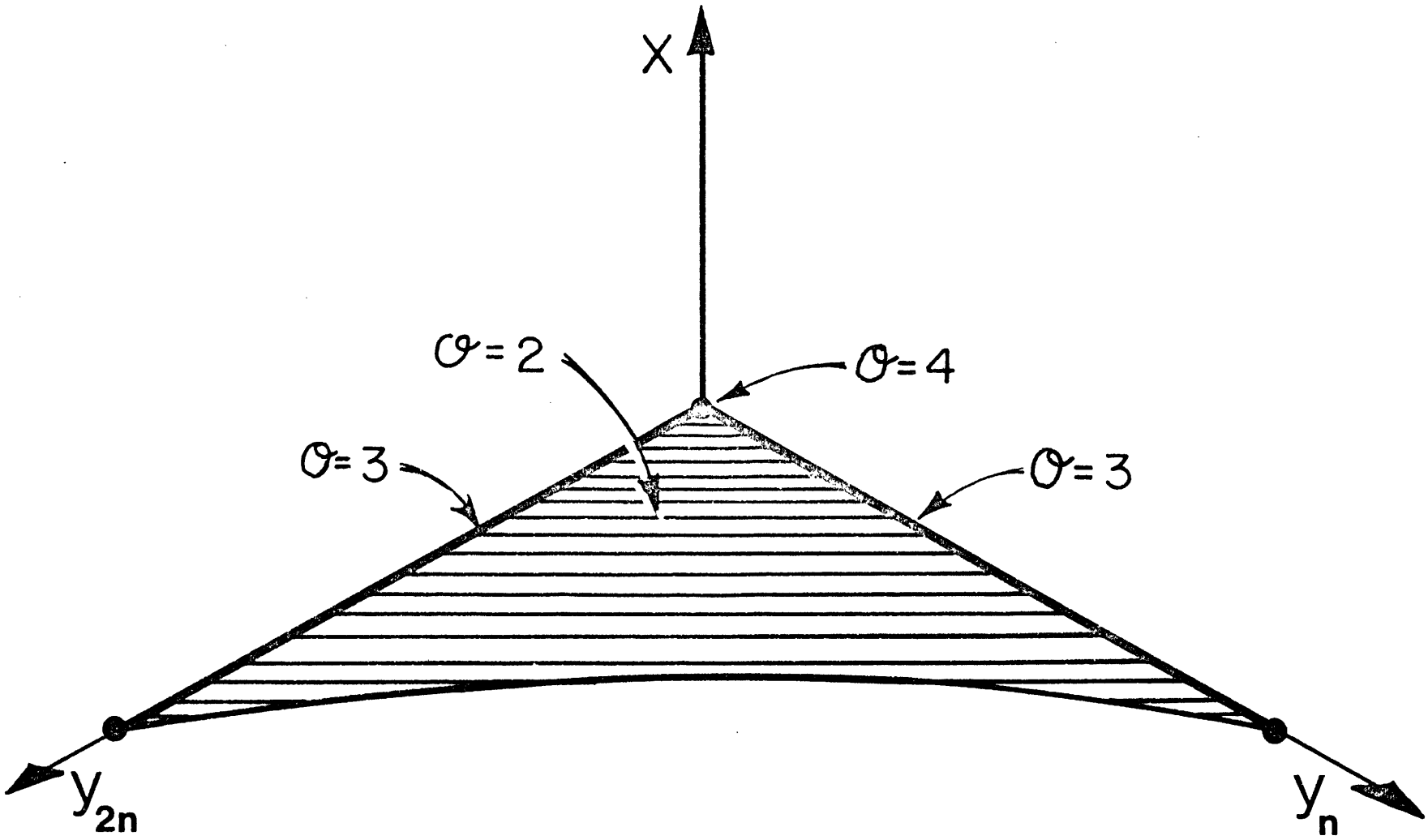

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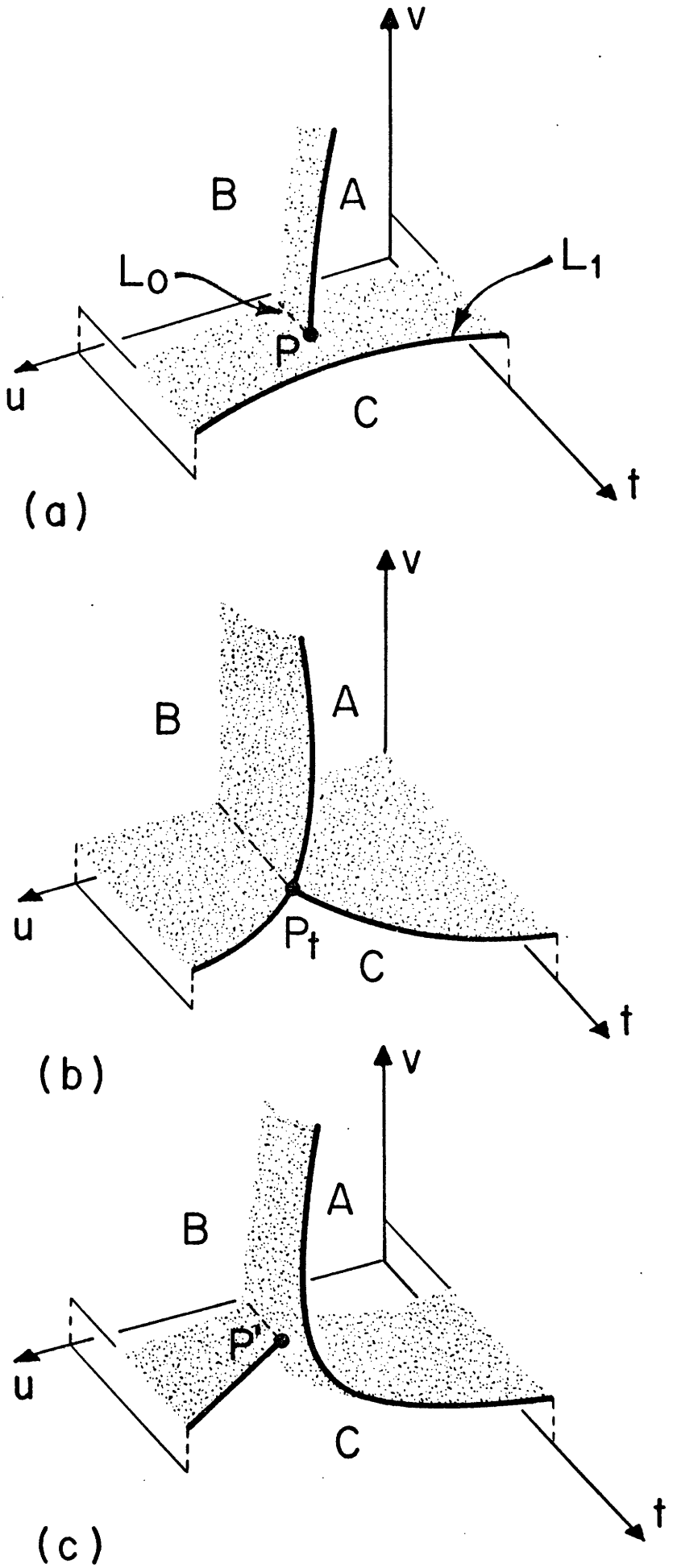

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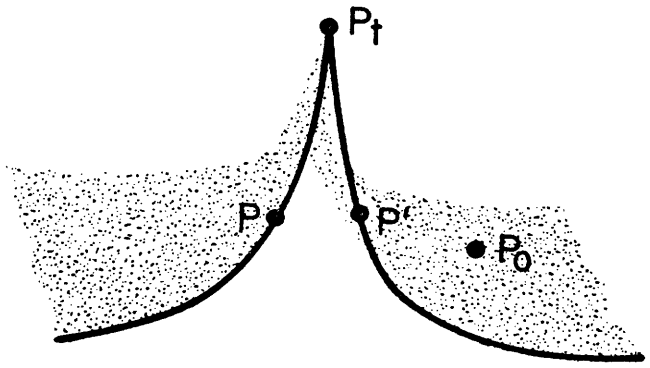

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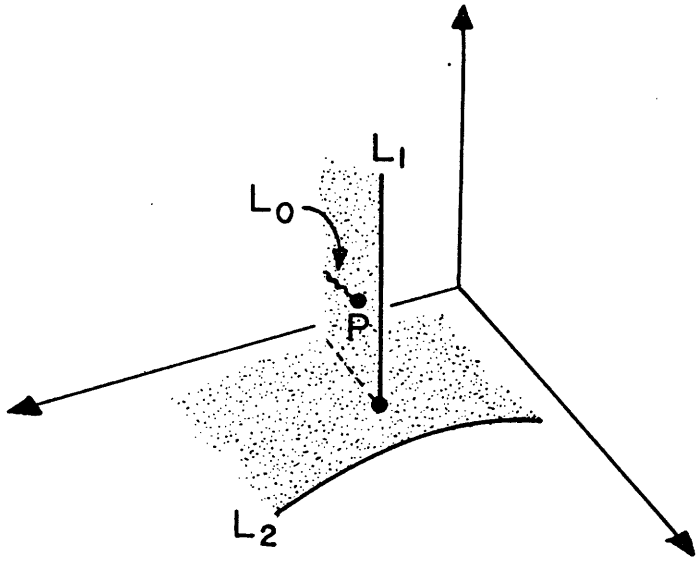

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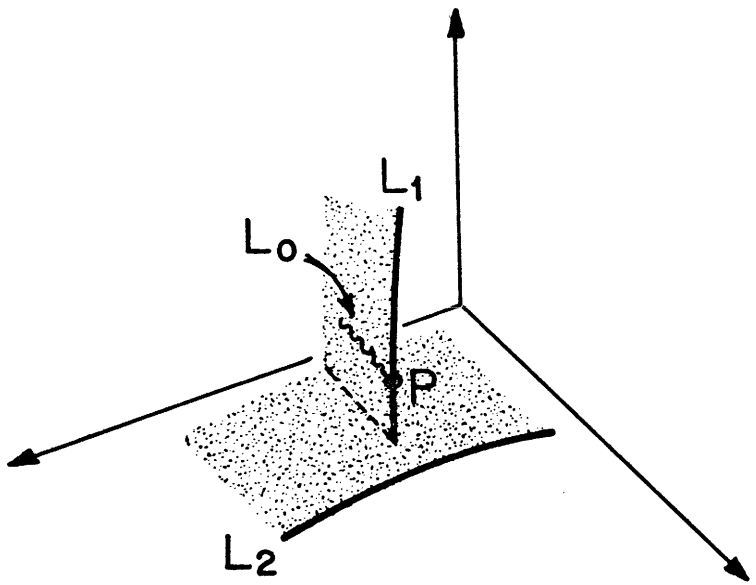

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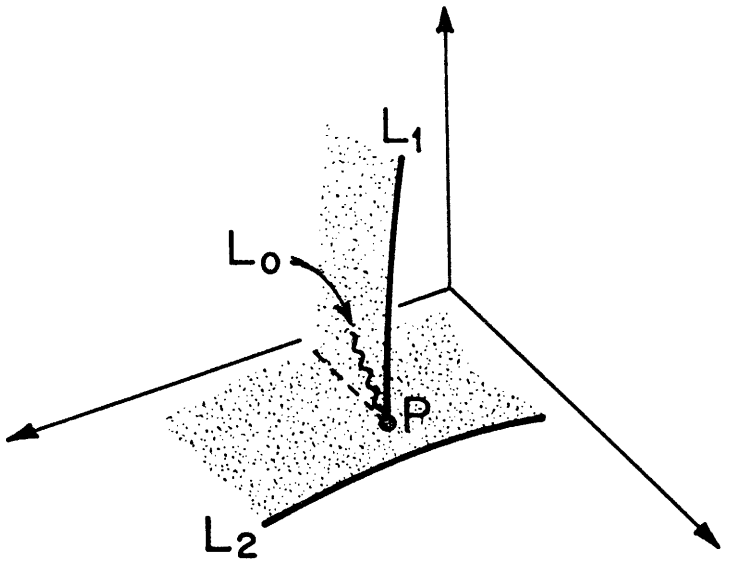
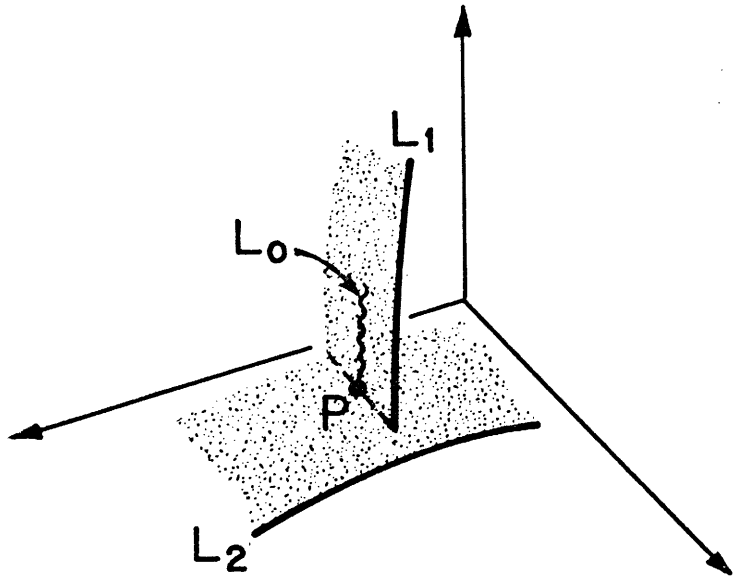


(c)

Figure 9



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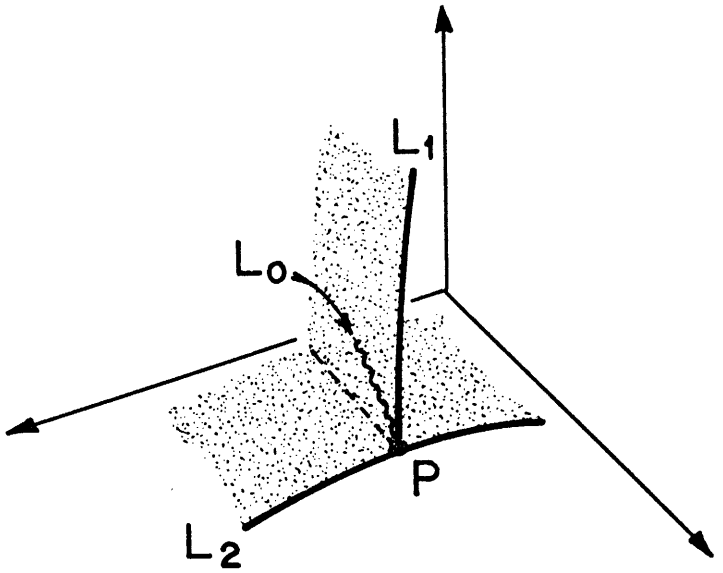

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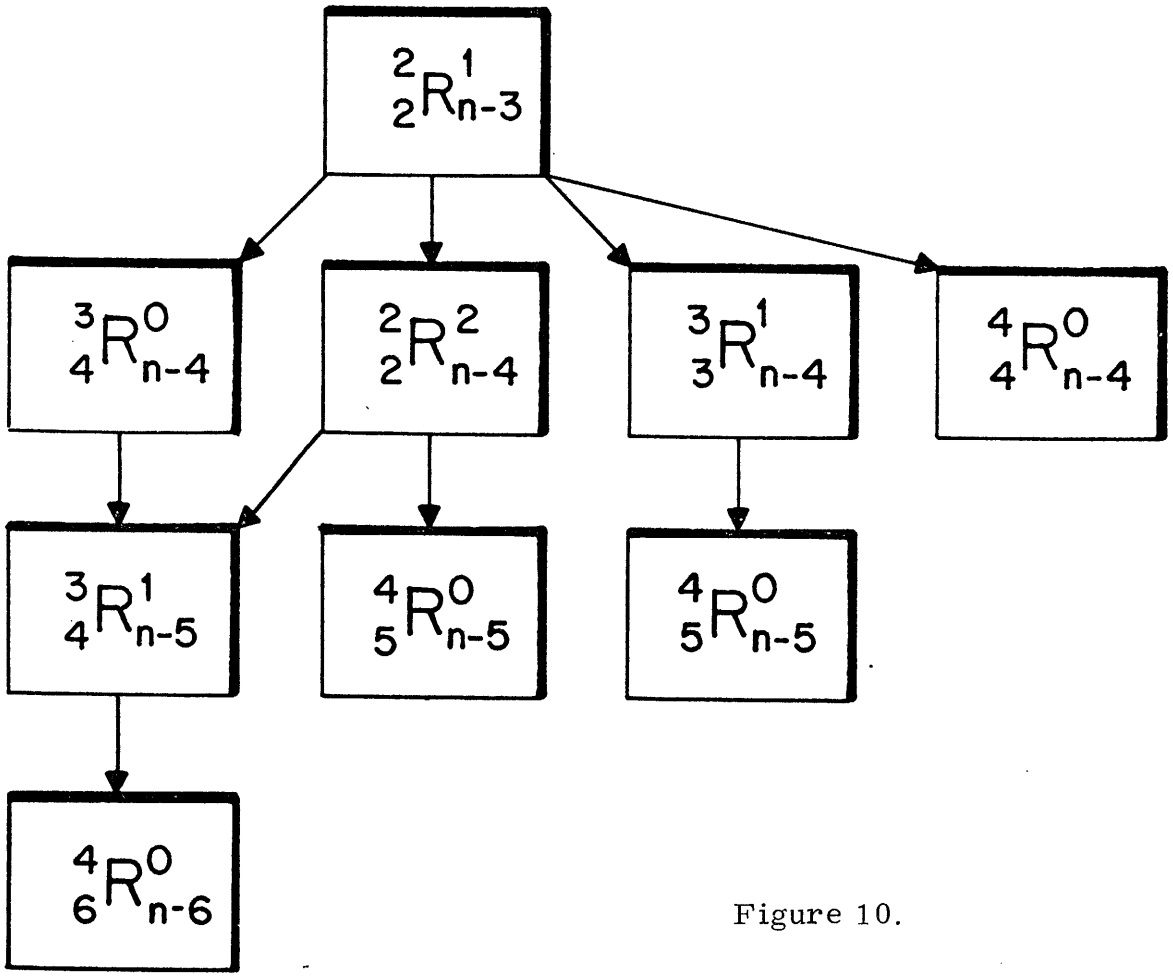
$^{2}_{2}R^{1}_{n-3}$
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$^{2}_{2}R^{2}_{n-4}$
$^{3}_{3}R^{1}_{n-4}$
$^{4}_{4}R^{0}_{n-4}$
$^{3}_{4}R^{1}_{n-5}$
$^{4}_{5}R^{0}_{n-5}$
$^{4}_{5}R^{0}_{n-5}$
$^{4}_{6}R^{0}_{n-6}$

Figure 10.

CHAPTER 4

RENORMALIZATION GROUP CALCULATION OF SCALING POWERS

I. Approximate Renormalization Group Based on the Wegner-Houghton Differential Generator*

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Approximate Renormalization Group Based on the Wegner-Houghton Differential Generator*

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We give an approximate renormalization-group formulation which parallels that of Wilson. The group generator represents the momentum-independent limit of the differential generator of Wegner and Houghton. The eigenfunctions near the Gaussian point are computed for all spin dimensions n and lattice dimensions d, including $d=2$. The nontrivial fixed-point Hamiltonian in dimensions near $d=2\mathcal{O}/(\mathcal{O}-1)$, together with the eigenvalues near that nontrivial fixed point, are found explicitly to first order in $\epsilon_{\mathcal{O}}\equiv\mathcal{O}(2-d)+d$ for *all* values of n and the order $\mathcal{O}$. Odd-dominated Ising systems and corresponding expansions in $\epsilon_{\mathcal{O}-1/2}$ are also treated.

The renormalization-group approach to the study of critical phenomena has had great initial success.[1-2] The renormalization group embodies in concrete mathematical form the scaling notions of Kadanoff[3] and provides a framework for explicit calculation. These calculations have usually been done by perturbative expansions, in analogy with similar problems in quantum field theory. All the difficulties of field theory have been incorporated into critical-phenomena calculations as well; the calculation of thermodynamic quantities involves complicated Feynman diagrams and divergent integrals.

Even in those cases where field-theoretic difficulties are not encountered, the perturbation techniques have been "brute force" in nature. For example, the calculation of critical-point exponents for higher-order[4] critical points has been hampered by the rapid increase of the number of equations which contribute.[5]

Many renormalization-group problems can be simplified by revising the perturbative techniques to conform as closely as possible to the structure of the renormalization group itself. It was noted by Wegner[6] that the eigenfunctions of Wilson's approximate renormalization group (when linearized around the Gaussian point[2]) are related to Laguerre polynomials. However, this observation has hitherto not been fully exploited. Here we show that by utilizing the structure of the renormalization group, a number of problems [see (i)–(iv) below] may be solved simply and explicitly.

To do this, we first write down an appropriate differential equation based upon the Wegner-Houghton[7] differential generator for the renormalization group. Their functional integrodifferential equations may be simplified if we consider them in the limit of vanishing "external" momenta.[2] We find that for n-dimensional isotropically interacting spins $\vec{s}$ on a d-dimensional lattice, the renormalization action on the reduced Hamiltonian H is given by

$$\dot{H}=dH+(2-d)x\frac{\partial H}{\partial x}+\frac{d}{2}\left[\left(1-\frac{1}{n}\right)\ln\left(1+\frac{\partial H}{\partial x}\right)+\frac{1}{n}\ln\left(1+\frac{\partial H}{\partial x}+2x\frac{\partial^2 H}{\partial x^2}\right)\right], \tag{1}$$

where the dot denotes differentiation with respect to the renormalization parameter l, and $x\equiv(\vec{s}\cdot\vec{s})/n$.[8] Since we have neglected the detailed momentum dependence in the renormalization group, we have set $\eta=0$.

(i) *The general $\epsilon_{\mathcal{O}}$ expansion.*—To solve (1), the Hamiltonian H can be expanded in terms of any complete set of functions; the expansion functions should be chosen to simplify the problem under consideration. A particularly useful set of functions are the eigenfunctions of (1) when (1) is linearized about

the Gaussian fixed point, $H=0$. These functions (not normalized) can be chosen to be

$$Q_p(x) \equiv [d/(2-d)n]^p L_p^{n/2-1}([(d-2)/d]nx), \tag{2}$$

where the conventions of Erdelyi[9] are used for the Laguerre polynomials, $L_p^{n/2-1}(z)$. The eigenvalue corresponding to Q_p is $\lambda_p = p(2-d)+d$. To illustrate the use of the Q_p, we have calculated the nontrivial fixed-point Hamiltonians, $H = H_\Theta{}^*$, corresponding to critical points of order Θ.[4] The fixed points of (1) are determined by setting $\dot{H}=0$. In analogy with the ϵ expansions introduced in Refs. 1 and 2, we calculate $H_\Theta{}^*$ as a perturbation expansion in $\epsilon_\Theta \equiv \Theta(2-d)+d$,[10] for $\Theta = 2, 3, 4, \ldots$ (the usual[1,2] ϵ is ϵ_2 in our notation). To first order in ϵ_Θ, $H_\Theta{}^* = \epsilon_\Theta v_\Theta Q_\Theta$, where v_Θ is given by

$$1 = \tfrac{1}{4} dv_\Theta \langle \mathfrak{D}(\Theta,\Theta) | \Theta \rangle. \tag{3a}$$

Here the bilinear functional $\mathfrak{D}(i,j)$ is given by

$$\mathfrak{D}(i,j) \equiv \left(1-\frac{1}{n}\right)\frac{dQ_i}{dx}\frac{dQ_j}{dx} + \frac{1}{n}\left((1-n)\frac{dQ_i}{dx} + (2i+n-2)Q_{i-1}\right)\left((1-n)\frac{dQ_j}{dx} + (2j+n-2)Q_{j-1}\right), \tag{3b}$$

and the inner product $\langle f | p \rangle$ for a function $f(x)$ is defined by

$$f(x) = \sum_{p=0}^{\infty} \langle f | p \rangle Q_p(x). \tag{3c}$$

Equation (1) can now be linearized around $H_\Theta{}^*$. The eigenfunctions will change slightly and so will the eigenvalues. If we denote by $\hat{\lambda}_l$ the eigenvalue of the new eigenfunction, which to zeroth order is Q_l, we find that to first order in ϵ_Θ

$$\hat{\lambda}_l = \lambda_l - 2\epsilon_\Theta \frac{\langle \mathfrak{D}(\Theta, l) | l \rangle}{\langle \mathfrak{D}(\Theta,\Theta) | \Theta \rangle}. \tag{4}$$

The evaluation of the bilinear coefficients in (4) is merely a problem in classical analysis. In fact, using the full renormalization-group equations, we have shown that (4) is exactly correct[11] to order ϵ_Θ.

For $n=1$ (Ising systems), (3b) simplifies considerably, the Q_p are related to Hermite polynomials, and (4) reduces to

$$\hat{\lambda}_l = [l(2-d)+d] - 2\epsilon_\Theta \left(\frac{(2l)!}{(2\Theta)!}\frac{\Theta!}{(2l-\Theta)!}\right). \tag{5}$$

These results are in agreement with the $\Theta=2$ calculations of Refs. 1 and 2, and the $\Theta=3,4$ calculations of Ref. 5. We note that (5) also contains the odd eigenvalues for $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

From (5) we immediately deduce several important consequences. (i) For $\epsilon_\Theta > 0$, the correction to the Gaussian eigenvalue is negative, so that the nontrivial fixed point always dominates the Gaussian fixed point sufficiently near the critical point. (ii) The correction to the Gaussian eigenvalue vanishes unless $2l \geq \Theta$. In particular, to order ϵ_Θ, $\hat{\lambda}_1 = 2$ for all $\Theta \neq 2$, independent of d. (iii) We note that $\hat{\lambda}_\Theta = -\lambda_\Theta = -\epsilon_\Theta$, so that if we examine the first Θ eigenvalues we find that at the Gaussian fixed point they are all positive, and at the nontrivial fixed point all but the last remain positive. The Gaussian point is unstable, and the nontrivial point is a generalized saddle point for $\epsilon_\Theta > 0$.[12]

We also note that the ordering field which couples directly to $\tilde{s}$ is entirely decoupled from the remainder of the renormalization-group transformations.[13] The eigenvalue $\lambda_{1/2}$, corresponding to the ordering field, is exactly $1+d/2$.

(ii) *Gaussian eigenfunctions for $d=2$.*—We next consider the behavior of (1) for $d=2$. The nontrivial fixed points at $d = 2\Theta/(\Theta-1)$ cluster densely around $d=2$ as $\Theta \to \infty$. By studying (1) with d set equal to 2 [or by examining the limit of (2) as $d \to 2$ with $p(2-d)$ fixed] we find the eigenfunctions around the Gaussian fixed point have a continuous set of eigenvalues, $\lambda \leq 2$. A complete orthonormal set of eigenfunctions is given by[14]

$$Q_\lambda(x) = (\tfrac{1}{2}n)^{1/2} x^{-(n/2-1)/2} J_{n/2-1}((4-2\lambda)^{1/2}(nx)^{1/2}), \tag{6a}$$

where $J_{n/2-1}$ denotes the Bessel function of the first kind, and

$$\int_0^\infty dx\, x^{n/2-1} Q_\lambda(x) Q_{\lambda'}(x) = \delta(\lambda - \lambda'). \tag{6b}$$

The Hamiltonian is expressible as an integral, $H=\int v_\lambda Q_\lambda d\lambda$, rather than a sum (for $d\neq 2$). In the discrete case, thermodynamic potentials are generalized homogeneous functions[15] of the expansion coefficients. In the continuum limit, they become generalized homogeneous functionals with similar properties. For example, the Gibbs potential satisfies

$$e^{dl}G(v_\lambda)=G(e^{\lambda l}v_\lambda). \tag{7}$$

The continuous nature of the eigenvalue spectrum leads, in general, to logarithmic factors multiplying the usual power-law dependence of generalized homogeneous functions.[15] Since the approximations made in deriving (1) require setting $\eta=0$ for consistency, one must be cautious in interpreting our results for $d=2$.

(iii) *Power-law expansions.*—The solution of (1) for other than ϵ_ϑ expansions is more difficult. For n arbitrary, the expansion of H in terms of Laguerre polynomials leads to equations coupled to all orders in the expansion parameters. If these cannot be assumed small, the equations are too complicated for immediate solution. If, however, H is expanded in powers of x, the resulting equations, while not appropriate for general ϵ_ϑ analysis, are essentially "triangular." That is, if we expand

$$H=\sum_{j=0}^{\infty}v_{2j}x^j/j\,!,$$

the generator for the v_{2p} equation is given by

$$\dot v_{2p}=[p(2-d)+d]v_{2p}+\frac{d}{2}\left(\frac{\partial}{\partial x}\right)^p\left[\left(1-\frac{1}{n}\right)\ln\left(1+\sum_{j=0}^{\infty}\frac{v_{2j}x^{j-1}}{(j-1)!}\right)+\frac{1}{n}\ln\left(1+\sum_{j=0}^{\infty}\frac{(2j-1)v_{2j}x^{j-1}}{(j-1)!}\right)\right]\Bigg|_{x=0}. \tag{8}$$

The linear structure has only one off-diagonal term, $d(1+2p/n)\hat v_{2p+2}/2$, and the nonlinear terms are at most of order p in the modified coupling constants $\hat v_{2j}=v_{2j}/(1+v_2)$. Furthermore, the nonlinear terms include no v_{2j} with $j>p$. In particular, for $n=-2m$, the first m equations decouple entirely from the remaining equations.[16]

We have used (8) to evaluate critical-point exponents for the ordinary and tricritical points ($\vartheta=2,3$). For $\vartheta=2$, our results agree with those of Refs. 1 and 2. For $\vartheta=3$ we find to order ϵ_3,

$$\hat\lambda_1=2,\quad \hat\lambda_2=1+[(6-n)/(3n+22)]\epsilon_3/2, \tag{9}$$

in agreement with the general formulas for $n=1$ given in (5).

(iv) *Odd-dominated Ising systems.*—In addition to the usual *even* fixed-point Hamiltonians described above, (1) admits (for $n=1$) fixed points which have leading *odd* terms. We may do $\epsilon_{\vartheta-1/2}$ expansions for $\vartheta=2,3,\ldots$ in this case as well. The fixed-point Hamiltonian is of order $(\epsilon_{\vartheta-1/2})^{1/2}$. We write the fixed-point Hamiltonian H^* as

$$H^*=(\epsilon_{\vartheta-1/2})^{1/2}v_\vartheta h_{2\vartheta-1}+\epsilon_{\vartheta-1/2}v_\vartheta{}^2f_e+(\epsilon_{\vartheta-1/2})^{3/2}v_\vartheta{}^3f_0+\cdots, \tag{10}$$

where $h_{2\vartheta-1}$ is an *odd* Hermite polynomial, and f_e is an even and f_0 an odd function of s. Solving (1) to first order in $\epsilon_{\vartheta-1/2}$, we find the fixed-point value v_ϑ and the perturbed eigenvalues to be given by

$$1=-\tfrac{1}{6}dv_\vartheta{}^2\langle\mathcal{J}(2\vartheta-1,2\vartheta-1)|2\vartheta-1\rangle, \tag{11a}$$

$$\hat\lambda_{l/2}-\lambda_{l/2}=-3\epsilon_{\vartheta-1/2}\frac{\langle\mathcal{J}(2\vartheta-1,l)|l\rangle}{\langle\mathcal{J}(2\vartheta-1,2\vartheta-1)|2\vartheta-1\rangle},\quad l=1,2,3,\ldots. \tag{11b}$$

The operator $\mathcal{J}$ in (11) is

$$\mathcal{J}(m,l)\equiv 64l(l-1)m(m-1)[(h_{m-2})^2h_{l-2}+h_{l-2}\mathcal{L}_m(h_{m-2})^2+2h_{m-2}\mathcal{L}_l(h_{l-2}\cdot h_{m-2})], \tag{12}$$

where $\mathcal{L}_l$ is defined by $\mathcal{L}_lh_p=[p(p-1)/p-l]h_{p-2}$ for all Hermite polynomials h_p. At least for $2\vartheta-1=3,5$, we have $v_\vartheta{}^2<0$; the Hamiltonian is real only if $\epsilon_{\vartheta-1/2}<0$. For $\epsilon_{\vartheta-1/2}>0$ the odd parts of the fixed-point Hamiltonian are purely imaginary.[17]

The Wegner-Houghton approximate renormalization group proposed here provides a straightforward framework in which to explore the consequences of the full renormalization group. As a differential representation, it is suited to investigations of nonlinear phenomena such as crossover competition between two or more fixed points. Elsewhere[18] we have solved (8) near $d=4$ for the *nonlinear* cross-

over between critical and Gaussian (mean-field) behavior. The extension to crossover from tricritical to mean-field behavior seems to be more difficult.

We would like to thank G. F. Tuthill and L. L. Liu for many helpful comments and suggestions and B. D. Hassard for discussions of the differential generator of the renormalization group.

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[8]In the special case $n=\infty$, Ref. 7 gives a derivation of a solution for (1). The zero-momentum requirement can be weakened somewhat in this case. If we write $v_{2j}(\vec{k}_1,\ldots,\vec{k}_{2j})$ for the momentum-dependent $2j$-spin coupling constant, Eq. (1) follows by restricting the $\vec{k}_j$ to cancel in pairs; that is, we consider only $v_{2j}(\vec{k}_1,-\vec{k}_1,\ldots,\vec{k}_j,-\vec{k}_j)$. We also note that the reduced Hamiltonian density H_W of Wilson (Ref. 2) has the form $H_W=|\nabla\vec{s}|^2+H(x)$. The gradient term is left unchanged by the renormalization group in the approximation employed here and is therefore not considered explicitly.

[9]A. Erdelyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 2, pp. 188 ff.

[10]Our definition of $\epsilon_{\mathcal{O}}$ differs slightly from that of Chang, Tuthill, and Stanley, Ref. 4. The convention adopted here has the advantage that the eigenvalue of $Q_{\mathcal{O}}$ is precisely $\epsilon_{\mathcal{O}}$.

[11]To see this, it is sufficient to note that the Q_j are eigenfunctions of the full linear renormalization-group operator. The powers of x in the Q_j are replaced by more complicated sums over momentum: $(nx)^p$ becomes

$$\sum_{k_1k_1'}\cdots\sum_{k_pk_p'}(\vec{s}_{k_1}\cdot\vec{s}_{k_1'})\cdots(\vec{s}_{k_p}\cdot\vec{s}_{k_p'})\delta_{k_1+k_1'+\cdots k_p+k_p',0}.$$

With these emendations, an examination of the full nonlinear renormalization-group equation of Ref. 7 shows that the fixed point and eigenvalues are correct to first order in $\epsilon_{\mathcal{O}}$, and $\eta_{\mathcal{O}}$ is $o(\epsilon_{\mathcal{O}}^2)$.

[12]Points (ii) and (iii) hold for general n; (i) cannot hold for *arbitrary* n {e.g., for $\mathcal{O}=2$, $\lambda_1=2-[(n+2)/(n+8)]\epsilon_2$}.

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II. Renormalization Group Calculation of the Critical Point Exponent η for a Critical Point of Arbitrary Order*

*(Phys. Rev. B (June, 1975))

Renormalization-Group Calculation of the Critical Point Exponent η For a Critical Point of Arbitrary Order*

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The critical point exponent η for a critical point of order O in dimensions less than $d_\sigma = 2\sigma/(\sigma-1)$ is calculated to leading non-vanishing order in the parameter $\epsilon_\sigma = d_\sigma - d$. The result is given for n-component isotopically interacting magnetic systems. For Ising systems, n=1, the result is $\eta_\sigma = \epsilon_\sigma^2 4(\sigma-1)^2/\binom{2\sigma}{\sigma}^3$. As σ increases, the coefficient of ϵ_σ^2 rapidly becomes very small, varying as $2^{-6\sigma}$ for σ large. In the limit of large n, η_σ for odd order points approaches a constant and, for even order points, is proportional to 1/n.

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The classification and study of critical points of "higher order" has been of recent interest.[1-7] The order of a critical point is defined by some authors to be the number of phases simultaneously critical at the critical point.[1-2] Thus, an ordinary critical point is an $\sigma=2$ point; tricritical points are $\sigma=3$ points. Although there are many kinds of higher order points[3], much of the work has concentrated on systems that in the mean field approximation could be represented by a Landau-Ginzberg form for the Hamiltonian density,

$$H(\vec{s}) = -\int d^d\vec{x}\,[\,\frac{1}{2}|\nabla\vec{s}(\vec{x})|^2 + \sum_{k=1}^{\sigma}\frac{u_{2k}}{(2k)!}(\vec{s}\cdot\vec{s})^k\,] \qquad (1)$$

where we have specialized to the "magnetic" case of an isotropically interacting n-component spin $\vec{s}(\vec{x})$.

The renormalization group approach to such systems was introduced by Wilson[8] for the case $\sigma=2$. Corrections to mean field behavior are calculated in a perturbation expansion in $\epsilon_2 \equiv 4-d$. The tricritical case, $\sigma=3$, has been studied by Riedel and Wegner[4] at d=3. Stephen and McCauley[5] and Chang, Tuthill, and Stanley[6] calculated exponents below three dimensions in an expansion in $\epsilon_3 \equiv 3-d$. Ref. 6 also gave explicit exponents to first order in $\epsilon_4 \equiv 8/3-d$ for the $\sigma=4$ case. The critical point exponents for the general σ case were given in Nicoll, Chang, and Stanley[7] to first order in $\epsilon_\sigma \equiv 2\sigma/(\sigma-1)-d$. The critical point exponent η was shown in Ref. 7 to be at most $o(\epsilon_\sigma^2)$. In this work, we complete the calculation of all critical point exponents to leading order by calculating η to $O(\epsilon_\sigma^2)$.

The ϵ_σ calculation of Ref. 7 were based on the differential renormalization group generator of Wegner and Houghton[9]. The calculation of η by this method is difficult and, therefore, through most of this work we will adopt a field-theoretic approach utilizing Feynman diagrams. However, we will extract the dependence of η on the number of spin components, n, by combining graph-counting with the solutions of Ref. 7.

Following the method used to locate fixed points[5-9], we assume u_{2k} to be $O(\epsilon_\sigma)$ for $k \leq \sigma$. It is then possible to carry out a self-consistent perturbation expansion in the parameters $u_4, u_6, \ldots u_{2\sigma}$ while applying a "mass counterterm"[8] so that the bare propagator is $(p^2+r)^{-1}$, with r the zero ordering-field susceptibility. The exponent η_σ is defined by a proportionality relation for the Fourier transform G of the spin-spin correlation function for small wave-number

$$G^{-1}(\vec{p}) \sim p^{2-\eta_\sigma} \sim p^2(1 - \eta_\sigma \ln p \ldots) \tag{2}$$

at the order σ point (r=0). We will now show that to $O(\epsilon_\sigma^2)$, the calculation of η_σ involves only two Feyman graphs to be evaluated in dimension d_σ

In the perturbation expansion for G^{-1} we may write

$$G^{-1}(\vec{p}, r) = p^2 + r + \sum(\vec{p}, r) \tag{3}$$

where $\sum$ represents the sum of all remaining graphs (with counter-term) displayed schematically in Fig. 1. The mass counterterm u_2-r cancels all p-independent terms in (3), in particular, all single vertex diagrams. The series may be further simplified by formally eliminating closed loops which include only one vertex and introducing r-dependent generalized vertices $\bar{u}_{2k}(r)$, defined by

$$\bar{u}_{2k}(r) = u_{2k} + \sum_{\ell=1}^{\sigma-k} \frac{u_{2k+2\ell}}{\ell!\, 2^\ell} (F_1(r))^\ell \tag{4}$$

Here, as in Ref. 5, $F_1(r)$ represents the loop integral $\int d^dk G(k, r)/(2\pi)^d$. With this change in notation and to $O(\epsilon_\sigma^2)$, the set of graphs in $\sum$ is reduced to those shown in Fig. 2.

Next, we note that $\bar{u}_{2k}(r=0) = 0$ for all $k < \sigma$. This follows from Wilson's scaling theorem[8] for 2k-point vertex functions

$$\Gamma_{2k}(p=0) \sim r^{[k - \frac{(k-1)d}{2-\eta}]} \tag{5}$$

For $d = d_\sigma - \epsilon_\sigma$, (5) requires that Γ_{2k} vanish at r=0 for all $k < \sigma$. Since the first order perturbation expansion for Γ_{2k} is just $\bar{u}_{2k}$, $\bar{u}_{2k}$ must vanish as well. At the critical point, therefore, all the graphs except the last shown in Fig. 2 are zero.

The combinatorial factor for this diagram may be evaluated by considering first the Ising case, in which it is simply $1/(2\sigma-1)!$. To determine the n-dependence, it suffices to note that a factor of (n+2N-2) is associated with the connection of two legs of a single 2N-leg vertex. Thus, the n-dependence of the $u_{2\sigma}^2$ diagram is given by $f_1(n)/(2\sigma-1)!$ where

$$f_1(n) = \prod_{\ell=1}^{\sigma-1} (n+2\ell)/(2\ell+1) \quad . \tag{6}$$

With this factor and denoting the $u_{2\sigma}^2$ integral by I_1, the correspondence between (2) and (3) gives

$$p^2(1 - \eta_\sigma \ell np) = p^2 - \frac{u_{2\sigma}^2 f_1(n)}{(2\sigma-1)!} [I_1(p, r=0)]_{p^2 \ell np \text{ part}} \tag{7}$$

Since $u_{2\sigma}$ is $O(\epsilon_\sigma)$, η_σ is clearly $O(\epsilon_\sigma^2)$.

The fixed point value of $u_{2\sigma}$ remains to be determined; it is chosen so that the vertex functions satisfy scaling laws. For $k=\sigma$ in (5) this gives

$$\Gamma_{2\sigma} \sim r^{[\epsilon_\sigma(\sigma-1)/2]} \sim 1 + \epsilon_\sigma \frac{\sigma-1}{2} \ell n r \ldots \tag{8}$$

The constant of proportionality must also be expanded as a series in ϵ_σ, so that

$$\Gamma_{2\sigma} = A + \epsilon_\sigma [\, A(\sigma-1)/2 \;\ell n r + B \,] \qquad (9)$$

with A and B constants.

In first order, $\Gamma_{2\sigma}$ is $u_{2\sigma}$, so that $A=u_{2\sigma}$. Second order terms all involve two-vertex diagrams $\bar{u}_{2\ell}\,\bar{u}_{2\ell'}$ (with $\ell, \ell' \leq \sigma$) and graphs with internal lines numbering $i \leq \sigma$, (cf. Fig. 3). The r-dependence of $\bar{u}_{2\ell}$ is given by the integral $F_1(r) - F_1(0)$, since by the remarks above $\bar{u}_{2\ell}(0)=0$ for $\ell < \sigma$.

$$F_1(r)-F_1(0) = \int \frac{d^d\vec{k}}{(2\pi)^d} \left[\frac{1}{(k^2+r)} - \frac{1}{k^2} \right] = -\frac{\Omega_d}{(2\pi)^d} r \int_0^\infty \frac{k^{3+d}\, dk}{(k^2+r)} \qquad ,(10)$$

where $\Omega_d = 2(\pi)^{d/2} / \Gamma(d/2)$ is the area of the unit sphere in dimension d. Changing variables, we have

$$F_1(r)-F_1(o) = -\frac{\Omega_d}{(2\pi)^d} r^{(d-2)/2} \int_0^\infty \frac{dx\, x^{d-3}}{1+x^2} \,. \qquad (11)$$

The integral converges for $2 < d < 4$ so that all r-dependence is in the prefactor; no $\ell n r$ factors are present.

Next, we examine the r-dependence of the graph of Fig.3. By power counting, this integral diverges like $r^{(i-\sigma)/(\sigma-1)}$ for small r. (For $i < \sigma$, the integral converges at large k without a momentum cutoff, and a change of variables similar to that in (11) shows that the diagram gives a prefactor of $r^{(i-\sigma)/(\sigma-1)}$ multiplied by a convergent integral.) Only for $i=\sigma$ will $\ell n r$ terms arise; the integral for this case is denoted as $I_2(r)$.

To compare with the scaling form (9), we note that the perturbation expansion gives

$$\Gamma_{2\sigma} = u_{2\sigma} - \frac{(2\sigma)!\, I_2(r)}{2(\sigma!)^2} u_{2\sigma}^2 \cdots \,. \qquad (12)$$

The resulting value for $u_{2\mathcal{O}}$ to first order is

$$u_{2\mathcal{O}} = -\frac{(\mathcal{O}-1)\,(\mathcal{O}!)^3 \epsilon_{\mathcal{O}}}{(2\mathcal{O}!)\,[I_2(r)]_{\ell nr \text{ part}}} \tag{13}$$

Combining (13) with (7), the expression for the exponent $\eta_{\mathcal{O}}$ (for n=1; n-dependence will be discussed below) to leading order is

$$\eta_{\mathcal{O}} = \epsilon_{\mathcal{O}}^2 \frac{(\mathcal{O}-1)^2\,(\mathcal{O}!)^6}{(2\mathcal{O}-1)!\,[(2\mathcal{O}!)]^2} \frac{[I_1(p)]_{p^2 \ell np}}{[I_2(r)]^2_{\ell nr}} \tag{14}$$

All that remains is the calculation of the two integrals

$$I_1 = \int d^d\vec{R}\, e^{i\vec{p}\cdot\vec{R}} \left[\int \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{R}}}{k^2}\right]^{2\mathcal{O}-1} \tag{15}$$

and

$$I_2 = \int d^d\vec{R} \left[\int \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{R}}}{(k^2+r)}\right]^{\mathcal{O}} , \tag{16}$$

where d and $\mathcal{O}$ are, of course, related by $d=d_{\mathcal{O}}= 2\mathcal{O}/(\mathcal{O}-1)$. Both integrals are divergent as written; I_1 diverges quadratically and I_2 diverges logarithmically. To extract the finite terms desired, we cut off the R integrations by integrating over $|R| > \Lambda^{-1}$.

From Bateman[10] we note that

$$\int d^d\vec{x}\, e^{i\vec{q}\cdot\vec{x}} = \int_0^\infty dx\, x^{d-1}\, \Omega_d \Gamma(d/2) J_\nu(xq)\,(xq/2)^{-\nu} , \tag{17}$$

where $\nu \equiv (d-2)/2$. Therefore, applying (17) to (15) we have

$$I_1 = \frac{[\Omega_d \Gamma(d/2)]^{2\mathcal{O}}}{(2\pi)^{d(2\mathcal{O}-1)}} \int_{1/\Lambda}^\infty R^{d-1} dR\, J_\nu(Rp)\left(\frac{Rp}{2}\right)^{-\nu} \left[\int_0^\infty k^{d-3} dk\, J_\nu(kR)\,\left(\frac{kR}{2}\right)^{-\nu}\right]^{2\mathcal{O}-1} \tag{18}$$

The inner integral can be evaluated exactly; after a change of variable (18) becomes

$$I_1 = \left[\frac{\Omega_d \Gamma(d/2)\, 2^{2\nu-1}\Gamma(\nu)}{(2\pi)^d}\right]^{2\Theta-1} \Omega_d \Gamma(d/2) p^2 \int_{p/\Lambda}^{\infty} dx\, x^{-3} J_\nu(x)\, (x/2)^{-\nu} \qquad (19)$$

The integral over the interval $[1,\infty)$ gives a finite contribution to the p^2 term. The integral over the interval $[0,1]$ can be evaluated by expanding the Bessel function in its Taylor series. We find that

$$I_1 = \left[\frac{\Omega_d \Gamma(d/2)\, 2^{2\nu-1}\Gamma(\nu)}{(2\pi)^d}\right]^{2\Theta-1} \frac{\Omega_d \Gamma(d/2)}{4\Gamma(d/2-1)}\, p^2 \ln p + O(\Lambda^2) \qquad (20)$$

The I_2 integral can be handled in the same way except that $r \neq 0$. Although the inner integral with nonzero r can be performed exactly, it is not necessary to do so explicitly. We merely note that

$$\int \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{R}}}{k^2 + r} = \frac{2^{2\nu-1}\Gamma(\nu)}{(2\pi)^d R^{d-2}} C(rR^2) \qquad (21)$$

where $C(x)$ is analytic at $x=0$, $C(0)=1$, and $C(x) \sim 1/x$ for x large. The I_2 integral is therefore

$$I_2 = \Omega_d \left[\frac{\Omega_d \Gamma(d/2)\, 2^{2\nu-1}}{(2\pi)^d} \Gamma(\nu)\right]^{\Theta} \int_{\frac{\sqrt{r}}{\Lambda}}^{\infty} \frac{dx}{x} C^{\Theta}(x) \qquad (22)$$

The integral over $[1,\infty)$ is a finite constant which we may discard. For the integral over $[0,1]$, $C(x)$ may be expanded in its Taylor series. Thus,

$$I_2 = \frac{\Omega_d}{2} \left[\frac{\Omega_d \Gamma(d/2)\, 2^{2\nu-1}\Gamma(\nu)}{(2\pi)^d}\right]^{\Theta} \ln r + O(\ln\Lambda) \qquad (23)$$

Combining (14), (20), and (23) we have for n=1 the following simple expression

$$\eta_\Theta = \frac{4(\Theta-1)^2 \epsilon_\Theta^2}{\binom{2\Theta}{\Theta}^3} \tag{24}$$

For the general n-case, more combinatorial factors must be computed. The n-dependent factor for the numerator is $f_1(n)$, given in (6). The combinatorial factor for the fixed point vertex $u_{2\Theta}$ is more complicated. In the differential equation formulation of Ref. 7, these same combinatorial complications determine the n-dependence of the fixed point. In Ref. 7, this n-dependence is given as an integral involving three generalized Laguerre polynomials. Performing this integral we find[11] the combinatorial factor for the numerator is $f_2(n)$ where

$$f_2(n) = \frac{\langle \mathcal{D}^n(0,0)|\Theta\rangle}{\langle \mathcal{D}^1(0,0)|\Theta\rangle} \tag{25a}$$

and the inner product $\langle \mathcal{D}^n(0,0)|\Theta\rangle$ is given by the double summation

$$\langle \mathcal{D}^n(0,0)|\Theta\rangle = \sum_{i,j} \binom{\Theta+\frac{n}{2}-1}{\Theta-i}\binom{\Theta+\frac{n}{2}-1}{\Theta-j}[n-1+(2i-1)(2j-1)]\binom{i+j-2}{i-1}\binom{i+j-1+\frac{n}{2}}{i+j-2-\Theta}(-1)^{i+j}. \tag{25b}$$

With these combinatorial factors, the result for general n and general Θ is

$$\eta_\Theta = \frac{4(\Theta-1)^2}{\binom{2\Theta}{\Theta}^3}\frac{f_1(n)}{f_2^2(n)}\epsilon_\Theta^2 \tag{26}$$

It is easy to check that (26) reduces to the previously calculated results for $\sigma=2$ [8] and $\sigma=3$ [5],

$$\eta_2 = \epsilon_2^2 \frac{n+2}{2(n+8)^2}$$

$$\eta_3 = \epsilon_3^2 \frac{(n+2)(n+4)}{12(3n+22)^2} \qquad (27)$$

We note that as σ increases the coefficient of ϵ_σ^2 rapidly becomes very small, $\sim 2^{-6\sigma}$ for σ large. In the limit of large n, η_σ for odd order points reduces to a constant and, for even order points, to zero.

For all $\sigma \geq 3$ we have $d_\sigma \leq 3$, and the mean field result $\eta_\sigma = 0$ therefore applies in three-dimensional systems. However, these results and those of Refs. 5-7 may apply to higher order critical points in two-dimensional systems. In any event, the previously obtained results[8] for ordinary critical points are placed in a broader theoretical context by the extension to general σ.

We wish to thank Prof. T. S. Chang for many useful discussions.

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Fig. 1 Perturbation series to $O(\epsilon_\Theta^2)$ for the function $\sum(\vec{p}, r)$, defined by Equation (3). Each diagram carries net momentum $\vec{p}$.

Fig. 2 The mass counterterm $u_2 - r$ cancels all $\vec{p}$-independent terms in $\sum$, and the use of the generalized vertices $\bar{u}_{2k}(r)$ eliminates all closed single-vertex loops.

Fig. 3 Typical second-order contribution to $\Gamma_{2\Theta}$. The 2Θ external lines carry no momenta.

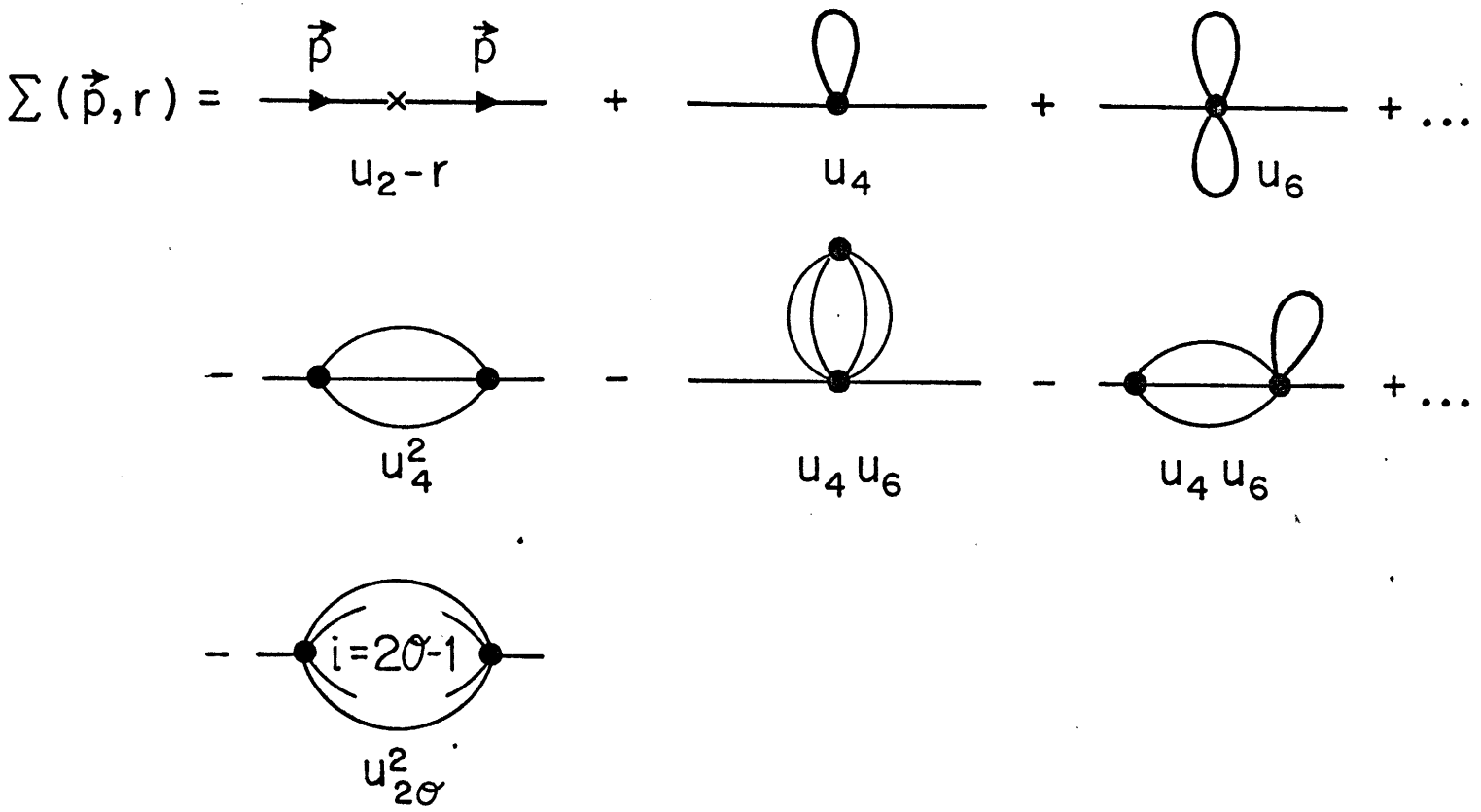

Figure 1.

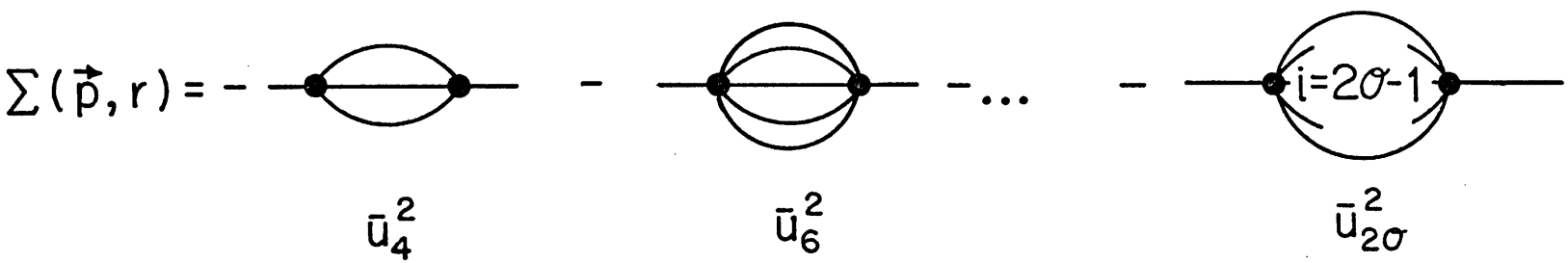

Figure 2.

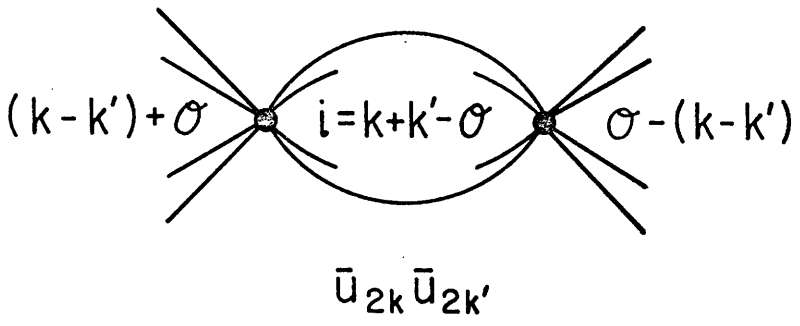

Figure 3.

III. Approximate Wilson Differential Generator and Higher-Order Critical Point Exponents for Systems with Long Range Forces*

*(submitted for publication)

Approximate Wilson Differential Generator and Higher-Order Critical Point Exponents for Systems with Long Range Forces

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We give a reformulation of Wilson's incomplete integration differential generator to simplify perturbational analysis and to facilitate comparison with the Wegner-Houghton generator. An approximate generator is then obtained in the form of a partial differential equation. The linear structure of the approximate generator introduced here is identical with that of our previous generator based on the Wegner-Houghton equation. The nonlinear structures, of course, are quite different. Using operator algebra based on an expansion in terms of Gaussians eigenfunctions, we derive in closed form the eigenvalues associated with a critical point of arbitrary order θ (θ phases simultaneously critical) and long range force exponent σ (interaction energy $\sim 1/r^{d+\sigma}$). These results are obtained to first order in the expansion parameter $\epsilon_\theta(\sigma) \equiv \theta(\sigma - d) + d$. We calculate explicitly specific inner products of Gaussian eigenfunctions. These inner products not only give the dependence of the eigenvalues on n (the number of spin components) for the first order corrections, but also are useful in calculating the n-dependence of the exponent η_θ to second order. This correction vanishes for $\sigma < 2$ for all θ so that $\eta_\theta = 2 - \sigma$. Previously, it has been demonstrated that to leading order the value of η_2 for $\sigma = 2$ is independent of the cutoff function in Wilson's original formulation. We give an alternative proof in terms of our modified exact generator.

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I. Introduction

The use of differential generators for the renormalization group [1,2] has several advantages over finite or recursive formulations. (i) In a recursive renormalization group, the equations will contain the renormalization factor explicitly. The eventually calculated critical point exponents are independent of this factor, which therefore represents an unneccessary complication. A differential generator avoids the difficulty. (ii) The differential equations obtained from a differential generator will, in general, have a far simpler form than the corresponding recursion relations. This is the case because the recursion relations must exhibit all the feedback that results from the finite amount of renormalization. (iii) Differential equations are amenable to more analytic solution techniques than recursion relations. This is particularly true of the nonlinear study of renormalization group equations[3-6].

There are presently two exact differential generators. The Wegner-Houghton generator[1] represents the differential limit of the finite generator of Wilson[2]. It gives the differential changes in Hamiltonian parameters when an infinitesimal shell of momenta is ingrated over. The Wilson differential generator[2] represents a partial integration in which the larger wave-vectors are more completely integrated than the small wave-vectors. In this paper, we compare and contrast the two generators. To do so, we reformulate the Wilson generator in such a way that it more closely resembles the Wegner-Houghton generator. We may then solve a large class of problems (to first order in a perturbation series) to show agreement between the two generators. For these calculations we utilize approximate versions of both the Wegner-Houghton and Wilson generators.

For many renormalization group studies the full structure of the renormalization group equations is not needed. The location and stability analysis of fixed points can be carried out to lowest order in a perturbation expansion. Studies of anisotropic systems[7], metamagnets[8] bicritical and

tetracritical points[9], tricritical points[10], critical points of arbitrary order[12,13], coupled order parameters [4,14] and nonlinear effects [3-5], to name a few, can be studied via approximations of the exact renormalization group equations. Detailed calculations of higher order correction to critical point exponents (such as the calculation of η_2[15,16]), of course require the full exact equations. However, in many cases the essential information is obtained in the lowest order expansion. Many of these results were or can be obtained with the approximate recursion formula of Wilson [2,17].

We recently introduced [13] an approximate differential generator based on the exact Wegner-Houghton generator. We wrote it in a form suitable for isotropically interacting systems: it is easily generalized for non-isotropic systems with long range forces (interaction $\sim 1/r^{d+\sigma}$ where σ is a "long range force exponent" [18]). It may be written as

$$\frac{\partial H}{\partial \ell} = dH + \left(\frac{\sigma - d}{2}\right)\vec{s}\cdot\vec{\nabla}H + \det\ln\left[I + \hat{H}\right] \qquad (1)$$

where $H(\vec{s}, \ell)$ is a function of a n-component spin vactor $\vec{s}$ in a d-dimensional space, ℓ is the renormalization parameter and $\hat{H}$ is the matrix of second partial derivatives of H with respect to the components of the spin: $\hat{H} = \| \partial^2 H/\partial s_i \partial s_j \|$. For $\sigma \geq 2$, the short range value $\sigma = 2$ is to be used.

A similar approximate generator based on the Wilson partial integration generator can be derived (see discussion below) and is given by

$$\frac{\partial H}{\partial \ell} = dH + \left(\frac{\sigma - d}{2}\right)\vec{s}\cdot\vec{\nabla}H + \nabla^2 H - \vec{\nabla}H\cdot\vec{\nabla}H \qquad (2)$$

In Section II, we introduce an ϵ_θ (σ) expansion for critical points of arbitrary order θ (θ phases simulataneously critical) with long range force exponent σ,. This generalizes our previous work[13] for short range forces (θ arbitrary, $\sigma \geq 2$) and the work of Fisher, Ma, and Nickel [18] for $\theta = 2$ and arbitrary σ. These calculations are made with the approximate generators (1) and (2) and are exact to 0 (ϵ_θ (σ)). We also find, by use of the method of Ref. 16, that η_θ "sticks" to the classical value of $2 - \sigma$ for all θ, in agreement with the $\theta = 2$ result of Ref. 18.

In Section III, we point out that the nonlinear structures of the two generators are very different as might be expected on general grounds[19].

In Section IV, we give a derivation of the approximate Wilson generator (2) from a reformulation of the exact Wilson generator. This reformulation is made to simplify the renormalization equations when λ expanded around the Gaussian fixed point solution.

The reformulated exact generator contains an arbitrary function of a different character than that of the original generator. The $\theta = 2$, $\sigma = 2$ critical point exponent has been shown to be independent of the arbitrary cutoff function of the original formulation.[20-22] In Section V, we show that this independence is characteristic of the reformulated generator as well.

II. The $\epsilon\sigma(\sigma)$ Expansion

Recently[20], it has been shown that the exact Wilson differential generator reproduces the earlier results in the 4-d expansion,[15] at least to lowest order. Subsequently, several authors [20-22] have shown that at least the exponent η_2 is correctly given to second order with this generator ($\eta_2 = (4-d)^2/54$ for Ising n=1 systems). With the approximate generator (2) we are able to show agreement between the Wilson and other renormalization group formulations for a wide class of critical systems.

We consider a Landau-like critical point of order σ for a single-order parameter at which σ phases are simulataneously critical. The mean-field theory for such a system is a polynomial of degree σ in $\vec{s}\cdot\vec{s}$. The renormalization group treatment of such a system is closely related to the "Gaussian" eigenfunctions of the renormalization group equation; in the case of (1) and (2) these are the eigenfunctions when the equations are linearized around H = 0. Both (1) and (2) have the same linearized structure and eigenfunctions. As noted by Wegner [7(b)] in the context of the Wilson approximate recursion formula [2,17], these eigenfunctions are products of generalized Laguerre polynomials and harmonic polynomials[23].

$$Q_{p,m} = L_p^{m-1+n/2}\left[(d-\sigma)s^2/4\right] P_m(\vec{s}). \tag{3}$$

with eigenvalue $\lambda_{p,m} = \left[d + (\sigma - d)\cdot(p + m/2)\right]$. For n-component spins, there are $(2m+n-2)\,(m+n-3)!/(m!\,(n-2)!)$ harmonic polynomials $P_m(s)$ of degree m; all of these are degenerate with respect to the linearized renormalization group equations. The eigenfunctions are also eigenfunctions of the exact Wegner-Houghton and Wilson differential generators when linearized around the Gaussian functional (cf. discussion below).

We restrict our attention to isotropic systems (m=0). The renormalization group study of an order σ system is simplified if the eigenvalue of the Laguerre polynomials of order σ is small. We therefore define an expansion parameter $\epsilon_\sigma(\sigma)$ by

$$\epsilon_\sigma(\sigma) = \lambda_{\sigma,0}$$
$$= d + \sigma(\sigma - d) \tag{4}$$

For $\sigma = 2$, this is the ϵ of Ref. 18; for $\sigma = 2$, this is the expansion parameter for higher order critical points discussed in Refs. 12-13.

We first locate the fixed point Hamiltonian. We expand the fixed point Hamiltonian as

$$H^* = a\,\epsilon_\sigma(\sigma)\,Q_\sigma + [\epsilon_\sigma(\sigma)]^2 H^{(2)} + \ldots \tag{5}$$

and substitute this into the fixed point equation $\partial H^*/\partial \ell = 0$. To the order required we may represent both generators (1) and (2) by

$$\frac{\partial H}{\partial \ell} = \mathcal{L} H + \mathcal{D}_{(p)}(H,H) \tag{6}$$

where $\mathcal{L}$ is a linear differential operator common to both (1) and (2) and $\mathcal{D}_{(1)}$ and $\mathcal{D}_{(2)}$ are the quadratic parts of generators (1) and (2) respectively. Upon inserting (5) into (6) we find

$$0 = aQ_\sigma + a^2 \mathcal{D}_{(p)}(Q_\sigma, Q_\sigma) + \mathcal{L} H^{(2)} + \ldots \tag{7}$$

We can choose $H^{(2)}$ orthogonal to Q_σ and take the inner product of (7) with Q_σ to determine a:

$$0 = \langle Q_\sigma | Q_\sigma \rangle = a \langle \mathcal{D}_{(p)}(Q_\sigma, Q_\sigma) \mid Q_\sigma \rangle \tag{8}$$

It is convenient to define $[i,j;k]$ by

$$[i,j;k] \equiv \langle \mathcal{D}_{(p)}(Q_i, Q_j) \mid Q_k \rangle / \langle Q_k \mid Q_k \rangle \tag{9}$$

Then we may write (8) as

$$1 = -a\,[\sigma,\sigma;\sigma] \tag{10}$$

We now determine the eignevalues of the new eigenfunctions when the generators are linearized around H^*. Since the fixed point Hamiltonian H^* differs from the Gaussian fixed point H = 0 only slightly, we expect the eigenfunctions and eigenvalues to be changed only by $O(\epsilon_\sigma(\sigma))$ amounts. We set

$$\bar{Q}_l = Q_l + \epsilon_\sigma(\sigma)\, q_l \tag{11a}$$

$$\bar{\lambda}_l = \lambda_l + \epsilon_\sigma(\sigma)\, \delta\lambda_l \tag{11b}$$

Inserting $H = H^* + \bar{Q}_l$ into (6) we find that

$$\delta\lambda_l Q_l = \mathcal{L} q_l + 2a\, \mathcal{D}_{(p)}(Q_\sigma, Q_l) + \ldots \tag{12}$$

Choosing q_ℓ orthogonal to Q_ℓ and taking the inner product of (12) with Q_ℓ gives

$$\delta\lambda_\ell = 2a[\sigma, \ell; \ell]. \tag{13a}$$

Using (10) to eliminate a ,

$$\bar{\lambda}_\ell = d + \ell(\sigma - d) - 2\epsilon_\sigma(\sigma)[\sigma, \ell; \ell]/[\sigma, \sigma; \sigma]. \tag{13b}$$

The evaluation of $[\sigma, \ell; \ell]$ involved the integration of the product of three Laguerre polynomials. For n=1, the Laguerre polynomials reduce to Hermite polynomials and we may evaluate (13) in closed form. For both (1) and (2) we find

$$\bar{\lambda}_\ell = d + \ell(\sigma - d) - 2\epsilon_\sigma(\sigma)\binom{2\ell}{\sigma}/\binom{2\sigma}{\sigma} \tag{14}$$

We have not been able to demonstrate that the generators agree for arbitrary n. However, we have checked that (1) and (2) agree for $\sigma = 2$ and $\sigma = 3$. We find

$$\bar{\lambda}_\ell = d + \ell(\sigma - d) - \epsilon_2(\sigma)\,\ell(6\ell + n - 4)/(n + 8) \tag{15a}$$

for $\sigma = 2$ and

$$\bar{\lambda}_\ell = d + \ell(\sigma - d) - \epsilon_3(\sigma)\,\ell(\ell-1)(10\ell + 3n - 8)/[3(3n + 22)] \tag{15b}$$

for $\sigma = 3$. For a general n and σ we can compute $[\sigma, \ell; \ell]$ as a sum:

$$[\sigma, \ell; \ell] = \sum_{i,j=1}^{\sigma-1} \binom{\sigma - 1 + n/2}{\sigma - 1 - j} (-1)^j \binom{j}{i} \binom{\ell}{i+1} \binom{\ell + j - i + n/2}{j-1} F_{ij} \tag{16}$$

where $F_{ij} = [i+(j+1)\ (1-2i)]$ for generator (1) and $F_{ij} = (j+1)$ for generator (2).

Equations (13) - (14) give the lowest order corrections to most critical point exponents. However, the shift of the critical point exponent η_σ from its classical value $\eta_\sigma = 2 - \sigma$, cannot be calculated from the approximate generators. In Ref. 16, η_σ is calculated by field theoretic techniques for critical points of order σ and $\sigma = 2$. The result is

$$\eta_\sigma = f\,[\epsilon_\sigma(2)]^2 \cdot 4 / \binom{2\sigma}{\sigma}^3 \tag{17a}$$

where f is given by

$$f = \frac{\left[\prod_{j=1}^{\sigma-1} (n+2j)\right] \left([\sigma,\sigma;\sigma]_{n=1}\right)^2}{(2\sigma-1)!! \cdot \left([\sigma,\sigma;\sigma]\right)^2} \tag{17b}$$

Thus, we have f = 1 for n = 1. By an extension of the method used in Ref. 16, we find that for $\sigma < 2$, η_σ "sticks" at the classical value $\eta_\sigma = 2 - \sigma$ to $O(\epsilon_\sigma(\sigma)^2)$, as found by Ref. 18 for the special case $\sigma = 2$.

III Comparison of Nonlinear Structures at (1) and (2)

Although the linear structures of (1) and (2) are identical and both seem to give the same corrected eigenvalues, the full nonlinear structures are very different[19]. For example, if we consider a Landau-Ginzberg-Wilson expansion of H for a single order parameter,

$$H = rs^2/2 + us^4/4! + vs^6/6! + \ldots \tag{18}$$

We may generate equations for r,u, and v from both (1) and (2). Setting n=1 for simplicity, we have

$$\dot{r} = 2r + u/(1+r) \tag{19a}$$

$$\dot{u} = (4-d)\ u + v/(1+r) - (3/2)\ u^2 / (1+r)^2 \tag{19b}$$

$$\dot{v} = (6-2d)\ v - 15uv/(1+r)^2 + 0\ (u^3), \tag{19c}$$

for the Wegner-Houghton based generator (1). On the other hand, the Wilson-based generator (2) gives

$$\dot{r} = 2r\ (1-r) + u \tag{20a}$$

$$\dot{u} = (4-d)\ u + v - 8ru \tag{20b}$$

$$\dot{v} = (6-2d)\ v - 12rv - 20u^2 \tag{20c}$$

The "propagator factors" of (1+r) characteristic of generator (1) are absent in (2). In recent work, we make a nonlinear transformation to remove the propagators and change the high temperature fixed point from r = ∞ to r = 1. This step is not necessary for (2), which incorporates this change automatically. Note, however, that (2) introduces more feedback terms than (1). For small r,u, and v, (19c) shows that v is 0 (u^3) and therefore v can be neglected in (19b). On the other hand, from (20c) we see that v is 0 (u^2), and therefore v must be considered in (20b).

The nonlinear structure of generator (1) has been explored for a variety of problems in Refs. 4-5. Because of the extra feeback terms, a similar analysis using (2) would be more difficult. Such a study would be extremely interesting, however, since the details of crossover solutions [4-5] would appear to depend on the details of the nonlinear structures of the generators.

IV Derivation of (2) from the Exact Wilson Equations

To derive an approximate differential generator from the Wilson partial integration generator, it is convenient to put the exact Wilson equations into

a form closely resembling the exact Wegner-Houghton equations. This reformulation is slightly more convenient for perturbation expansions from the Gaussian fixed point solution, as illustrated below.

We begin with the exact Wilson equation for Ising like spins (n = 1),

$$\dot{\mathcal{H}} = d\mathcal{H} + \int_k s_k [(2-d-\eta)/2 + \beta(k^2) - k\frac{\partial}{\partial k}] \frac{\delta \mathcal{H}}{\delta s_k}$$

$$+ \int_k [1-\eta/2 + \beta(k^2)] \left\{ \frac{\delta^2 \mathcal{H}}{\delta s_k \delta s_{-k}} - \frac{\delta \mathcal{H}}{\delta s_k} \frac{\delta \mathcal{H}}{\delta s_{-k}} \right\} \qquad (21)$$

This expression differs from that given in Ref. 2 since in (21) the operator $k\frac{\partial}{\partial k}$ does not act on the s_k or momentum conserving delta-functions in the expansion of $\mathcal{H}$, while in Ref. 2 $k\frac{\partial}{\partial k}$ acts only on the s_k. The difference is simply an integration by parts. The function β is an arbitrary (increasing) function of k^2 (e.g., $\beta = k^2$).

The presence of β in the first integral in (21) as well as in the second integral is an inconvenience for some calculations. If we make the change of variable

$$s_k \longrightarrow s_k \sqrt{\frac{1-\eta/2 + \beta(k^2)}{C(k^2)}} \qquad (22)$$

Eq. (21) can be rewritten as

$$\dot{\mathcal{H}} = d\mathcal{H} + \int_k s_k [(2-d-\eta)/2 + q(k^2) - k\frac{\partial}{\partial k}] \frac{\delta \mathcal{H}}{\delta s_k}$$

$$+ \int_k C(k^2) \left[\frac{\delta^2 \mathcal{H}}{\delta s_k \delta s_{-k}} - \frac{\delta \mathcal{H}}{\delta s_k} \frac{\delta \mathcal{H}}{\delta s_{-k}} \right] \qquad (23a)$$

where q is related to β and C by

$$q(k^2) = \beta(k^2) + \frac{k}{2}\frac{\partial}{\partial k} \ln[(1-\eta/2+\beta(k^2))/C(k^2)] \quad (23b)$$

The function C (k^2) is a cutoff function in the usual sense; for example, C = $\theta(1-k^2)$, a Brillouin zone cutoff, or C= exp ($-k^2$), a smooth cutoff. In passing from (21) to (23) we have increased the number of arbitrary functions from one (β) to two (q and C). We reduce the number to one again by examining the Gaussian fixed point solution.

The Gaussian solution is defined as the fixed point solution of the form

$$\mathcal{H}_G = \frac{1}{2}\int_k W(|k|) S_k S_{-k} \quad (24a)$$

The function w satisfied

$$(\sigma + 2q - k\frac{\partial}{\partial k}) W = 2C(k^2) W^2 \quad (24b)$$

where we have set $\eta = 2 - \sigma$. If we expand around the Gaussian solution, $\mathcal{H} = \mathcal{H}_G + \mathcal{H}'$, we have

$$\delta\eta\, \mathcal{H}_G + \dot{\mathcal{H}}' = d\mathcal{H} + \int_k S_k [(2-d-\eta)/2 + q - 2wC - k\frac{\partial}{\partial k}] \frac{\delta\mathcal{H}'}{\delta S_k}$$

$$+ \int_k C(k^2) \left[\frac{\delta^2\mathcal{H}'}{\delta S_k \delta S_{-k}} - \frac{\delta\mathcal{H}'}{\delta S_k}\frac{\delta\mathcal{H}'}{\delta S_{-k}} \right] \quad (25)$$

where $\delta\eta = \eta - 2 + \sigma$.

Equation (25) is simplified if q and C are related by q = 2 wC. Combining this with (14b) we have

$$W = \frac{|k|^{\sigma}}{\int_{k^2}^{\infty} C(x)\, x^{\frac{\sigma-2}{2}}\, dx} \tag{26}$$

In (26) the upper limit of the integral has been chosen so that q and β are increasing functions of |k| for large k .

In this formulation, the generator (25) when linearized around the Gaussian fixed point has the same form as the similarly linearized Wegner-Houghton generator

$$\dot{\mathcal{H}} = d\mathcal{H} + \int_k S_k[(\sigma-d)/2 - k\frac{\partial}{\partial k}]\frac{\delta\mathcal{H}}{\delta S_k} + \int_k C(k^2)\frac{\delta^2\mathcal{H}}{\delta S_k \delta S_{-k}} \tag{27}$$

For the Wegner-Houghton generator, the cutoff function C is replaced by $\delta(|k| - 1)$. Equation (27) admits solutions with momentum independent expansion coefficients, in contrast with the usual formulations of the Wilson generator[2, 19-22, 26].

If we now set $\delta\eta = 0$ in (25) and neglect momentum dependence by considering the limit of all k's $\to 0$, we have the following equation for H (s, ℓ)

$$\dot{H} = dH + \left(\frac{\sigma-d}{2}\right)s\frac{\partial H}{\partial s} + \frac{\partial^2 H}{\partial s^2} - \left(\frac{\partial H}{\partial s}\right)^2 \tag{28}$$

where we have normalized C by $\int C = 1$. The linearized eigenfunctions of (28) are eigenfunctions of (27) if we identify s^m with $\int\cdots\int\delta(\vec{k}_1 + \ldots \vec{k}_m)\, s_{k_1}\cdots s_{k_m}$. The above discussion may be repeated for general n, leading to the approximate generator (2).

V. Lack of Dependence on C of η.

The reformulated equation (25), just as the original generator (21) may be used for momentum dependent calculation. As discussed in Refs. 20-22, the leading dependence of η_2 (for $\alpha = 2$, $\sigma = 2$) is independent of the function β in (21). Thus it is to be expected that η_2 is independent of the choice of C in (25).

This is somewhat easier to demonstrate than the independence of β because of the simplification of the equation given by the constraint $q = 2wC$ used in (25). It is straightforward to express η in terms of the function C. The method is similar to that of Ref. 20 and will not be detailed here. The result is (for $\sigma = 2$)

$$\eta_2 = -(\epsilon_2^2/q)\left[\int_0^\infty C(x)dx\right]^{-3} \cdot I, \quad \text{where} \tag{29a}$$

$$I = \int_0^1 dt \int_0^\infty p^2 dp \int_0^\infty q^2 dq \int_0^\pi \frac{2}{\pi} \sin^2\theta \, d\theta \; C(p^2) C(q^2) C'(p^2 + tq^2 + 2pq\sqrt{t}\cos\theta), \tag{29b}$$

with $C'(x) \equiv dC(x)/dx$.

To compute η_2, we rewrite the multiple integrals of (29b) in terms of the Fourier transform of C, $\hat{C}(z) = \int_0^\infty C(x) \exp(-izx)dx$.

Performing the angular integration gives a Bessel function and (29) becomes

$$I = \frac{i}{(2\pi)^3} \int dz_1 dz_2 dz_3 dt dx dy \; \hat{C}(z_1)\hat{C}(z_2)\hat{C}(z_3)(xy/t)^{1/2} \cdot J_1(2z_3\sqrt{t}xy) \exp\{i[xz_1 + yz_2 + (x+ty)z_3]\}. \tag{30}$$

where $x \equiv p^2$, $y \equiv q^2$. We now assume (as in Ref. 22) that the z-integrals can be deformed off the real axis so that each z has a small positive imaginary part. With the aid of this convergence factor, the integral is a well-known Bessel function integral [24] giving

$$I = -\frac{i}{(2\pi)^3}\int dz_1\, dz_2\, dz_3\, dt\, dy\, y\, \frac{z_3}{(z_1+z_2)^2}\cdot$$

$$\cdot\, \hat{C}(z_1)\,\hat{C}(z_2)\,\hat{C}(z_3)\exp\left[iy z_2 + i t y z_1 z_3/(z_1+z_3)\right] . \tag{31}$$

The y and t integrations are now elementary and we obtain (after symmetrizing in z_1, z_2 and z_3)

$$I = \frac{i}{6(2\pi)^3}\left[\int dz\, \hat{C}(z)\right]^3 \tag{32}$$

Because C (x) $\equiv$ 0 for x$<$0, $\hat{C}(z)$ is analytic in the lower half plane. We may therefore, close the contour down and write

$$I = -(1/6)\left[\hat{C}(0)\right]^3 \tag{33}$$

However, $\hat{C}(0) = \int C(x)dx$, Therefore, from (29) we find $\eta_2 = \epsilon_2^2/54$ independent of the cutoff function C.

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CHAPTER 5

GLOBAL PROPERTIES OF NONLINEAR RENORMALIZATION GROUP EQUATIONS

I. Nonlinear Solutions of Renormalization Group Equations*

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Nonlinear Solutions of Renormalization-Group Equations*

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We give the first nonlinear solution of renormalization-group equations. This solution, based on the differential generator of Wegner and Houghton, exhibits an explicit mixing of (or crossover between) critical and mean-field behavior. The solution is given for all values of the spin dimension n and to first order in $\epsilon \equiv 4-d$, where d is the lattice dimensionality.

Recently[1,2] much work has been devoted to the renormalization-group equations linearized around various fixed points. Each fixed-point Hamiltonian governs a particular class of critical phenomena. The linearized equations about a fixed point have solutions which represent scaling equations of state, with critical-point exponents which are simply calculated from the eigenvalues of the linearized renormalization-group equations. The physically measurable exponents are those of the dominant fixed point. The analysis of a single fixed point is therefore sufficient to discuss the critical behavior *asymptotically close to* the critical point. However, *at finite distances from the critical point*, the competing influences of the many other fixed points may become important. This competition between fixed points is loosely described as "crossover"; the physical system passes from the domination of one fixed point to the domination of another.

Riedel and Wegner,[3] using a semimicroscopic model which simulates renormalization-group crossover, have discussed the competition between tricritical and critical behavior. Here we present the first cross over solution based *directly* on the nonlinear renormalization-group equations. The solution given describes the transition from true critical behavior near the critical point to mean-field–like behavior at higher temperatures.[4,5]

To preface the discussion of the *nonlinear* solution itself, we will first give a general abstract description of the solution of a *linear* renormalization-group equation. This will also serve to establish our notation. Generally a renoralization-group representation near a fixed point can be written as a set of linear differential equations. For example, a model Hamiltonian parametrized by variables p and q might be described by the equations.

$$\dot{p} = 2p, \tag{1a}$$

$$\dot{q} = \epsilon q, \tag{1b}$$

where the dot denotes the derivative with respect to the renormalization parameter l and $\epsilon \equiv 4-d$, where d is the lattice dimension. The fundamental equation defining the renormalization parameter itself is given by the renormalization trajectory for the correlation length, $\xi(p, q)$,

$$\dot{\xi} = -\xi. \tag{2}$$

The solutions of Eq. (1) are

$$p = \text{const} e^{2l}, \tag{3a}$$

$$q = \text{const} e^{\epsilon l}. \tag{3b}$$

The solution of Eq. (2) is a generalized homogeneous function,

$$\xi(\lambda^2 p, \lambda^\epsilon q) = \lambda^{-1}\xi(p, q). \tag{4}$$

The correlation-length solution is more usually written as

$$\xi(p, q) = p^{-1/2} P(p^\epsilon/q^2), \tag{5}$$

where P is any arbitrary function which, however, is assumed to be regular and nonzero at $p = 0$. We call p and q scaling fields. They play the same role in Eq. (4) as the scaling variables of the usual scaling theory. In this case, the critical-point exponent $\nu = \frac{1}{2}$.

More generally, Eqs. (1) will have nonlinear terms as well as linear ones. However, there will still be functions of p and q (not simply equal to p and q) which have a simple exponential dependence on the renormalization parameter. We will call these functions the nonlinear scaling fields.[6] The correlation length is again a generalized homogeneous function, not of p and q, but of the corresponding nonlinear scaling fields.

Wegner and Houghton[1] have suggested a differential generator for the renormalization group which reproduces the results of Wilson's finite-difference generator. For nonlinear solutions good to first order in ϵ, the momentum-indepen-

dent equations of Ref. 1 reduce to

$$\dot{r} = 2r + \frac{u}{1+r}\frac{d}{2}\frac{n+2}{n}, \tag{6a}$$

$$\dot{u} = (4-d)u - \frac{u^2}{(1+r)^2}\frac{d}{2}\frac{n+8}{n}, \tag{6b}$$

where r and u are the momentum-independent two- and four-spin coefficients in Wilson's reduced Hamiltonian.[2]

The character of Eqs. (6) is more easily seen after a transformation which maps the solution trajectories of interest into a finite region. We define new variables $\bar{r}$ and $\bar{u}$ by

$$\bar{r} \equiv r/(1+r), \tag{7a}$$

$$\bar{u} \equiv u/(1+r)^2. \tag{7b}$$

The fundamental equations now take the form

$$\dot{\bar{r}} = 2(1-\bar{r})[\bar{r} + \bar{u}d(n+2)/4n], \tag{8a}$$

$$\dot{\bar{u}} = \bar{u}[\epsilon - \bar{u}(3d/2n)(n+4) - 4\bar{r}]. \tag{8b}$$

There are three fixed points of physical interest ($u \geq 0$): the "finite" Gaussian point at $\bar{r} = \bar{u} = 0$; the "infinite" Gaussian point at $\bar{r} = 1$, $\bar{u} = 0$; and the Wilson-Fisher[7] point at $\bar{r} = -\epsilon(n+2)/2(n+8)$, $\bar{u} = \epsilon 2n/d(n+8)$.

Equations (8) are already in diagonal form around the infinite Gaussian fixed point ($\bar{r} = 1$, $\bar{u} = 0$). It is also useful to diagonalize (8) around the finite Gaussian fixed point ($\bar{r} = \bar{u} = 0$). Defining new variables x and y by

$$x \equiv \bar{r} + [\bar{u}/(2-\epsilon)][d(n+2)/2n], \tag{9a}$$

$$\epsilon y \equiv \bar{u}d(n+8)/2n, \tag{9b}$$

we rewrite Eqs. (8) as

$$\dot{x} = 2x\{1 - x - [(n+2)/2(n+8)]\epsilon y\}, \tag{10a}$$

$$\dot{y} = y[\epsilon(1-y) - 4x]. \tag{10b}$$

We have neglected terms of order $\epsilon^2 y^2$ in (10) consistent with (6). This approximation puts (8) and (10) into the same form. We also note (cf. Fig. 1) that the various fixed points are located at $x = y = 0$ (finite Gaussian); $x = 1$, $y = 0$ (infinite Gaussian); and $x = 0$, $y = 1$ (Wilson-Fisher).

We may write the solutions to Eqs. (8) in terms of two functions R and U, which satisfy the equations

$$\dot{R} = 2(1-\bar{r})R, \quad \dot{U} = d\bar{u}U. \tag{11}$$

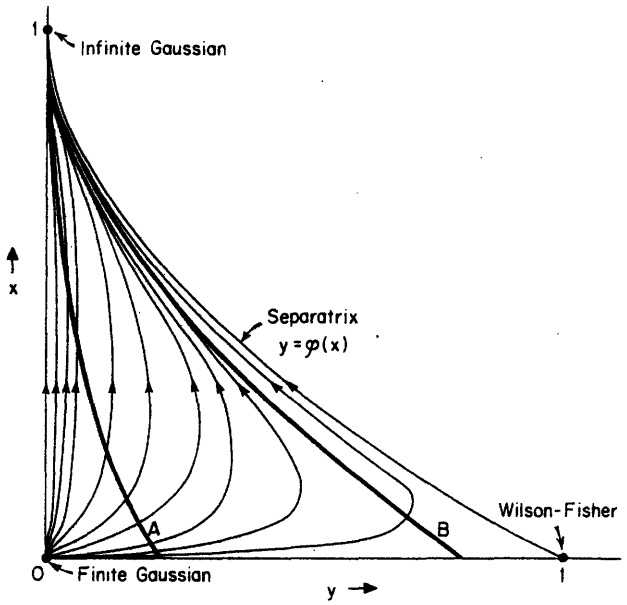

FIG. 1. Qualitative behavior of renormalization-group and temperature trajectories. The light lines depict the renormalization-group trajectories for the parameters x and y [cf. Eqs. (9) and (10)]. The heavy lines labeled A and B depict temperature trajectories for different system Hamiltonians [cf. Eqs. (24)].

The solutions are given by the scaling fields

$$(\bar{u}/R^2)U^{3(n+4)/2n} = \text{const}e^{-dl}, \tag{12a}$$

$$[(1-\bar{r})/R]U^{(n+2)/2n} = \text{const}e^{-2l}. \tag{12b}$$

The advantage of this formulation becomes apparent when we perform a similar calculation for Eqs. (10). Defining F and G through the equations

$$\dot{F} = -2xF, \tag{13a}$$

$$\dot{G} = -\epsilon y G, \tag{13b}$$

we discover that the scaling fields can be written as

$$y/GF^2 = \text{const}e^{(4-d)l}, \tag{14a}$$

$$x/FG^{(n+2)/(n+8)} = \text{const}e^{2l}. \tag{14b}$$

Since both sets of scaling fields describe the same solutions, we may match them to reduce the number of unknown functions. Noting that $U = G^{-2n/(n+8)}$, we find that

$$F = 1 - \bar{r}, \tag{15a}$$

$$R = xG^{-2(n+2)/(n+8)}. \tag{15b}$$

All that remains is the calculation of G. The partial differential equation for G can be solved

in terms of the separatrix connecting the Wilson-Fisher point with the infinite Gaussian point; this separatrix is indicated as $y=\varphi(x)$ in Fig. 1. The function φ satisfies

$$2x\{(1-x)-[(n+2)/2(n+8)]\epsilon\varphi\}d\varphi/dx = \varphi[\tfrac{1}{2}(1-\varphi)-4x]. \quad (16)$$

On this separatrix G is identically zero. Using Eqs. (16) we may write G as

$$G=(1-y/\varphi)e^{g}, \quad (17a)$$

where g satisfies

$$\dot{g}=-\frac{n+2}{n+8}\epsilon x\frac{y}{\varphi}\frac{d\varphi}{dx}. \quad (17b)$$

Solving Eqs. (16) and (17b) together we find (to order ϵ)

$$\varphi=(1-x)^{d/2}\exp[\tfrac{1}{2}\epsilon x(4-n)/(n+8)], \quad (18a)$$

$$G=\left(1-\frac{y}{\varphi}\right)\exp\left[\frac{n+2}{n+8}\epsilon x\frac{y}{\varphi}\right]. \quad (18b)$$

Equations (6) are now completely solved (to order ϵ). We define the Gaussian and Wilson-Fisher scaling fields by

$$S_G=xG^{-(n+2)/(n+8)}/(1-\bar{r}), \quad (19a)$$

$$S_{WF}=xy^{-(n+2)/(n+8)}/(1-\bar{r})^{(4-n)/(n+8)}. \quad (19b)$$

The behavior of any function whose renormalization behavior is known can be expressed in terms of a generalized homogeneous function. If Q is a function that satisfies the renormalization transformation

$$\dot{Q}=a_Q Q, \quad (20)$$

then Q satisfies

$$Q(\lambda^{a_H}H, \lambda^{a_G}S_G, \lambda^{a_{WF}}S_{WF}) = \lambda^{a_Q}Q(H, S_G, S_{WF}), \quad (21)$$

where H is the ordering field, and[8]

$$a_H=1+d/2, \quad a_G=2, \quad a_{WF}=2-\epsilon(n+2)/(n+8). \quad (22)$$

In particular, the correlation length satisfies (20) with $a_Q=-1$; the Gibbs potential satisfies (20) with $a_Q=d$.[9] An example of a correlation length which satisfies (21) is

$$\xi=\left[\frac{y^{(n+2)/(n+8)}}{x}\frac{(1-\bar{r})^{(4-n)/(n+8)}}{x}\right]^{1/a_{WF}} + A\left[\frac{(1-\bar{r})G^{(n+2)/(n+8)}}{x}\right]^{1/a_G}. \quad (23)$$

For any nonzero y (at the critical temperature), the Wilson-Fisher term will dominate asymptotically near the $x=0$ ($T=T_c$) singularity [provided that $a_{WF}<a_G$, i.e., $\epsilon(n+2)/(n+8)>0$], giving $\nu=\frac{1}{2}+\epsilon(n+2)/4(n+8)$. However, for finite x ($T\neq T_c$) the Gaussian term may become important. This would give mean-field behavior, characterized by the exponent $\nu=\frac{1}{2}$. The "rate" of the crossover (between critical and mean-field behavior) depends on the magnitude of the constant A and on the explicit temperature dependences of x and y.

The temperature dependence of the two- and four-spin coefficients r and u will vary from model to model. For the case of two-spin interaction models, for which the four-spin term is introduced as a phase-space weight factor, the only temperature dependence is in the two-spin term, $r(T)$. It is straightforward to show that, in this case, the temperature trajectories are

$$1-x=(1+r_c)^{-1}(y/y_c)^{1/2}[1+r_c(y/y_c)^{1/2}], \quad (24a)$$

where r_c is the value of r at the critical temperature,

$$\frac{r_c}{1+r_c}=-\frac{\epsilon y_c}{2-\epsilon}\frac{n+2}{n+8}, \quad (24b)$$

and y_c is the value of y at the critical temperature. Two temperature trajectories are shown by the heavy lines labeled A and B in Fig. 1. It is clear that, for a given change of x, temperature trajectory A crosses more renormalization-group trajectories than does temperature trajectory B. To make this more quantitative, the renormalization trajectories can be labeled by the renormalization invariant I:

$$I=x(1-\bar{r})^{d}G^{a_{WF}}/y^{2}. \quad (25)$$

The invariant I is zero on the separatrices passing through the Wilson-Fisher fixed point [$x=0$ and $y=\varphi(x)$]. It is infinite on the limiting integral curve ($y=0$) joining the finite Gaussian fixed point to the infinite Gaussian fixed point. It may therefore be used as a measure of the criticality of a system. A small invariant characterizes a system dominated by the Wilson-Fisher fixed point, while a large invariant indicates that the system is dominated by the Gaussian or mean-field behavior. The crossover of a system from critical to mean-field behavior is governed by the rate of growth of the invariant. For the two-spin systems under consideration [temperature trajectories given by (24)] and $n=-2$ (for simplic-

ity)[10]

$$I(T)=\frac{1}{y_c^{\,2}}\left(\frac{x(T)}{1-x(T)}\right)^{\epsilon}\left(1-\frac{[1-x(T)]^{\epsilon/2}y_c}{e^{\epsilon x(T)/2}}\right)^{2}. \quad (26)$$

For small y_c, $I(T)$ is a rapidly varying function of $x(T)$; for y_c near 1, $I(T)$ varies very slowly. For $x(T)$ monotonically increasing, $I(T)$ is also monotonic in T, cutting each renormalization-group trajectory exactly once. Similar behavior holds for general n.

If $x(T)\to 1$ as $T\to\infty$, the temperature trajectories all pass through the infinite Gaussian point at $x=1$, $y=0$. This requires that $r(T)\to\infty$ for $T\to\infty$. For realistic Hamiltonians, $r(T)$ has a finite limit at infinite temperature,[11] and the formal crossover properties of the renormalization-group equations are not completely realized. Moreover, even before the limiting values of x and y are approached (whether these limits are at the infinite Gaussian point or not) the correlation length and other thermodynamic functions will be dominated by their high-temperature behavior, rather than by the limiting behavior of an expression such as Eq. (23).

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II. Global Nonlinear Renormalization Group Analysis for Magnetic Systems*

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GLOBAL NONLINEAR RENORMALIZATION GROUP ANALYSIS FOR MAGNETIC SYSTEMS

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ABSTRACT

Recent applications of the renormalization group to critical phenomena in magnetic systems have been based mainly on local linear arguments. It has been implicitly assumed that the global nonlinear effects are important only in crossover effects and that the behavior asymptotically close to the critical point is determined by the stablest fixed point alone. We have given a nonlinear analysis which incorporates the crossover between the Wilson-Fisher and mean field behavior. We point out in this paper that this competition expresses itself in globally valid solutions which can upset the dominance presumed from the linear stability analysis unless certain regularity conditions are imposed.

In a recent paper[1], we gave a solution describing the crossover between Gaussian and Wilson-Fisher[2] (WF) critical behavior of a set of renormalization group equations within an ϵ expansion ($\epsilon \equiv 4-d$). Briefly we considered a reduced Hamiltonian density of the form (for an n-component spin $\vec{s}$)

$$H = |\nabla \vec{s}|^2 + r\vec{s}^2 + u(\vec{s}^2)^2 \qquad (1)$$

Applying an approximate form[3] of the differential generator of Wegner and Houghton[4] we obtained nonlinear differential equations for the renormalization behavior of the parameters r and u. We found, that nonlinear scaling fields can be written to $O(\epsilon)$ as

$$S_G = x\bar{G}^{-(n+2)/(n+8)}(1+r) \qquad (2a)$$

$$S_{WF} = xy^{-(n+2)/(n+8)}(1+r)^{(4-n)/(n+8)} \qquad (2b)$$

where $x \equiv r/(1+r) + u/(1+r)^2((n+2)/n)$ and $\epsilon y \equiv u/(1+r)^2(2(n+8)/n)$. The function $\bar{G}$ is given by

$$\bar{G} = (1-y/\phi)\ \exp\{((n+2)/(n+8))\varepsilon xy/\phi\} \qquad (2c)$$

where $y=\phi(x)$ is the equation of the separatrix connecting the WF to the infinite Gaussian (high temperature) fixed point (cf. Fig. 1).

Many thermodynamic functions can now be expressed as generalized homogeneous functions of the scaling fields given in (2). These scaling fields are not unique since a generalized homogeneous function of any pair of scaling fields is again a suitable scaling field. On the other hand, if we require that the scaling fields be proportional to the variable x for small $x(x\propto T-T_c)$, the scaling fields given in (2) are unique up to multipication by arbitrary functions of the renormalization invariant.

$$I = x^{\varepsilon}\bar{G}^{2-\varepsilon(n+2)/(n+8)}/y^2(1+r)^d \qquad (3)$$

which are nonzero at I=0. Note that I vanishes on the separatrix $y=\phi(x)$ as well as on the line x=0; I=∞ on the pure Gaussian trajectory y=0.

We may, however, use the scaling fields (2) without loss of generality since we have not specified the form of any generalized homogeneous function. For the zero field correlation length we can choose as an example the simple form

$$\xi = S_G^{-1/a_G} + S_{WF}^{-1/a_{WF}} \qquad (4)$$

where the scaling powers of S_G and S_{WF} are $a_G=2$ and $a_{WF}=2 - \varepsilon(n+2)/(n+8)$. Since the two scaling fields appear symmetrically in (4), this form has the virtue of reducing to the appropriate linear solution as either of the two fixed points is approached. For x fixed and $y\to 0$, $S_{WF}\to\infty$, and the Wilson-Fisher term vanishes. Similarly, as the separatrix is approached, $S_G\to\infty$, and the Gaussian term vanishes. For intermediate values of y, both singularities contribute, giving the expected nonlinear crossover.

A more complicated behavior is exhibited by the Gibbs potential G. In addition to the spin-dependent terms, an additive constant v_o in the Hamiltonian density also contributes. This spin-independent term grows at the rate of $e^{d\ell}$. To first order in ε, we find

$$G(h,S_G,S_{WF}) = e^{-d\ell}G(he^{a_h\ell},S_Ge^{a_G\ell},S_{WF}e^{a_{WF}\ell}) + (dn/2)\int_0^{\ell}d\ell' \; \ell n\{1+r(\ell')\}e^{-d\ell'} \qquad (5a)$$

where h is the ordering field and $a_h=1+d/2$. As the renormalization average proceeds ($\ell\to\infty$), information about the Gibbs potential passes from the first term on the right hand side of (5a) to the second term. In some circumstances, (in particular, zero magnetization) it may be possible to take the limit $\ell\to\infty$ and consider only the second term. This method has been utilized by some authors[5,6,7] to calculate an approximate Gibbs potential. For our case, the result would be

$$G = (n/2)\ell n\{1+r(0)\} + n\int_0^{\infty}e^{-d\ell}x(\ell)d\ell \qquad (5b)$$

However, for fixed ℓ, the second term contains information that should be unimportant for critical behavior. Accordingly, we will deal with the homogenous term only in our discussions. Thus, when discussing the Gibbs potential and its temperature like derivatives, we will confine our attention to $x\to o$, even though the solutions for the nonlinear scaling fields are valid for all $x<1$. The difficulty does not arise when studying the derivatives of the Gibbs potential with respect to the ordering field h (such as the magnetization and susceptibility) since the second term in (5a) is not dependent on h and does not contribute. We could, therefore, phrase our discussion of crossover in terms of these functions; we will discuss the Gibbs potential to allow the closest connection between this work and other phenomenological discussions of crossover.

In general, $G(h, S_G, S_{WF})$ will generate critical point exponents that do <u>not</u> satisfy exponent inequalities as equalities. This is to be expected since G depends on <u>three</u> distinct scaling powers. The usual scaling equalities which relate three exponents are satisfied because there are only <u>two</u> independent scaling powers. An example of a Gibbs potential which is a nonscaling global solution of the renormalization group equations is given by

$$G = G_G(h,S_G) + G_{WF}(h,S_{WF}) \tag{6}$$

where G_G and G_{WF} are separately generalized homogeneous functions. Each piece of the Gibbs potential generates its own singularities with exponents that satisfy equalities. However, since $\gamma_G<\gamma_{WF}$ it follows that $\alpha_G>\alpha_{WF}$. The measured exponents would be γ_{WF} and α_G. Therefore, $\alpha+2\beta+\gamma>2$.

A solution of the form (6) cannot, however, be matched to the expected linear solutions near the two fixed points since as the scaling fields diverge the Gibbs potential becomes infinite. To show this more explicitly, we consider the h=0 potential. We may write it in two ways,

$$G = \{S_G\}^{2-\alpha_G} f_G(I) \tag{7a}$$

or

$$G = \{S_{WF}\}^{2-\alpha_{WF}} f_{WF}(I) \tag{7b}$$

If the asymptotically valid value of α were α_G, then it follows that $f_G(0)$ would be a finite constant. By (3), $f_G(I)$ is also well behaved as the separatrix $y=\phi(x)$ is approached. Since S_G is singular on the separatrix, the Gibbs potential would be singular there as well. On the other hand, if the asymptotically valid value of α is α_{WF}, then $f_{WF}(0)$ is finite. It is also finite on the separatrix. However, as $y\to 0$, the invariant $I\to\infty$. Therefore, the divergence in S_{WF} as $y\to 0$ may be cancelled by an appropriate behavior of f_{WF}. An example which has this property is given by

$$G = G_G(h,S_G)G_{WF}(h,S_{WF})/(G_G+G_{WF}) \tag{8}$$

It is easy to check that (8) generates exponents that agree with those of G_{WF} alone; the linear analysis is thereby justified by the global results. If $\varepsilon<0$, the arguments given above are precisely reversed and the Gaussian fixed point (in this case the stable)

determines the asymptotically valid critical point exponents.

The analysis presented above gives a theoretical understanding of a mechanism for possible non-scaling critical behavior as global renormalization group solutions with singularities on portions of the boundary of the solution region. This possibility and further nonlinear analysis will be explored in a separate paper.[7]

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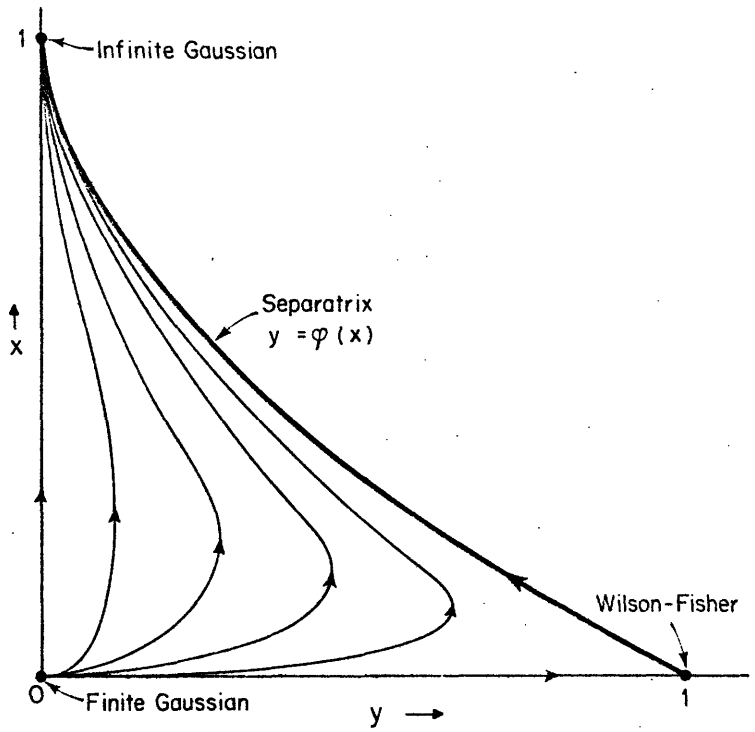

Figure 1 Nonlinear Renormalization Group Trajectories in the x-y Plane

III. Global Features of Nonlinear Renormalization Group Equations*

*(Phys. Rev. B (July, 1975))

Global Features of Nonlinear Renormalization Group Equations*

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The analysis of selected nonlinear problems in the renormalization group is found to show striking contrasts between the usual local linearized fixed point analysis and the properties of global solutions of nonlinear equations derived from an approximation of the Wegner-Houghton differential formulation. The competition between various fixed points that is incorporated in general global solutions can upset the asymptotically valid critical behavior deduced from the local analysis. In general, the critical point exponents of such a solution will not satisfy equalities, but rather the corresponding inequalities. However, these non-scaling solutions have extraneous singularities that are not related to the thermodynamic singularities of the system. If singularities of this type are excluded, then the global solution has the same critical point exponents as the local solution derived by linearizing around the stablest fixed point.

It is shown that in this case the critical surface in the Hamiltonian space is closely related to the surface of order two critical points in a thermodynamic field space. The boundaries of this surface are correspondingly related to the critical points of higher order in this thermodynamic space. The nonlinear global solution predicts multiple power scaling behavior from a single scaling equation deduced form the renormalization group. Previously such behavior was obtained by postulating the simultaneous validity of two of more "linear" scaling hypotheses.

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I. Introduction

The renormalization group approach to the study of critical phenomena is a mathematical expression of certain heuristic ideas of Kadanoff.[1] Kadanoff argued that sufficiently near the critical point, the correlation length was so large that even crude averages over small groups of spins would not alter the physics in an unmanageable way, but would only change the parameters slightly. If the transformation of the parameters is assumed to be of a particular form (the "scaling hypothesis"), then many valid and useful predictions of critical behavior follow.[1-2] In the renormalization group approach,[3-5] a particular form of Kadanoff averaging is carried out explicitly. If the system Hamiltonian is characterized by some set of parameters $\{p_i\}$ the renormalization group equations provide a definite transformation on the parameter space.

The fixed points of the renormalization group are just the fixed points of this transformation in the parameter space. As is well known from the study of nonlinear finite-difference and differential equations,[6] the qualitative and much of the quantitative properties of a set of transformations are determined by the location and study of the fixed points of those transformations. This is the rationale of the renormalization group approach: to study the transformation properties (via the fixed points) in order to deduce the properties of the partition function and other thermodynamic quantities.

A formulation of a renormalization group may be of a recursive character with a "finite-difference" generator, or it may have a differential generator. For example, if we consider a system with discrete spins localized on lattice sites, we could construct a renormalization group which replaced each spin by an average of that spin and the spins of its neighbors.[7] After averaging, the parameters of the Hamiltonian would, in general, change. The new parameters would

be given by relations of the form $p_j' = P_j(\{p_i\})$ for some functions P_j. Thus the renormalization group equations in the parameter space would take the form of finite difference equations coupling all the parameters together. A second case of a finite-difference formulation is the well-known renormalization group of Wilson.[5] It treats a system of continuum spins; the renormalization average is performed by averaging over a finite fraction[8] of the momenta in the space of the Fourier transform of the spin density.

Finite difference equations, however, are clumsy to manipulate in the large, i.e., over large domains of the variables. A differential generator, which performs an average over an infinitesimal number of degrees of freedom is far more convenient. A differential generator gives a smooth transformation of the p_j's, of the form $dp_j/d\ell = P_j(\{p_i\})$ where ℓ is a parameter describing the progress of the renormalization averaging. Various differential generators have been proposed; e.g. Wilson[5] has proposed a "partial integration" generator, while Wegner and Houghton[9] have proposed a differential generator which averages over an infinitesimal shell of momentum.

In applications of the renormalization group to critical phenomena, it is customary to perform an average which corresponds to a simple scale change of the correlation length as in Kadanoff scaling. For a <u>finite difference</u> generator we expect that the renormalization equation is of the form $\xi_{n+1} = (\text{constant})\xi_n$.

For a differential generator the parameter ℓ is usually normalized so that the renormalization equation for ξ is $\dot{\xi} = -\xi$, where the dot denotes differentiation with respect to ℓ.

With a few exceptions[10-11], the work devoted to the application of the renormalization group to critical phenomena has been confined to the location and linearized analysis of fixed points. For example, we could consider a set of two parameters p and q with renormalization group equations

$$\dot{p} = 2p\left(1 - p - \epsilon\frac{A}{2}q\right) ; \tag{1.1a}$$

$$\dot{q} = q\left[\epsilon(1-q) - 4p\right] . \tag{1.1b}$$

These equations have several fixed points. One fixed point is at $p = q = 0$. If we linearize around $p = q = 0$ we obtain the elementary solutions $p = p_0 \exp[\lambda\ell]$ and $q_0 \exp[\epsilon\ell]$.

In terms of p and q, the equation for the correlation length $\dot{\xi} = -\xi$ becomes

$$\frac{\partial \xi}{\partial p}\dot{p} + \frac{\partial \xi}{\partial q}\dot{q} = -\xi . \tag{1.2}$$

If we make the linearized approximation for $\dot{p}$ and $\dot{q}$, (1.2) is of the form

$$\sum_i a_i p_i \frac{\partial F}{\partial p_i} = a_F F . \tag{1.3}$$

The solutions of an equation such as (1.6) are generalized homogeneous functions (GHFs)[12]. That is, they satisfy the functional relationship

$$F(\{\lambda^{a_c} p_c\}) = \lambda^{a_F} F(\{p_c\}) . \tag{1.4}$$

The constants a_i and a_F are termed the scaling powers of the variables p_i and the function F, respectively. #To see that (1.3) implies (1.4) it is sufficient to examine the case at hand. We write

$$\xi(p_0 e^{2\ell}, q_0 e^{\epsilon\ell}) = \xi(p_0, q_0) e^{-\ell} . \tag{1.5}$$

which is just (1.4) with $\lambda = e^{\ell}$, $a_p = 2$, $a_q = \epsilon$, and $a_\xi = -1$.
Eq. (1.5) is equivalent to

$$\xi(p,q) = |p|^{-\frac{1}{2}} \xi(\mathrm{sgn} p, q/|p|^{\epsilon/2}) . \tag{1.6}$$

The quantity $q/|p|^{\epsilon/2}$ is a <u>renormalization invariant</u> of the linearized equations, as is easily checked from (1.1).

The correspondence between the form for the correlation length and the usual scaling hypothesis leads to the definition of p and q as <u>(linear) scaling fields</u>. If we make the identification $p \sim T-T_c$ we derive the value of the critical point exponent $\nu = \frac{1}{2}$. However, we linearized (1.1) to obtain this solution. In principle this analysis might only be valid <u>locally</u>, infinitesimally close to the fixed point $p=0, q=0$ (cf. Fig 1).

We can examine other fixed points. A second fixed point is located at $p=0$, $q=1$. At this fixed point, we have a different pair of <u>linear</u> scaling fields, $p' = p$ and $q' = (q-1) + [4/(2-\epsilon\Delta)]\, p$, with the new <u>linearized</u> renormalization equations

$$\dot{p}' = (2-\epsilon\Delta)\, p' \,, \qquad (1.7a)$$

$$\dot{q}' = -\epsilon\, q' \,. \qquad (1.7b)$$

We again obtain a GHF but with variables p' and q' and scaling powers $a_{p'} = 2-\epsilon\Delta$ and $a_{q'} = -\epsilon$. Thus,

$$\xi(\lambda^{2-\epsilon\Delta} p', \lambda^{-\epsilon} q') = \lambda^{-1}\, \xi(p', q') \,, \qquad (1.8a)$$

or, equivalently

$$\xi(p', q') = |p'|^{-1/(2-\epsilon\Delta)}\, \xi\left(\mathrm{sgn}\, p',\ q'\,|p'|^{\epsilon/(2-\epsilon\Delta)}\right). \qquad (1.8b)$$

Again, since the fundamental equations(1.1) were linearized, the solution given in (1.8) is, in principle, valid only infinitesimally near the fixed point p=0,q=1.

Thus, by locating <u>two</u> fixed points and analyzing the behavior of the linearized equations in a neighborhood of each fixed point we have produced two <u>competing</u> forms for the correlation length with <u>different critical exponents,</u> $\nu = \frac{1}{2}$ and $\nu' = 1/(2-\epsilon\Delta)$. If we are not to be confined to infinitesimal regions about <u>one</u> of the fixed points, the effects of <u>both</u> fixed points must be incorporated. This obviously requires a solution of the <u>nonlinear equations</u> to give a solution valid at each fixed point and at every point between the two fixed points. To include more than a single fixed point, the <u>local</u>, <u>linear</u> anaylsis must be replaced by a <u>global</u>, <u>nonlinear</u> analysis(cf. Fig 1).

From the example treated above it is easy to see that any set of <u>linearized</u> renormalization group equations confirm the scaling ideas of Kadanoff: thermodynamic

functions are GHFs of suitable linear combinations of the parameters $\{p_i\}$. Since many fixed points may be included in a global analysis, three questions must be answered:

(i) Which fixed point should be choosen to represent the true scaling behavior of the system?

(ii) Can solutions derived from linearizing the renormalization group equations around various fixed points be matched together in such a way as to form a globally valid solution?

(iii) Does the class of global solutions include behavior that is drastically different than the behavior deduced from the linearized solutions?

Question (i) has been traditionally answered by the criterion of relative stability. If two fixed points can be considered as important for a particular system, we examine them to determine whether a trajectory in the parameter space connects them. If such a path exists, and under the action of the renormalization equations it passes from fixed point A to fixed point B, we say that A is unstable with respect to B. It is assumed that the fixed point which is least unstable (of those fixed points which lie on the "critical surface", cf.Sec.III) is the dominant or controlling fixed point. The asymptotically valid scaling behavior is assumed to be that given by the linearization about that point. For example, the isotropic Heisenberg fixed point is unstable with respect to an anisotropy along one spin axis. In the renormalization group parameter space, two paths lead out of the Heisenberg point, connecting it to a point of XY character and a point of Ising character.[13] We see that the idea that the slightest bit of anisotropy turns the system into either an XY-*like* or Ising-*like* system is supported by this notion of relative stability (and seems to be confirmed by high temperature series analysis).[14]

The procedure of checking relative stability requires first finding the fixed points of the renormalization group transformations and this is an exceedingly nontrivial task. Numerous perturbation expansions have been developed to discover those fixed points that are "close" to some fixed point located by inspection (the ϵ-expansions of Ref 4-5 and the ϵ_c expansions of critical points of higher "order" of Ref. 15-16 are examples).

Question (ii) must be answered individually for each renormalization group and probably for each problem within any one renormalization group. At least for groups with differential generators, one supposes that the solutions for the thermodynamic functions are again GHFs with revised arguments. That is, instead of linear combinations of the parameters $\{p_i\}$ [17], certain nonlinear functions of the parameters (called nonlinear scaling fields) will be the arguments of the GHF. The equations for the nonlinear scaling fields will be first order partial differential equations with coefficients that are nonlinear in the p_j. For any particular case, the general theory of such equations can be envoked to determine whether solutions to these equations exist in the large.[18] A further question is whether every global solution for the thermodynamic function matches onto the local solutions at all the fixed points (or at least at all the fixed points we have found); that is, are all the global solutions sufficiently regular (in a sense particular to each problem) near each fixed point. (Of course, the nonlinear solution for the $\{p_i\}$ themselves always match.) In general, the answer is no; not every global solution matches onto the linearized solution at each fixed point. Many global solutions exist that have singularities that are unrelated to the physical thermodynamic singularities.[19] If we require that a global solution match smoothly at each fixed point, then the set of global solutions will be restricted, but the global solution is still not in general uniquely determined.

Question (iii) can only be answered by explicit construction of the nonlinear scaling fields and some class of global solutions. In this work we will consider two cases which illustrate that the answer is yes: global solutions can be radically different than what might be presumed from the local analysis. However, we will also show that the global solutions that violate the local analysis have extra singularities on the boundary or some portion of the boundary of the solution region. These singularities are apparently unrelated to usual thermodyhamic singularities. If we require that the solution be well-behaved everywhere on the boundary of the solution region, then the only global solutions that are acceptable support the local linearized analysis.

In Sec. II we review the nonlinear solution given in Ref. 11 for the crossover or competition between the Gaussian fixed point (which has mean-field exponents) and the Wilson-Fisher fixed point (which has non-classical exponents). We show that a general global solution is not dominated by the stabler fixed point. Such solutions, however, have singularities on the separatrix which emerges from the stabler fixed point. This separatrix also forms part of the boundary of the solution region. The exclusion of those solutions with singularities on the separatrix leaves only solutions which are dominated by the local linearized behavior of the stabler fixed point.

In Sec. III we discuss a three-parameter crossover problem. The system considered consists of two internally isotropic n-spin subsystems which are coupled together through a biquadratic term. The competition is among a fixed point of dubious spin and the usual Gaussian, n-spin, and 2n-spin fixed points. In this case it is again true that a global solution which is not dominated by the linear behavior of the stablest fixed point has singularities on the boundary of the solution region. However, the singularities do not cover the entire bounding surface (which is two-dimensional) but are confined to the line emerging from the stablest

fixed point. The removal of the singularity along this separatrix again restricts the class of global solutions to those dominated by the behavior of the stablest fixed point. The properties of these restricted global solutions strongly resemble the crossover behavior of systems which contain several different types of critical points, including critical points of higher order.[20] In particular, the "double power law" scaling behavior characteristic of critical to tricritical crossover[21] is an automatic consequence of the nonlinear renormalization group solutions.

In Sec. IV we discuss the general properties of global renormalization group solutions as illustrated in Secs. II and III. We discuss the possibility of accepting the global solutions which are singular on some separatrix. The critical point exponents of such systems are more complicated than the more regular global solutions, but are still characterized by double-power law expressions. These systems, which are a generalization of scaling systems, share many properties with the simpler systems: elsewhere, we have given a partial classification and discussion of such systems and termed them "critically ordered" systems.[23]

II. Two-parameter Crossover

In this section we review the crossover solution given in Ref. 11 for a set of nonlinear renormalization group equations involving two parameters. This solution describes the crossover between Gaussian and Wilson-Fisher (WF) critical behavior,[4] and is obtained within an ϵ-expansion approximation. The techniques used in the solution of this problem parallel those that are used in Sec. III for a three-parameter crossover problem and, in fact, the three-parameter problem reduces to the two-parameter problem on special surfaces in the Hamiltonian parameter space.

The properties of the global solutions given in Ref. 11 are more fully developed in this section. In particular, we show that not every global solution can match the local, linearized solutions at both of the fixed points. More precisely, a general global solution may be singular on particular trajectories leading from the Gaussian or WF fixed points. If the global solution is to avoid such singularities, then the class of admissible solutions is reduced. In fact, it can be shown that the local, linearized analysis is now justified; the global solution matches the local solutions formed at both the WF and Gaussian points and the asymptotically valid critical behavior is determined by the stabler fixed points.

In this section and in Sec. III we use an approximate renormalization group based on the momentum independent limit of the differential generator derived by Wegner and Houghton.[9] The use of a differential formulation is far more convenient for the global study of nonlinear equations than a iteration equation such as the Wilson approximate recursion formula,[5] since it allows the use of many techniques familiar from the general theory of differential equations.[6,18] A discussion of the Wegner-Houghton approximate renormalization group (WHARG) is given in Appendix A.

We consider a Wilson reduced Hamiltonian density of the form

$$\mathcal{H} = |\nabla s|^2 + r\vec{s}^2 + \frac{u}{2}(\vec{s}^2)^2 + \vec{h}\cdot\vec{s} \quad , \tag{2.1}$$

for a continuum spin vector s with n-components. The variables r and u are constant interaction parameters and $\vec{h}$ is the magnetic field. As discussed in Ref. 5, the Hamiltonian (2.1) models a short range interaction between spins on a lattice. The approximation of such a system by a continuous spin allows the renormalization average to be performed more easily. The WHARG equations for the isotropically interacting n-spin system of (2.1) in a lattice of dimension d are

$$\dot{r} = 2r + \frac{u}{1+r}\left[\tfrac{1}{2}d(n+2)\right] \tag{2.2a}$$

$$\dot{u} = (4-d)u - \frac{u^2}{(1+r)^2}\left[\tfrac{1}{2}d(n+8)\right] \tag{2.2b}$$

A natural change of variable maps all the fixed points of interest into a finite region of parameter space. We write

$$\bar{r} \equiv r/(1+r) \quad ; \quad \epsilon y_n \equiv \frac{u}{(1+r)^2}\left[\frac{d}{2}(n+8)\right] . \tag{2.2c}$$

The WHARG equations in terms of these variables are [23]

$$\dot{\bar{r}} = 2(1-\bar{r})\left[\bar{r} + \epsilon y_n (n+2)/2(n+8)\right] ; \tag{2.3a}$$

$$\dot{y}_n = y_n\left[\epsilon\left(1 - 3y_n(n+4)/(n+8)\right) - 4\bar{r}\right] . \tag{2.3b}$$

The three fixed points of interest are the "finite Gaussian" point ($\bar{r} = y_n = 0$), the infinite Gaussian" point ($\bar{r}=1$, $y_n=0$), and the Wilson-Fisher point ($\bar{r}=-\epsilon(n+2)/(2(n+8))$, $y_n=1$). The term "Gaussian" is applied to the first two points since the nonquadratic terms in the effective Hamiltonian are zero. We distinguish between the two Gaussian points by the value of r_0 which is zero at the finite Gaussian point and infinite at the infinite Gaussian point.

The equations given in (2.3) are already diagonalized about the infinite Gaussian point. If we diagonalize them around the finite Gaussian point, we have the complementary set of equations

$$\dot{x} = 2x\left(1 - x - \epsilon \frac{\Delta_n}{2} y_n\right) \tag{2.4a}$$

and

$$\dot{y}_n = y_n\left(\epsilon(1 - y_n) - 4x\right), \tag{2.4b}$$

where

$$x \equiv \bar{r} + y_n \epsilon \Delta_n/(2\epsilon), \quad \Delta_n \equiv (n+2)/(n+8). \tag{2.4c}$$

In passing from (2.3) to (2.4) we have discarded terms proportional to $\epsilon^2 y_n$ and $\epsilon^2 y_n^2$. Thus, (2.3) and (2.4) are formally equivalent to $0(\epsilon)$ (see, however, Appendix B). The advantage of this approximation is that the surface of critical Hamiltonians is now the straight line x=0. We are justified in neglecting the ϵ^2 terms if y_n is always of 0(1). Since the original WHARG equations (2.2) are only good to $0(\epsilon)^{16}$ this approximation is self-consistent. The restriction on the size of y_n limits us to global solutions for which the renormalization trajectories are bounded in y_n.

With these approximations, (2.3) and (2.4) are of the same form. The three fixed points have been placed at the canonical locations $x=y_n=0$ (the finite Gaussian point), $x=0, y_n=1$ (the Wilson-Fisher point), and $x=1, y_n=0$ (the infinite Gaussian point) (see Fig. 2).

The solution of (2.3) and (2.4) for x and y_n can be conveniently expressed in terms of three functions R,X, and Y_n which are defined through the equations

$$\dot{R} = 2(1-\bar{r})R \; ; \tag{2.5a}$$

$$\dot{X} = -2x X \; ; \tag{2.5b}$$

$$\dot{Y}_n = -\epsilon y_n Y_n \; . \tag{2.5c}$$

Employing (2.7a) and (2.7c) in Eq. (2.3) we have

$$\frac{y_n}{R^2 Y_n}^{3(n+4)/(n+8)} = \text{const } e^{-d\ell} \; ; \tag{2.6a}$$

$$\frac{1-\bar{r}}{R\, Y_n^{\Delta_n}} = \text{const } e^{-2\ell} \; . \tag{2.6b}$$

Using (2.5b) and (2.5c) in the complementary equations (2.4) we also have

$$\frac{y_n}{X^2 Y_n} = \text{const } e^{\epsilon \ell} \; ; \tag{2.6c}$$

$$\frac{x}{X\, Y_n^{\Delta_n}} = \text{const } e^{2\ell} \; . \tag{2.6d}$$

The four expressions on the left hand sides of (2.6a)-(2.6d) are termed nonlinear scaling fields[17] since they have a purely exponential dependence on the renormalization parameter ℓ. Equation (2.6a)-(2.6i) cannot all be independent since there can only be two independent scaling fields. It is easy to see that

$$X = 1 - \bar{r} \tag{2.7a}$$

$$R = x\, Y_n^{-2\Delta_n} \tag{2.7b}$$

All that remains is finding the solution of (2.5c) for Y_n. We are interested in the solution that can be written as

$$Y_n = \exp\left(-\epsilon \int_{-\infty}^{\ell} y_n(\ell')\, d\ell'\right). \tag{2.8}$$

By (2.8) we mean that the value of $Y_n(x,y_n)$ is to be determined by performing the indicated integral along the unique renormalization trajectory that passes through the point (x,y_n). Thus, $Y_n(0,y_n)=1-y_n$ by direct integration of (2.5). The separatrix connecting the WF point $x=0, y_n=1$ and the infinite Gaussian point $x=1$, $y_n=0$ is denoted by $y=\varphi_n(x)$ in Fig. 2. Since the renormalization solution along this trajectory reaches the WF fixed point only in the limit $\ell \to -\infty$, at each point $y=y_0(\ell_0)$ on the separatrix, $y > y_0$ for all $\ell < \ell_0$. The integral in the exponent of (2.8) diverges and, therefore Y_n has a zero on the separatrix $y=\varphi_n(x)$.

Since we have the exact solution for $x \to 0$, $Y_n = 1-y_n$, it is easy to show that Y_n can be written as (to first order in ϵ, cf. Appendix B),

$$Y_n = \mathcal{Y}(x, y_n, \Delta_n) \equiv (1 - y_n/\varphi_n) \exp(\epsilon x \Delta_n y_n/\varphi_n), \tag{2.9a}$$

where the separatrix function $\varphi_n(x)$ shown in Fig. 2. is given by

$$\varphi_n = \Phi(x, \Delta_n) \equiv (1-x)^{3/2} \exp[\epsilon x(1-2\Delta_n)/2] \quad (2.9b)$$

(The derivation of the functions Ψ and Φ is given in Appendix B; they are used with different arguments in Sec. III). We can now write down the globally valid nonlinear scaling fields appropriate to the Gaussian and WF fixed points. That is, the nonlinear fields that embody the behavior characteristic of the renormalization equations when linearized around the two fixed points. The simplest forms of these fields are given by

$$S_{gau} = \frac{x\,\Psi_n^{-\Delta_n}}{X} \quad ; \qquad S_n = \frac{x\,y_n^{-\Delta_n}}{X^{1-2\Delta_n}} \quad . \quad (2.10a,b)$$

With these scaling fields, we may describe the global behavior of any function whose renormalization transformation behavior is known. If f is a function that satisfies the renormalization equation $\dot{f} = a_f f$,

then f is a GHF of the magnetic field h, and the two scaling fields S_{gau} and S_n,

$$f(\lambda^{a_h} h, \lambda^{a_{gau}} S_{gau}, \lambda^{a_n} S_n) = \lambda^{a_f} f(h, S_{gau}, S_n) \quad (2.11)$$

Here the scaling powers a_h [24], a_{gau}, and a_n are given by

$$a_h = 1 + d/2 \; , \quad a_{gau} = 2 \quad a_n = 2 - \epsilon \Delta_n \; . \tag{2. }$$

For example the correlation length satisfies (2.1) with $a_f=-1$; the Gibbs potential satisifies (2.1) with $a_f=d$(see, however, discussion following Eq. (2.15)).

The renormalization group equations do not determine the form at the GHF (2.1) [25] and they also do not determine the scaling fields uniquely. Since any GHF remains a GHF under any transformation of varibles which is itself a GHF we may choose new scaling fields which are arbitrary GHFs of S_{gau} and S.

Thus, we can choose scaling fields with any scaling powers. This freedom is reduced by considering the fact that the separatrix connecting the finite Gaussian point and the WF point ($x=0$, $0 \leq y \leq 1$) corresponds exactly to the surface of critical Hamiltonians. That is, $x=0$ if and only if $T=T_c$. Making the usual Taylor series expansion we assume that $x \simeq T-T_c$ for sufficiently small $T-T_c$. If we require that the scaling fields themselves be proportional to x for T near T_c and

that they reduce to the natural linearized scaling fields at the Gaussian and WF fixed points, then the most general fields are given by those of (2.12) multiplied by arbitrary functions of the renormalization invariant

$$I \equiv \frac{x^{\epsilon}}{y_n^{\lambda}} \, Y_n^{a_n} \, X^{d} \tag{2.13}$$

which do not vanish at x=0. Note that I=0 on the separatrix $y=\phi_n(x)$ as well as when x=0; I=∞ along the pure Gaussian trajectory $y_n = 0$.

This freedom in the choice of nonlinear scaling fields is illusory since we have not specified the GHFs for which the scaling fields are arguments. Any change in the scaling fields induces a corresponding change in the form of the GHFs. We may, therefore, choose the nonlinear scaling fields at our convenience.

Without loss of generality, we will use the scaling fields defined in (2.10). With these nonlinear scaling fields a particularly simple example for the h=0 correlation length is

$$\xi = A(I) \, S_{\text{gau}}^{-1/a_{\text{gau}}} \, B(I) \, S_n^{-1/a_n} \tag{2.14}$$

where A(I) and B(I) are smoothly varying functions of the renormalization invariant (finite both at I=0 and I=∞). Since both of the scaling fields appear in (2.14) symmetrically, this form has the virtue of reducing to the appropriate local solution

as either of the limiting trajectories ($y_n = 0$ or $y_n = \phi_n(x)$ is approached. For x fixed and $y_n \to 0$, $S_n \to \infty$ and the WF term vanishes. Similarly, as the $y_n = \phi_n$ separatrix is approached, $S_{gau} \to \infty$ and the Gaussian term $\to$ 0. For intermediate values of y_n, both singularities compete, giving the expected nonlinear crossover.

A more complicated behavior is exhibited by the Gibbs potential. In addition to the spin dependent terms, an additive constant in the Hamiltonian density contributes to G. Therefore, we can write

$$G(\{p_i\}) + v_o = e^{-d\ell} G(\{p_i(\ell)\}) + e^{-d\ell} v_o(\ell) \quad . \tag{2.15a}$$

The WHARG equation for v_o can be easily integrated to give

$$G(\{p_i\}) = e^{-d\ell} G(\{p_i(\ell)\}) + \frac{dn}{2} \int^{\ell} d\ell' f_n(1+r(\ell')) e^{-d\ell'} \quad . \tag{2.15b}$$

As the renormalization average proceeds ($\ell \to \infty$), information about the Gibbs potential passes from the first term on the right hand side of (2.15b) to the second term. In some circumstances (in particular, zero magnetization) it may be possible to take the limit $\ell \to \infty$ and consider only the second term.[3] This method has been utilized by some authors[26,27] to calculate approximate Gibbs potentials in zero ordering field (i.e., h=0). However, for fixed ℓ, the second term contains information would be unimportant for critical behavior. Accordingly, we will in our discussions drop the second term of (2.15b) and deal only with the homogeneous term. Thus,

when discussing the Gibbs potential and its temperature like derivatives, we will confine our attention to $x \to 0$, even though the solutions for the nonlinear scaling fields are valid for all $x < 1$. The difficulty does not arise when studying the derivatives of the Gibbs potential with respect to the ordering field h (such as the magnetization and suseptibility) since the second term in (2.15b) is independent of h and does not contribute. We could therefore phrase our discussion of crossover in terms of these functions; we discuss the Gibbs potential to allow the closest connection between this work and phenomenological discussions of crossover.[20-21] A brief discussion of the limit $\ell \to \infty$ in the second term is given in Appendix D.

Within this approximation, therefore, the Gibbs potential is a GHF of the ordering field h and the two nonlinear scaling fields S_n and S_{gau}. In general, $G(h,S_{gau},S_n)$ will generate critical point exponents that do <u>not</u> satisfy exponent inequalities as equalities. This is to be expected on general grounds simply because $G(h,S_{gau},S_n)$ depends on three scaling fields with three distinct scaling powers. The usual scaling equalities which relate <u>three</u> exponents are satisfied because there are only <u>two</u> independent scaling powers. An example of a Gibbs potential which is a global solution of the renormalization equations which does not give exponent equalities is

$$G(h, S_{gau}, S_n) = G_{gau}(h, S_{gau}) + G_h(h, S_n), \qquad (2.16)$$

where G_{gau} and G_n are both GHFs. Each piece of the Gibbs potential generates its own singularities with exponents that satisfy exponent equalities. However, since $\gamma_n > \gamma_{gau}$, it is immediate that $\Delta_{gau} > \Delta_n$. The measured exponents would be γ_n and Δ_{gau}, and therefore $\alpha+2\beta+\gamma > 2$!

However, a solution of the form given in (2.16) cannot be matched to the expected local solutions near the two limiting trajectories. As $y_n \to 0$, $S_n \to \infty$ and therefore $G_n \to \infty$. Similarly, as the separatrix is approached, $S_{gau} \to \infty$ and G_{gau}

$\rightarrow \infty$. If we replace G_n by $G_n/(1+I^2)$ the divergence at $y_n=0$ is removed. However, to remove the divergence on the separatrix we would have to multiply G_{gau} by a power of the invariant. Since the invariant is proportional to a power of x(cf. equation (2.13)), this weakens the singularities generated by G_{gau}. In fact, it is easy to see that it weakens G_{gau} just enough to ensure that the measured α will be α_n. Thus, if we require that the global solution match the expected local solution on both boundaries a splitting of the Gibbs potential as in (2.16) does not lead to the violation of exponent equalities since the G_{gau} term must be discarded. The critical behavior is determined entirely by the WF point.

To show this in another way, consider the h=0 Gibbs potential. We can write it in two ways,

$$G = [S_{gau}]^{2-d_{gau}} f_{gau}(I) \qquad (2.17a)$$

or

$$G = [S_n]^{2-d_n} f_n(I) \qquad (2.17b)$$

If the asymptotically valid value of d is d_{gau} then $f_{gau}(0)$ is a finite constant. However, I=0 on the $y_n=\varphi_n(x)$ separatrix as well as at x=0; therefore, as the separatrix is approached, f_{gau} is well behaved and, since S_{gau} is singular at $Y_n=0$, G has a singularity on the separatrix. On the other hand, if the asymptotically valid value of d is d_n, then $f_n(0)$ is finite. It is also finite, therefore, on the separatrix. However, as $y_n \rightarrow 0$, the invariant $I \rightarrow \infty$. Therefore, the divergence in S_n as $y_n \rightarrow 0$ may be cancelled by an appropriate behavior of $f_n(I)$ as $I \rightarrow \infty$. An example is given by

$$G = \frac{S_n^{2-d_n} S_{gau}^{2-d_{gau}}}{S_n^{2-d_n} + S_{gau}^{2-d_{gau}}} . \qquad (2.18)$$

The form given in (2.18) has the virtue of reducing to the expected local solutions on both bounding trajectories. A form valid for non-zero h which corresponds to (2.18) is

$$G = \frac{G_n(h, S_n)\; G_{0au}(h, S_{0au})}{G_n(h, S_n) + G_{0au}(h, S_{0au})}$$

If $\epsilon < 0$, the argument given above is precisely reversed so that the Gaussian fixed point (which in this case is the stabler fixed point) does dominate the global solution. Thus, in this two-parameter example, the stabler fixed point is always dominant globally.

III. Three-parameter Crossover; Coupled Order Parameters.

In this section we describe the solution to nonlinear renormalization group equations which involve three parameters. These equations model a system involving two interacting order parameters. There are several realistic systems whose phase diagram may be understood in terms of a model Hamiltonian encompassing the interaction between two (or more) coupled order parameters.

One simple example is provided by the phase diagram of the mixed crystals $(Fe_xMn_{1-x})WO_4$ near the quadruple point.[28] In these crystals (which possess a monoclinic wolframite structure), the oxygen ions form a distorted hexagonal close packed pattern; half the octahedra spaces are filled with Fe or Mn ions and the other half are filled with W ions. The magnetic structure of $FeWO_4$ (ferberite) is antiparallel in alternate planes (↑↓). The magnetic cells for $MnWO_4$ (huebnerite) on the other hand, is quadrupled in the a-direction and doubled in the b and c directions (↑↑ ↓↓). Wegner[29] has shown that near the quadrauple point of such substances (defined to be the point at which the paramagnetic phase, Fe ordered phase, Mn ordered phase, and a mixed phase are simulataneously in coexistence) the free energy may be represented by a model involving two order parameters with reflection symmetry and a biquadratic coupling term. Depending on the various interaction strengths, the two ordered phases are either separated by a first-order transition or by an additional phase. Other examples of magnetic materials exhibiting similar quadruple point include $Fe(Pd,Pt)_3$.[30]

Ther order-disorder tansitions in the ammonium halides provide further examples of systems with coupled order parameters. At sufficiently low temperatures, the NH_4 tetrahedra can have two different types of ordering, parallel and anti-parallel, in the cubic structure of the halide ions. The coupling between the two types of ordering is, however, not direct; it is probably

mediated by non-ordering effects (e.g., magnetoelastic interactions). Model Hamiltonians involving the interaction of the parallel ordering and magnetoelactic effects[31] have been used to simulate the order-disorder phenomena of NH_4Cl, leading to a renormalization group prediction of classical tricritical behavior. A model Hamiltonian which treats magnetoelastic effects and both the parallel and antiparallel ordering is more complicated and will be treated in a separate paper.[32] Because the elastic distortion breaks the reflection symmetry, the model Hamiltonians for NH_4Cl are generally assumed to contain coupling terms different from the simple biquadratic term employed in the mixed crystal examples discussed above.

There exist a number of systems that can be described by model Hamiltonians with biquadratic coupling terms.[33] These include the metamagnets (such as $FeCl_2$)[34], systems involving spin-flop transitions[35], and displacive transitions in perovskite crystals[36]. It has been shown that such a model provides descriptions not only of classical tricritical points but also "bicritical" and "tetracritical" points[34-36] in the "physical plane". We will demonstrate in the latter part of this section that this model also contains the type of "higher order critical points" exemplified by the intersection of critical subspaces (as first proposed by Ref. 20) when the phase diagrams are viewed in the multi-parameter Hamiltonian space.

A general model Hamiltonian with a biquadratic coupling term has five interaction parameters[33] (as discussed in Appendix A). In this section, we consider the special case of a system in which the two order parameters play precisely equivalent roles. The system considered is a generalization to n-component spins of the anisotropic Hamiltonian discussed in Ref. 4 for n=1. The close relationship of this three-parameter system to the two parameter system solved in Sec. II allows many of the solution methods of the simpler problem to be applied to its generalization in this section.

We consider two internally isotropic n-component spin subsystems, $\vec{s}_1$ and $\vec{s}_2$ which interact through a quartic term,

$$\mathcal{H} = |\nabla \vec{s}_1|^2 + |\nabla \vec{s}_2|^2 + r(\vec{s}_1^{\,2} + \vec{s}_2^{\,2}) + \frac{u}{2}\left[(\vec{s}_1^{\,2})^2 + (\vec{s}_2^{\,2})^2\right] + w\,\vec{s}_1^{\,2}\,\vec{s}_2^{\,2} + \vec{h}\cdot\vec{s}_1 + \vec{h}\cdot\vec{s}_2 \quad . \tag{3.1}$$

This Hamiltonian can be viewed as the sum of two n-spin Wilson Hamiltonians of the form given in Eq. (2.1) with a *biquadratic* interaction term $w\,\vec{s}_1^{\,2}\,\vec{s}_2^{\,2}$. On the other hand, it can also be considered as the Wilson Hamiltonian for a single 2n-component spin system $\vec{s} = (\vec{s}_1, \vec{s}_2)$ with an "anisotropy-like" interaction $(w-u)\,\vec{s}_1^{\,2}\,\vec{s}_2^{\,2}$ (cf Fig. 3). These two descriptions of the single Hamiltonian (3.1) are reflected in the discussion of the global renormalization properities of (3.1) as will be shown below.

The WHARG equations are given by (see Appendix A for details)

$$\begin{aligned}
\dot{\bar{r}} &= 2(1-\bar{r})\left[\bar{r} + \frac{\epsilon}{2}\left(\frac{n+2}{n+8}\,y_n + \frac{n+1}{n+4}\,y_{2n}\right)\right] \quad ; \\
\dot{y}_n &= y_n\left[\epsilon\left(1 - \frac{3(n+4)}{(n+8)}\,y_n - 2\,y_{2n}\right) - 4\,\bar{r}\right] \quad ; \\
\dot{y}_{2n} &= y_{2n}\left[\epsilon\left(1 - 3\left(\frac{n+2}{n+4}\right)y_{2n} - 4\left(\frac{n+2}{n+8}\right)y_n - 4\bar{r}\right]\right. ,
\end{aligned} \tag{3.2}$$

where the variables $\bar{r}$, y_n, and y_{2n} are defined by the relations

$$\begin{aligned}
\bar{r} &\equiv r/(1+r) \quad ; \\
\epsilon\, y_n &\equiv \tfrac{1}{2}(n+8)(u-w)/(1+r)^2 \quad ; \\
\epsilon\, y_{2n} &\equiv d(n+4)\,w/(1+r)^2 \quad .
\end{aligned} \tag{3.3}$$

The two descriptions mentioned above of the Hamiltonian (3.1) have been incorporated in the choice of the variables y_n and y_{2n} employed here. When $y_n=0$ the system is equivalent to an isotropic 2n-component spin system; on the other hand, when $y_{2n}=0$ the system breaks into two non-interacting n-component spin systems.

Eqs. (3.2) are already diagonalized around the infinite Gaussian fixed point, $\bar{r}=1$, $y_n=y_{2n}=0$. If, as in Sec. II, we diagonalize around the finite Gaussian point $\bar{r}=y_n=y_{2n}=0$, we obtain the equations

$$\dot{x} = 2x\left[1-x-\frac{\epsilon}{2}\left(\Delta_n y_n+\Delta_{2n} y_{2n}\right)\right] ; \tag{3.4a}$$

$$\dot{y}_n = y_n\left[\epsilon\left(1-y_n-\frac{6}{n+4}y_{2n}\right)-4x\right] ; \tag{3.4b}$$

$$\dot{y}_{2n} = y_{2n}\left[\epsilon\left(1-y_{2n}-2\Delta_n y_n-4x\right)\right] ; \tag{3.4c}$$

where $x \equiv \bar{r}+\epsilon\left(\Delta_n y_n+\Delta_{2n} y_{2n}\right)/2$, $\Delta_n \equiv (n+2)/(n+8)$, $\Delta_{2n} \equiv (n+1)/(n+4)$. Terms such as $\epsilon^2 y_n$, $\epsilon^2 y_n^2$, $\epsilon^2 y_{2n}$, and $\epsilon^2 y_{2n}^2$ have been neglected. The considerations of Sec. II apply here as well; only global solutions bounded in y_n and y_{2n} are acceptable (see again Appendix B).

The surface of critical Hamiltonians is the plane x=0. There are five fixed points at which x, y_n, and y_{2n} are $O(1)$ (there are other fixed points at which x, y_n, and y_{2n} are $0(1/\epsilon)$; these cannot be subsumed in this perturbation

analysis). Four are located on the plane x=0, and one at x=1:

(i) the finite Gaussian point, $y_n=y_{2n}=x=0$;

(ii) the usual n-spin WF point, $y_n=1, y_{2n}=x=0$;

(iii) the usual 2n-spin WF point, $y_{2n}=1$, $y_n=x=0$;

(iv) a point of no particular spin (unless n=1) which we will call the z-point, $x=0$, $y_n=y_{no}\equiv (n-2)(n+8)/(n^2+8)$, $y_{2n}=y_{2no}\equiv (16-n^2)/(n^2+8)$;

(v) the infinite Gaussian point, $x=1, y_n=y_{2n}=0$.

These fixed points and the integral curves and surfaces connecting them form a finite region of the parameter space which is invariant under the action of the renormalization group.

The most relevant eigenvalue (the eigenvalue of the x equation (3.4a)) for the four fixed points in the x=0 plane are given by

$$\begin{aligned} a_{gau} &= 2 \\ a_n &= 2-\epsilon\Delta_n \\ a_{2n} &= 2-\epsilon\Delta_{2n} \\ a_z &= 2-\epsilon\Delta_z \end{aligned} \qquad (3.5)$$

where $\Delta_z \equiv 3n/(n^2+8)$. Since we identify x with $T-T_c$ for sufficiently small x, the critical point exponent ν is given by the inverse of the eigenvalues of (3.5); for example, $\nu_{gau}=\frac{1}{2}$.

The existence of the z-point shows that the Hamiltonian (3.1) contains a third symmetry relation similar to the two discussed above which are represented by the n-spin and 2n-spin fixed points[4]. The relative stability of the n-spin, 2n-spin and z-points depends on the spin dimension n as determined by (3.4).

For $n<2$, the 2n-spin point is stablest (cf. Fig. 4a). For $2\leq n\leq 4$, the z-point is stablest (Fig. 4b). Finally, for $n>4$, the n-spin point is stablest (Fig. 4c). The finite Gaussian point is always unstable with respect to all the other x=0 fixed points; all the x=0 fixed points are unstable with respect to the infinite Gaussian fixed point.

From (3.4) we note that the trajectory which joins the finite Gaussian point ($y_n = y_{2n} = 0$) to the z-point ($y_n = y_{no}$, $y_{2n} = y_{2no}$) is a straight line. Hence, we may supplement (3.4) by

$$\dot{z} = z\left[\epsilon\,(1 - y_n - y_{2n}) - 4x\right] \,, \tag{3.6}$$

where $z \equiv y_n y_{2no} - y_{2n} y_{no}$. Note that the z-point is $y_n = y_{no}$, $x = z = 0$. Eq (3.6) is not independent of Eqs (3.4), but the redundant information expressed in (3.6) will be very helpful in the solution of the original equations (3.4).

Proceeding as in Sec. II, we define functions Y_n, Y_{2n}, X, R, by the equations

$$\dot{R} = 2(1-\bar{r})R \,; \tag{3.7a}$$

$$\dot{X} = -2x\,X \,; \tag{3.7b}$$

$$\dot{Y}_n = -\epsilon\, y_n Y_n \,; \tag{3.7c}$$

$$\dot{Y}_{2n} = -\epsilon\, y_{2n} Y_{2n} , \tag{3.7d}$$

By inspection of Eqs. (3.2), (3.4) and (3.6), we write down non-linear scaling fields

$$\frac{x}{X\, Y_n^{\Delta_n} Y_{2n}^{\Delta_{2n}}} = \text{const}\; e^{2\ell} , \tag{3.8a}$$

$$\frac{1-\bar{r}}{R\, Y_n^{\Delta_n} Y_{2n}^{\Delta_{2n}}} = \text{const}\; e^{-2\ell} , \tag{3.8b}$$

$$\frac{y_n}{R^2 Y_{2n}^2 Y_n^{3(n+4)/(n+8)}} = \text{const}\; e^{-d\ell} , \tag{3.8c}$$

$$\frac{y_{2n}}{R^2 Y_n^{4\Delta_n} Y_{2n}^{3(n+2)/(n+4)}} = \text{const}\; e^{-d\ell} , \tag{3.8d}$$

$$\frac{y_n}{X^2 Y_{2n}^{6/(n+4)} Y_n} = \text{const}\; e^{\epsilon\ell} , \tag{3.8e}$$

$$\frac{y_{2n}}{X^2 Y_n^{2\Delta_n} Y_{2n}} = \text{const}\; e^{\epsilon\ell} , \tag{3.8f}$$

$$\frac{z}{X^2 Y_n Y_{2n}} = \text{const}\, e^{\epsilon \ell} \tag{3.8g}$$

Matching scaling fields gives $X=(1-\bar{r})$ and $R=xY_n^{-2\Delta_n} Y_{2n}^{-2\Delta_{2n}}$ in analogy with (2.7). Eq (3.8c-g) cannot be independent since z is a linear combination at y_{2n} and y_n. Comparing the solutions for $\ell \to -\infty$, where Y_n and $Y_{2n}=1$ gives the following relationship between Y_n and Y_{2n}:

$$y_n y_{2nc} - y_{2n} y_{nc} = y_n y_{2n0} Y_{2n}^{(n-2)/(n+4)} - y_{2n} y_{n0} Y_n^{(4-n)/(n+8)} \tag{3.9}$$

Thus, for $n \neq 2,4$, the complete solution of the three-parameter crossover problem depends only on the solution of (3.7) for either Y_n or Y_{2n}. Unfortunately (except for n=0 and $n=\infty$, see Appendix C), we have not been able to derive the form of Y_n or Y_{2n} even in an ϵ-expansion. The essential difficulty is in the x=0 plane, where there is no small parameter and the nonlinear equations must be solved exactly. However, a great deal can be learned about the solutions by comparison with the two-parameter solutions of Sec. II on appropriate two-dimensional surfaces. Whenever y_n, y_{2n}, or z vanishes the problem reduces to a two-parameter problem. We have, therefore, the following partial results:

$$Y_n(x, y_n, y_{2n}=0) = \mathcal{Y}(x, y_n, \Delta_n), \tag{3.10a}$$

$$Y_{2n}(x, y_n=0, y_{2n}) = \mathcal{Y}(x, y_{2n}, \Delta_{2n}), \tag{3.10b}$$

$$Y_n(x, y_n, y_{2n})\big|_{z=0} = \left[\gamma(x, y_n/y_{no}, \Delta z)\right]^{y_{no}} \qquad (3.10c)$$

$$Y_{2n}(x, y_n, y_{2n})\big|_{z=0} = \left[\gamma(x\ y_n/y_{no}, \Delta z)\right]^{y_{2no}} \qquad (3.10d)$$

To discuss the boundary conditions that apply to Y_n and Y_{2n} we must generalize the notion of separatrix. A fixed point at which not all the eigenvalues of the linearized equations are of the same sign is a saddle point in the appropriate space. The family of trajectories leaving a saddle point in a two-dimensional space is a one-dimensional line (a separatrix). In higher dimensional spaces, this family of trajectories may be of higher dimension, and we will call the corresponding surface a "separ-surface". On the separ-surface leaving the n-spin point, there is a boundary condition on Y_n; on the separ-surface leaving the 2n-spin point, there is a boundary condition on Y_{2n}; on the separ-surface leaving the z-point, boundary conditions on both Y_n and Y_{2n} apply. As in the discussion following (2.8), the Y on which the condition is imposed must be identically zero or infinite on the appropriate surface. These boundary values are shown in Fig. 4 for the three ranges of n, $n \leq 2$, $2 \leq n \leq 4$, and $n \geq 4$. The general character of Y_n and Y_{2n} is established, even though the solutions cannot be given explicitly. We will proceed as if Y_n and Y_{2n} were known (for n=0 and $n=\infty$, Y_n and Y_{2n} are known, see Appendix C).

Since there are three parameters (x, y_n, y_{2n}) there are two independent renormalization invarients. It will be convenient to write down several non-independent invariants for compactness in later discussion. There are three ways of combining (3.8c)-(3.8g):

$$I_{y_n y_{2n}} \equiv \frac{y_n\ Y_{2n}^{(n-2)/(n+4)}}{y_{2n}\ Y_n^{(4-n)/(n+8)}} \qquad (3.11a)$$

$$I_{z\,y_n} \equiv \frac{z}{y_n\, Y_{2n}^{(n-2)/(n+4)}}, \tag{3.11b}$$

$$I_{z\,y_{2n}} \equiv \frac{z}{y_{2n}\, Y_n^{(4-n)/(n+8)}} \tag{3.11c}$$

which are connected through (3.9). Combining (3.8a) with (3.8e)-(3.8g) we have three invariants involving x explicitly

$$I_{x\,y_{2n}} \equiv \frac{x^{\epsilon} X^{d} Y_{2n}^{a_{2n}} Y_n^{a_n - 2(4-n)/(n+8)}}{y_{2n}^{2}}, \tag{3.12a}$$

$$I_{x\,y_n} \equiv \frac{x^{\epsilon} X^{d} Y_n^{a_n} Y_{2n}^{a_{2n} - 2(n-2)/(n+4)}}{y_n^{2}} \tag{3.12b}$$

$$I_{xz} \equiv \frac{x^{\epsilon} X^{d} Y_n^{a_n} Y_{2n}^{a_{2n}}}{z^{2}}. \tag{3.12c}$$

Finally we combine (3.11) and 3.12 to give an invariant which does not contain either Y_{2n} or Y_n)

$$I_0 \equiv \frac{x^{\epsilon} X^{d} z^{a_z (n^2+8)/(4-n)(n-2)}}{y_n^{a_n (n+8)/(4-n)}\, y_{2n}^{a_{2n}(n+4)/(n-2)}} \tag{3.13}$$

The invariant I_0 of (3.13) is useful because it distinguishes between different trajectories on the curved separ-surfaces where either Y_n or Y_{2n} is identically

zero or infinite. The invariants (3.11) and (3.12), which involve Y_n and Y_{2n} explicitly, are constant on these separ-surfaces, while (3.13) varies. For n=2 and n=4 it is impossible to form an invariant that contains neither Y_n nor Y_{2n}. However, in these cases it is not necessary to eliminate both of the Y's since only one of them has singular behavior on the separ-surface. For n=2, $Y_n=0$ on the separ-surface and we can choose the invariant

$$I_2 \equiv \frac{x^\epsilon X^d Y_{2n}^{a_{2n}}}{y_{2n}^{a_n (n+8)/(4-n)} z^{2 - a_n (n+8)/4-n)}} \tag{3.13b}$$

in this case. For n=4, $Y_{2n}=0$ on the separ-surface, and the invariant

$$I_4 \equiv \frac{x^\epsilon X^d Y_n^{a_n}}{y_n^{a_{2n}(n+4)/(n-2)} z^{2 - a_{2n}(n+4)/(n-2)}} \tag{3.13c}$$

distinguishes trajectories. For the remainder of this work it is simplest to assume that $n \neq 2,4$, although the analysis carries over to those cases, via slightly modified arguments.

We may choose the nonlinear scaling fields to be given by

$$S_{gau} \equiv \frac{x}{X Y_n^{\Delta_n} Y_{2n}^{\Delta_{2n}}} \tag{3.14a}$$

$$S_z \equiv \frac{x}{X^{1-2\Delta_z} y_{2n}^{\Delta_z}} \left[\frac{y_n}{y_{2n}} \right]^{(n^2-4)/(n^2+8)} \tag{3.14b}$$

$$S'_n \equiv \frac{x}{X^{1-2\Delta_n} y_n^{\Delta_n} Y_{2n}^{(n-1)/(n+8)}} \qquad (3.14c)$$

$$S'_{2n} \equiv \frac{x\, Y_n^{(n^2-4)/(n+4)(n+8)}}{X^{1-2\Delta_{2n}} y_{2n}^{\Delta_{2n}}} \qquad (3.14d)$$

The choice of the scaling fields is (as in Sec. II) not unique, but it is a convenient choice for the case $2 \leq n \leq 4$ which is discussed in detail below as a concrete example. In this case, both y_n and y_{2n} are both positive throughout the solution region.

The invariant I_o also contains information regarding relative stability. It is zero in the x=0 plane and also in that plane which contains the stablest fixed point. For example, in the case $2 \leq n \leq 4$, the z-point is stablest and $I_0=0$ whenever z=0. The invariant I_0 is infinite in those planes which contain the relatively unstable points. For $2 < n < 4$, $I_0 = \infty$ when $y_n=0$ or $y_{2n}=0$.

As in Sec. II we can show that global solutions which do not generate the exponents of the stablest fixed point have extraneous singularities. For concreteness, we will consider $2 \leq n \leq 4$; a similar analysis can be made for any value of n. If we were to suppose that the asymptotically valid value of α were that given by the n-spin fixed point, rather than that of the z-point, it would be appropriate to write the Gibbs free energy as

$$G = S_n^{d/a_n} F(\text{invariants}) \qquad (3.15)$$

Since $d/a_n = 2 - d_n$

It is presumed in writing the free energy in the form (3.15) that F(invariants) is well behaved at x=0 and in the $y_{2n}=0$ plane (where we definitely expect n-spin point behavior). The amplitude of the free energy singularity is given by

$$\lim_{x\to 0} G / x^{d/a_n} = y_n^{-\Delta_n d/a_n} \gamma_{2n}^{-\frac{d}{a_n}(n-1)/(n+8)} F(\text{invariants})\Big|_{x=0} \quad (3.16)$$

If we consider a path in the x=0 plane that is a renormalization trajectory (cf Fig. 5), (3.11) is again constant. The function F is therefore constant and we see that the amplitude of the free energy singularity diverges as the stablest fixed point is approached along any renormalization trajectory ($Y_{2n} \to 0$). The singularity can be removed along any particular trajectory or finite number of trajectories but it cannot be removed everywhere in the x = 0 plane. Similar difficulties are encountered if we assume that the 2n-spin point dominates the critical behavior of G.

If, however, we assume that the z-point dominates the exponents everywhere (except in the $y_n=0$ and $y_{2n}=0$ planes), then we can avoid singularities. We may define a scaling field

$$S_z' \equiv \frac{S_z \left(1 + S_z^{-1} S_{2n}^{a_z/a_{2n}}\right)}{1 + S_z S_n^{-a_z/a_n}} \quad (3.17)$$

S_z' reduces to S_z as $x \to 0$, for $y_n \neq 0$, $y_{2n} \neq 0$; as $y_n \to 0$, $x \neq 0$, $S_z' \to S_n^{a_z/a_n}$; as $y_n \to 0$, $x \neq 0$, $S_z' \to S_{2n}^{a_z/a_{2n}}$. Thus a global solution for the Gibbs potential which has z-point like exponents (except in the special symmetry planes $y_n=0$ and $y_{2n}=0$) could be given by

$$G = \frac{G_{gau}(h S_{gau})\, G_z(h, S_z')}{G_{gau}(h, S_{gau}) + G_z(h, S_z')} \quad (3.18)$$

The linear local analysis is again supported.

We may describe the crossover involved in the scaling field (3.17) and the Gibbs potential (3.18) by noting that if $x \to 0$ with y_n and y_{2n} fixed and nonzero, then only the magnetic field h and S_z "scale". Near the special planes $y_n=0$ or $y_{2n}=0$, an additional field is important (S_{2n} or S_n, respectively) and "scales".

It would be more customary to describe this in an alternate manner. Away from the special symmetry planes, S_z scales and y_n and y_{2n} are truly irrelevant variables. Thus, at each point of the critical surfac, x=0, we may consider x to be a scaling variable with y_n and y_{2n} being irrelevant variables which have only an insigificant effect on the thermodynamic functions. Near the junction of the plane x=0 and one of the symmetry planes (for example, $y_{2n}=0$), y_{2n} is clearly important. Phenomenologically, we might expect x and y_{2n} to scale, while y_n remains irrelevant. To see that this is the case, note that S_z can be written as

$$S_z = S_n \left[\frac{y_{2n}}{Y_{2n} X^2 Y_n^{2\Delta_n}} \right]^{\Delta_n - \Delta_z} I_{y_n y_{2n}}^{\Delta_n + (n^2-4)/(n^2+8)} \qquad (3.19)$$

Thus, the scaling invariant combintion $S_z S_n^{-a_z/a_n}$ in (3.19) can be written (apart from the irrelevant quantities y_n and y_{2n}) as

$$S_z S_n^{-a_z/a_n} \approx \left[\frac{x^{\epsilon(4-n)/(n+8)}}{y_{2n}^{a_n}} \right]^{[(n+4)(n-1)/(n^2+8)]/a_n} \qquad (3.20)$$

This is exactly the expected scaling invariant involving x and y_{2n} providing that, near the junction of the x=0 and y_{2n} =0 planes, we identify the scaling power of y_{2n} as $a_y \equiv \epsilon(4-n)/(n+8)$ and the scaling power of x as $a_x \equiv a_n$. An examination of (3.4) shows that these are the scaling powers that are obtained by linearizing around the n-spin fixed point. The nonlinear scaling field S_z' embodies the behavior of both fixed points in such a way as to generate the "double-power" scaling

laws used to describe crossover in anisotropic systems[37] and in higher order systems.[20-21] For example, the zero field Gibbs potential is given by

$$G = \left[\frac{X}{Y_n^{(n+4)(n-1)/(n^2+8)} + \left(X^{a_y/a_x}\right)^{(n+4)(n-1)/(n^2+8)}} \right]^{2-\alpha_z} . \qquad (3.21)$$

Greek alpha

We note that a_y/a_x is a "crossover exponent" and $(2-\alpha_z)(n^2+8)/(n+4)(n-1)$ is an "amplitude exponent".[38]

In Eq. (3.21) the dependence on the unimportant quantities y_n, X, Y_n, and Y_{2n} has been ignored. Precisely analogous behavior is found near the junction of the x=0 and y_n=0 planes, where the variables y_n and x appear in a scaling combination with scaling powers derived by linearization around the 2n-spin fixed point.

The analogy between the nonlinear effect incorporated in the nonlinear scaling fields and the crossover effects in systems with critical points of higher order can be extended. Fig. 6a shows a three-dimensional section of the four-dimensional phase diagram of the "Ising metamagnet"[39] with interactions J in the plane and $\mathcal{R}$J between planes ($\mathcal{R}<0$); the staggered field H' is zero in Fig. 6a. By varying the strength of the interaction parameter $\mathcal{R}$, the line of ordinary critical points of a simple metamagnet sweeps out a surface of critical points. The tricritical points which marked the transition between the second order critical behavior and the first order transition become lines of tricritical points. At $\mathcal{R}$=0, the system reduces to a set of two-dimensional ferromagnetic systems, and the tricritical lines meet on the T axis at the d=2 critical temperature. As $\mathcal{R} \to -\infty$, the system becomes a one-dimensional antiferromagnet, and there is no phase transition. The surface of ordinary critical points shrinks and the tricritical lines

Script R

meet at T=0. The classification system of Ref. 20 terms the surface of ordinary critical points a surface of critical points of order three. The special point at $\mathcal{R}$=0 is a critical point of order four. This notion of order corresponds both to the number of phases which are simultaneously critical at the critical point and to the number of variables which "scale" at that point. Thus, Ref. 20 proposes that on the surface of order two points, two variables scale; that is, the singular part of the Gibbs potential is a GHF of two of the variables (while the dependence on the remaining variables is smooth and non-singular). At a point on one of the lines of order three points, three variables scale. Finally, at the fourth order point, all four variables scale.

To compare this to the coupled-order parameter system described in this section, we note that near the plane x=0 (but not near the lines y_n =0 or y_{2n} =0) only the variable x and the magnetic field h scale; that is, the leading singular behavior of any function derived from the Gibbs potential (3.18) depends only slightly on the variables y_n and y_{2n} (and the nonlinear functions X, Y_n, and Y_{2n}). However, near the junction of the x=0 and y_{2n}=0 planes, y_{2n} does not appear merely as a smoothly varying parameter in the amplitudes of the thermodynamic function, but rather in an important, characteristically "double-power" scaling manner. Similarly, near the Gaussian fixed point, all the variables x, y_n, and y_{2n} appear in a "triple-power" scaling formula. We may tentatively relabel the Hamiltonian parameter space as in Fig. 6b. The x=0 plane is a surface of order two critical points; the lines y_n=0 and y_{2n}=0 are "tricritical" lines or lines of order three; finally, the Gaussian fixed point is a point of order four.

In determining the domain of influence of a tricritical point, scaling invariants of the form (3.2c) are natural choices for the "crossover cones".[21] The crossover effects contained in a double-power scaling formula are derived in the context

the fourth order point both being equal to ϵ. A similar situation does occur at the fourth order point of the metamagnet. In this case, the scaling powers of the direct and staggered fields are equal and the coexistence volumes do not merge together smoothly.

We also remark that the attainment of a point of order four in a space of four demensions is achievable in both the metamagnet and the renormalization group examples only due to their highly symmetric nature. In general, a space of demension six is required to observe a point at which four phases are simultaneously critical.[40] Indeed, a less symmetric version of (3.1) discussed in Appendix A is expressed in a six-parameter space.

IV. Discussion

In Secs. II and III we were able to explicitly carry out the solution of the nonlinear WHARG equations. Summarizing the features of these solutions which we believe are of a general nature, we have shown that

(a) Global solutions can be found in a limited but finite region of the Hamiltonian parameter space. This region includes the competing fixed points and is bounded by separ-surfaces emanating from the fixed points.

(b) Global solutions which do not yield the critical-point exponents derived from the local, linearized analysis of the stablest of the fixed points (on the critical surface) are singular on the separatrix leaving that fixed point.

Ther properties of the "regular" global solutions are closely analogous to those properites proposed in a phenomenological manner for crossover behavior between various critical points of higher order.[20] For the former, it is the nonlinear character of the scaling equation and scaling fields that embodies several types of ordering and critical behavior. For the latter, the simultaneous validity to lowest order of several linear scaling equations with linear scaling fields is presumed.[21] In both cases, the borders of a region where m variables scale is a region where (m+1) variables scale.

As shown in Sec. III for a three-parameter example, there is some surface (of dimension greater than two, in general) of order two critical points. The exponents everywhere on this surface are determined by linearization about the stablest fixed point, which is located somewhere on that surface. This surface will be partially bordered by special symmetry "planes" on which the renormalization group equations involve fewer parameters. In these special surfaces, another fixed point determines the critical behavior. Near the junction of the symmetry plane and the critical surface the two fixed points are in sharp competition. On the border of the order two critical points, three variables will scale in a characteristically "double-

power " law scaling form (cf. Eq. (3.21)). If the special symmetry planes associated with each such "tricritical" line intersect (as in Sec. III), more parameters are removed from the renormalization group equations, a new fixed point controls the exponents, and characteristic "triple-power" scaling behavior results. As in the phenomenological studies, this process can be continued indefinitely. As more and more symmetry restrictions are placed on the Hamiltonians, fixed points of weaker and weaker stability determine the critical point exponents. Since in the immediate neighborhood of a truly unstable fixed point there must be regions controlled by more stable fixed points, the crossover effects get more and more complicated as the order of the critical point increases. All of these crossover effects are automatically incorporated into the nonlinear scaling fields.

The above discussion gives reassuring support both to the usual local linearized fixed point analysis and also to the phenomenological descriptions of crossover. However, there are farther questions about the behavior of real systems modeled by renormalization group equations.

The solutions developed in Sec. II and III are only valid in a specified region of the parameter space. For the two-parameter problem (at the critical temperature), $0 \leq y_n \leq 1$. For the three-parameter problem, y_n and y_{2n} are confined to the region enclosed by the lines $y_{2n}=0$, $y_n=0$ and the two separatrices joining the n-spin and 2n-spin points to the stabler z-point (for $2 \leq n \leq 4$). Portions of these boundaries can be understood on a physically intuitive level. In the two-parameter case the restriction $y_n \geq 0$, is necessary for thermodynamic stability. The parameter y_n is proportional to the coefficient of the quadratic term in the Hamiltonian density; since the quadratic term is the term of highest degree in the Hamiltonian its coefficient must be positive. In the three-parameter problem the stability requirements are $u > 0$ and $w > -u$. These are not necessarily the "tricritical" lines (for $2 \leq n \leq 4$, $y_n=0$ and $y_{2n}=0$ are the order three lines; these restriction

are in the lines u=0 and w=u). However, if these lines do mark the boundary of a region of first order transition, as in the metamagenetic analog, they do form natural borders for the scaling behavior. However, the portion of the boundary formed by the restriction $y_n < 1$ in the two-parameter case and by the separatrices in the three-parameter case have no such intuitive explanation.

Within the confines of the region of scaling behavior, the renormalization trajectories are bounded. We may imagine changing the values of the parameters y_n and y_{2n} until the system Hamiltonian lies outside the solution region. In this case, the parameters y_n and y_{2n}, which (except near the "tricritical" lines) did not significantly affect the critical behavior, have unbounded renormalization trajectories. The approximations employed when $\epsilon y_n << 1$ and $\epsilon y_{2n} << 1$ are no longer valid. The problem immediately becomes far more complicated and it is no longer possible to discuss the renormalization group solutions within a simple perturbative scheme. Therfore, we can only speculate that the nonlinear solutions will involve many new fixed points and qualitatively different behavior.

Thus, although we have seen that thermodynamic stability requirements and possible first order transitions may account for some portion of the bounding surface of the solution region, the boundary formed by the separ- surfaces is more complex and possibly marks a transition to vastly altered behavior. However, it is precisely on this surface that regularity conditions were imposed on the global solutions. This is not an obvious step.

The requirement of regularity everywhere on the bounding surface corresponds to the notion that we can smoothly move the Hamiltonian (in particular, the critical Hamiltonian) to the "singular separatrix" [41] (in particular, the stablest fixed point). It might seem reasonable to requre that such a

procedure have a finite limit (as in Sec. III.) This resolution has several difficulties, among which are the following:

(i) The bounding separ-surfaces divide the region where even a general global solution is well-behaved from a region in which (as argued above) the global behavior may be radically different. It is perhaps over-optimistic to ascribe to the boundary between two such regions all the properties of one or the other of the regions.

(ii) It may not be possible by any application of an external field to move the parameters arbitrarily close to the boundary and the singular separatrix. This difficulty is unlikely in the particular cases treated in Secs. II and III since the fixed point values of the parameters are small, but it is not an impossible occurence. For example, we showed in Ref. 11 that temperature trajectories for the two-parameter problem usually terminate at a finite value of x (with $x < 1$) instead of reaching the infinite Gaussian fixed point x=1.

A more cogent example is provided by the five-parameter crossover model described in Appendix A. We consider a system with n+m spin components which decomposes into an n-spin system and an m-spin system[13]. For n=1 and m=2, this is a description of a Heisenberg ferromagnet with a single axis of anisotropy. The stablest fixed point is either Ising-like (one "easy axis") or XY-like (one "hard axis"). However, these fixed points consists of the XY subsystem parameters taking on their usual WF fixed point values, while the Ising subsystem is at its infinite Gaussian fixed point. Loosely, one subsystem is at its critical temperature while the second subsystem is at "infinite" temperature. No physical system with finite Hamiltonian parameters can be <u>at</u> such a fixed point; at best, it represents a limit of realizable systems. Therefore, we might expect that any singularity in a thermodynamic function would not manifest itself in the physical space.

Thus, in some cases (not necessarily those treated in Sec. II and III), the requirement of regularity everywhere on the bounding surface may be too stringent. If this requirement is relaxed, non-scaling behavior results. For example, in the three-parameter problem discussed in Sec. III, we might allow free competition among the n-spin fixed point, the 2n-spin fixed point, and the z-point, regardless of stability. These solutions, as shown in Sec. III, will be singular on the separatrix leaving the stablest of the x=0 fixed points. If this singularity is tolerated, the asymptotically valid critical point exponents will be determined by the fixed point which contributes the largest singularity to the thermodynamic quantity considered. Therefore, ν and γ will be determined by the fixed point which gives the smallest eigenvalue in the renormalization equation for x(3.4a); on the other hand α and β will be determined by the fixed point which yields the largest such eigenvalue. Since the competition is among three non-trival fixed points, we need only consider the relative sizes of a_n, a_{2n}, and a_z.

If we confine our attention to $n > 0$, for concreteness, we have $\Delta_{2n} \geq \Delta_n$ and $\Delta_{2n} \geq \Delta_z$. For $1 \leq n \leq 4$, we also have $\Delta_z \geq \Delta_n$. Thus for $n > 0$ the correlation length exponent will be that of the 2n-spin point, while α and β are given by their n-spin point values for $1 \leq n \leq 4$, and the z-point otherwise. Instead of equality in the relationship $\alpha + 2\beta + \gamma = 2$, we have that

$$\alpha + 2\beta + \gamma = 2 + \frac{\epsilon\, 3n}{2(n+4)(n+8)}, \quad 1 \le n \le 4. \qquad (4.1a)$$

$$\alpha + 2\beta + \gamma = 2 + \frac{\epsilon\,(n-2)^2(n+2)}{2(n+4)(n^2+8)}, \quad n<1,\ n>4. \qquad (4.1b)$$

If the finite Gaussian fixed point is allowed to compete as well, $\sqrt{}$ is as ... above; however $\alpha = \epsilon/2$ and $\beta = 1/2 - \epsilon/4$, independent of n. The Rushbrooke inequality is

$$\alpha + 2\beta + \gamma \quad 2 + \frac{\epsilon(n+1)}{2(n+4)} \qquad (4.1c)$$

The crossover effects embodied in a general global solution are similarly more complicated than those of the regular solutions exemplified in (3.19). It is reasonsible to impose the condition that, in the symmetry planes $y_n=0$ $y_{2n}=0$, and z=0 (where the renormalization equations involve only two parameters) any global solution should reduce smoothly to a two-parameter solution (except at the intersection of the separ-surfaces and the symmetry planes). This can always be done by using the invariants (3.10). The crossover near such a symmetry plane is between one set of exponents determined by the free competition of the fixed points in the <u>three-parameter space</u> and another set of exponents derived from a similar competition in a two-parameter space. Neither set of

exponents need satisfy scaling equalities. This is in contrast to the situation described in Sec. III; in that case, both sets of exponents separately satisfy scaling equalities.

Even though they do not satisfy exponent equalities, general global solution share other properties with the regular solutions (3.18). The eigenvalue of the magnetic field a_h is larger than a_n, a_{2n}, or a_z. Differentiation with respect to h increases the singularity of the Gibbs potential more rapidly than differentiation with respect to the temperature T. Therefore, we may still describe h as a "strong" direction and T as a "weak" direction in the sense of Griffiths and Wheeler[42]. Elsewhere[23], we have considered systems which have this property that derivatives taken in different directions have different well-defined relative strengths. We term such systems "critically ordered" and have shown that under certain conditions, the geometric postulates of Griffiths and Wheeler are satisfied for these non-scaling systems. Using this terminology, we can restate the distinction between the general and regular solutions for the thermodynamic functions. The former represents a system that is only critically ordered, while the latter has a true asymptotic scaling form.[43]

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APPENDIX A. The Wegner-Houghton Approximate Renormalization Group

Wegner and Houghton[9] have introduced an exact differential generator for the renormalization group as applied to continuum spin Hamiltonian densities. It takes the form of a highly nonlinear functional integrodifferential equation. The Hamiltonian density $\mathcal{H}(\vec{s})$ must be written as a functional of the Fourier transform of the spin density s(x). Thus it is conventional to write an expansion for the Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H} = \sum_{j=0}^{\infty} \sum_{k_1 \cdots k_j} \sum_{\vec{\alpha}} v_j^{\vec{\alpha}}(\vec{k}_1 \cdots \vec{k}_j)\, s_{\alpha_1}(\vec{k}_1) \cdots s_{\alpha_j}(\vec{k}_j) \qquad (A1)$$

where $\vec{s}(\vec{k})$ is the Fourier transform of $\vec{s}(\vec{x})$ and α is a j-component index, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_j)$.

The coefficent functions $v_j^{\vec{\alpha}}(k)$ are also to be considered as functions of the renormalization parameter, ℓ. The renormalization group equations become nonlinear integrodifferential equations coupling the $v_j^{\vec{\alpha}}(k)$. Equations of this form are nearly intractable. Following the lead of Wilson's approximate recursion formula, we hope that a certain "momentum-independent" or "zero-momentum" limit of the full renormalization group structure will preserve the basic content of the renormalization group approach. We force the mometum dependent coupling constants to be of the form

$$v_j^{\vec{\alpha}} \qquad v_j^{\vec{\alpha}}(0) + \delta_{j,2}\delta_{\alpha_1,\alpha_2} k^2 \qquad (A2)$$

This is equivalent to choosing a Wilson reduced Hamiltonian density of the form

$$\mathcal{H}(S) = |\vec{\nabla} S(\vec{x})|^2 \cdot H(\vec{S}(\vec{x})) , \tag{A3}$$

where $H(\vec{S}) = \sum_{j=0}^{\infty} \sum_{\alpha} V_j^{\alpha} S_{\alpha_1} \cdots S_{\alpha_j}$.

Hamiltonian densities such as (A3) are, of course, not renormalization group invariant. Thus, terms which arise from the exact group equations must be discarded if they do not retain the form (A3). This requires, for example, that the critical point exponent η be set equal to zero since it cannot be determined from the resulting equations.

For the momentum independent part of the Hamiltonian density (A3), H(s) we find that the Wegner-Houghton Approximate Renormalization Group (WHARG) equation is given by

$$\dot{H} = d H + \frac{(2-d)}{2} s \cdot \vec{\nabla}_s H + \frac{d}{2} \ln \det[1 + \hat{H}/2] , \tag{A4}$$

where $\hat{H}$ is the matrix of second partial derivatives

$$\hat{H}_{ij} \equiv \partial^2 H / \partial s_i \partial s_j , \tag{A5}$$

and d is the lattice dimension. Expanding the function ln det,(A4) can also be written as

$$\dot{H} = dH + \frac{(2-d)}{2}\vec{s}\cdot\vec{\nabla}_s H + \frac{d}{2}\left[\frac{1}{2}\mathrm{tr}\hat{H} - \frac{1}{8}\mathrm{tr}\hat{H}^2 + \frac{1}{24}\hat{H}^3 - \cdots\right] . \quad \text{(A6)}$$

This however, may not be the best form of the expansion (see below).

The solutions of the linearized form of (A6), in which the traces of $\hat{H}^2$, $\hat{H}^3$ and so on are discarded, are the Gaussian fixed point eigenfunctions. They are given by products of generalized Laguerre polynomials and harmonic polynomials just as in the Wilson approximate recursion formulae[5,44]

$$Q_{m,K} = L_m^{K-1+n/2}\left(\frac{d-2}{4}s^2\right) P_K(\vec{s}) , \quad \text{(A7)}$$

where $P_k(\vec{s})$ is a harmonic polynomial of degree k, and n the spin dimension For a system in dimension n, there are $(2k+n-2)(k+n-3)!/(k!(n-2)!)$ harmonic polynomials of degree k.[45] All of these polynomials, moreover, are degenerate with respect to the renormalization group since the eigenvalue of the linearized renormalization group equation depends only on m and k:

$$\begin{aligned}\dot{Q}_{m,K} &= dQ_{m,K} + \frac{2-d}{2}\vec{s}\cdot\nabla Q_{m,K} + \frac{d}{4}\nabla^2 Q_{m,K} \\ &= \left[d + (2-d)(m+K/2)\right] Q_{m,K}\end{aligned} \quad \text{(A8)}$$

Elsewhere,[16] we have given a study of the fixed points determined by perturbation

from the Gaussian fixed points for the isotropic cases (m arbitrary, k=0) and for the special case n=1. These problems were considered only from the viewpoint of location of, and linearization about, particular fixed points.

It is easy to check that the solutions of the linearized WHARG equations are solutions of the full Wegner-Houghton momentum dependent equations when similarly linearized (for η =0; if we wish to insert an a priori determined the (d-2) in the argument of the Laguerre polynomial changes to d-2+ η). Thus if the Hamiltonian is "small", the error in using the WHARG equations is of second order of smallness. As a parenthetical remark we note that the class of solutions to the linearized momentum-dependent equations is very large. Each eigenfunction that is at most a polynomial of degree r in the spins has as the coefficient of the highest power of the spins an arbitrary homogeneous function of the momentum vectors. If this coefficient function satisfies $\lambda^q f(k_1,k_2..k_r)=$ $f(\lambda k_1, \lambda k_2, \ldots \lambda k_r)$ then the eigenvalue of such an eigenfunction is $d+(2-d)(r/2-q)$.

Fully nonlinear problems must be considered individually. For use in Sec II and Sec III we consider a system of spin dimension n+m, which breaks into two internally isotropic blocks; that is, the Hamiltonian is a function of $x \equiv \sum_{i=1}^{n} S_i^2$ and $y \equiv \sum_{i=n+1}^{n+m} S_i^2$ alone. The WHARG equations are

$$\dot{H} = dH + (2-d)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{d}{2}\left[n \ln\left(1+\frac{\partial H}{\partial x}\right) + m \ln\left(1+\frac{\partial H}{\partial y}\right) + \ln\left\{1 + \frac{2x\frac{\partial^2 H}{\partial x^2}}{1+\partial H/\partial x} + \frac{2y\frac{\partial^2 H}{\partial y^2}}{1+\frac{\partial H}{\partial y}} + \frac{4xy\left[\frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2\right]}{\left(1+\frac{\partial H}{\partial x}\right)\left(1+\frac{\partial H}{\partial y}\right)}\right\}\right]. \quad \text{(A9)}$$

It is more compact to leave the nonlinear structure inside the logrithm in this case. Now we write H as

$$H = r_x X + r_y Y + U_{xx}\frac{X^2}{2} + U_{yy}\frac{Y^2}{2} + U_{xy} XY, \quad \text{(A10)}$$

and make the change of variables

$$
\begin{aligned}
r_n &\equiv r_x / (1+r_x) \\
r_m &\equiv r_y / (1+r_y) \\
U_n &\equiv U_{xx} / (1+r_x)^2 \\
U_m &\equiv U_{yy} / (1+r_y)^2 \\
\omega &\equiv U_{xy} / (1+r_x)(1+r_y)
\end{aligned}
\tag{A12}
$$

Then the WHARG equations are

$$\dot{r}_n = (1-r_n)\left[2r_n + \frac{d}{2}\left((n+2)u_n + mw\right)\right]$$

$$\dot{r}_m = (1-r_m)\left[2r_m + \frac{d}{2}\left((m+2)u_m + nw\right)\right]$$

$$\dot{u}_n = \epsilon u_n - \frac{d}{2}\left[(n+8)u_n^2 + mw^2\right] - 4u_n\left[r_n + \frac{d}{4}\left((n+2)u_n + mw\right)\right]$$

$$\dot{u}_m = \epsilon u_m - \frac{d}{2}\left[(m+8)u_m^2 + nw^2\right] - 4u_m\left[r_m + \frac{d}{4}\left((m+2)u_m + nw\right)\right] \qquad \text{(A12b)}$$

$$\dot{w} = \epsilon w - \frac{d}{2}w\left[4w + (n+2)u_n + (m+2)u_m\right] - 2w\left[r_n + r_m + \frac{d}{4}\left((n+m)w + (n+2)u_n + (m+2)u_m\right)\right].$$

It is easy to check that coefficients of higher powers of the spin are $O(\epsilon^3)$.

These equations are already diagonalized around the infinite Gaussian point, $r_n = r_m = 1$, $u_n = u_m = w = 0$. Diagonalizing around the finite Gaussian point $r_n = r_m = u_n = u_m = w = 0$, we make the further change of variables

$$X_n = r_n + \frac{d}{2(2-\epsilon)}\left[(n+2)u_n + mw\right],$$

$$X_m = r_m + \frac{d}{2(2-\epsilon)}\left[(m+2)u_m + nw\right], \qquad \text{(A13)}$$

and write the WHARG equations as (neglecting terms like $\epsilon^2 u_n^2$ and $\epsilon^2 u_m^2$)

$$\dot{X}_n = 2X_n\left[1 - X_n - (n+2)u_n\right] - 2mX_m w,$$

$$\dot{X}_m = 2X_m\left[1 - X_m - (m+2)u_m\right] - 2nX_n w,$$

$$\dot{u}_n = \epsilon u_n - \frac{d}{2}\left[(n+8)u_n^2 + mw^2\right] - 4u_n X_n,$$

$$\dot{u}_m = \epsilon u_m - \frac{d}{2}\left[(m+8)u_m^2 + nw^2\right] - 4u_m X_m, \qquad \text{(A14)}$$

$$\dot{w} = \epsilon w - \frac{d}{2}w\left[4w + (n+2)u_n + (m+2)u_m\right] - 2w(X_n + X_m).$$

The equations given in (A14) do not lend themselves easily to global analysis. There are 32 fixed points of (A14), many of which are not particularly interesting. Points of particular interest in the three-dimensional subspace $x_n=x_m=0$ are the Gaussian point at $u_n=u_m=w=0$; the n-spin point at $u_m=w=0, u_n=\epsilon/(2(n+8))$; the m-spin point at $u_n=0=w, u_m=\epsilon/(2(m+8))$; the coexisting but uncoupled (n,m) spin point at $w=0, u_n=\epsilon/(2(n+8))$, $u_m=\epsilon/(2(m+8))$; and the n+m spin point at $w=u_n=u_m=\epsilon/(2(n+m+8))$. The usual n and m spin points are unstable with respect to the uncoupled (n,m) point which is unstable with respect to the n+m spin point. Thus, if these fixed points were sufficient to describe the system, the n+m spin point would be the most stable.

However, there are also fixed points for non-zero x_n and x_m. The most important are the "isolated" n-spin point at $x_n=0$ $x_m=1$ (!), $u_m=w=0$, $u_n=\epsilon/(2(m+8))$ and the "isolated" m-spin point at $x_m=0, x_n=1$ (!), $u_n=w=0$, $u_m=\epsilon/(2(m+8))$. Recalling that x_n (respectively, x_m) =1 implies that $r_n=\infty$ (respectively, $r_m=\infty$), we see that these isolated points correspond to systems for which one subsystem is at its critical temperature while the other subsystem is effectively at "infinite" temperature. It is clear that no real Hamiltonian can be said to be "at" these fixed points. However, the n+m isotropic spin fixed point, which is the stablest of the $x_n=x_m=0$ fixed points is unstable with respect to these isolated n-spin and m=spin points. These isolated points are unstable only with respect to the infinite Gaussian fixed point $x_n=x_m=1$, $u_n=u_m=w=0$. The full five-dimensional space is thus partitioned in such a way that one four-dimensional manifold is generated by the isolated m-spin fixed point; a second four-dimensional manifold is generated by the isolated n-spin point; these manifolds intersect in the three-dimension manifold $x_n=x_m=0$, which is generated by the n+m spin point (by generated we mean that the fixed point in question is a stable node with respect to all trajectories lying entirely within the manifold, that is, the fixed point

is the stablest fixed point contained within the manifold.). The field-like variable carrying the system from one of the four-dimensional manifolds through the three dimensional boundary to the other four-dimensional manifold is essentially the anisotropy field. Linearizing about the n+m spin fixed point gives the usual determination of the crossover exponent[13], $\phi = 1 + \epsilon (n+m)/2(n+m+8)$.

The complicated geometry of the four-dimensional manifolds precludes any direct attack on the full five-dimensional problem. For this reson, Sec. III of the text considers the special case of n=m and $x_n = x_m$, $u_n = u_m$. Specializing equations (A12-14) to this case, one obtains (3.1-3).

APPENDIX B. CALCULATION OF Φ AND γ.

In this appendix we will discuss the calculation of the two parameter separatrix and crossover functions, Φ and γ, which are used extensively in both Sec. II and III. Consider the coupled first order differential equations,

$$\dot{x} = 2x(1-x-\epsilon\Delta y/2), \tag{B1}$$

$$\dot{y} = y(\epsilon(1-y)-4x). \tag{B2}$$

We are interested in the form of the separatrix $y=\Phi(x)$, joining the point x=0, y=1 to the point x=1, y=0 (cf Fig. 1). We may form an equation for $\Phi(x)$ by noting that $\Phi'(x)\dot{x}(x,\Phi) = \dot{y}(x,\Phi)$ or

$$\Phi'(x)\,2x(1-x-\epsilon\Delta\Phi/2) = \Phi(\epsilon(1-\Phi)-4x). \tag{B3}$$

Near x=1 we expect $\Phi \ll 1-x$. Examining (B3) in that limit, it reduces to the linear equation

$$2\Phi'(1-x) = -\Phi d, \tag{B4}$$

which has solutions proportional to $(1-x)^{d/2}$. If we set $\bar{F} = (1-x)^{d/2} e^{g}$ we obtain an equation for g. This equation is rather messy and it is better within the context of the WHARG equations to perform an ϵ expansion. A form for g which contains all $O(\epsilon)$ corrections exactly is

$$\bar{\Phi} = (1-x)^{d/2} \exp\left[\epsilon \times (1-2\Delta)/2\right] \qquad \text{(B5)}$$

However, since this is arrived at in terms of an expansion in ϵ, it is not immediately clear that (B5) is the best or most suitable way of writing the solution of (B3) to $O(\epsilon)$. We have checked the suitability of such an expansions to show that no

ridiculous singularities would be introduced at that order.

We now turn our attention to $\mathcal{Y} \equiv \exp\left(-\int y \, d\ell\right)$. On the line x=0, $\mathcal{Y} = 1-y$ by explicit integration. We also recall that $\mathcal{Y}$ vanishes on the separatrix $y=\bar{\Phi}(x)$.

We therefore try a solution of the form $\mathcal{Y} = (1-y/\bar{\Phi}) e^{\epsilon g}$.

To first order in ϵ, we find

$$\mathcal{Y} = (1 - y/\bar{\Phi}) \exp\left[\epsilon \Delta \times y / \bar{\Phi}\right. \qquad \text{(B6)}$$

We have also checked that this form is suitable.

Thus we have shown that the forms given in (B5) and (B6) are suitable $O(\epsilon)$ approximate solutions. Since they are used in a variety of contexts in Sec. II and Sec. III we repeat their expressions here and display their dependence on Δ as an additional argument.

$$\Phi(x,\Delta) = (1-x)^{d/2} \exp(\epsilon x(1-2\Delta)/2) , \tag{B7}$$

$$\psi(x,y,\Delta) = (1 - y/\Phi(x,\Delta)) \exp(\epsilon\Delta x y/\Phi) . \tag{B8}$$

It is important to stress that (B7) and (B8) give the solutions to (B1)-(B2) to $O(\epsilon)$. Thus, in Sec. II, the solution given in (2.9) is the proper $O(\epsilon)$ solution of (2.5). However, it is not a proper solution to (2.3) except formally. If the solution given in (2.9) is tested with the transformations given in (2.3) one of the $O(\epsilon^2)$ terms which is formally dropped is proportional to $\epsilon^2 y_n^2/x$. This is to be expected since (2.3) and (2.4) have different locations for their singularities. In passing to (2.4), the separatrix connecting the finite Gaussian and WF points was approximated by a straightline. It is easy to see that if terms up to y_n^m are kept in the definition of x (x=0 defines the critical separatrix) the terms to be formally discarded include one proportional to $\epsilon^{m+1} y_n^{m+1}/x$. Thus to avoid this inconvenient singularity in the solution of (2.3) we would have to go to arbitrary

order in ϵ. Although we will argue below that such an effort is bootless, we will sketch briefly the results of such a solution.

We make the exact change of variable $x=\bar{r}+\rho(\epsilon y_n)$ in (2.3) and demand that x=0 represent the $T=T_c$ separatrix. The resulting equations are

$$\dot{x} = 2x\left[1-x-\epsilon y_n \Delta_n/2 + 2\left(\rho - y_n \, d\rho/dy_n\right)\right], \tag{B9}$$

$$\dot{y}_n = y_n\left[\epsilon\left(1-y_n(1+2\Delta_n)\right) - 4x + 4\rho\right]. \tag{B10}$$

where ρ satisfies

$$0 = 2(1+\rho)\left(\epsilon y_n \Delta_n/2 - \rho\right) + y_n \frac{d\rho}{dy_n}\left[\epsilon\left(1-y_n(1+2\Delta_n)\right)+4\rho\right] \tag{B11}$$

Matching the solution to (2.3) and (B9) and (B10) will involve R, X, Y_n, and two new functions Y_n' and Y_n'' defined by $\dot{Y}_n' = 4\rho Y_n'$ and $\dot{Y}_n'' = 4\, y_n (d\rho/dy_n) Y_n''$ respectively. Y_n' and Y_n'' are simple powers of Y_n if the linear approximation for ρ is made as in (2.4). An immediate consequence of (B9)-(B11) is that the singularity of Y_n at x=0 is changed from $(1-y_n)$ to $(1-y_n)^{1+a}$, where

$$a = 2\Delta_n - 4\frac{d\rho}{dy_n} \approx -\epsilon\Delta_n .$$

The effect of keeping the curvature of the separatrix is to introduce at least a cubic term into the y_n equation (B11). However we have already discarded from the y_n equation any six-spin interaction term which "feeds back" from the higher order equations. By examining the WHARG equations one finds that the six-spin coupling constant is $O(y_n^3)$. Thus, we cannot keep any curvature in the separatrix without including the six-spin terms. If we were to use the exact separatrix we would have to solve the infinite set of WHARG equations. Furthermore, the use of the WHARG equations could not be justified since the momentum dependence has been neglected.

We also note that logrithmic corrections of the sort described by Wegner[17] do not appear in an ϵ-expansion. Ref. 17 gives a general procedure to extract nonlinear scaling fields and shows that the method may fail if the eigenvalues satisfy certain integral relations. In the examples in Sec. II or Sec. III, Wegner would predict logrithmic corrections if $2/\epsilon = N$, an integer. These corrections will never appear in any ϵ-expansion. The source of these terms is the vanishing of the denominator of some coefficient in a tentative power series expansion for the scaling fields of the form $1/(N\epsilon - 2)$. In a consistent ϵ-expansion this denominator must be expanded as $-1/2 - 1/4 N\epsilon$ ... Thus, to any fixed order in ϵ no difficulty is encountered.

This rather unhappy resolution is closely related to a similar situation in the field theoretic approach to the ϵ-expansion. The Feyman diagram illustrated in Fig. 7a diverges as $P^{-\epsilon}$ for small P. This divergence is not troublesome in a few simple diagrams, but by concatenating N such simple loops the divergence is increased to $P^{-N\epsilon}$ (Fig. 7b). The multiple loop diagram can replace the single loop diagram in any Feyman diagram, and no further "renormalization" removes this divergence. To handle this, a consistent ϵ-expansion is performed so that $P^{-N\epsilon} = 1 - N\epsilon \ln P + \ldots$ The divergence of each log term is

sufficiently weak to be incorporated into the remaining diagramatic calculations and renormalization proceedure. Thus, in both cases a real singularity is removed by the use of a self-consistent ϵ-expansion.

We also note that the expansion of the nonlinear scaling fields in power series is limited in usefulness by the fact that the fields contain singularities. The zero of the function $\tilde{y}$,for example, is hard to locate in a power series expansion. The delineation of the basic singularities of the nonlinear scaling fields enables the series expansions of Ref. 17 to be partially summed to give faithful representations of the scaling fields

APPENDIX C. The Special Cases n=0, ∞, 2,4.

In a few special cases, the solution of the equations of Sec. III can be carried further. For n=0 and n= ∞ completely explicit solutions can be obtained within the context of an ϵ -expansion. For n=2 and n=4, the renormalization trajectories in the x=0 plane can be obtained exactly.

For n=0 and n=∞ , we may take $z=y_n+y_{2n}$. The x equation and the z equation decouple from the remaining equation. We have

$$\dot{x} = 2x(1-x-\epsilon z/8) \tag{C1}$$

for n=0, and

$$\dot{x} = 2x(1-x-\epsilon z/2) \tag{C2}$$

for n= ∞;

while in both cases the z equation is

$$\dot{z} = z(\epsilon(1-z)-4x) \tag{C3}$$

Combining the solution of the two dimensional problem with the information given in (3.9) we have immediately for n = 0

$$Y_{2n}^{1/2} = \frac{y_n + y_{2n} Y_0^{1/2}}{y_n + y_{2n}} , \tag{C4}$$

$$Y_n^{1/2} = \frac{Y_0^{1/2} (y_n + y_{2n})}{y_n + y_{2n} Y_0^{1/2}} , \tag{C5}$$

where $Y_0 = \mathcal{Y}(x, y_n + y_{2n}, 1/4)$. Similarly for $n=\infty$, we have

$$Y_n = \frac{y_n Y_\infty + y_{2n}}{y_n + y_{2n}} \tag{C6}$$

$$Y_{2n} = \frac{Y_\infty (y_n + y_{2n})}{Y_\infty y_n + y_{2n}} , \tag{C7}$$

where $Y_\infty = \mathcal{Y}(x, y_n + y_{2n}, 1)$.

For the case n=2 and n=4 (where (3.9) fails to provide any information), some extra information is available in the form of the renormalization trajectories in the x=0 plane. In principle, Y_n and Y_{2n} can be obtained in the x=0 plane by one integration. The form of Y_n and Y_{2n} at x=0 is probably sufficient since we are interested in $x \to 0$ and we know how the boundary conditions for Y_n and Y_{2n} depend on x (see Sec. III). For n=2, the trajectories are given

implicitly by

$$\left[\frac{y_{2n}^5}{y_n^4} - \left(\frac{y_{2n}}{y_n}\right)^4 + \frac{4}{5}\left(\frac{y_{2n}}{y_n}\right)^3 - \frac{12}{25}\left(\frac{y_{2n}}{y_n}\right)^2 + \frac{24}{125}\,\frac{y_{2n}}{y_n} - \frac{24}{625} \right] e^{5\,y_{2n}/y_n} = K \quad \text{(C8)}$$

while, for n=4, the corresponding equation is

$$\left[\frac{y_n^4}{y_{2n}^3} - \left(\frac{y_n}{y_{2n}}\right)^3 + \frac{3}{4}\left(\frac{y_n}{y_{2n}}\right)^2 - \frac{3}{8}\,\frac{y_n}{y_{2n}} + \frac{3}{32} \right] e^{4\,y_n/y_{2n}} = K' \quad \text{(C9)}$$

APPENDIX D. Zero-Fields Gibbs potential

As discussed in Sec. II, the zero field Gibbs potential (for $T>T_c$) can be written as[26,27]

$$G = \frac{dn}{2} \int_0^\infty e^{-d\ell} \ln(1+r(\ell))\, d\ell \tag{D1}$$

In (D1) we have set $\eta=0$. Integrating this equation by parts we have that

$$G = \frac{n}{2} \ln(1+r(0)) + \frac{n}{2} \int_0^\infty e^{-d\ell} \frac{\dot{r}(\ell)}{1+r(\ell)}\, d\ell \tag{D2}$$

Examination of Eqs. (2.2) and (2.4) for the two-parameter problem or Eqs. (3.2) and (3.3) for the three parameter problem shows that to $O(\epsilon)$ we may replace $\dot{r}/(1+r)$ by $2x$ and $(1+r)$ by X^{-1}. Eq. (D2) now reads

$$G = -\frac{n}{2} \ln X(0) + n \int_0^\infty e^{-d\ell} x(\ell)\, d\ell \tag{D3}$$

The Gibbs potential can now in principle be evaluated by expressing x in terms of the nonlinear scaling fields given in (2.10) and (3.14) for the two-parameter and three parameter problems respectively, and performing the integration in (D3).

For example, in the two-parameter case we may write the integral in (D3) as

$$n\int_0^\infty e^{-d\ell}\chi(\ell)d\ell = \frac{n}{2}\, S_n^{2-d_n}(x,y_n)\int_x^1 \frac{S_n'^{\,d_n-2}(x',y_n(x'))\,dx'}{\left[1-x'-\frac{\epsilon d_n}{2}y_n(x')\right]} \qquad \text{(D4)}$$

In (D4), $y_n(x')$ denotes the value of y_n at $x=x'$ on the renormalization trajectory passing through (x,y_n). We may consider two limiting cases of (D4).

First we restrict (x,y_n) to lie on the separatrix $y_n = \varphi_n(x)$. In this case, using (2.9b) we have

$$n\int_0^\infty e^{-d\ell}\chi(\ell)d\ell = \frac{x^{2-d_n}}{(1-x)^{d/2}}\frac{n}{2}\int_x^1 \frac{(1-x')^{d/2-1}}{(x')^{2-d_n}}\frac{dx'}{(1-\epsilon d_n(1-x)/2)} \qquad \text{(D5)}$$

As a second example we consider (x,y_n) such that $y_n \ll (1-x)^2$ all along the renormalization trajectory. In this case we may neglect Y_n and write

$$n\int_0^\infty e^{-d\ell}\chi(\ell)d\ell = \frac{x^{d/2}}{(1-x)^{d/2}}\frac{n}{2}\int_x^1 \frac{(1-x')^{\frac{d}{2}-1}}{x'^{\,d/2}}dx' \qquad \text{(D6)}$$

The condition $y_n \ll (1-x)^2$ cannot be satisfied as $x \to 0$ unless y_n is identically zero; in all other cases, the trajectories sweep toward the separatrix so that the free energy more closely approximates the solution given in (D5). We note that the prefactor of the integral in (D6) is simply $S_{gau}^{2-\alpha_{gau}}$ with Y_n set =1.

For the general case, the trajectories $y_n(x')$ are given implicitly in the renormalization invariant (2.13). For further discussions and model calculations, see Ref. 26-27.

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Figure Captions

Fig. 1. The (p,q) plane of Eq. (1.1). Only two fixed points are shown. Local integral curves for the fixed points (0,0) and (0,1) are sketched as shown. The local regions of validity of the linearized approximations to the correlation length (cf. Eq. (1.6) and (1.8)) are indicated, as well as the nonlinear global region considered in Sec. II.

Fig. 2. The solution region for the two-parameter problem of Sec. II is shown. The region includes three fixed points, denoted as the finite Gaussian, infinite Gaussian, and Wilson-Fisher fixed points. The separatrix connecting the Wilson-Fisher and infinite Gaussian fixed points is labeled $y_n=\varphi_n(x)$. The line x=0 corresponds to the surface of critical Hamiltonians.

Fig. 3. Diagramatic representation of the Hamiltonian density (3.1). The squares and circles represent the n-components subsystems $\vec{s}_1$ and $\vec{s}_2$ respectively. This system can be regarded as either (a) possessing a biquadratic interaction between the two subsystems, or (b) possessing an anisotropic self-interaction of a single 2n-component spin $\vec{s} \equiv (\vec{s}_1, \vec{s}_2)$. In case (a), the intrasystem interaction strength is w while the intersystem interaction is u/2. In case (b), the super-spin interactions is u/2 while the anisotropic interaction is w-u.

Fig. 4. The solution region for the three-parameter problem of Sec. III is shown for various values of the spin-dimension n. There are five fixed points: the finite Gaussian, point, the infinite Gaussian point, the n-spin point at x=0, $y_{2n}=0$, $y_n=1$, the 2n-spin point at x=0, $y_n=0$, $y_{2n}=1$ and the "z-point" whose location depends on n as indicated in the text. Boundary conditions on the functions Y_n and Y_{2n}

are indicated for the separ-surfaces. In (a), the solution region is depicted for $n \leq 2$; in (b), for $2 \leq n \leq 4$; in (c), $n \geq 4$.

Fig. 5. The solution trajectories in the x=0 plane are shown. The renormalization trajectories sweep toward the separatrices joining the relatively unstable fixed points to the stablest fixed points before moving to the satblest fixed point. This behavior, although illstrated for $n \approx 1$ holds for all n.

Fig. 6. Comparison of the phase diagram of an Ising metamagnet and the three-parameter crossover problem of Sec. III.

(a) The phase diagram of the metamagnet is shown in zero staggared field. A coexistence volume is capped by a surface of ordinary critical points (σ=2). This surface is bounded by two lines of tricritical points (σ=3). The tricritical lines intersect at the σ=4 point H=0, $\mathcal{R}$=0, $T=T_2$, the two-dimensional Ising critical temperature.

(b) The solution region for the three parameter problem for $2 \leq n \leq 4$ is shown. The finite Gaussian point corresponds to the point of order four; the lines $y_n=0$ and $y_{2n}=0$ correspond to the σ=3 lines; The portion of the x=0 plane bounded by the "tricritical" lines and the separatrices joining the 2n-spin point and the n-spin point to the z-point correspond to the surface of ordinary critical points (σ=2).

Fig. 7. Feynman diagrams for simple loop divergence.

(a) A single loop with momenta $\vec{p}_1$ and $\vec{p}_2$ diverges like $P^{-\epsilon}$ where $\vec{P}=\vec{p}_1+\vec{p}_2$.

(b) N such loops linked together give a diagram which diverges like $P^{-N\epsilon}$, which is of arbitrary order for sufficiently large N.

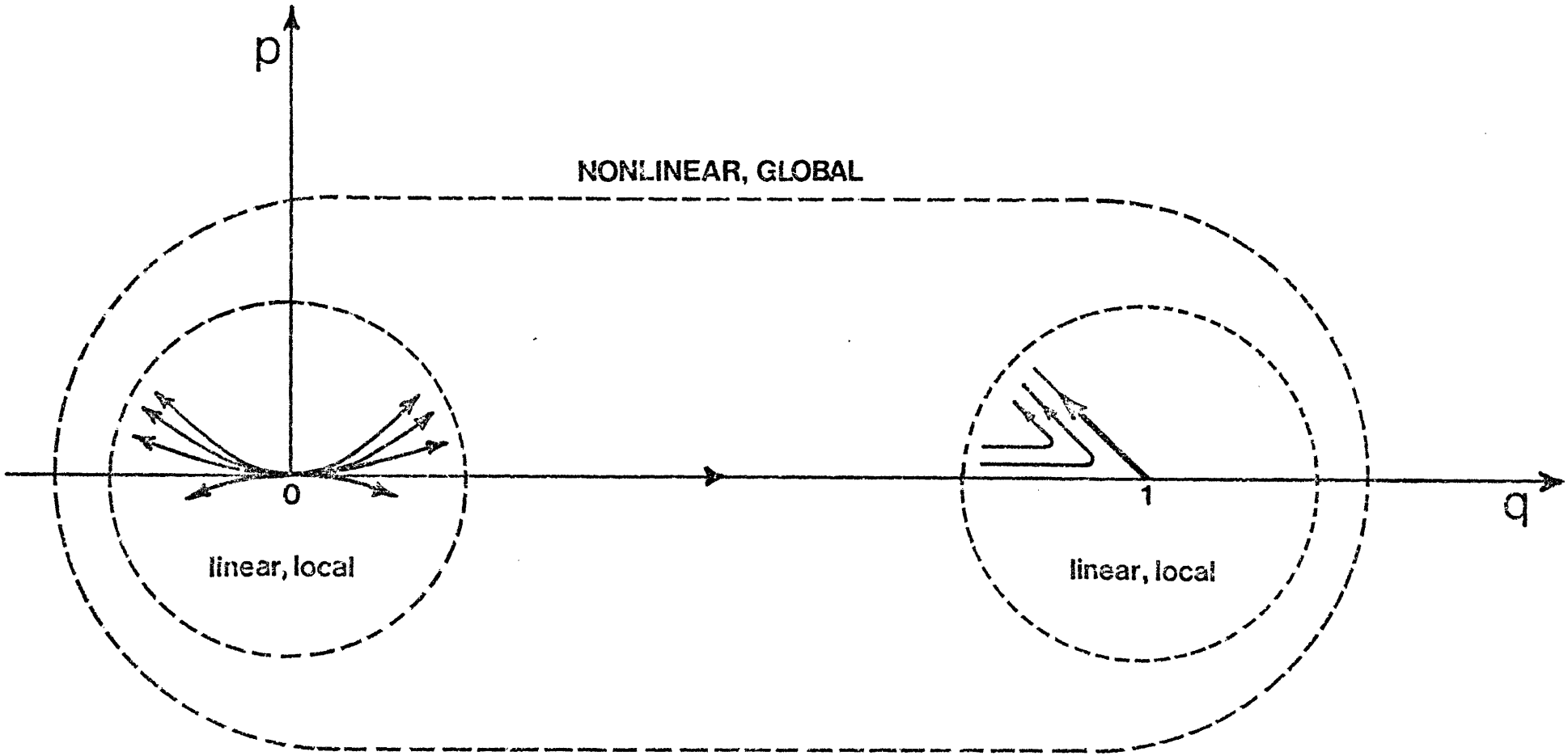
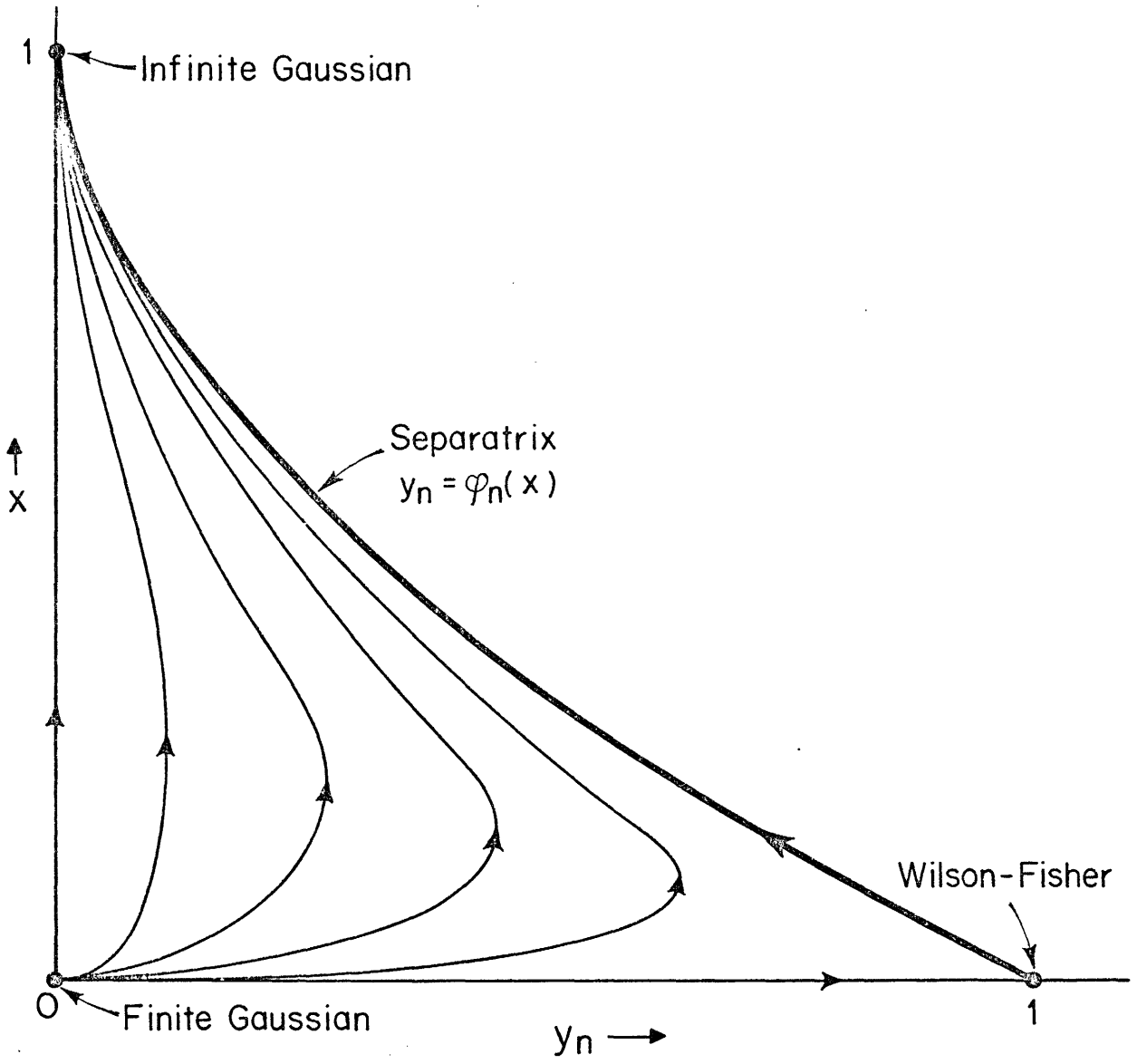


Fig. 1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Separatrix
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Finite Gaussian
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y_n

Fig. 2

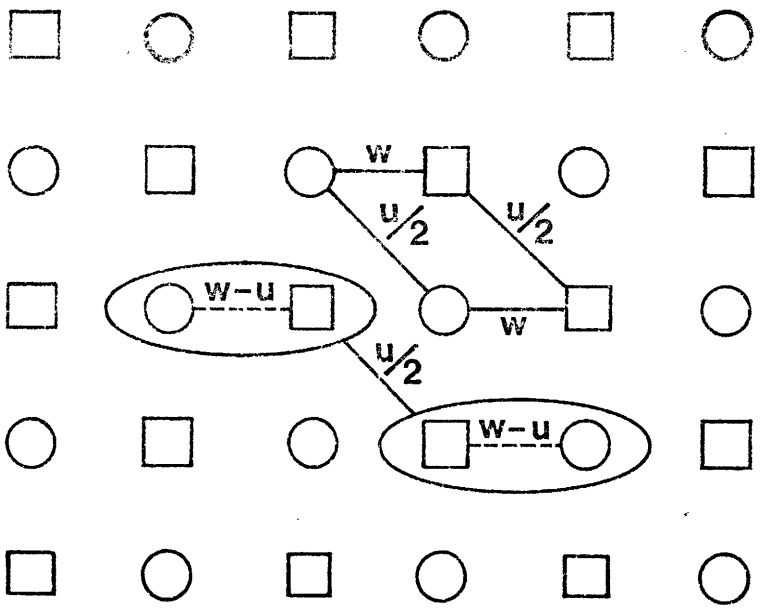

Fig. 3

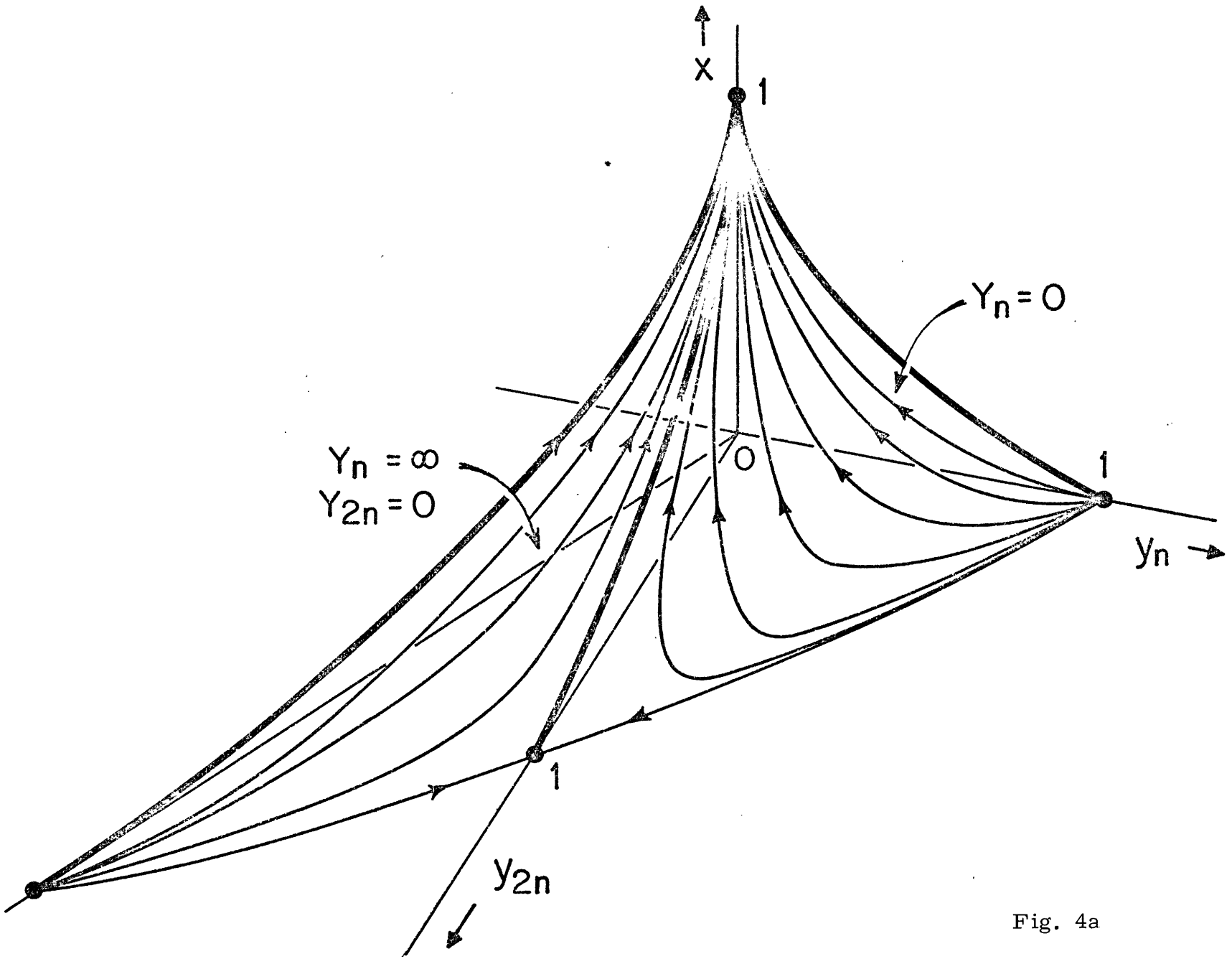

Fig. 4a

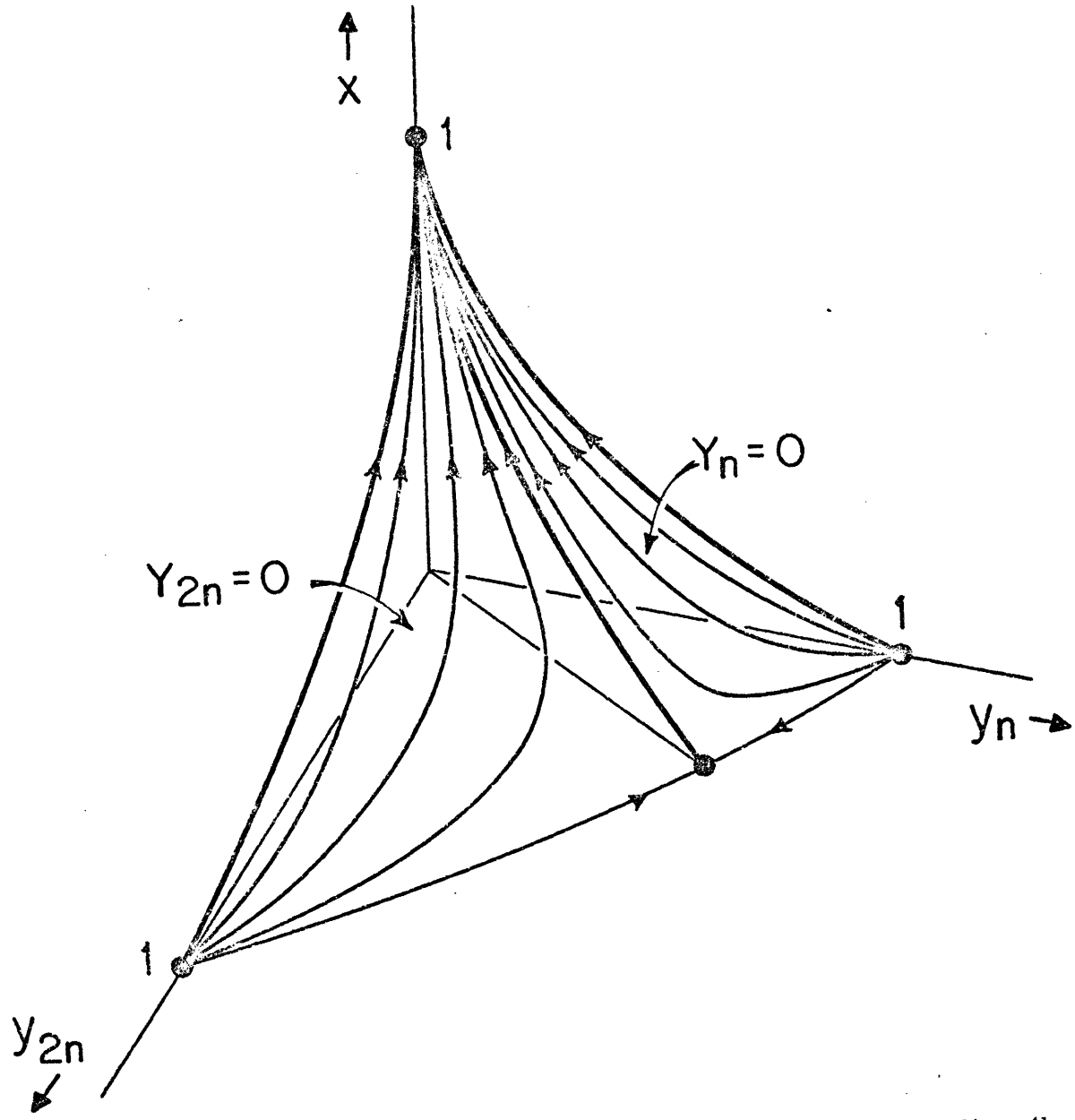

Fig. 4b

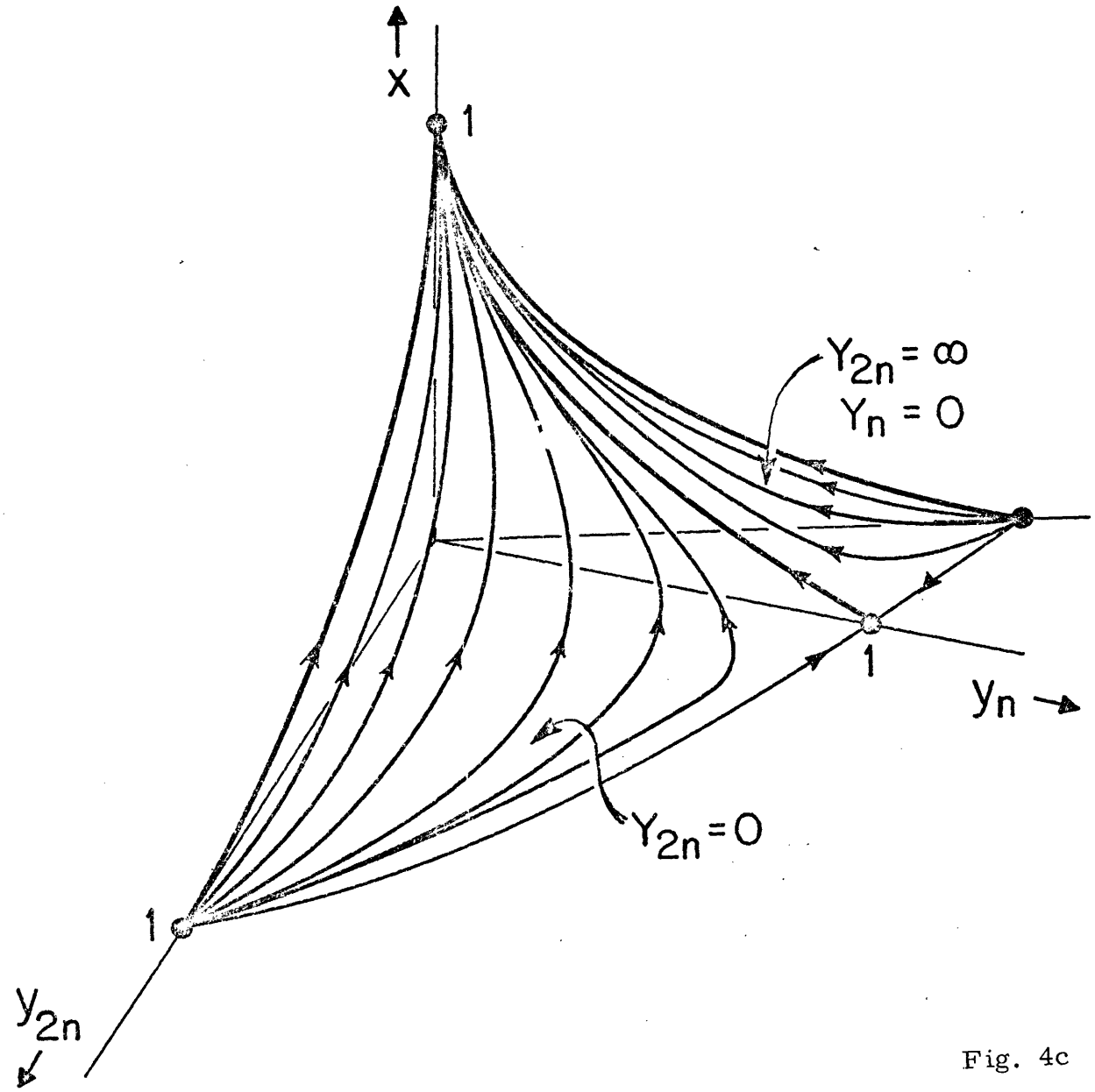

Fig. 4c

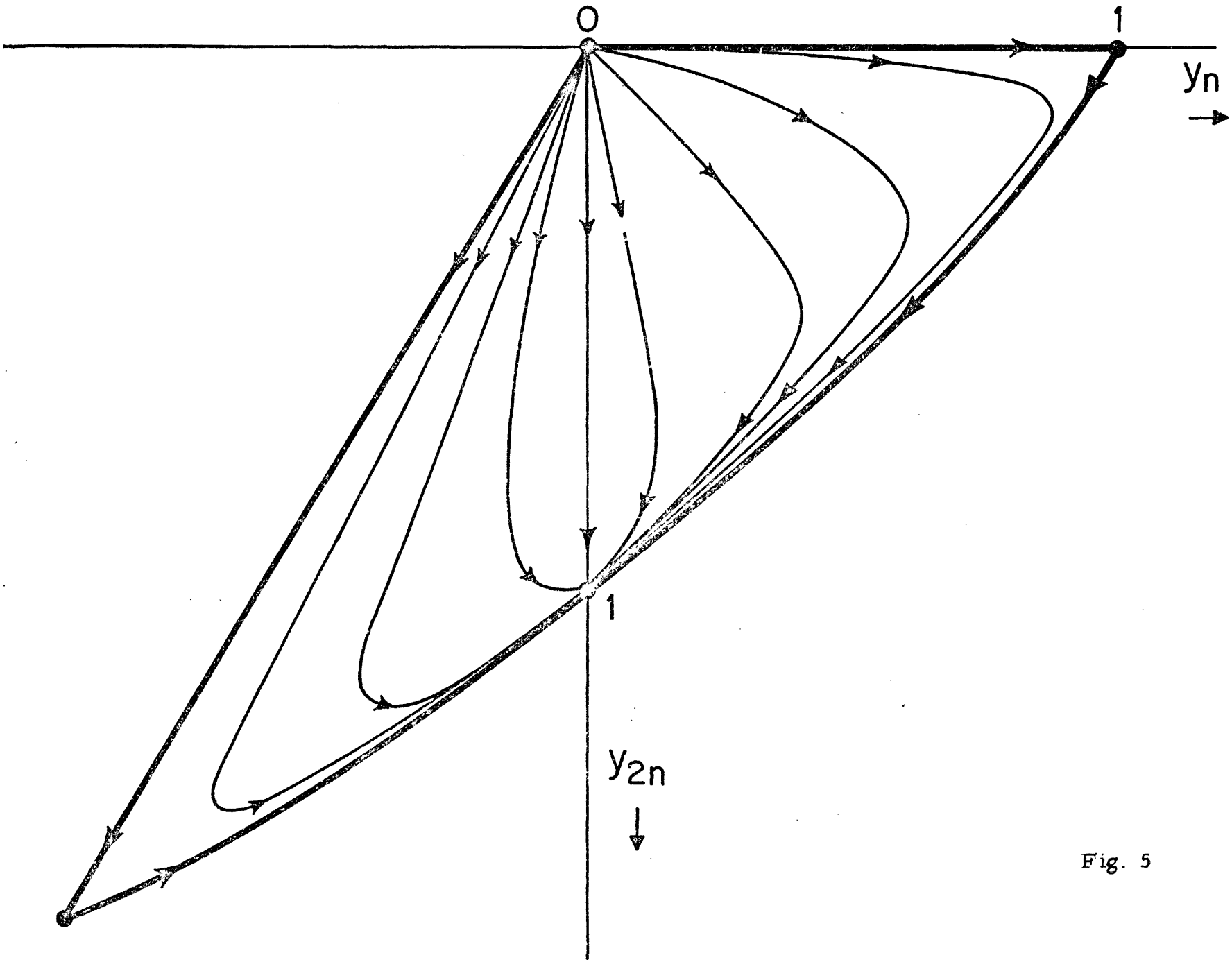

Fig. 5

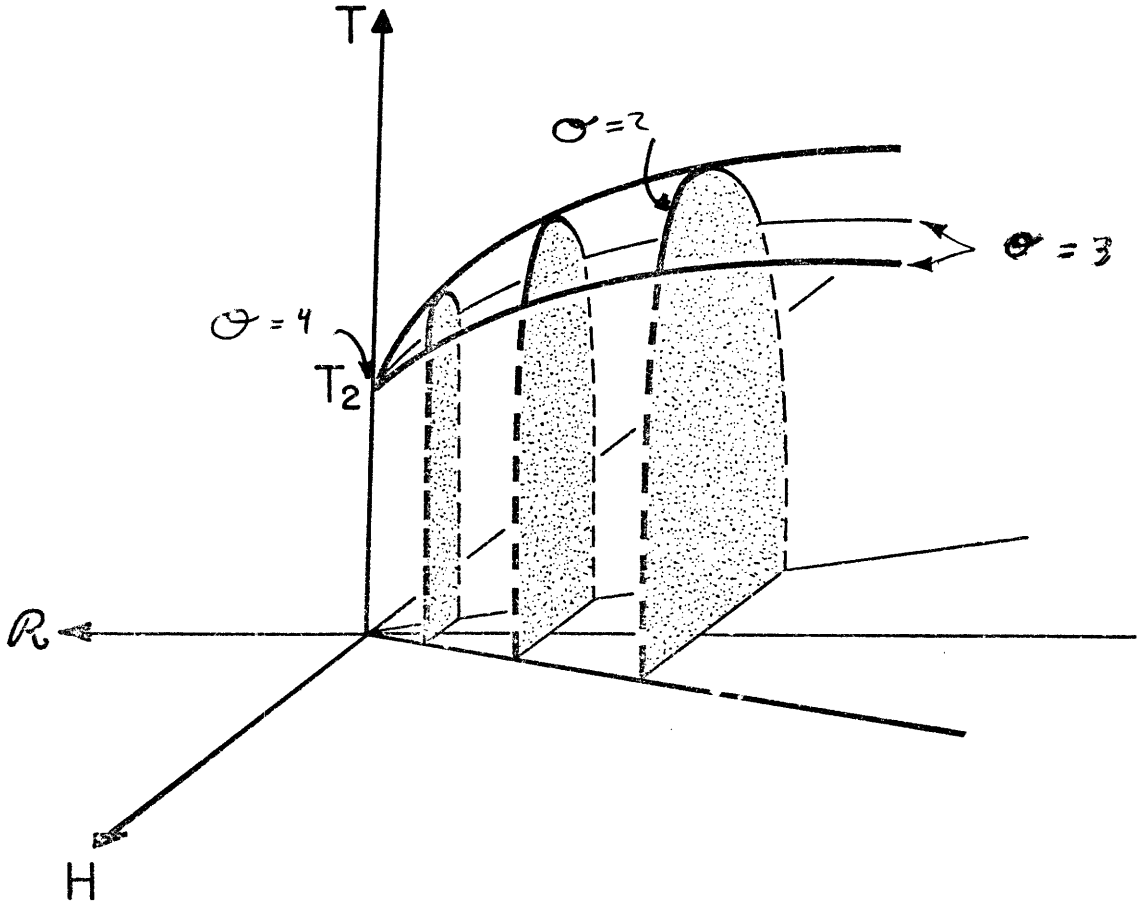

Fig. 6a

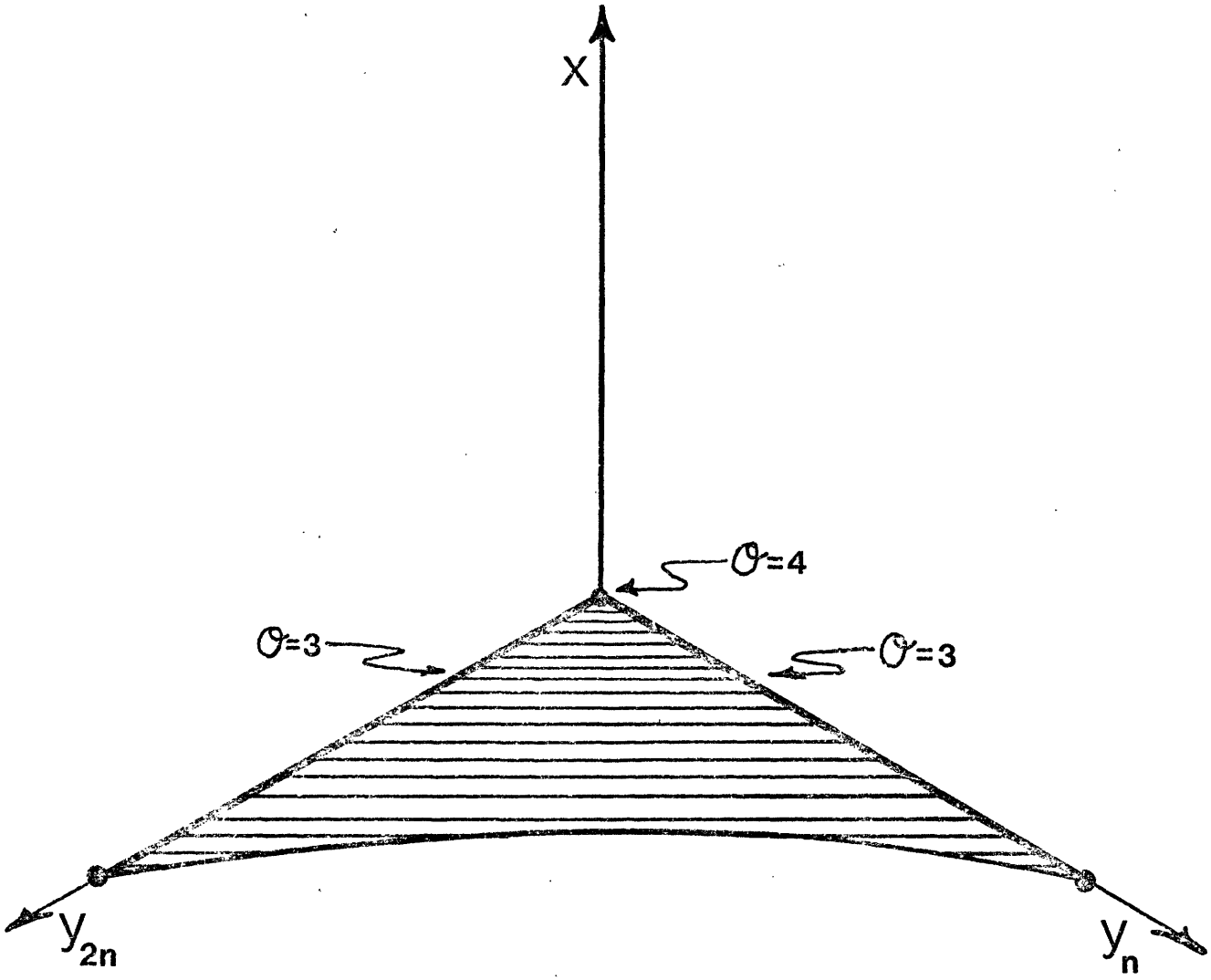

Fig. 6b

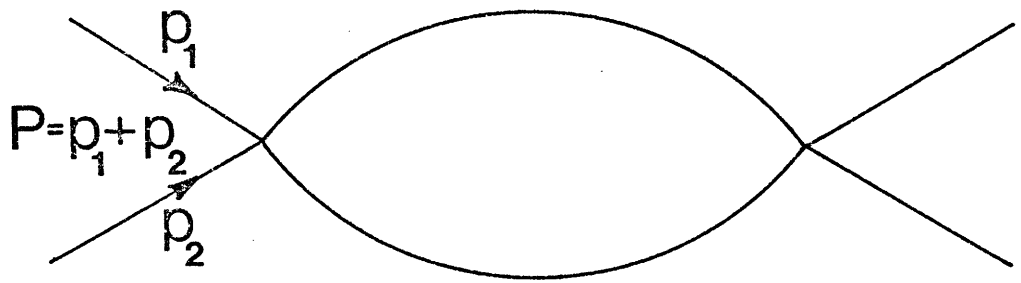

Fig. 7a

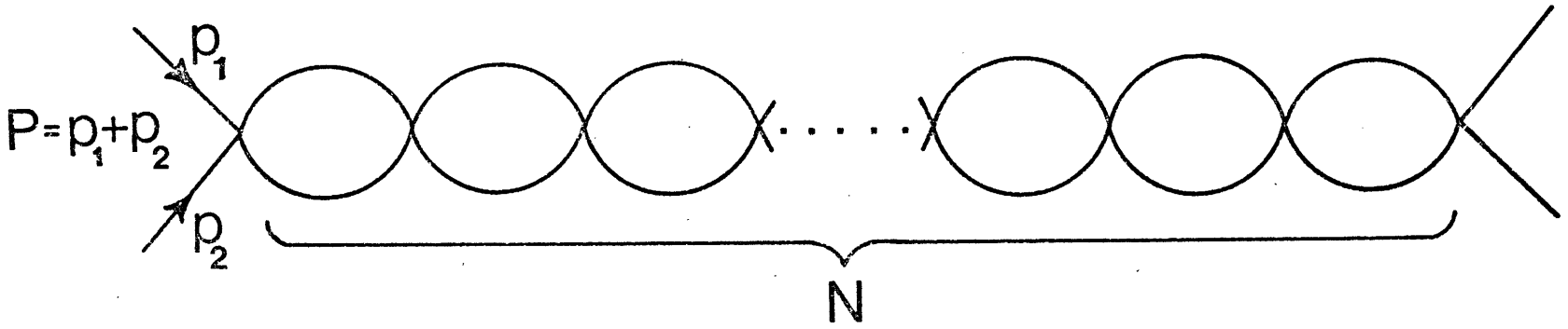

Fig. 7b

BIOGRAPHICAL NOTE

Jeffrey F. Nicoll was born on February 25, 1948, in Washington, D.C. He was educated in Virginia public schools until entering M.I.T. In 1970 he recieved B.S. degrees in physics, mathematics, and electrical engineering. His graduate study in physics at M.I.T. has been supported by teaching and research assistantships. He is married to Michaele Harrington.

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