A New Completely Integrable System on the Symmetric Periodic Toda Lattice Phase Space

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1995

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Abstract

Let $\mathbf{g}$ be a simple Lie algebra, with corresponding connected Lie group $\mathbf{G}$. Suppose $\mathbf{g}$ has a vector space splitting into a sum of subalgebras $\mathbf{g} = \mathbf{k} + \mathbf{b}$, so that one can identify $\mathbf{g}^* \cong \mathbf{k}^* + \mathbf{b}^*$. By the Kostant-Symes involution theorem, the invariant functions on $\mathbf{g}^*$, when restricted to $\mathbf{b}^*$, Poisson commute there, and in particular Poisson commute on co-adjoint orbits of $\mathbf{B} = \exp \mathbf{b}$.

Let $L\mathbf{g}$ denote the loop algebra of smooth maps from S^1 to $\mathbf{g}$, whose Fourier series converge absolutely with respect to some weight function w. Using the Killing form of $\mathbf{g}$, one can construct a canonical central extension $\tilde{L}\mathbf{g}$ of $L\mathbf{g}$. For suitably chosen w, there exists a group $\tilde{L}\mathbf{G}$ corresponding to $\tilde{L}\mathbf{g}$, as well as Iwasawa decompositions at the algebra and group levels: $\tilde{L}\mathbf{g} = \tilde{\mathbf{k}} + \tilde{\mathbf{a}} + \tilde{\mathbf{n}}$, and $\tilde{L}\mathbf{G} = \tilde{\mathbf{K}}\tilde{\mathbf{A}}\tilde{\mathbf{N}}$.

We apply the Kostant-Symes theorem to the splitting $\tilde{L}\mathbf{g} = \tilde{\mathbf{k}} + \tilde{\mathbf{b}}$, (where $\tilde{\mathbf{b}} = \tilde{\mathbf{a}} + \tilde{\mathbf{n}}$). Co-adjoint orbits in $(\tilde{L}\mathbf{g})^*$ are parameterized by conjugacy classes in $\mathbf{G}$, and so the class functions of $\mathbf{G}$ give rise to invariant functions on $(\tilde{L}\mathbf{g})^*$. When restricted to co-adjoint orbits of $\tilde{\mathbf{B}} = \exp \tilde{\mathbf{b}}$, these functions Poisson commute. In this paper we realize the symmetric periodic Toda phase space associated with $\mathbf{g}$, as a co-adjoint orbit of $\tilde{\mathbf{B}}$, and show that the restricted invariant functions result in a completely integrable system on this orbit.

Thesis Supervisor: Victor Guillemin
Title: Professor

Acknowledgments

I wish to express special thanks to Stephanie Singer. The work contained in this thesis was a collaborative effort with Stephanie, and were it not for her many brilliant and creative insights, would probably not have progressed very far at all. It was a pleasure working with her, and my hope is that I will have the opportunity to do so again in the future.

I would also like to thank my advisor Victor Guillemin, whose guidance and encouragement were invaluable to both Stephanie and myself, as we tried to unravel some of the mysteries of the Toda lattice.

Finally, I want to thank most of all the love of my life, Laura, who was more supportive and patient than can be reasonably expected of any person.

A NEW COMPLETELY INTEGRABLE SYSTEM

MALCOLM QUINN

1. INTRODUCTION

This paper is the result of a collaboration with S. Singer, and presents a new completely integrable system, whose phase space coincides with that of the real symmetric periodic Toda lattice. Following the scheme used by Singer [Si] to construct complex action variables for the complexified non-periodic Toda lattice, we arrive at an integrable system defined for any simple Lie algebra, and prescribed entirely in terms of Lie-theoretic data. Despite the analogy with Singer's earlier work, the functions defining our system are not action variables, (their flows do not seem to be periodic), nor is the system the usual Toda lattice, (the functions do not appear to commute with the standard Toda Hamiltonian). These discrepancies arise because our system is real, and suggest that by looking at a suitable complexification one might recover complex action variables for a Toda-like Hamiltonian. Our original goal was to give a geometric explanation of the construction of action variables for the real symmetric periodic Toda lattice, given by Flaschka and McLaughlin [FM], for the lattice obtained from the algebra $sl(n)$. However, the present system is of interest in its own right, because of its very natural description.

Toda lattice models belong to a general class of integrable systems, associated to vector space splittings of Lie algebras. Suppose a Lie algebra $\mathfrak{g}$, (with corresponding connected Lie group $\mathbf{G}$), splits as a vector space into $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$, where $\mathfrak{k}, \mathfrak{b}$ are subalgebras, (corresponding to subgroups $\mathbf{K}, \mathbf{B}$ respectively). Then one can naturally identify $\mathfrak{g}^* \cong \mathfrak{k}^* + \mathfrak{b}^*$, and so co-adjoint orbits of $\mathbf{B}$ can be thought of as sitting inside $\mathfrak{g}^*$. The invariant functions on $\mathfrak{g}^*$, when restricted to a particular $\mathbf{B}$ co-adjoint orbit, give a family of Poisson commuting functions on that orbit, and generate flows there described by Lax pair equations; this is the content of the Kostant-Symes involution theorem, ([Ko],[Sy]). And so provided the dimensions are right, by restricting the

invariant functions on $\mathfrak{g}^*$, one can generate interesting completely integrable systems on the co-adjoint orbits of $\mathbf{B}$. The classical Toda lattices arise in this way from various splittings of $sl(n,\mathbb{R})$ and $sl(n,\mathbb{C})$, and their construction generalizes to arbitrary simple Lie algebras, (see for instance [Pe]).

Given an invariant function I, one can describe the flows it generates in $\mathfrak{b}^*$ quite explicitly. Taking an $x_0 \in \mathfrak{b}^*$, one factorizes $\exp t\nabla I(x_0) = k(t)^{-1}b(t)$, where $k(t) \in \mathbf{K}, b(t) \in \mathbf{B}$ and $k(0) = \mathrm{Id} = b(0)$. Then the flow generated by I through x_0 is given by

$$x(t) = \mathrm{Ad}^*_{k(t)}x_0 = \mathrm{Ad}^*_{b(t)}x_0 \ .$$

Singer observed that when $\mathbf{K} \cap \mathbf{B} = \{\mathrm{Id}\}$, periodicity of $\exp t\nabla I$ implies periodicity of the flows generated by I. In particular if ∇I is constant on some cross-section of the co-adjoint action, and $\exp t\nabla I$ is periodic, then I generates a torus action on $\mathfrak{b}^*$.

This idea suggests a method of finding action variables on co-adjoint orbits in $\mathfrak{b}^*$, (though in practice the construction will only be local). One first looks for some natural cross-section of $\mathfrak{g}^*$ transverse to the co-adjoint orbits, and then defines invariant functions by pulling back to $\mathfrak{g}^*$ functions defined on this cross-section. One then tries to identify those functions I such that ∇I satisfies the periodicity conditions described. By this method Singer constructs action variables for the complexified non-periodic Toda lattice.

In the present paper the algebra considered is an infinite dimensional loop algebra $L\mathfrak{g}$, (actually, a central extension of this), derived from some finite dimensional $\mathfrak{g}$. By a well-known construction, (e.g. [GW1], [RS], [RSF]), we realize the real symmetric periodic Toda phase space, as a co-adjoint orbit associated with a splitting of this algebra. Because of the remarkable fact that co-adjoint orbits in $L\mathfrak{g}^*$ are parameterized by conjugacy classes in the finite dimensional group $\mathbf{G}$, a natural choice of invariant functions is available by way of the class functions on $\mathbf{G}$. Using local arguments, we show that this family of functions generates a completely integrable system on our phase space.

The construction closely parallels that used by Singer in the case of complexified non-periodic Toda, although more technical machinery is needed, to deal with the infinite dimensional setting. And in contrast to the non-periodic case, the locally defined functions used to prove integrability are not readily recognizable as having

periodic flows, nor do they seem to commute with the standard Toda Hamiltonian. Our hope is that once the complexified system is analyzed, our method will yield complex action variables, periodic in complex time, making the analogy with nonperiodic Toda complete.

2. Background Material

2.1. The classical periodic Toda lattice. The Toda lattice was first considered by Toda in 1967, as a system of particles on the line with exponential interaction of nearest neighbors. The periodic Toda lattice deals with the case where the particles lie on a circle with the potential $V(x) = e^{-x} - 1 + x$ connecting neighboring particles. For an $(n+1)$-particle system the phase space is $T^*\mathbb{R}^{n+1}$, and in terms of canonical coordinates $\{q_i, p_i\}$ the Hamiltonian is

$$H = H(q,p) = \frac{1}{2} \sum_{k=1}^{n+1} p_k^2 + \sum_{k=1}^{n+1} e^{-(q_k - q_{k-1})}, \quad q_0 \equiv q_{n+1}.$$

The rotational symmetry of the system means that the total momentum $\sum p_i$ is conserved, so moving to "center of mass" coordinates on the circle one can take as fixed $\sum q_i = 0 = \sum p_i$. The transformation of Flaschka:

$$a_k = \frac{1}{2} e^{(q_k - q_{k+1})/2}, \quad b_k = -\frac{1}{2} p_k \qquad k = 1, 2, \ldots, n+1$$

transforms to phase space to $\{(a_k, b_k) : \Sigma b_k = 0, \ a_k > 0 \text{ with } \Pi a_k = 2^{-n-1}\}$, with symplectic form $\omega = -4 \sum_{j=1}^{n+1} \sum_{k=j}^{n+1} \frac{da_k}{a_k} \wedge db_j$. The Hamiltonian becomes

$$H = H(a,b) = 2 \sum_{k=1}^{n+1} b_k^2 + 4 \sum_{k=1}^{n+1} a_k^2 ,$$

and the associated Hamiltonian equations of motion are

$$\dot{a}_k = a_k(b_{k+1} - b_k)$$
$$\dot{b}_k = 2(a_k^2 - a_{k-1}^2)$$

The crucial observation is that these equations may be written in Lax pair form. If we set

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_{n+1} \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_n \\ a_{n+1} & \cdots & 0 & a_n & b_{n+1} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & -a_{n+1} \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & -a_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_n \\ a_{n+1} & \cdots & 0 & -a_n & 0 \end{pmatrix},$$

then the equations of motion can be written

$$\dot{L} = [B, L], \qquad (*)$$

and so B, L form a Lax pair. We can also write the Hamiltonian as

$$H(L) = \frac{1}{2}\mathrm{Tr}(L^2).$$

To show the system is completely integrable one considers the functions

$$I_k(L) = \frac{1}{k}\mathrm{Tr}(L^k), \qquad k = 2, 3, \ldots, n+1.$$

To see that these functions are constants of motion suppose $L = L(t)$ is a solution to the equation $(*)$. By induction $(\dot{L^k}) = [B, L^k]$ for $k = 2, 3 \ldots, n+1$, from which it follows that $\mathrm{Tr}(\dot{L^k}) = 0$, and so $\dot{I}_k(L) \equiv 0$ along $L(t)$. These functions are independent almost everywhere in the phase space, essentially because together they generate the symmetric polynomials in the eigenvalues of L, and generic L has distinct eigenvalues.

Poisson commutativity is a consequence of the Kostant-Symes involution theorem, (which we describe later in this section). Since the dimension of the phase-space is $2n$, and we have n a.e. functionally independent commuting constants of motion, the system is completely integrable.

The phase space of matrices of the form of L is contained in $sl(n+1, \mathbb{C})$. Later we shall give the general construction of this periodic Toda phase space, for any simple algebra $\mathfrak{g}$, and present a new completely integrable system on this space.

2.2. Hamiltonian systems on co-adjoint orbits.

We now recall some well known facts about Hamiltonian systems on Lie algebra duals.

For any finite dimensional lie algebra $\mathfrak{g}$ there is a canonical Poisson structure on $\mathfrak{g}^*$. The Poisson bracket of two functions $f, g \in C^\infty(\mathfrak{g}^*)$ is defined to be

$$\{f, g\}(\mu) = \langle \mu, [\nabla f, \nabla g] \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, and $\nabla f(\mu), \nabla g(\mu)$ are the gradients at μ, of f, g respectively. Casimirs of this bracket, i.e. functions for which $\{f, \cdot\} \equiv 0$, are precisely those functions which are invariant under the co-adjoint action of the group associated to $\mathfrak{g}$. We shall denote this ring of invariant functions by $I(\mathfrak{g}^*)$.

Given a function $H \in C^\infty(\mathfrak{g}^*)$, we define the associated Hamiltonian vector field to be $\Xi_H = \{\cdot, H\}$, which has a corresponding flow described by the equation

$$\dot{\xi} = -\text{ad}^*_{\nabla H(\xi)} \xi \ .$$

The right hand side is the infinitesimal co-adjoint action of $\nabla H(\xi) \in \mathfrak{g}$ on $\xi \in \mathfrak{g}^*$, showing that the flows are contained in co-adjoint orbits. In particular the Hamiltonian vector fields are tangent to the co-adjoint orbits, which are precisely the symplectic leaves of the Poisson foliation arising from the bracket $\{\cdot, \cdot\}$. They are symplectic manifolds, and the inclusion of an orbit $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ is a Poisson map.

If in addition $\mathfrak{g}$ admits a non-degenerate invariant bilinear form, then one can identify $\mathfrak{g} \cong \mathfrak{g}^*$, the co-adjoint action corresponds to the adjoint action, and the equations of motion can be written in "Lax pair form":

$$\dot{x} = \text{ad}_{\nabla H(x)} x \ .$$

2.3. The involution and factorization theorems. A map $R \in \text{End}(\mathfrak{g})$ is said to be an "R-matrix", if the associated R-bracket, defined by

$$[X, Y]_R = \frac{1}{2}\big([RX, Y] + [X, RY]\big),$$

is a Lie bracket. When this is the case, the R-bracket gives rise to an associated R-Poisson bracket $\{\cdot, \cdot\}_R$ on $\mathfrak{g}^*$.

Involution Theorem. *Functions from $I(\mathfrak{g}^*)$ are in involution with respect to $\{\cdot, \cdot\}_R$.*

Proof. We have

$$\{f, g\}_R(\mu) := \langle \mu, [\nabla f, \nabla g]_R \rangle = \frac{1}{2}\langle \mu, [R\nabla f, \nabla g] + [\nabla f, R\nabla g] \rangle.$$

If f, g are invariant then $\text{ad}^*_{\nabla f} \equiv 0 \equiv \text{ad}^*_{\nabla g}$, and the result follows. $\square$

Suppose now that the Lie algebra $\mathfrak{g}$, which corresponds to a connected Lie group $\mathbf{G}$, has a vector space splitting $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, where $\mathfrak{g}_+, \mathfrak{g}_-$ are subalgebras corresponding to connected subgroups $\mathbf{G}_+, \mathbf{G}_-$ respectively. Let $\Pi_\pm$ denote the projection operators onto $\mathfrak{g}_\pm$ along $\mathfrak{g}_\mp$, so for each $X \in \mathfrak{g}$ we have $X = X_+ + X_-$, where $X_\pm = \Pi_\pm X$. Let R denote the endomorphism of $\mathfrak{g}$ given by

$$R = \Pi_+ - \Pi_- \ ;$$

this is an R-matrix, whose corresponding R-bracket is

$$[X, Y]_R = [X_+, Y_+] - [X_-, Y_-].$$

This bracket is of particular importance because of the following proposition:

Proposition. *The equations of motion induced by a function $H \in I(\mathfrak{g}^*)$ with respect to the above R-matrix have the form*

$$\frac{d\xi}{dt} = -ad^*_{M_\pm}\xi, \quad where \quad \xi \in \mathfrak{g}^*, \quad M_\pm = \pm\Pi_\pm\nabla H(\xi).$$

If $\mathfrak{g}$ admits a non-degenerate invariant bilinear form, so that one can identify $\mathfrak{g} \subset \mathfrak{g}^*$, and $ad^* = ad$, then these equations can be written in Lax-pair form:

$$\frac{dx}{dt} = [x, M_\pm]$$

Proof. The vector field associated to H by the R-bracket is $\Xi_H = \{\cdot, H\}_R$, and so

$$(\Xi_H)_\mu F = \langle \mu, [\nabla F, \nabla H]_R \rangle$$
$$= \frac{1}{2}\langle \mu, [\nabla_+ F, \nabla_+ H] - [\nabla_- F, \nabla_- H] \rangle.$$

Since H is invariant, $ad^*_{\nabla H} \equiv 0$, so that we get $\Xi_H = -ad^*_{\nabla_+ H} = ad^*_{\nabla_- H}$, which proves the result. □

Remarkably, a simple recipe exists for solving the equations of motion associated to H by the R-bracket:

Factorization Theorem. *Let $g_\pm(t)$ be the smooth curves in $\mathbf{G}_\pm$ which solve the factorization problem:*

$$\exp t\nabla H(\xi_0) = g_+(t)^{-1}g_-(t), \quad g_\pm(0) = Id.$$

Then the R-matrix flow of $H \in I(\mathfrak{g}^*)$ through the point $\xi_0 \in \mathfrak{g}^*$ is given by

$$\xi(t) = Ad^*_{g_\pm(t)}\xi_0 \ .$$

Proof. Note that because of the splitting of $\mathfrak{g}$, the factorization described will exist, at least for t near zero. Differentiating the proposed solution with respect to t gives $\dot{\xi} = ad^*_{\dot{g}_\pm g_\pm^{-1}}\xi_0$, so we need to show that $\dot{g}_\pm g_\pm^{-1} = -M_\pm$. Set $X = \nabla H(\xi_0)$ and take the derivative of $g_+ \exp tX = g_-$. This gives $dR_{\exp tX}\dot{g}_+ + dL_{g_+ \exp tX}X = \dot{g}_-$, where R, L denote the right and left multiplication operators on $\mathbf{G}$. Since $\exp tX = g_+^{-1}g_-$, this becomes

$$\dot{g}_+ g_+^{-1} + Ad_{g_-}X = \dot{g}_- g_-^{-1}.$$

But $Ad_{g_-}X = Ad_{g_-}\nabla H(\xi_0) = \nabla H(Ad^*_{g_-}\xi_0) = \nabla H(\xi(t))$, using the invariance of H, so that

$$\nabla H(\xi(t)) = \dot{g}_- g_-^{-1} - \dot{g}_+ g_+^{-1} \ .$$

Now apply the projections $\Pi_\pm$, and we get the desired identity. $\square$

Notice that if $\mathbf{G}_- \cap \mathbf{G}_+ = \{Id\}$, and for some t_0 one has $\exp t_0 \nabla H(\xi_0) = Id$, then by uniqueness of the factorization, it must be that $g_\pm(t_0) = Id$, and so the flow generated by H is periodic. This observation can be very useful when one is trying to find action variables, which by definition must generate periodic flows.

2.4. The Kostant-Symes Involution Theorem.

We can now express the usual Kostant-Symes involution theorem as a consequence of these theorems. Since $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, we have $\mathfrak{g}^* = Ann(\mathfrak{g}_+) + Ann(\mathfrak{g}_-)$, where $Ann(\mathfrak{g}_\pm)$ are the annihilators of $\mathfrak{g}_\pm$ in $\mathfrak{g}^*$. Because one can identify $Ann(\mathfrak{g}_\mp) \cong \mathfrak{g}^*_\pm$, we get $\mathfrak{g}^* \cong \mathfrak{g}^*_+ + \mathfrak{g}^*_-$.

Now note that from the R-bracket induced by the splitting, there is an associated R-adjoint action $^R Ad$ of $\mathbf{G}$, which we can express in terms of the ordinary adjoint action:

$$^R Ad_{\exp X}Y = Ad_{\exp X_+}Y_+ + Ad_{\exp -X_-}Y_- \ .$$

Taking $\beta \in \mathfrak{g}^*_+ \cong Ann(\mathfrak{g}_-)$ we have that

$$\langle \beta, ^R Ad_{\exp -X}Y \rangle = \langle \beta, Ad_{\exp -X_+}Y_+ \rangle$$

$$= \langle Ad^*_{\exp X_+}\beta, Y_+ \rangle$$

$$= \langle \Pi_{\mathfrak{g}^*_+} Ad^*_{\exp X_+}\beta, Y \rangle \ .$$

Thus the R-co-adjoint action of $\exp X$ on β is given by $\Pi_{\mathfrak{g}_+^*} \mathrm{Ad}^*_{\exp X_+} \beta$, which is precisely the ordinary co-adjoint action of $\exp X_+ \in \mathbf{G}_+$ on $\beta \in \mathfrak{g}_+^*$. This shows that the R-co-adjoint action preserves $\mathfrak{g}_+^* \subset \mathfrak{g}^*$, and that the R-co-adjoint orbits through elements of $\mathfrak{g}_+^*$ are precisely the ordinary co-adjoint orbits under the action of $\mathbf{G}_+$.

We note also that if $\lambda \in \mathfrak{g}^*$ satisfies $\langle \lambda, [\mathfrak{g}_+, \mathfrak{g}_+] \rangle = 0 = \langle \lambda, [\mathfrak{g}_-, \mathfrak{g}_-] \rangle$, (for example if λ is a character of $\mathfrak{g}_-$), then λ is fixed by the R-co-adjoint action.

Combining these facts we get the involution theorem usually attributed to Kostant and Symes:

Kostant-Symes Involution Theorem. *Suppose* $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ *is a vector space splitting into two subalgebras, and that* $\lambda \in \mathfrak{g}^*$ *satisfies* $\langle \lambda, [\mathfrak{g}_+, \mathfrak{g}_+] \rangle = 0 = \langle \lambda, [\mathfrak{g}_-, \mathfrak{g}_-] \rangle$.

(1) *Functions on* $\mathfrak{g}_+^* \cong Ann(\mathfrak{g}_-)$ *of the form* $F_\lambda(\beta) = F(\beta + \lambda)$, *where* F *ranges over the invariant functions on* $\mathfrak{g}^*$, *are in involution with respect to the Poisson bracket on* $\mathfrak{g}_+^*$.

(2) *The equations of motion on* $\mathfrak{g}_+^*$ *induced by a function of the form* F_λ *can be written*

$$\frac{d\xi}{dt} = -ad^*_{M_\pm} \xi$$

where $\xi = \beta + \lambda$, $\quad M_\pm = \pm \Pi_\pm \nabla F(\beta + \lambda)$.

In particular we shall consider generalised Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, for special infinite dimensional algebras, and take $\mathfrak{g}_+ = \mathfrak{a} + \mathfrak{n}$, $\mathfrak{g}_- = \mathfrak{k}$. The symmetric Toda lattice phase space will be identified as a (character shifted) co-adjoint orbit of $\mathbf{G}_+$, and then the Kostant-Symes involution theorem will be used to find Poisson commuting functions on the phase space.

2.5. Application to infinite dimensional algebras.

The involution theorem follows from a formal algebraic manipulation, and so with suitable care can be applied to infinite dimensional algebras. Suppose $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ is infinite dimensional, and admits an invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$, so that $\mathfrak{g}$ embeds in $\mathfrak{g}^*$ by the mapping $X \mapsto \langle X, \cdot \rangle$. The image is called the "smooth part of the dual", and we restrict attention to this subspace. We also consider only smooth functions F on this subspace which have a gradient, i.e. for which there exists a smooth map

$\nabla F \colon \mathfrak{g} \to \mathfrak{g}$ with $dF_X Y = \langle Y, \nabla F(X) \rangle$, for all $X, Y \in \mathfrak{g}$. Then the Poisson bracket on such functions is defined by the usual formula,

$$\{F, H\}(X) = \langle X, [\nabla F, \nabla H](X) \rangle,$$

although $\{F, H\}$ need not possess a gradient, so this is not a Lie bracket in general. Now one can show by the usual argument that if F is invariant under the action of $\mathbf{G}$ on $\mathfrak{g} \subset \mathfrak{g}^*$, then F is a Casimir for the bracket, i.e. $\{F, H\} \equiv 0$, and the involution theorem goes through as in the finite dimensional case. The form $\langle \cdot, \cdot \rangle$ also gives an embedding of $\mathfrak{g}_-^\perp \subset \mathfrak{g}_+^*$, and so when we come to speak of the Poisson structure on $\mathfrak{g}_+^*$, and the co-adjoint orbits in $\mathfrak{g}_+^*$, we similarly restrict attention to the invariant subspace $\mathfrak{g}_-^\perp$, and interpret the brackets and so on, in the appropriate fashion.

2.6. Some Lie Theory.

We now fix some notation, and recall a few standard facts from Lie theory, (see for example [Kn]). These facts are worth keeping in mind, as the analogous results for infinite dimensional loop algebras, (particularly the Iwasawa decomposition), will be important later on.

Take $\mathfrak{g}$ to be a rank n complex simple Lie algebra, and let $\mathbf{G}$ denote the corresponding connected complex Lie group. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$, with corresponding Cartan subgroup $\mathbf{H}$, and let $\triangle$ denote the set of roots associated to $(\mathfrak{g}, \mathfrak{h})$, $\triangle_+ \subset \triangle$ a fixed choice of positive roots, and $\Pi = \{\alpha_1, \ldots \alpha_n\} \subset \triangle_+$ the corresponding simple roots. Let $\alpha_* \in \triangle_+$ denote the highest root, so that $\alpha_* = \sum k_i \alpha_i$ for some non-negative integers k_i.

Let $\mathfrak{g}_\alpha$ denote the root space corresponding to $\alpha \in \triangle$, and choose a Chevalley basis $\{E_\alpha\}_{\alpha \in \triangle} \cup \{H_{\alpha_i} : 1 \leq i \leq n\}$ for $\mathfrak{g}$. Here $\mathfrak{g}_\alpha = \mathbb{C} E_\alpha$, and $H_\alpha = [E_\alpha, E_{-\alpha}]$ is normalised so that $\alpha(H_\alpha) = 2$. With respect to the Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, one has that $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ for $\alpha + \beta \neq 0$, with $\mathfrak{g}_0 \equiv \mathfrak{h}$.

Take $\mathfrak{k}$ to be the compact real form of $\mathfrak{g}$ determined by this basis:

$$\mathfrak{k} = \operatorname{Span}_{\mathbb{R}} \{iH_\alpha, \ E_\alpha - E_{-\alpha}, \ i(E_\alpha + E_{-\alpha}) : \quad \alpha \in \triangle\},$$

and let $\mathbf{K}$ be the real compact immersed subgroup of $\mathbf{G}$ corresponding to $\mathfrak{k}$. Take $\theta \colon \mathfrak{g} \to \mathfrak{g}$ to be the Cartan involution of $\mathfrak{g}$ defined by $\mathfrak{k}$, so that

$$\theta(H_\alpha) = -H_\alpha, \quad \theta(E_\alpha) = -E_{-\alpha},$$

and θ extends by conjugate linearity to the rest of $\mathfrak{g}$. Set $\mathfrak{p} = i\mathfrak{k}$, then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, into $+1$ and -1 eigenspaces for θ, and is orthogonal with respect to the Killing form of $\mathfrak{g}$ regarded as a real Lie algebra, which is given by $2\mathrm{Re}\langle\cdot,\cdot\rangle$. At the group level there is a corresponding factorization of $\mathbf{G}$ into $\mathbf{G} = \mathbf{K}\exp\mathfrak{p}$.

The elements of $\mathfrak{k}$ and $\mathfrak{p}$ are semisimple, $(\mathrm{ad}_X$ for $X \in \mathfrak{k}$, (resp. $X \in \mathfrak{p}$), is anti-symmetric, (resp. symmetric), with respect to the form $B(X,Y) = -\mathrm{Re}\langle X, \theta(Y)\rangle)$. Take $\mathfrak{a} = \mathrm{Span}_{\mathbb{R}}\{H_\alpha\}$ and note that this is a maximal abelian subalgebra of $\mathfrak{p}$, consisting of semisimple elements of $\mathfrak{g}$, thus we can decompose $\mathfrak{g}$ with respect to the ad-action of $\mathfrak{a}$. From the relative root system of $(\mathfrak{g}, \mathfrak{a})$, one arrives at a root space decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}_+$, where $\mathfrak{m} = \mathrm{Span}_{\mathbb{R}}\{iH_\alpha\}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are respectively the sum of the positive and negative root spaces. Because of our choice of $\mathfrak{a}$, $\mathfrak{n}_\pm = \mathrm{Span}_{\mathbb{R}}\{E_\alpha, iE_\alpha : \pm\alpha \in \Delta\}$.

The resulting Iwasawa decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}_+$. Let $\mathbf{A}$ and $\mathbf{N}$ be the immersed subgroups of $\mathbf{G}$ corresponding to the subalgebras $\mathfrak{a}$ and $\mathfrak{n}_+$ respectively. Then $\mathbf{A}$ is a vector group, $\mathbf{N}$ is a nilpotent Lie group, and the Iwasawa decomposition for $\mathbf{G}$ is given by the factorization $\mathbf{G} = \mathbf{KAN}$.

Example 2.6.1. Take $\mathfrak{g} = sl(n{+}1, \mathbb{C})$, with invariant bilinear form given by $\langle X, Y\rangle = \mathrm{Tr}(XY)$. Let E_{ij} denote the matrix with a 1 in the (i,j)-th position, and zeroes everywhere else. Take $\mathfrak{h}$ to be the standard Cartan subalgebra spanned by the diagonal matrices $E_{i,i} - E_{i+1,i+1}$, $1 \le i \le n$, and define $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(E_{jj}) = \delta_{i,j}$. Take the standard positive roots given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i \le n$. The Cartan involution is negative conjugate transpose $\theta(X) = -\bar{X}^t$, which gives the Cartan decomposition

$$sl(n + 1, \mathbb{C}) = su(n + 1) + isu(n + 1),$$

i.e. $sl(n + 1, \mathbb{C})$ decomposes as a sum of skew-Hermitian and Hermitian matrices. To see that this decomposition is orthogonal with respect to $\mathrm{Re}\langle\cdot,\cdot\rangle$, take $X, Y \in sl(n + 1, \mathbb{C})$ to be respectively skew-Hermitian and Hermitian, so that $\bar{X}^t = -X$ and $\bar{Y}^t = Y$. Then

$$\mathrm{Tr}(XY) = -\mathrm{Tr}(\bar{X}^t\bar{Y}^t) = -\overline{\mathrm{Tr}(YX)} = -\overline{\mathrm{Tr}(XY)},$$

and consequently $\text{Tr}(XY)$ is pure imaginary, so that $\text{Re}\,\text{Tr}(XY) = 0$, and the decomposition is orthogonal. Taking $\mathfrak{a} \subset isu(n+1)$ to be the algebra of real diagonal matrices, the resulting Iwasawa decomposition is

$$sl(n+1,\mathbb{C}) = su(n+1) + \{\text{real diag trace } 0\} + \{\text{strictly upper triangular}\}.$$

At the group level $SL(n+1,\mathbb{C}) = SU(n+1)\mathbf{AN}$, where $\mathbf{A}$ is the group of diagonal determinant one matrices with positive entries, and $\mathbf{N}$ is the group of upper triangular matrices with ones on the diagonal.

3. LOOP ALGEBRAS

Consider the algebra $\mathfrak{g}[z, z^{-1}] = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$, of Laurent polynomials with coefficients in $\mathfrak{g}$. This so-called "(polynomial) loop algebra" has bracket $[Xz^m, Yz^n] = [X, Y]z^{m+n}$, and possesses an invariant non-degenerate bilinear form, which by an abuse of notation we also denote $\langle \cdot, \cdot \rangle$:

$$\left\langle \sum_m X_m z^m, \sum_n Y_n z^n \right\rangle = \sum_m \langle X_m, Y_{-m} \rangle.$$

The form makes sense because the Laurent polynomials include only finitely many non-zero terms. Note that we can think of elements of this algebra as finite Fourier series, and can equivalently write the form as

$$\langle \phi, \psi \rangle = \frac{1}{2\pi i} \oint_{S^1} \langle \phi(z), \psi(z) \rangle \frac{dz}{z}.$$

3.1. A Cartan decomposition.

We extend the Cartan involution θ of $\mathfrak{g}$ to $\mathfrak{g}[z, z^{-1}]$ by defining:

$$\tilde{\theta}(\phi)(z) = \theta(\phi(\bar{z}^{-1})), \quad \text{so that} \quad \tilde{\theta}(Xz^k) = \theta(X)z^{-k}.$$

Remark. Notice that the map $z \mapsto \bar{z}^{-1}$, is an involution of $\mathbb{C}^\times$ with fixed point set the compact real form S^1.

The real form corresponding to $\tilde{\theta}$ is

$$\tilde{\mathfrak{k}} = \text{Span}_\mathbb{R}\{iH_\alpha(\frac{z^m + z^{-m}}{2}), \ E_\alpha z^m - E_{-\alpha}z^{-m}, \ i(E_\alpha z^m + E_{-\alpha}z^{-m}) : \alpha \in \Delta, m \in \mathbb{Z}\},$$

and we get the Cartan decomposition $\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{k}} + i\tilde{\mathfrak{k}}$. Note that $\tilde{\mathfrak{k}}$ consists of those elements of $\mathfrak{g}[z, z^{-1}]$ whose image is in the compact form $\mathfrak{k}$, for all $z \in S^1$. We also have that $\tilde{\mathfrak{k}}$ and $i\tilde{\mathfrak{k}}$ are orthogonal with respect to the form $\text{Re}\langle \cdot, \cdot \rangle$.

There is also a generalised Iwasawa decomposition for $\mathfrak{g}[z, z^{-1}]$, which we construct after first giving the "principal gradation" for $\mathfrak{g}[z, z^{-1}]$.

3.2. The principal gradation.

The S^1 action $(e^{i\theta} \cdot \phi)(z) = \phi(e^{i\theta}z)$, and the adjoint action of the Cartan subgroup $\mathbf{H}$, are commuting actions on $\mathfrak{g}[z, z^{-1}]$, which give rise to the natural eigenspace decomposition $\mathfrak{g}[z, z^{-1}] = \bigoplus_{\alpha, k} \mathfrak{g}_\alpha \otimes z^k$, (where $\mathfrak{g}_0 \equiv \mathfrak{h}$). We now generate this decomposition algebraically, as well as presenting a standard gradation, and a generalised Iwasawa decomposition, for $\mathfrak{g}[z, z^{-1}]$.

Let $\mathbb{R}D$ denote the trivial real algebra generated by the derivation $D = iz\frac{d}{dz}$, and note that this is the infinitesimal generator of the S^1 action described above. Take the semidirect product $\mathcal{G} = \mathbb{R}D \ltimes \mathfrak{g}[z, z^{-1}]$, which has commutator

$$[(a, X), (b, Y)] = (0, [X, Y] + aDY - bDX).$$

Now take $\mathfrak{a}$ as in section 2.6 to be the real span of $\{H_{\alpha_i}\}$, and set $\mathcal{A} = \mathbb{R}D \ltimes \mathfrak{a}$. This is a commutative subalgebra of $\mathcal{G}$, and we get a root space decomposition corresponding to the pair $(\mathcal{G}, \mathcal{A})$. An element $\tilde{\alpha} \in \mathcal{A}^*, \tilde{\alpha} \neq 0$, is said to be a root if the corresponding root space

$$\mathcal{G}_{\tilde{\alpha}} = \{\check{X} \in \mathcal{G} : \mathrm{ad}_{\check{H}}\check{X} = \tilde{\alpha}(\check{H})\check{X} \quad \forall \check{H} \in \mathcal{A}\},$$

is non-zero. One can easily verify that the roots of $(\mathcal{G}, \mathcal{A})$ are given by

$$\tilde{\Delta} = \Big\{\alpha + n\gamma \ : \alpha \in \Delta, n \in \mathbb{Z}\Big\} \cup \Big\{n\gamma \ : n \in \mathbb{Z} \setminus \{0\}\Big\}$$

where $\gamma \in \mathcal{A}^*$ is defined by $\gamma(D) = 1, \gamma|_{\mathfrak{a}} \equiv 0$. The corresponding root spaces are

$$\mathcal{G}_{\alpha+n\gamma} = \mathfrak{g}_\alpha \otimes z^n, \quad \mathcal{G}_{n\gamma} = \mathfrak{h} \otimes z^n.$$

Notice that because we began with a Chevalley basis we have $\alpha(H_\beta) \in \mathbb{R}$, and consequently we can regard the roots $\alpha \in \mathfrak{g}^*$, (complex dual), of $(\mathfrak{g}, \mathfrak{h})$, as roots $\alpha \in \mathfrak{a}^*$, (real dual), of $(\mathfrak{g}, \mathfrak{a})$. Correspondingly we think of $\mathfrak{g}_\alpha$ as a real two-dimensional space spanned by $\{E_\alpha, iE_\alpha\}$. We arrive at the (real) decomposition

$$\mathcal{G} = \mathbb{R}D + \bigoplus_k \mathfrak{h} \otimes z^k + \bigoplus_{\alpha,k} \mathfrak{g}_\alpha \otimes z^k$$

which gives the desired decomposition for $\mathfrak{g}[z, z^{-1}]$.

Define a root $\tilde{\alpha} = \alpha + n\gamma$ to be positive if $n > 0$, or if $n = 0$ and α is positive in Δ. Then a system of simple roots is given by

$$\tilde{\Pi} = \{\alpha_1, \ldots, \alpha_n, \gamma - \alpha_*\},$$

(where $\alpha_* = \sum k_i\alpha_i$ is the highest root of $(\mathfrak{g}, \mathfrak{a})$).

Let H_0 be the unique element of $\mathfrak{a}$ satisfying $\alpha_i(H_0) = 1$ for $1 \leq i \leq n$, take $c := 1 + \alpha_*(H_0) = 1 + \Sigma k_i$ to be the Coxeter number of the root system $(\mathfrak{g}, \mathfrak{a})$, and set $\check{H} = cD + H_0 \in \mathcal{A}$. By construction $\mathrm{ad}_{\check{H}}$ has integer eigenvalues. Let $\mathcal{G}_m$

denote the eigenspace corresponding to the eigenvalue $m \in \mathbb{Z}$, and so now we have $\mathcal{G} = \mathbb{R}D + \mathfrak{h} + \oplus_{m \neq 0} \mathcal{G}_m$, with

$$\mathcal{G}_m = \left\{ \mathfrak{h} \otimes z^j \; : jc = m \right\} \bigoplus \left\{ \mathfrak{g}_\alpha \otimes z^k \; : ht(\alpha) + kc = m \right\},$$

where $ht(\alpha)$ is the height of a root α with respect to the simple base Π. By inspection $[\mathcal{G}_m, \mathcal{G}_n] \subset \mathcal{G}_{m+n}$, and we get the so-called "principal gradation" of $\mathfrak{g}[z, z^{-1}]$:

$$\mathfrak{g}[z, z^{-1}] = \mathfrak{h} + \bigoplus_{m \neq 0} \mathcal{G}_m.$$

From invariance of the form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}[z, z^{-1}]$ defined at the beginning of this section, one also has that $\langle \mathcal{G}_m, \mathcal{G}_n \rangle = 0$ if $m \neq -n$, and the Cartan involution maps $\tilde{\theta}(\mathcal{G}_m) \subset \mathcal{G}_{-m}$. Define $\tilde{\mathfrak{n}}_\pm = \oplus_{m>0} \mathcal{G}_{\pm m}$, so we can write the decomposition as

$$\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{n}}_- + \mathfrak{m} + \mathfrak{a} + \tilde{\mathfrak{n}}_+ \, ,$$

where $\mathfrak{m} = i\mathfrak{a}$. We define the generalised Iwasawa decomposition to be

$$\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{k}} + \mathfrak{a} + \tilde{\mathfrak{n}}_+$$

or simply $\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{k}} + \tilde{\mathfrak{b}}$ where $\tilde{\mathfrak{b}} = \mathfrak{a} + \tilde{\mathfrak{n}} = \mathfrak{a} + \mathfrak{n} + z\mathfrak{g}[z]$.

Example 3.2.1. Consider $\mathfrak{g} = sl(n + 1, \mathbb{C})$. Let E_{ij} denote the matrix with a 1 in the (i, j)-th position, and zeroes everywhere else. Take $\mathfrak{h}$ to be the standard Cartan subalgebra spanned by the diagonal matrices $E_{i,i} - E_{i+1,i+1}$, $1 \leq i \leq n$, and define $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(E_{jj}) = \delta_{i,j}$. Take the standard positive roots given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq n$. The highest root is $\alpha_* = \epsilon_1 - \epsilon_{n+1}$, and the Coxeter number is $c = n + 1$. The element that generates the principal gradation is $\tilde{H} = (n + 1)D + H_0$, where H_0 is given by

$$H_0 = \begin{pmatrix} n/2 & & \cdots & & 0 \\ & n/2 - 1 & & & \\ \vdots & & \ddots & & \vdots \\ & & & -n/2 + 1 & \\ 0 & & \cdots & & -n/2 \end{pmatrix},$$

where the entries are anti-symmetric about the anti-diagonal.

Let D_k be the span of $\{E_{ij} : j - i = k\}$, so that D_k denotes those elements of $sl(n+1, \mathbb{C})$ non-zero only along the kth off-diagonal, $-n \leq k \leq n$. Then the eigenspaces appearing in the principal gradation are given by

$$\mathcal{G}_{(n+1)j} = \mathfrak{h} \otimes z^j \qquad j \in \mathbb{Z},$$

$$\mathcal{G}_{(n+1)j+\ell} = D_\ell \otimes z^j + D_{\ell-(n+1)} \otimes z^{j+1} \qquad j \in \mathbb{Z}, \ 0 < \ell \leq n.$$

Thus $\oplus_{m>0}\mathcal{G}_m$ consists of polynomials from $sl(n+1, \mathbb{C})[z]$, (i.e. without z^{-1} terms), whose constant term is strictly upper-triangular. The resulting Iwasawa decomposition is $sl(n+1, \mathbb{C})[z, z^{-1}] = \tilde{su}(n+1) + \tilde{\mathfrak{b}}$, where $\tilde{\mathfrak{b}}$ is given by

$$\tilde{\mathfrak{b}} = \mathrm{Span}_{\mathbb{R}}\{E_{ii} - E_{i+1,i+1}\} + \mathrm{Span}_{\mathbb{C}}\{E_{ij} : j > i\} + z \cdot sl(n+1, \mathbb{C})[z].$$

That is, $\tilde{\mathfrak{b}}$ consists of polynomials in z with coefficients from $sl(n+1, \mathbb{C})$, whose constant term is upper triangular, with real entries on the diagonal.

3.3. A special co-adjoint orbit.

Consider the decomposition $\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{k}} + \tilde{\mathfrak{b}}$. Each of these spaces is infinite dimensional, and so their duals are rather unwieldy. For the time being we shall ignore questions of topology, and consider only algebraic duals. The non-degenerate form $\langle \cdot, \cdot \rangle$ gives a natural inclusion $\tilde{\mathfrak{k}}^\perp \subset \tilde{\mathfrak{b}}^*$, and from the Cartan decomposition, $\tilde{\mathfrak{k}}^\perp$ is $i\tilde{\mathfrak{k}}$. We shall restrict attention to this subspace of $\tilde{\mathfrak{b}}^*$.

Consider the element $\phi_o \in i\tilde{\mathfrak{k}}$ given by

$$\phi_o(z) = \sum_i (E_{\alpha_i} + E_{-\alpha_i}) + E_{\alpha_*}z^{-1} + E_{-\alpha_*}z.$$

(Recall that $\alpha_* = \sum k_i \alpha_i$ is the highest root of $\mathfrak{g}$). We shall realize the Toda phase space as a co-adjoint orbit of $\exp \mathfrak{b}$, or more precisely as a co-adjoint orbit of $\exp \tilde{\mathfrak{b}}$, through ϕ_o. Of course we have not yet associated a group to $\tilde{\mathfrak{b}}$, and must take a completion of $\tilde{\mathfrak{b}}$ before we can do so, so that the meaning of $\exp \tilde{\mathfrak{b}}$ is not clear. However, we can make sense of this as follows.

Take $X \in \tilde{\mathfrak{b}}$. Formally we know that since $\tilde{\mathfrak{b}}$ is a subalgebra of $\mathfrak{g}[z, z^{-1}]$, the action of $^{\mathfrak{b}}\mathrm{ad}_X^*$ on ϕ_o, can be found by taking the action of ad_X^* on ϕ_o in $\mathfrak{g}[z, z^{-1}]$, and projecting onto $\tilde{\mathfrak{k}}^\perp \subset \tilde{\mathfrak{b}}^*$ along $\tilde{\mathfrak{b}}^\perp$. Invariance of $\langle \cdot, \cdot \rangle$ means that under the identification of $\tilde{\mathfrak{k}}^\perp \subset \tilde{\mathfrak{b}}^*$, ad^* becomes ad, and we get

$$^{\mathfrak{b}}\tilde{\mathrm{ad}}_X \phi_o = \Pi_{\tilde{\mathfrak{k}}^\perp} \tilde{\mathrm{ad}}_X \phi_o .$$

Now $\tilde{\mathfrak{b}}$ is generated by $\mathfrak{b}$, together with the element $E_{-\alpha_*}z$, and one easily finds that because of the projection occurring in this formula, and the form of ϕ_o, the entire infinitesimal orbit is generated by the action of $\mathfrak{b}$ alone. For $X \in \mathfrak{b}$ we have $\mathrm{Ad}_{\exp(X)} = e^{\mathrm{ad}_X}$, and so the co-adjoint action of $\exp(X)$ on ϕ_o can be written

$$\exp(X) \cdot \phi_o = \Pi_{\tilde{\mathfrak{k}}^\perp} e^{\mathrm{ad}_X} \phi_o \ .$$

For general $X \in \tilde{\mathfrak{b}}$ this also makes sense, provided we regard the projection as killing all $O(z^2)$ terms appearing in the potentially infinite series for $e^{\mathrm{ad}_X} \phi_o$. This will be precisely what happens when we pass to a completion for $\tilde{\mathfrak{b}}$ in section 4.6, and so this is how we shall intrepret the co-adjoint action of $\exp \tilde{\mathfrak{b}}$ on ϕ_o. The important thing to observe is that the action effectively reduces to an action of the finite dimensional group $\exp \mathfrak{b} = \mathbf{B}$, and so the corresponding orbit is finite dimensional.

Proposition. *The co-adjoint orbit $\mathcal{O}_o$ of $\exp \tilde{\mathfrak{b}}$ through ϕ_o consists of points of the form*

$$H + \sum_i q_i(E_{\alpha_i} + E_{-\alpha_i}) + q_*(E_{\alpha_*}z^{-1} + E_{-\alpha_*}z)$$

where $q_i, q_* > 0$ satisfy $q_* \cdot \prod q_i^{k_i} = 1$, and H is in the real span of $\{H_{\alpha_i}\}$. For any fixed H set $\psi_H = \langle H, \cdot \rangle$. Then the Poisson structure on $\mathcal{O}_o$ is given by

$$\{\psi_H, q_i\} = \alpha_i(H)q_i, \quad \{\psi_H, q_*\} = -\alpha_*(H)q_*.$$

Proof. The algebra $\tilde{\mathfrak{b}} = \mathfrak{a} + \mathfrak{n}_+ + z\mathfrak{g}[z]$ is generated by $\mathbb{R}\{H_{\alpha_i}\} + \mathbb{C}\{E_{\alpha_i}\} + \mathbb{C}E_{-\alpha_*}z$, and it suffices to find the action of these generators. To calculate the projection $\Pi_{\tilde{\mathfrak{k}}^\perp}$ we note that according to the splitting $\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{k}}^\perp + \tilde{\mathfrak{b}}^\perp$ we have

$$\mu H = \mathrm{Re}(\mu)H + \mathrm{Im}(\mu)H,$$
$$E_{\alpha_i} = 0 + E_{\alpha_i}, \quad E_{-\alpha_i} = \left(E_{-\alpha_i} + E_{\alpha_i}\right) - E_{\alpha_i},$$
$$E_{-\alpha_*}z = 0 + E_{-\alpha_*}z, \quad E_{\alpha_*}z^{-1} = \left(E_{\alpha_*}z^{-1} + E_{-\alpha_*}z\right) - E_{-\alpha_*}z \ .$$

First we calculate $\exp(H) \cdot \phi_o$, where H is in the real span of $\{H_{\alpha_i}\}$:
Consider $e^{\mathrm{ad}_H}(E_\alpha + E_{-\alpha}) = e^{\alpha(H)}E_\alpha + e^{-\alpha(H)}E_{-\alpha}$. Projecting on to $\tilde{\mathfrak{k}}^\perp$ along $\tilde{\mathfrak{b}}^\perp$ gives

$$\exp(H) \cdot (E_\alpha + E_{-\alpha}) = e^{-\alpha(H)}(E_\alpha + E_{-\alpha}),$$

and likewise

$$\exp(H) \cdot (E_{\alpha_*} z^{-1} + E_{-\alpha_*} z) = e^{\alpha_*(H)}(E_{\alpha_*} z^{-1} + E_{-\alpha_*} z).$$

Thus the action of $\exp(H)$ on ϕ_o is given by

$$\exp(H) \cdot \phi_o = \sum_i q_i (E_{\alpha_i} + E_{-\alpha_i}) + q_*(E_{\alpha_*} z^{-1} + E_{-\alpha_*} z), \qquad (1)$$

where $q_i = e^{-\alpha_i(H)}$ and $q_* = e^{\alpha_*(H)}$. Since we started with a Chevalley basis, $\alpha_i(H_{\alpha_j})$ is an integer for each i and j, and by assumption H is in the real span of $\{H_{\alpha_j}\}$, so that q_i, q_* are real-valued and positive. In addition $\alpha_* = \Sigma k_i \alpha_i$, which implies that $q_* \cdot \Pi q_i^{k_i} = 1$.

Similarly, taking $\mu \in \mathbb{C}$, and $\beta \in \Pi$, we get

$$\exp(\mu E_\beta) \cdot \phi_o = \mathrm{Re}(\mu) H_\beta + \phi_o \qquad (2)$$

$$\exp(\mu E_{-\alpha_*} z) \cdot \phi_o = \mathrm{Re}(\mu) H_{\alpha_*} + \phi_o \ . \qquad (3)$$

Repeated applications of the generators to (1), (2) and (3) gives the required orbit.

To calculate the Poisson brackets on $\mathcal{O}_o$ note that the q's can be regarded as linear functionals on $\tilde{\mathfrak{k}}^\perp \subset \tilde{\mathfrak{b}}^*$, with

$$q_\alpha = \frac{\langle E_\alpha, \cdot \rangle}{\langle E_\alpha, E_{-\alpha} \rangle}, \quad q_* = \frac{\langle E_{-\alpha_*} z, \cdot \rangle}{\langle E_{\alpha_*}, E_{-\alpha_*} \rangle} \ .$$

Fix H in the real span of $\{H_{\alpha_i}\}$ and set $\psi_H = \langle H, \cdot \rangle$. The gradients of these linear functionals are given by

$$\nabla q_\alpha = \frac{E_\alpha}{\langle E_\alpha, E_{-\alpha} \rangle}, \quad \nabla q_* = \frac{E_{\alpha_*} z^{-1}}{\langle E_{\alpha_*}, E_{-\alpha_*} \rangle}, \quad \nabla \psi_H = H.$$

(Note that these gradients are normalised to belong to $\tilde{\mathfrak{b}}$). The Poisson brackets are now easy to calculate. For example,

$$\{\psi_H, q_\alpha\}(X) = \langle X, [\nabla \psi_H, \nabla q_\alpha](X) \rangle = \langle X, [H, \frac{E_\alpha}{\langle E_\alpha, E_{-\alpha} \rangle}] \rangle$$

$$= \alpha(H) \cdot \langle X, \frac{E_\alpha}{\langle E_\alpha, E_{-\alpha} \rangle} \rangle$$

$$= \alpha(H) \cdot q_\alpha(X).$$

So $\{\psi_H, q_\alpha\} = \alpha(H) \cdot q_\alpha$, and similarly $\{\psi_H, q_*\} = -\alpha^*(H) \cdot q_*$. All other brackets are zero. $\square$

3.4. The main example: $\mathfrak{g} = sl(n + 1, \mathbb{C})$.

We can now construct the classical symmetric periodic Toda lattice. We have

$$\phi_o(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & z^{-1} \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ z & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and the co-adjoint orbit of $\exp \tilde{\mathfrak{b}}$ through $\phi_o(z)$ consists of matrices of the form

$$\phi(z) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_{n+1}z^{-1} \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_n \\ a_{n+1}z & \cdots & 0 & a_n & b_{n+1} \end{pmatrix}$$

where $a_i, b_j \in \mathbb{R}$, $a_i > 0$, and $\Sigma b_i = 0$, $\Pi a_i = 1$. The Poisson structure is given by

$$\{b_i - b_{i+1}, a_i\} = 2a_i, \qquad \{b_{i+1} - b_{i+2}, a_i\} = -a_i,$$

which reduces to

$$\{a_i, b_j\} = (\delta_{i+1,j} - \delta_{ij})a_i.$$

From the form $\langle \Sigma \phi_k z^k, \Sigma \psi_j z^j \rangle = \Sigma \mathrm{Tr}\, \phi_k \psi_{-k}$, we get the Hamiltonian

$$H(\phi) = \frac{1}{2} \langle \phi, \phi \rangle = \frac{1}{2} \sum_{k=1}^{n+1} b_k^2 + \sum_{k=1}^{n+1} a_k^2,$$

and so recover the classical real symmetric periodic Toda lattice.

3.5. A Natural Central Extension.

The operator $D = iz\frac{d}{dz}$ is a derivation on $\mathfrak{g}[z, z^{-1}]$, and maps $Xz^k \mapsto ikXz^k$. To this we can associate a 2-cocycle ω via the form $\langle \cdot, \cdot \rangle$, with

$$\omega(\phi, \psi) = \langle \phi, D\psi \rangle,$$

or in terms of Fourier series

$$\omega(\phi, \psi) = i \sum_k \langle \phi_{-k}, \psi_k \rangle.$$

This is a $\mathbb{C}$-valued skew-symmetric bilinear form on $\mathfrak{g}[z, z^{-1}]$, which satisfies

$$\omega([\phi, \psi], \delta) + cyclic = 0,$$

and so can be used to construct a central extension of $\mathfrak{g}[z, z^{-1}]$, which we denote $\tilde{\mathfrak{g}}[z, z^{-1}]$:

$$0 \to \mathbb{C} \to \tilde{\mathfrak{g}}[z, z^{-1}] \to \mathfrak{g}[z, z^{-1}] \to 0.$$

As a vector space $\tilde{\mathfrak{g}}[z, z^{-1}]$ is $\mathfrak{g}[z, z^{-1}] + \mathbb{C}$, and its commutator is given by:

$$\left[(\phi, p), (\psi, q)\right] = \left([\phi, \psi], \omega(\phi, \psi)\right).$$

This extended loop algebra, and its completions, will play a key role from now on.

4. The Completed Loop Algebras

We now consider various completions of the algebras $\mathfrak{g}[z, z^{-1}]$ and $\tilde{\mathfrak{g}}[z, z^{-1}]$. The goal is to do this in such a way that the existence of corresponding Lie groups is assured, along with group factorizations corresponding to the Iwasawa decompositions of the algebras.

4.1. The smooth loop algebra.

Perhaps the most natural completion of $\mathfrak{g}[z, z^{-1}]$ is the smooth loop algebra $L_s\mathfrak{g}$, which is the space of smooth maps from S^1 to $\mathfrak{g}$. By taking Fourier series one can identify this algebra as

$$L_s\mathfrak{g} = \{\phi = \sum_{-\infty}^{\infty} \phi_k z^k : \phi_k \in \mathfrak{g}, \sum_k k^n \phi_k z^k \text{ unif. cvgt. on } |z| = 1 \ \forall n \geq 0\}.$$

The bracket on $\mathfrak{g}$ induces one on $L_s\mathfrak{g}$ making it into an infinite dimensional Lie algebra. With the topology of uniform convergence

$$\phi_n, \phi \in L_s\mathfrak{g} \ \ \phi_n \to \phi \iff \left\| \frac{d^N}{dz^N}(\phi_n - \phi) \right\| \to 0 \text{ uniformly } \forall N \geq 0$$

$L_s\mathfrak{g}$ has the structure of a Fréchet space; it is a complete Hausdorff metrizable locally convex topological vector space, (for the details see [Ha]). The corresponding infinite dimensional Lie group is the smooth loop group $L_s G$ of smooth maps from S^1 to G. The definitions of the Killing form $\langle \cdot, \cdot \rangle$, the derivation $D = iz\frac{d}{dz}$, and the 2-cocycle ω for the algebra $\mathfrak{g}[z, z^{-1}]$, all pass without problem to $L_s\mathfrak{g}$, and give rise in particular to a central extension $\widetilde{L_s\mathfrak{g}}$ of $L_s\mathfrak{g}$ by $\mathbb{C}$, which is nothing but the smooth completion of the extension $\tilde{\mathfrak{g}}[z, z^{-1}]$.

The problem with choosing the smooth completion of $\tilde{\mathfrak{g}}[z, z^{-1}]$ is that in general there is no guarantee of the existence of a group corresponding to $\widetilde{L_s\mathfrak{g}}$, (see for example [PS] Theorem 4.4.1), and so the factorization method cannot be used to generate solutions for Lax-pair equations in $\widetilde{L_s\mathfrak{g}}$. Because of this we are led to consider other completions of $\mathfrak{g}[z, z^{-1}]$.

4.2. w-completions of the affine algebras.

A function $w \colon \mathbb{Z} \to (0, \infty)$ is called a *weight* if $w(k+m) \leq w(k)w(m)$. Given a weight w there is an associated norm $\|\cdot\|_w$ on $\mathfrak{g}[z, z^{-1}]$ given by

$$\left\| \sum_k X_k z^k \right\|_w = \sum_k |X_k| w(k).$$

Let $L_w\mathfrak{g}$ denote the completion of $\mathfrak{g}[z, z^{-1}]$ relative to this norm. That is, $L_w\mathfrak{g}$ consists of all those infinite Fourier series with coefficients in $\mathfrak{g}$, that are uniformly convergent with respect to $|| \cdot ||_w$. The norm extends to $\tilde{\mathfrak{g}}[z, z^{-1}]$ by setting $||\phi(z) + p||_w = ||\phi(z)||_w + |p|$, and the corresponding completion is just $L_w\mathfrak{g} + \mathbb{C}$. Note that if $w(k) \geq C|k|^{\frac{1}{2}}$ for some constant $C > 0$, then these completions are Banach Lie algebras, ([GW2]).

Definition 4.2.1. (a) w is said to be *symmetric* if $w(k) = w(-k)$.

(b) w is of *non-analytic type* if $\lim_{k\to\infty} w(k)^{1/k} = 1$.

(c) w is said to be *rapidly increasing at infinity* if there exists λ with $1 < \lambda < 2$, and $\lim_{k\to\infty} |k|^{-1/\lambda} \log w(k) = \infty$.

Remark. Note that if w is rapidly increasing at infinity, then $\lim_{k\to\infty} w(k) \cdot k^{-p} = \infty$ for all $p > 0$, and from this one can easily deduce that the elements of $L_w\mathfrak{g}$ are smooth, so that $L_w\mathfrak{g} \subset L_s\mathfrak{g}$.

Example 4.2.1. Take $w(k) = \exp(|k|^{2/3})$. Then w is symmetric, of non-analytic type, and rapidly increasing at infinity, (e.g. take $\lambda = 7/4$ in part (c) of the definition).

4.3. Associated Lie Groups. Suppose w is a symmetric weight of non-analytic type. Let $L_w G$ denote the set of Fourier series with coefficients in G, which are uniformly convergent with respect to the weight w. Then $L_w G$ is a Banach-Lie group, whose corresponding Lie algebra is $L_w\mathfrak{g}$. As noted in section 4.1 the extension $\widetilde{L_s\mathfrak{g}}$ may not correspond to any Lie group, but for the central extension of $L_w\mathfrak{g}$ we have the following theorem from [GW2]:

Theorem 1. *If the symmetric weight w is of non-analytic type, and rapidly increasing at infinity, then there is a complex Banach-Lie group $\widetilde{L_w G}$, whose Lie algebra is $\widetilde{L_w\mathfrak{g}} = L_w\mathfrak{g} + \mathbb{C}$, and which is a central extension of $L_w G$ by $\mathbb{C}^\times$.*

Henceforth we shall assume that we have fixed some choice of weight w of symmetric, non-analytic type, that increases rapidly at infinity, and that we have constructed the completions described above. As a notational convenience we shall drop any reference to w from now on, so that $L\mathfrak{g}$ shall mean the w-completion of the Laurent

polynomial algebra, which in general will be strictly smaller than the full smooth loop algebra. The corresponding group will be denoted LG, and the associated central extensions will be denoted $\widetilde{L\mathfrak{g}}$, and $\widetilde{LG}$, respectively.

4.4. A generalised Iwasawa decomposition.

Earlier we constructed the decomposition $\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{k}} + i\tilde{\mathfrak{k}}$, where the algebra $\tilde{\mathfrak{k}}$ is the real form of the Cartan involution θ, extended to $\mathfrak{g}[z, z^{-1}]$ by $(\tilde{\theta}(\phi))(z) = \theta(\phi(z))$. $\tilde{\mathfrak{k}}$ consists of those elements of $\mathfrak{g}[z, z^{-1}]$ whose image is in the compact form $\mathfrak{k}$ for all $z \in S^1$. Via the principal gradation we came to the Iwasawa decomposition

$$\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{k}} + \mathfrak{a} + \mathfrak{n}_+ + z\mathfrak{g}[z]$$
$$= \tilde{\mathfrak{k}} + \tilde{\mathfrak{b}}$$

where $\mathfrak{a}$ is the real span of $\{H_\alpha\}$, and $\mathfrak{n}_+$ is the sum of the positive root spaces. We note that $\tilde{\mathfrak{b}}$ consists of polynomial loops with no z^{-1} terms, and whose constant term lies in $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}_+$.

The definition of $\tilde{\theta}$ makes sense on $L\mathfrak{g}$, and leads to the decomposition $L\mathfrak{g} = \tilde{\mathfrak{k}} + i\tilde{\mathfrak{k}}$, where now $\tilde{\mathfrak{k}}$ denotes those w-smooth loops invariant under $\tilde{\theta}$, (and so whose image is in $\mathfrak{k}$ for all $z \in S^1$). The generalised Iwasawa decomposition is

$$L\mathfrak{g} = \tilde{\mathfrak{k}} + \mathfrak{a} + \tilde{\mathfrak{n}}_+$$
$$= \tilde{\mathfrak{k}} + \tilde{\mathfrak{b}}$$

where $\tilde{\mathfrak{b}}$ is the w-completion of the corresponding algebra for $\mathfrak{g}[z, z^{-1}]$, hence consists of those w-convergent Fourier series with no z^{-1} terms, and constant term in $\mathfrak{b}$.

Finally, $\tilde{\theta}$ extends to a conjugation of $\widetilde{L\mathfrak{g}}$, by setting $\tilde{\theta}(\phi(z), p) = (\theta(\phi(z)), \bar{p})$, which leads to the decomposition $\widetilde{L\mathfrak{g}} = (\tilde{\mathfrak{k}} + \mathbb{R}) + i(\tilde{\mathfrak{k}} + \mathbb{R})$, and the Iwasawa decomposition

$$\widetilde{L\mathfrak{g}} = (\tilde{\mathfrak{k}} + \mathbb{R}) + (\tilde{\mathfrak{b}} + i\mathbb{R}).$$

The justification for calling these splittings Iwasawa decompositions, comes from the work of Goodman and Wallach [GW2], which shows that there are corresponding factorizations at the group level, very much like the finite dimensional case. Paraphrasing [GW2] theorems 5.5, 6.4 and 6.5 we have:

Theorem 2. *There are Lie subgroups $\tilde{K}, \tilde{B}$ of LG, whose Lie algebras are $\check{\mathfrak{k}}$ and $\tilde{\mathfrak{b}}$, such that the map $\tilde{K} \times \tilde{B} \to LG$: $(k, b) \mapsto kb$ is a real-analytic manifold isomorphism. Similarly, there are Lie subgroups $\hat{K}, \hat{B}$ of $\widetilde{LG}$, whose Lie algebras are $\check{\mathfrak{k}} + \mathbb{R}$ and $\tilde{\mathfrak{b}} + i\mathbb{R}$, such that the map $\hat{K} \times \hat{B} \to \widetilde{LG}$: $(\hat{k}, \hat{b}) \mapsto \hat{k}\hat{b}$ is a real-analytic manifold isomorphism. The group $\hat{K}$ is a central extension of $\tilde{K}$ by S^1, and $\hat{B} = \tilde{B} \cdot \mathbb{R}$.*

As a matter of convenience we shall consider the splitting

$$\widetilde{L\mathfrak{g}} = (\check{\mathfrak{k}} + \mathbb{C}) + \tilde{\mathfrak{b}} \ .$$

Because elements of $\tilde{\mathfrak{b}}$ have no powers of z^{-1} in their Fourier series, $\omega(\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}) = 0$, so that the terms in this splitting are (real) subalgebras of $\widetilde{L\mathfrak{g}}$. The corresponding group factorization is $\widetilde{LG} = (\hat{K} \cdot \mathbb{R})\hat{B}$.

4.5. The gauge-coadjoint action. The algebra $\widetilde{L\mathfrak{g}} = \widetilde{L\mathfrak{g}} + \mathbb{C}$ has commutator

$$\left[(\phi, p), (\psi, q)\right] = \left([\phi, \psi], \omega(\phi, \psi)\right).$$

Because $(0, p)$ is in the center of the algebra, one can think of as an adjoint action of $L\mathfrak{g}$ on $\widetilde{L\mathfrak{g}}$:

$$\tilde{\mathrm{ad}}_\phi\left(\psi, q\right) = \left(\mathrm{ad}_\phi \psi, \omega(\phi, \psi)\right).$$

This in turn comes from an adjoint action of the group LG on $\widetilde{L\mathfrak{g}}$:

Lemma 4.5.1. *For $\gamma \in LG$, and $(\psi, q) \in \widetilde{L\mathfrak{g}}$*

$$\tilde{Ad}_\gamma\left(\psi, q\right) = \left(Ad_\gamma \psi, q - \langle \gamma^{-1}D\gamma, \psi \rangle\right).$$

Proof. One can easily verify that this is an action, so it suffices to show that the derivative at the identity of this action is $\tilde{\mathrm{ad}}$. Suppose $\gamma_t = \exp t\phi$ for some $\phi \in L\mathfrak{g}$. Then

$$D\gamma_t = \frac{d}{d\theta}\exp t\phi(\theta) = \frac{d}{ds}\Big|_{s=0}\exp t\phi(\theta + s),$$

so that

$$\frac{d}{dt}\Big|_{t=0}\gamma_t^{-1}D\gamma_t = \frac{d^2}{dt\,ds}\Big|_{s,t=0}\exp-t\phi(\theta)\cdot\exp t\phi(\theta+s)$$

$$= \frac{d}{ds}\Big|_{s=0}[-\phi(\theta)+\phi(\theta+s)]$$

$$= \frac{d}{d\theta}\phi(\theta)$$

$$= D\phi.$$

This shows $\frac{d}{dt}\big|_{t=0}\langle\gamma_t^{-1}D\gamma_t,\psi\rangle = -\omega(\phi,\psi)$, and so $\frac{d}{dt}\big|_{t=0}\tilde{\mathrm{Ad}}_{\gamma_t}(\psi,q)$ is $\tilde{\mathrm{ad}}_\phi(\psi,q)$ as required. $\square$

From now on we shall treat $\widetilde{L\mathfrak{g}} = L\mathfrak{g}+\mathbb{C}$ as a real Lie algebra. We introduce a nondegenerate real-bilinear pairing $\langle\langle\cdot,\cdot\rangle\rangle$ on $\widetilde{L\mathfrak{g}}$, defined by

$$\big\langle\big\langle(\phi,p),(\psi,q)\big\rangle\big\rangle = \mathrm{Re}\big\{\langle\phi,\psi\rangle+pq\big\}.$$

Remark. Note that this form is *not* invariant. In fact if $\mathfrak{g}$ is semi-simple then $\widetilde{L\mathfrak{g}}$ does not possess a non-degenerate invariant form. This is because $[L\mathfrak{g},L\mathfrak{g}] = L\mathfrak{g}$, and consequently a central element $(0,p)$ paired with any other element of $\widetilde{L\mathfrak{g}}$ by an invariant form will be zero.

Using this form we can embed $\widetilde{L\mathfrak{g}} \subset \widetilde{L\mathfrak{g}}^*$ as a real subspace. The image of the embedding is the so-called "smooth part of the dual"; it is invariant under the co-adjoint action of LG as the following lemma shows:

Lemma 4.5.2. *If $\gamma \in LG$ and $(\phi,p) \in \widetilde{L\mathfrak{g}} \subset (\widetilde{L\mathfrak{g}})^*$, then the co-adjoint action of γ on (ϕ,p) is given by*

$$\tilde{\mathrm{Ad}}_\gamma^*\big(\phi,p\big) = \big(\mathrm{Ad}_\gamma\phi + pD\gamma\gamma^{-1},p\big).$$

Note: We shall at times use the abbreviation $\gamma\cdot(\phi,p)$ for $\tilde{\mathrm{Ad}}_\gamma^*(\phi,p)$.

Proof. The verification is straightforward:

$$\left\langle\!\!\left\langle (\phi, p), \tilde{\mathrm{Ad}}_{\gamma^{-1}}(\psi, q)\right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle (\phi, p), (\mathrm{Ad}_{\gamma^{-1}}\psi, q - \langle \gamma D\gamma^{-1}, \psi\rangle)\right\rangle\!\!\right\rangle$$

$$= \mathrm{Re}\left\{\langle \phi, \mathrm{Ad}_{\gamma^{-1}}\psi\rangle + pq - p\langle \gamma D\gamma^{-1}, \psi\rangle\right\}$$

$$= \mathrm{Re}\left\{\langle \mathrm{Ad}_{\gamma}\phi + pD\gamma\gamma^{-1}, \psi\rangle + pq\right\}$$

$$= \left\langle\!\!\left\langle (\mathrm{Ad}_{\gamma}\phi + pD\gamma\gamma^{-1}, p), (\psi, q)\right\rangle\!\!\right\rangle$$

This completes the proof. $\square$

Definition 4.5.1. This action is called the *gauge co-adjoint action*, or simply the *gauge action*, of $\gamma \in LG$ on $(\phi, p) \in \widetilde{L\mathfrak{g}}$.

Remark. The reason for this nomenclature is the following. For a principal G-bundle $P \to X$, the gauge group $\mathcal{G}$ of the bundle is defined to be the group of fiber preserving automorphisms of P, that commute with the G action. This group has a natural action on the affine space $\mathcal{A}$ of connections on the bundle. For the trivial G-bundle $S^1 \times G \to S^1$ there is a natural identification of $\mathcal{G} \cong L_s G$, and $\mathcal{A} \cong L_s \mathfrak{g}$, and the action of a gauge group element $\gamma \in L_s G$ on a connection $\phi \in L_s \mathfrak{g}$ is given by $\gamma \cdot \phi = \mathrm{Ad}_{\gamma}\phi + D\gamma\gamma^{-1}$.

4.6. The Toda phase space. In section 3.3 the Toda phase space was constructed from the splitting $\mathfrak{g}[z, z^{-1}] = \tilde{\mathfrak{k}} + \tilde{\mathfrak{b}}$, by looking at the co-adjoint orbit of $\exp \tilde{\mathfrak{b}}$ through an element $\phi_o \in \tilde{\mathfrak{k}}^{\perp} \subset \tilde{\mathfrak{b}}^*$. We argued that even though $\exp \tilde{\mathfrak{b}}$ was ill-defined at that point, one could make sense of the action by restricting attention to $\exp \mathfrak{b} = \mathbf{B}$.

Now we do have groups corresponding to our algebras, and by considering the splitting $\widetilde{L\mathfrak{g}} = (\tilde{\mathfrak{k}} + \mathbb{C}) + \tilde{\mathfrak{b}}$ we can examine the co-adjoint orbit of $\exp \tilde{\mathfrak{b}} = \tilde{\mathbf{B}}$ through $(\phi_o, 0) \in (\tilde{\mathfrak{k}} + \mathbb{C})^{\perp} \subset \tilde{\mathfrak{b}}^*$. The orbit consists of all points of the form

$$\Pi_{(\tilde{\mathfrak{k}}+\mathbb{C})^{\perp}} \gamma \cdot (\phi_o, 0) = (\Pi_{\tilde{\mathfrak{k}}^{\perp}} \mathrm{Ad}_{\gamma}\phi_o, 0),$$

where $\gamma \in \tilde{\mathbf{B}}$. The first term is just the w-completion of the orbit $\mathcal{O}_o$ found in the $\mathfrak{g}[z, z^{-1}]$ case, which is just $\mathcal{O}_o$ itself. Thus the co-adjoint orbit through $(\phi_o, 0)$ is $(\mathcal{O}_o, 0)$. Note that to generate the full orbit it suffices to consider only the action of $\mathbf{B} \subset \tilde{\mathbf{B}}$, which is consistent with our comments in 3.3.

We will identify the Toda phase space not as $(\mathcal{O}_o, 0)$, but as a character shifted version of this.

Claim. $(0, i)$ is a character of $\tilde{\mathfrak{k}} + \mathbb{C}$.

Certainly $(0, i) \in \tilde{\mathfrak{b}}^{\perp} \subset (\tilde{\mathfrak{k}} + \mathbb{C})^{*}$, so it suffices to check that $\langle \langle (0, i), [\tilde{\mathfrak{k}} + \mathbb{C}, \tilde{\mathfrak{k}} + \mathbb{C}] \rangle \rangle = 0$. But this follows from the fact that $\omega(\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}) \in \mathbb{R}$, which one can verify by first checking that $D\tilde{\mathfrak{k}} \subset \tilde{\mathfrak{k}}$, and then showing that $\langle \tilde{\mathfrak{k}}, \tilde{\mathfrak{k}} \rangle \in \mathbb{R}$.

We henceforth identify the

$$\text{Toda phase space} \quad \cong \quad (\mathcal{O}_{o}, 0) + (0, i),$$

and shall apply the Kostant-Symes involution theorem to this (character shifted) co-adjoint orbit.

5. Invariant functions on $\widetilde{L\mathfrak{g}}$

We now endeavor to construct a family of Poisson commuting functions on the Toda phase space $(\mathcal{O}_o, i)$, by an application of the Kostant-Symes involution theorem. To do so we require invariant functions on $(\widetilde{L\mathfrak{g}})^*$, (or more precisely on $\widetilde{L\mathfrak{g}} \subset (\widetilde{L\mathfrak{g}})^*$).

5.1. The monodromy. Roughly speaking, coadjoint orbits in $\widetilde{L\mathfrak{g}} \subset (\widetilde{L\mathfrak{g}})^*$ correspond to conjugacy classes in **G**. A natural way to get invariant functions on $\widetilde{L\mathfrak{g}}$ is to take class functions on **G**, and pull them back to $\widetilde{L\mathfrak{g}}$ via this correspondence. We make this precise as follows.

Definition 5.1.1. Given $(\phi, p) \in \widetilde{L\mathfrak{g}}, (p \neq 0)$, consider the differential equation

$$\frac{dh(t)}{dt} h(t)^{-1} = \frac{1}{p} \phi(e^{it}), \quad h(0) = \mathrm{Id},$$

where $h \colon \mathbb{R} \to \mathbf{G}$. We shall refer to this as the *monodromy equation* associated to (ϕ, p). The *monodromy* $M_{(\phi,p)}$ of the element $(\phi, p) \in \widetilde{L\mathfrak{g}}$ is defined by $M_{(\phi,p)} = h(2\pi)$, and we call $M \colon (\phi, p) \mapsto M_{(\phi,p)}$ the *monodromy map*. An element (ϕ, p) will be said to be *loop-regular* if its monodromy is regular, that is, if the maximal abelian subgroup of **G** containing $M_{(\phi,p)}$ is conjugate to the Cartan subgroup.

The monodromy map is a smooth tame map of Fréchet spaces, ([Ha] Theorem 3.2.2), and consequently the set of loop-regular elements is open in $\widetilde{L\mathfrak{g}}$.

Remark. The monodromy has a simple geometric interpretation. As noted earlier, we can identify $L\mathfrak{g}$ with the space of connections on the trivial principal bundle $S^1 \times \mathbf{G} \to S^1$. Given a connection $\frac{1}{p}\phi$, the horizontal lift of the curve $c \colon \mathbb{R} \to S^1 \colon c(t) = e^{it}$, through the point $(1, \mathrm{Id})$, is $(c(t), h(t))$ where $h(t)$ is determined by the differential equation in the definition. $M_{(\phi,p)}$ is thus the monodromy, (or equivalently the holonomy), of the connection $\frac{1}{p}\phi$. This interpretation shows that the solution $h(t)$ exists for all $t \in \mathbb{R}$.

The following proposition gives a parameterization of the gauge co-adjoint orbits, and will allow us to define invariant functions on $\widetilde{L\mathfrak{g}} \subset (\widetilde{L\mathfrak{g}})^*$.

Proposition. *The monodromy classifies the gauge co-adjoint orbits in the following sense:*

(1) *If $\gamma \in LG$, then the monodromy of $\gamma \cdot (\phi, p)$ is $\gamma(1)M_{(\phi,p)}\gamma(1)^{-1}$.*

(2) *If the monodromy of each of $(\phi, p), (\tilde{\phi}, p)$ belong to the same conjugacy class in G, then there exists $\gamma \in LG$ with $\gamma \cdot (\phi, p) = (\tilde{\phi}, p)$.*

Hence for fixed $p \neq 0$, the monodromy map gives a 1-1 correspondence between coadjoint orbits of LG in $L\mathfrak{g} \times \{p\}$, and conjugacy classes of G.

Proof. (1) If $\gamma \cdot (\phi, p) = (\tilde{\phi}, p)$, one can easily verify that $\tilde{h}(t) = \gamma(e^{it})h(t)\gamma(1)^{-1}$ solves the monodromy equation for $(\tilde{\phi}, p)$, and the result follows.

(2) Suppose $h, \tilde{h}$ are solutions to the monodromy equation for $(\phi, p), (\tilde{\phi}, p)$ respectively, and that $g \in G$ conjugates $M_{(\phi,p)}$ to $M_{(\tilde{\phi},p)}$. Take $\gamma(e^{it}) = \tilde{h}(t)gh(t)^{-1}$, then $\gamma \cdot (\phi, p) = (\tilde{\phi}, p)$. $\square$

Hence by composing class functions on G with the monodromy map, we get a natural invariant family of functions on $\widetilde{L\mathfrak{g}} \subset (\widetilde{L\mathfrak{g}})^*$. By the Kostant-Symes theorem, these functions will Poisson commute when restricted to the Toda orbit. The rest of the paper will be spent demonstrating that this gives a completely integrable system on the orbit.

5.2. Locally defined invariant functions.

Consider a family of generators of the class functions of some matrix group. The values of these functions, evaluated at a particular matrix A, determine the eigenvalues of that matrix. If the matrix is regular, (i.e. has distinct eigenvalues), one can define "eigenvalue functions" near A, in terms of the generators. Essentially this involves conjugating matrices near A to diagonal form, and taking the diagonal entries; locally this map is well-defined, up to a choice of ordering of the eigenvalues. It is also invariant under conjugation.

We now give an analogous construction of invariant locally defined functions on $\widetilde{L\mathfrak{g}}$, defined near loop-regular elements. First observe that loop-regular elements may be conjugated to $\mathfrak{h} \times \mathbb{C}^\times$ under the gauge action:

Lemma 5.2.1. *If (ϕ, p) is loop-regular, then there exists $\gamma \in LG$ and a constant loop $\mu \in \mathfrak{h} \subset L\mathfrak{g}$ such that $\gamma \cdot (\phi, p) = (\mu, p)$.*

Proof. By loop regularity $M_{(\phi,p)}$ is conjugate to some $h \in \exp \mathfrak{h}$. Suppose $h = \exp H$ and set $\mu(z) = \frac{p}{2\pi} H \in L\mathfrak{g}$. Then $M_{(\mu,p)} = h$, so $M_{(\phi,p)}, M_{(\mu,p)}$ are conjugate, and there exists $\gamma \in LG$ with $\gamma \cdot (\phi, p) = (\mu, p)$. $\square$

For a loop regular (ϕ, i) in the Toda phase space $\tilde{\mathcal{O}}_o$, more can be said, as the following lemma, which will be used later on, shows.

Lemma 5.2.2. *If (ϕ, i) is a loop-regular element in the Toda phase space, there exists $\gamma \in \check{K} \subset \widetilde{LG}$, and $\mu \in \mathfrak{a} \subset \mathfrak{h}$ such that $(\phi, i) = \gamma \cdot (\mu, i)$.*

Proof. The monodromy equation for (ϕ, i) is $\dot{f} f^{-1} = -i\phi(e^{it})$. Since $\phi(z) \in i\tilde{\mathfrak{k}}$, this shows that $f(t) \in K$ for all t, and in particular the monodromy $f(2\pi) = M_{(\phi,i)}$ is in K. By loop regularity, and since K is the compact form of G, $M_{(\phi,i)}$ can be conjugated in K to an element of the maximal torus $H \cap K$ of K: $kM_{(\phi,i)}k^{-1} = h \in H \cap K$, for some $k \in K$. But h can be written $h = \exp -2\pi i\mu$ for some $i\mu \in \mathfrak{h} \cap \mathfrak{k} = i\mathfrak{a}$, and is the monodromy of (μ, i). Because the monodromies are conjugate in K, and the solutions of the monodromy equations have images in K, there exists $\gamma \in \check{K}$: $\gamma \cdot (\mu, i) = (\phi, i)$. $\square$

We now show that locally, gauge-conjugation to $\mathfrak{h} \times \mathbb{C}^\times$ is a well defined map. Specifically we prove:

Proposition. *Fix a loop-regular (ϕ, p), and a $(\mu, p) \in \mathfrak{h} \times \mathbb{C}^\times$ gauge conjugate to it. Then there exists a gauge invariant loop-regular neighborhood $U \subset \widetilde{L\mathfrak{g}}$ of (ϕ, p), and a well-defined smooth gauge invariant map $\mu \colon U \to \mathfrak{h}$ such that $\mu(\phi, p) = \mu$.*

Proof. This proposition follows from a standard implicit function theorem argument, although in this setting we need the Nash-Moser inverse function theorem for tame Fréchet spaces, (see [Ha] for more details). We give a sketch of the argument.

Let $\nu \colon G \to \mathbb{C}^l$ denote the vector of character maps, corresponding to the fundamental weights of $\mathfrak{g}$. These characters freely generate the ring of class functions, (see [St], pg.87). Define $\chi \colon \widetilde{L\mathfrak{g}} \to \mathbb{C}^l$ to be the composite of this map with the monodromy map: $\chi = \nu_o M$, and let $\tilde{\chi}$ denote the restriction of χ to $\mathfrak{h} \times \mathbb{C}^\times$. Define $F \colon \widetilde{L\mathfrak{g}} \times (\mathfrak{h} \times \mathbb{C}^\times) \to \widetilde{L\mathfrak{g}} \times \mathbb{C}^l$ by $F = (pr_{\widetilde{L\mathfrak{g}}}, \chi - \tilde{\chi})$, and note that F is a smooth tame map.

Claim. F is locally invertible at $((\phi,p),(\mu,p))$

This follows from the Nash-Moser theorem, after one shows that $(D\tilde{\chi})^{-1}$ and $D\chi$ are smooth tame maps near $((\phi,p),(\mu,p))$, which is an easy consequence of the definitions. Hence by the standard argument one constructs an open neighborhood $\tilde{U} \subset \widetilde{L\mathfrak{g}}$ of (μ,p), and a smooth map $\tilde{\mu}\colon \tilde{U} \to \mathfrak{h}\times\mathbb{C}^l$ which satisfies $F((\psi,q),\tilde{\mu}(\psi,q)) = ((\psi,q),0)$, and $\tilde{\mu}(\phi,p) = (\mu,p)$. (Note that since the set of loop-regular elements is open in $\widetilde{L\mathfrak{g}}$, we can without loss of generality take all the open sets constructed to be loop regular).

Now because (μ,p) is loop regular, its stabiliser in the affine Weyl group is trivial, and so we can find a neighborhood $N \subset \mathfrak{h}\times\mathbb{C}^l$ of it in which no two elements are conjugate by the Weyl group action. By the continuity of $\tilde{\mu}$, shrinking $\tilde{U}$ if necessary, we can assume that $\tilde{\mu}(\tilde{U}) \subset N$. Now set $U = LG \cdot \tilde{U}$, the image of $\tilde{U}$ under the gauge action of LG. Define $\mu\colon U \to \mathfrak{h}$ to be the map which first gauge conjugates $u \in U$ to $\tilde{U}$, then maps to $\mathfrak{h}\times\mathbb{C}^\times$ by $\tilde{\mu}$, then projects to $\mathfrak{h}$. By construction μ is well defined, and has the desired properties. $\square$

We can now define, at least locally, invariant functions on the set of loop-regular elements in $\widetilde{L\mathfrak{g}}$. We simply use the map μ constructed above to gauge conjugate the loop-regular (ϕ,p) to (h,p) in the Cartan subalgebra $\mathfrak{h}\times\mathbb{C} \subset \widetilde{L\mathfrak{g}}$, and then evaluate $\alpha(h)$, for roots $\alpha \in \Delta$. In particular we use the simple roots:

Definition 5.2.1. For $j = 1,\ldots,n$, define $I_j\colon U \to \mathbb{R}\colon (\phi,p) \mapsto \mathrm{Re}[\alpha_j(\mu(\phi,p))]$.

The goal is to show that these functions generate a completely integrable system, when restricted to the Toda orbit. Notice that despite the local definition of these functions, the flows that they generate on the orbit are defined for all time. That is, they never leave the region of definition of the functions. This is because from the factorization theorem, the flows are effected by a co-adjoint action on the starting point, and our functions are defined on co-adjoint invariant neighborhoods.

Of course we must first show that they are even defined on the Toda orbit, by proving the existence of loop-regular elements there. We must also prove that when restricted to the orbit, the I_j are functionally independent. Poisson commutativity is guaranteed by the Kostant-Symes theorem, since by construction the functions are invariant on $L\mathfrak{g}$.

Once these things have been proven, we will also have shown that the globally defined functions obtained by composing class functions of **G** with the monodromy map, define a completely integrable system on the orbit. This follows because from the construction of μ, one sees that these global functions determine the values of the local I_j.

Before we go on to the proofs we present an example of what is involved in calculating the functions I_j.

Example 5.2.1. Take $\mathfrak{g} = sl(n+1,\mathbb{C})$. The typical form of an element in the orbit is given by (ϕ, i) where

$$\phi(z) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_{n+1}z^{-1} \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_n \\ a_{n+1}z & \cdots & 0 & a_n & b_{n+1} \end{pmatrix}$$

The monodromy equation is $\dot{h}(t)h(t)^{-1} = -i\phi(e^{it})$ with $h(0) = \mathrm{Id}$. Because ϕ is Hermitian, the solution $h(t)$ will be unitary, and in particular the monodromy $M(\phi, i) = h(2\pi)$ belongs to $SU(n+1)$, and so has modulus one eigenvalues: $\{e^{i\lambda_1}, \ldots, e^{i\lambda_{n+1}}\}$, where each λ_i belongs to $\mathbb{R}$. The regularity assumption means that these eigenvalues are distinct. The λ_i's are determined only up to a choice of their ordering, and up to addition of an integer multiple of 2π. (This corresponds to the action of the affine Weyl group on $\mathfrak{h} \times \mathbb{C}$). Making a choice of λ_i's, define the map μ so that it takes (ϕ, i) to $\mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1})$. Then the functions I_j corresponding to this choice of μ map $I_j : (\phi, i) \mapsto \lambda_j - \lambda_{j+1}$.

5.3. The gradient of I_j. We finish this section with a calculation of the gradients of the functions I_j. The following lemma will make the calculation relatively easy:

Lemma 5.3.1. *The gauge orbits are transverse to* $\mathfrak{h} \times \mathbb{C}$.

Proof. This follows from noting that

$$\left\langle\!\left\langle \tilde{ad}^*_\phi(h,p),(h',p')\right\rangle\!\right\rangle = \left\langle\!\left\langle (h,p),-\tilde{ad}_\phi(h',p')\right\rangle\!\right\rangle$$
$$= -\left\langle\!\left\langle (h,p),([\phi,h'],0)\right\rangle\!\right\rangle$$
$$= -\mathrm{Re}\langle h,[\phi,h']\rangle,$$

which is zero using the invariance of the Killing form, and the commutativity of $\mathfrak{h}$. But $\tilde{ad}^*_\phi(h,p)$ is tangent at (h,p) to the gauge orbit through (h,p), which proves the result. $\square$

For each $\alpha_j \in \Pi$ take $h_j \in \mathfrak{h}$ to be the unique element satisfying $\alpha_j(h) = \langle h,h_j\rangle$ for every $h \in \mathfrak{h}$. Note in particular that h_j is in the real span of the H_{α_i}'s. The gradients of the I_j's are now straightforward to calculate:

Proposition. *The gradient of I_j at $(\phi,p) \in U$ is given by*

$$\nabla I_j(\phi,p) = \tilde{Ad}_\gamma(h_j, -\frac{1}{p}\alpha_j(\mu(\phi,p))),$$

where $\gamma \in LG$ satisfies $(\phi,p) = \tilde{Ad}^*_\gamma(\mu(\phi,p),p)$.

Proof. We first suppose that $(h,p) \in U \cap (\mathfrak{h} \times \mathbb{C})$, with $\mu(h,p) = (h,p)$, and evaluate $\nabla I_j(h,p)$. By the lemma, the tangent space at (h,p) decomposes into an $\mathfrak{h} \times \mathbb{C}$ component, and a transversal component in the direction of the orbit through (h,p). If (ψ,q) is in the latter direction then $(\psi,q) = \tilde{ad}^*_\phi(h,p)$ for some $\phi \in L\mathfrak{g}$, and

$$\left\langle\!\left\langle (\psi,q),\nabla I_j(h,p)\right\rangle\!\right\rangle = \frac{d}{dt}\Big|_{t=0} I_j((h,p)+t\tilde{ad}^*_\phi(h,p)),$$

which is zero since I_j is invariant under the gauge action. For a vector (h',p') in the $\mathfrak{h} \times \mathbb{C}$ direction we get

$$\left\langle\!\left\langle (h',p'),\nabla I_j(h,p)\right\rangle\!\right\rangle = \frac{d}{dt}\Big|_{t=0} I_j((h,p)+t(h',p'))$$
$$= \frac{d}{dt}\Big|_{t=0}\mathrm{Re}\,\alpha_j(\mu(h+th',p+tp')).$$

Now $(h+th',p+tp')$ has monodromy $\exp 2\pi\frac{h+th'}{p+tp'}$, which is also the monodromy of $(p\frac{h+th'}{p+tp'},p)$. This element can be made arbitrarily close to (h,p) by taking t sufficiently

small, and so μ of it will just be $p\frac{h+th'}{p+tp'}$ for small t, since μ fixes a neighborhood of (h, p) in $U \cap (\mathfrak{h} \times \mathbb{C})$. Hence we get

$$\left\langle\!\left\langle (h', p'), \nabla I_j(h, p) \right\rangle\!\right\rangle = \frac{d}{dt}\Big|_{t=0} I_j((h, p) + t(h', p'))$$

$$= \operatorname{Re}\frac{d}{dt}\Big|_{t=0}\left[p\frac{\alpha(h) + t \cdot \alpha(h')}{p + tp'}\right]$$

$$= \operatorname{Re}\left\{-\frac{p'}{p}\alpha(h) + \alpha(h')\right\}$$

$$= \left\langle\!\left\langle (h', p'), (h_j, -\frac{1}{p}\alpha(h)) \right\rangle\!\right\rangle$$

Thus $\nabla I_j(h, p) = (h_j, -\frac{1}{p}\alpha(h))$. Now take a general $(\phi, p) \in U$, so there exists $\gamma \in LG$ such that $(\phi, p) = \tilde{\mathrm{Ad}}^*_\gamma(h, p)$ where $h = \mu(\phi, p)$. Then

$$\left\langle\!\left\langle (\psi, q), \nabla I_j(\phi, p) \right\rangle\!\right\rangle = \frac{d}{dt}\Big|_{t=0} I_j(\tilde{\mathrm{Ad}}^*_\gamma(h, p) + t(\psi, q))$$

$$= \frac{d}{dt}\Big|_{t=0} I_j((h, p) + t\tilde{\mathrm{Ad}}^*_{\gamma^{-1}}(\psi, q))$$

$$= \left\langle\!\left\langle \tilde{\mathrm{Ad}}^*_{\gamma^{-1}}(\psi, q), \nabla I_j(h, p) \right\rangle\!\right\rangle$$

$$= \left\langle\!\left\langle (\psi, q), \tilde{\mathrm{Ad}}_\gamma \nabla I_j(h, p) \right\rangle\!\right\rangle$$

And so $\nabla I_j(\phi, p) = \tilde{\mathrm{Ad}}_\gamma(h_j, -\frac{1}{p}\alpha_j(h))$ as claimed. $\square$

This calculation will be used in section 7 to show that the functions I_j, when restricted to the Toda orbit, are functionally independent there.

Notice that despite the simple form of ∇I_j, the behavior of $\exp t\nabla I_j$ is still unclear, and periodicity is by no means certain. This should be compared with Singer's work on the complexified non-periodic Toda lattice, where exp of the corresponding quantity *is* plainly periodic, and one gets complex action variables, [Si].

Another discrepancy with Singer's work is that commutativity with the Toda Hamiltonian also seems to break down. Consider $H : \widetilde{L\mathfrak{g}} \to \mathbb{R}$ defined by

$$H(\psi, p) = \frac{1}{2}\operatorname{Re}\langle\psi, \psi\rangle .$$

This function gives the usual Toda Hamiltonian when restricted to the Toda orbit, but since it is not invariant under the co-adjoint action, we can't use the Kostant-Symes theorem to deduce that it commutes with the functions I_j.

Instead we calculate the bracket directly. One can easily verify that the gradient of H is $\nabla H(\psi, q) = (\psi, 0)$. Applying the R-bracket formulation to the splitting of $\widetilde{L\mathfrak{g}} = (\tilde{\mathfrak{k}} + \mathbb{C}) + \tilde{\mathfrak{b}}$ we get

$$\left\{H, I_j\right\}_R(\phi, i) = -\left\langle\left\langle(\phi, i), [\nabla H, (\nabla I_j)_{\tilde{\mathfrak{b}}}]\right\rangle\right\rangle,$$

where we have used the invariance of I_j, and are projecting ∇I_j onto $\tilde{\mathfrak{b}}$ along $\tilde{\mathfrak{k}} + \mathbb{C}$. Plugging in our expressions for the gradients we get

$$\left\{H, I_j\right\}_R(\phi, i) = -\mathrm{Re}\left\{\langle\phi, [\phi, (\mathrm{Ad}_\gamma h_j)_{\tilde{\mathfrak{b}}}]\rangle + i\omega(\phi, (\mathrm{Ad}_\gamma h_j)_{\tilde{\mathfrak{b}}})\right\}$$
$$= -\mathrm{Re}\, i\omega(\phi, (\mathrm{Ad}_\gamma h_j)_{\tilde{\mathfrak{b}}})$$

Using the general form for a point (ϕ, i) in the Toda orbit, and a Fourier series expansion for $(\mathrm{Ad}_\gamma h_j)_{\tilde{\mathfrak{b}}}$, this further reduces to

$$\left\{H, I_j\right\}_R(\phi, i) = q_*\mathrm{Re}\langle E_{\alpha_*}, [(\mathrm{Ad}_\gamma h_j)_{\tilde{\mathfrak{b}}}]_1\rangle$$

where $[(\mathrm{Ad}_\gamma h_j)_{\tilde{\mathfrak{b}}}]_1$ denotes the coefficient of z in the Fourier series for $(\mathrm{Ad}_\gamma h_j)_{\tilde{\mathfrak{b}}}$. Unfortunately, it seems unlikely that this quantity is zero for each point in the phase space at which I_j is defined, (recall that γ is determined by (ϕ, i) via $(\phi, i) = \gamma \cdot (\mu, i)$). Thus our functions seem not to commute with the Toda Hamiltonian on the Toda phase space. Nevertheless, commutativity is certainly "close", suggesting that it may be possible to find some slightly perturbed Toda Hamiltonian which does commute.

For the present paper, we shall content ourselves with showing that the functions I_j do generate a completely integrable system on the Toda phase space.

6. Existence of Loop-regular Elements in the Toda Phase Space

We now show that there are loop-regular elements in the Toda phase space, so that it makes sense to restrict the functions $\{I_j\}$ there, and consider the flows they generate.

If we consider only elements $(\phi, p) \in \widetilde{L\mathfrak{g}}$ with $p = i$, we can regard the monodromy map as mapping $M : L\mathfrak{g} \to G : \phi \mapsto f(2\pi)$, where $f \colon \mathbb{R} \to G$ solves the equation $\dot{f}f^{-1} = \frac{1}{i}\phi(e^{it})$, and $f(0) = \mathrm{Id}$. This definition easily extends to any $\phi \in C^\infty(\mathbb{R}, \mathfrak{g})$. Given a smooth function $\phi \colon \mathbb{R} \to \mathfrak{g}$, define the "lift equation" associated to ϕ to be

$$\frac{df(t)}{dt}f(t)^{-1} = \frac{1}{i}\phi(t) , \quad f(0) = \mathrm{Id} ,$$

where $f \colon \mathbb{R} \to G$; that a unique solution exists follows from the smoothness of ϕ. Abusing notation, we define the "lift map" $M \colon C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ by $M(\phi) = f(2\pi)$. Thinking of $L\mathfrak{g}$ as sitting inside $C^\infty(\mathbb{R}, \mathfrak{g})$ as the subspace of smooth 2π-periodic functions, the lift map of an element $\phi \in L\mathfrak{g}$ is just the ordinary monodromy of $(\phi, i) \in \widetilde{L\mathfrak{g}}$.

Notice that if we consider the trivial principal bundle $G \to \mathbb{R} \times G \to \mathbb{R}$, (whose space of connections can be identified with $\Omega^1(\mathbb{R}) \otimes \mathfrak{g} \cong C^\infty(\mathbb{R}, \mathfrak{g})$), then the horizontal lift of the curve $c(t) \colon t \mapsto t$ through $(1, \mathrm{Id})$, determined by the connection $\frac{1}{i}\phi$, is given by $(c(t), f(t))$, where $f(t)$ solves the lift equation for ϕ. This interpretation shows that the solution $f(t)$ exists for all $t \in \mathbb{R}$.

Lemma 6.0.2. *If f solves the lift equation for $\phi \in C^\infty(\mathbb{R}, \mathfrak{g})$, and $\psi \in C^\infty(\mathbb{R}, \mathfrak{g})$, then*

$$M(\phi + \psi) = M(\phi)M(Ad_{f^{-1}}\psi) .$$

Proof. Suppose g solves the lift equation for $\mathrm{Ad}_{f^{-1}}\psi$. Then

$$\frac{d(fg)}{dt}(fg)^{-1} = \dot{f}f^{-1} + f\dot{g}g^{-1}f^{-1} = \frac{1}{i}(\phi + \psi) .$$

Thus $M(\phi + \psi) = f(2\pi)g(2\pi) = M(\phi)M(\mathrm{Ad}_{f^{-1}}\psi)$. $\square$

We can now prove the following proposition:

Proposition. *The set of loop-regular elements in the Toda phase space is non-empty.*

Proof. The idea is simple. First we construct a sequence of elements $\phi_s + \psi_s$ lying in the Toda phase space, and satisfying:

(1) $M(\phi_s) = M_0$ is constant, and regular in **G**.

(2) The solution f_s to the lift equation for ϕ_s, has image in the compact subgroup **K** $\subset$ **G**.

(3) $\psi_s \to 0$ as $s \to \infty$.

Properties (2) and (3) force $\mathrm{Ad}_{f_s^{-1}} \psi_s \to 0$ as $s \to \infty$, and so by continuity of M one has $M(\mathrm{Ad}_{f_s^{-1}} \psi_s) \to \mathrm{Id}$. (In this instance, rather than using the fact that M is a "smooth tame map" in the sense of Fréchet spaces, since we know f_s and ψ_s are bounded we can appeal to a more robust theorem from the standard theory of o.d.e's to deduce the desired continuity of M. See for example [J], section 2.3).

Applying lemma 6.0.2, and using property (1), we get $M(\phi_s + \psi_s) \to M_0$. Since M_0 is regular, and the set of regular elements in **G** is open, this implies that for sufficiently large s the element $M(\phi_s + \psi_s)$ will be regular, and hence $\phi_s + \psi_s$ will be loop-regular.

So the problem becomes one of finding a sequence $\phi_s + \psi_s$ satisfying (1), (2) and (3). To do this, we need a couple of simple lemmas.

Let $\{\epsilon_i : i = 1, \ldots, n\} \subset \mathfrak{h}$ be a dual basis to the set of simple roots, so that $\alpha_i(\epsilon_j) = \delta_{i,j}$ for all i, j, and set $H = 2\sum_i \epsilon_i$.

Lemma 6.0.3. $H = \sum_{\alpha > 0} H_\alpha$.

Proof. It suffices to show that $\alpha_i(\sum_{\alpha > 0} H_\alpha) = 2$, for each simple root α_i. This is equivalent to showing that $\langle \rho, \alpha_i \rangle = 1$, where $\rho = \frac{1}{2}\sum_{\alpha > 0} \alpha$. Let r_i denote the Weyl group reflection generated by α_i. Since $r_i(\alpha_i) = -\alpha_i$, and $r_i(\rho) = \rho - \alpha_i$, using invariance of the killing form we have:

$$\langle \rho, \alpha_i \rangle = \langle r_i(\rho), r_i(\alpha_i) \rangle = \langle \rho - \alpha_i, -\alpha_i \rangle = -\langle \rho, \alpha_i \rangle + \langle \alpha_i, \alpha_i \rangle$$

Since $\langle \alpha_i, \alpha_i \rangle = 2$, we get $\langle \rho, \alpha_i \rangle = 1$ as required. $\square$

Now $\{H_\alpha : \alpha \in \triangle\}$ forms a root system, with base given by the simple co-roots $\{H_{\alpha_i}\}$. Thus H can also be written as a positive integer linear combination of the

H_{α_i}, so we have three expressions for H:

$$H = 2\sum_{i=1}^{n} \epsilon_i = \sum_{\alpha>0} H_\alpha = \sum_{i=1}^{n} c_i H_{\alpha_i} \, ,$$

where the coefficients c_i belong to $\mathbb{Z}_{>0}$. Define

$$E = \sum_{i=1}^{n} \sqrt{c_i} E_{\alpha_i}, \quad F = \sum_{i=1}^{n} \sqrt{c_i} E_{-\alpha_i} \, .$$

Using the three expressions for H the following lemma is easy to verify:

Lemma 6.0.4. *The triple H, E, F generates an algebra isomorphic to $sl(2, \mathbb{C})$, with $[E, F] = H$, $[H, E] = 2E$, $[H, F] = -2F$, and so there exists $g \in G$ such that $E + F = Ad_g H$.*

We can now define the sequence $\phi_s + \psi_s$. Set $p(s) = \frac{1+2cs}{2c}$, where c is the Coxeter number of the system, and take

$$\phi_s = p(s)(E + F) = p(s) \sum_{i=1}^{n} \sqrt{c_i}(E_{\alpha_i} + E_{-\alpha_i}) \, ,$$

$$\psi_s = \frac{1}{\prod_{i=1}^{n}(p(s)\sqrt{c_i})^{k_i}}(E_{\alpha_*}z^{-1} + E_{-\alpha_*}z) \, ,$$

where the coefficients k_i come from the highest root $\alpha_* = \sum k_i \alpha_i$.

By construction $\phi_s + \psi_s$ belongs to the Toda phase space for each positive integer s, and $\psi_s \to 0$ as $s \to \infty$. The solution to the lift equation for ϕ_s is $f_s(t) = \exp \frac{t}{i}\phi_s$, and since $\frac{1}{i}\phi_s$ belongs to $\mathfrak{k}$, $f_s(t)$ lies in **K** for all t. Because $E + F = Ad_g H$, the monodromy of ϕ_s is

$$M(\phi_s) = \exp -2\pi i \phi_s = g \exp(-2\pi i \frac{1+2cs}{2c} H)g^{-1}$$

$$= g \exp(-2\pi i \frac{1}{2c}H)\exp(2\pi i H)^{-s} g^{-1} \, .$$

But $\exp(2\pi i H_\alpha) = \text{Id}$, (in the adjoint representation H_α is diagonal with integer entries), and so $\exp(2\pi i H) = \text{Id}$. One also has that for each positive root α,

$$\alpha(\frac{1}{2c}H) = \frac{1}{c}\alpha(\sum \epsilon_i) = \frac{1}{c}ht(\alpha) \in \mathbb{Q} \cap (0,1),$$

so that $\exp(-2\pi i \frac{1}{2c} H)$ is regular in $\mathbf{G}$. This implies that we have regularity of the monodromy $M(\phi_s) = g \exp(-2\pi i \frac{1}{2c} H) g^{-1} = M_0$. Thus the requirements (1), (2) and (3) are satisfied, and the proof is complete. $\square$

Example 6.0.1. In $sl(4, \mathbb{C})$ one finds:

$$\phi_s = \frac{1+8s}{8} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = \frac{1+8s}{8} g \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} g^{-1}$$

where $g \in O(4)$, and

$$\psi_s = \left(\frac{8}{1+8s}\right)^3 \begin{pmatrix} 0 & 0 & 0 & \frac{1}{6} z^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{6} z & 0 & 0 & 0 \end{pmatrix},$$

and the monodromy of ϕ_s is given by

$$M(\phi_s) = M_0 = g \begin{pmatrix} e^{-\frac{3\pi i}{4}} & 0 & 0 & 0 \\ 0 & e^{-\frac{\pi i}{4}} & 0 & 0 \\ 0 & 0 & e^{\frac{\pi i}{4}} & 0 \\ 0 & 0 & 0 & e^{\frac{3\pi i}{4}} \end{pmatrix} g^{-1} .$$

7. Independence of the variables

Independence of the functions I_j on the Toda orbit $\tilde{\mathcal{O}}_o = (\mathcal{O}_o, i)$ means linear independence of the differentials dI_j, at points where the functions are defined. This is equivalent to independence of the associated Hamiltonian vector fields, obtained using the symplectic structure of $\tilde{\mathcal{O}}_o$. These vector fields will be independent at a point, provided the flow through that point, generated by any non-trivial linear combination of the vector fields, is itself non-trivial. Hence to demonstrate the desired functional independence, we prove the following proposition:

Proposition. *Consider a (real) linear combination $I = \sum c_j I_j$. The flow generated by I through any loop-regular $(\phi, i) \in \tilde{\mathcal{O}}_o$ is trivial, if and only if all the c_j's are zero.*

Proof. If $I \equiv 0$, the associated flow is obviously trivial. Conversely, suppose the flow through some point (ϕ, i) is trivial. From the factorization theorem, the flow is given by

$$(\phi(t), i) = \tilde{\mathrm{Ad}}^*_{k(t)}(\phi, i) = \tilde{\mathrm{Ad}}^*_{b(t)}(\phi, i)\,,$$

where $k(t), b(t)$ solve the factorization problem $\exp t\nabla I(\phi, i) = k(t)^{-1}b(t)$, induced by $\widetilde{LG} = (\hat{K}\mathbb{R})\tilde{B}$, (see section 4.4). Triviality of the flow means for all times t:

$$\tilde{\mathrm{Ad}}^*_{b(t)}(\phi, i) = (\phi, i)\,.$$

We can write $(\phi, i) = \tilde{\mathrm{Ad}}^*_\gamma(\mu, i)$, where $\gamma \in \tilde{K}$ and $\mu \in \mathfrak{a}$, (see lemma 5.2.2), which implies that

$$\tilde{\mathrm{Ad}}^*_{\gamma^{-1}b(t)\gamma}(\mu, i) = (\mu, i).$$

Now for small times $\gamma^{-1}b(t)\gamma$ is close to the identity, and hence in the image of the exponential map. Write $\gamma^{-1}b(t)\gamma = \exp \nu(t)$, for some curve $\nu(t)$ in $L\mathfrak{g}$, and fix t for the time being. One has that $\tilde{\mathrm{Ad}}^*_{\exp s\nu}(\mu, i) = (\mu, i)$, for all s, and differentiating this with respect to s at $s = 0$ gives $\tilde{\mathrm{ad}}^*_\nu(\mu, i) = (0, 0)$.

Lemma 7.0.5. $\tilde{\mathrm{ad}}^*_\nu(\mu, i) = (0, 0) \implies \nu \in \mathfrak{h}$

Proof. From the formula for $\tilde{\mathrm{ad}}^*$, we have $\mathrm{ad}_\nu\mu + iD\nu = 0$. Write ν as a Fourier series $\nu = \sum \nu_k z^k$, and decompose each ν_k according to the root-space decomposition as

$\nu_k = \nu_k^0 + \sum_\Delta \nu_k^\alpha$. Using the fact that $\mu \in \mathfrak{h}$, this gives

$$\sum_{k,\alpha} \left[(\alpha(\mu) + k)\nu_k^\alpha + k\nu_k^0 \right] z^k = 0 \ .$$

Since by assumption μ is regular, $\alpha(\mu)$ is non-integer for every root α, hence $\nu_k^\alpha = 0$ for all k, α. It follows that $\nu = \nu_0^0$, which is in $\mathfrak{h}$, as required. $\square$

Hence $\gamma^{-1}b(t)\gamma = \exp\nu(t) \in \mathbf{H}$ for small t. For small times $b(t)$ is also in the image of the exponential map, and so can be written $b(t) = \exp\beta(t)$, for some curve $\beta(t) \in \tilde{\mathfrak{b}}$. It follows that $\beta(t) = \mathrm{Ad}_{\gamma^{-1}}\nu(t)$

Lemma 7.0.6. $\beta(t)$ is in $\mathfrak{a} \subset \mathfrak{h}$, and consequently $b(t)$ is a constant loop, (i.e. independent of z), in $A \subset H$.

Proof. Since ν is a constant loop independent of z, the equation $\beta = \mathrm{Ad}_{\gamma^{-1}}\nu$ implies that the eigenvalues of $\beta(z)$ are constant for $z \in S^1$, and are determined by $\nu \in \mathfrak{h}$. Because $\beta(z)$ is in $\tilde{\mathfrak{b}}$, the coefficients of the characteristic polynomial for $\beta(z)$ are holomorphic on $|z| \leq 1$. They have constant values on S^1, and hence are constant for all $|z| \leq 1$, and so the eigenvalues of $\beta(z)$ are constant.

But $\beta(0)$ belongs to $\mathfrak{a} + \mathfrak{n}_+$, and can be realised as an upper triangular matrix with real diagonal entries. In particular the eigenvalues of $\beta(0)$, and hence of $\beta(z)$, are real. As the eigenvalues of $\beta(z)$ are determined by ν, this shows that $\nu \in \mathfrak{a} \subset \mathfrak{h}$.

Since $\gamma \in \check{\mathbf{K}}$, this implies that $\mathrm{Ad}_{\gamma^{-1}}\nu \in i\tilde{\mathfrak{k}}$, and so $\beta(t,z) \in \tilde{\mathfrak{b}} \cap i\tilde{\mathfrak{k}} = \mathfrak{a}$, which completes the proof. $\square$

Thus $b(t) = \exp\beta(t)$ is a constant loop, (i.e. independent of z), and the triviality of the flow $\tilde{\mathrm{Ad}}^*_{b(t)}(\phi, i)$ implies that $\mathrm{Ad}_{\exp\beta(t)}\phi = \phi$. However, if one substitutes for ϕ the general form of a point in the Toda orbit, found in section 3.3, one sees that this last equality is only possible if $\alpha_i(\beta(t)) = 0$, for each simple root α_i. This forces $\beta(t) \equiv 0$, and so $b(t) \equiv \mathrm{Id}$ for small times.

We can now deduce that $I \equiv 0$. We have $\exp t\nabla I(\phi, i) = k(t) \in \check{\mathbf{K}\mathbb{R}}$, and so $\nabla I(\phi, i)$ is in the corresponding Lie algebra $\tilde{\mathfrak{k}} + \mathbb{C}$. But $\nabla I(\phi, i) = (\mathrm{Ad}_\gamma h, *)$ also belongs to $i\tilde{\mathfrak{k}} + \mathbb{C}$, as $\gamma \in \mathbf{K}$ and $h = \sum c_j h_j \in \mathfrak{a}$. Thus $\mathrm{Ad}_\gamma h \in \tilde{\mathfrak{k}} \cap i\tilde{\mathfrak{k}} = \{0\}$, hence h is zero, and so the coefficients c_j are zero, by independence of the elements h_j.

Thus a trivial flow is possible only if we take a trivial linear combination of the I_j's; a non-trivial combination will generate a non-trivial flow, hence the I_j are functionally independent as required. $\square$

Conclusion. This shows that, as we originally asserted, the local functions I_j generate a completely integrable system on the Toda phase space. Hence we have also achieved our goal of showing that the functions formed by composing the class functions of **G** with the monodromy map, (which locally generate the functions I_j), give a completely integrable system on the Toda phase space as well.

Quinn Malcom — "A new
completely integrable system
on the symmetric period
Toda latice phase space",
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REFERENCES

[FM] H. Flaschka and D. W. McLaughlin, *Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions*, Prog. Theoret. Phys., 55, 438–456, (1976).

[GW1] R. Goodman and N. R. Wallach, *Classical and quantum mechanical systems of Toda lattice type (II)*, Commun. Math. Phys., 94, 177–217, (1984).

[GW2] R. Goodman and N. R. Wallach, *Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle*, Jour. Reine Angew. Math., (1984), 69–133.

[Ha] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7 (1982), 65–222.

[J] F. John, *Ordinary Differential Equations*, Courant Institute of Mathematical Sciences, New York University, (1965).

[Ka] V. Kac, *Infinite Dimensional Lie Algebras*, Cambridge Univ. Press, Cambridge, (1990), 3rd edition.

[Ko] B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. Math., 39, 195–338, (1979).

[Kn] A. Knapp, *Representation Theory of Semisimple Groups; An Overview Based On Examples*, Princeton University Press, (1986).

[PS] A. Pressley and G. Segal, *Loop Groups*, Clarendon Press, Oxford, (1988).

[Pe] A. M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Vol. I, Birkhäuser, (1990).

[RS] A. G. Reyman and M. A. Semenov–Tian–Shansky, *Group–theoretical methods in the theory of finite–dimensional integrable systems*, Dynamical Systems VII, ed.'s V. I. Arnol'd and S. P. Novikov, Springer–Verlag, Berline, (1987).

[RSF] A. G. Reiman, M. A. Semenov-Tian-Shansky and I. E. Frenkel, *Graded Lie Algebras and Completely Integrable Dynamical Systems*, Soviet Math. Dokl., 20, (1979), 811–814.

[Si] S. Singer, Private communication, (1995).

[St] R. Steinberg, *Conjugacy Classes in Algebraic Groups*, Lecture Notes in Mathematics, 366, Springer-Verlag, (1974).

[Sy] W. Symes, *Systems of Toda type, inverse spectral problems and representation theory*, Invent. Math., 59, 13–53, (1980).