METHODS OF INTRODUCING FUNCTIONAL RELATIONS AUTOMATICALLY ON THE DIFFERENTIAL ANALYSER

By Joseph Guerrieri
B.S. Massachusetts Institute of Technology

A thesis submitted in partial fulfillment
of the requirements for a degree of

Master of Science
Massachusetts Institute of Technology
Department of Electrical Engineering

June 1932

Signed

Approved

Chairman of Graduate Committee

Acknowledgement

To Mr. S. H. Caldwell I am very much indebted for the suggestion of the topic of this thesis and for his valuable help thruout the course of the work. I wish also to express my gratitude to Dr. V. Bush for some important ideas which he contributed, to Mr. A. J. McLennan for his help with the details of some of the problems, and to the many other friends and acquaintances who have in any way assisted.

Contents

METHODS OF INTRODUCING FUNCTIONAL RELATIONS AUTOMATICALLY ON THE DIFFERENTIAL ANALYSER

I- The Problem

In one of the recent publications of the Massachusetts Institute of Technology[1] will be found a complete description of the differential analyser and its manner of operation. It will be assumed that the reader is acquainted with the contents thereof. In that bulletin is outlined a method of solution of differential equations in which functional relations are introduced into the machine by means of input tables which are not automatic in their operation and which require the services of an attendant for their use. That this method has proven to be entirely reliable and satisfactory is evidenced by the numerous successful results which it already has to its credit. But the suggestion arises that the process might possibly be made entirely automatic; that since the turning of a crank according to a specified motion is an operation which is essentially mechanical in nature, the machine should of itself be able to do this. It is the aim of this thesis to determine some of the means available for this purpose, particularly those

(1) See Bibliography

using no apparatus other than additional units of the types already existing. Altho originally it was intended to include a study of new mechanisms especially designed for these operations, the field soon broadened to such an extent that a separate investigation of this phase was thought advisable. Other workers are at present bending their energies towards successful completion of this other part of the problem, leaving to this paper a study of the interconnections of integrators, adders and shafts for producing the more common elementary functions.

II- Historical

Sir William Thomson in a paper presented before the Royal Society of London[1] in 1876 showed, in connection with a description of the integrating device, that by proper interconnection of parts it would theoretically solve any differential equation of any order and degree. His method differs from that in use at present in two respects; he proposed to introduce functional relations by means of cams and curved surfaces; also in solving equations his device could supposedly be driven equally well either as a differentiator or as an integrator, and was, therefore, less restricted in its choice of method, of interconnection. On the other hand, the torque amplifiers, which are so important for the operation of the present machine, impose the limitation that all relations must be introduced in the form of equations using integrals explicitly. The handicap is not as serious as it would at first appear, for, when the advantages of inherent accuracy are considered, and when it is realized that practical equations are seldom of a complicated nature (difficult as they may be of formal solution), it becomes evident that the drawback is somewhat academic.

But returning now to Lord Kelvin, one finds a sometimes useful suggestion in his original method for handling linear equations. It is applicable for second order equations only (altho in theory third and possibly higher order equations could be so treated). The theorem is this; that given a linear equation of the form

$$-----Q_3 \frac{d^3y}{dx^3} + Q_2 \frac{d^2y}{dx^2} + Q_1 \frac{dy}{dx} + Qy = Q_0 \qquad (1)$$

in which the Q's are functions of x only, one can throw this into the form

$$-----\frac{d}{dx} \frac{1}{P_2} \frac{d}{dx} \frac{1}{P_1} \frac{dy}{dx} = Py + P_0$$

by proper choice of an integrating factor. (For the second order equation the factor is $\frac{1}{Q}e^{\int \frac{Q_1}{Q_2} dx}$). Then the corresponding integral equation becomes

$$y = \int P_1\, dx \int P_2\, dx \int ----- \int (Py + P_0)\, dx$$

or writing $d(P_1')$ for $P_1\, dx$

$$y = \int d(P_1') \int d(P_2') \int ----- \int (y\, d(P') + dP_0')$$

The integrators are simply connected in series; the output of the first displaces the second, the output of the second displaces the third, and so on, while the disks are driven by shafts which are turned according to the functions P_1', P_2', etc. His suggestion of the use of properly designed cams for introducing these functions does not appear to be feasible.

Another method which he proposed to use on the equation (1) was to connect the group of integrators in series as before, but having all the disks driven by the independent variable, x. In this way he obtained $y = \int \frac{dy}{dx} dx$, $\frac{dy}{dx} = \int \frac{d^2y}{dx^2} dx$, etc. He then multiplied $\frac{dy}{dx}$ by the function, Q_1, $\frac{d^2y}{dx^2}$ by the function, Q_2, etc., and made other interconnections so as to satisfy the equation (1). But he did not show how the Q's were to be formed.

From that date until the recent attack of the problem by the Electrical Engineering Department of the Massachusetts Institute of Technology, no recorded work has been done in this field. As Kelvin rightfully pointed out, until the machine should be actually constructed, an investigation of the details connected with its operation would be premature. Of course there have been many calculating machines invented in that time concerning which much interesting information can be found in a number of seminar papers recently written,(3) but none have had sufficient resemblance to the Thomson integrator to be of any use in the present study.

III- In General

One of the possible ways of setting up the differential analyser for the solution of a single equation in two variables consists of interconnecting the machine so that the following relations obtain:

$$y = \int \frac{dy}{dx}\, dx$$

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\, dx$$

$$\frac{d^{n-1}y}{dx^{n-1}} = \int \frac{d^n y}{dx^n}\, dx$$

and $$\frac{d^n y}{dx^n} = f\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}, --- \frac{d^{n-1}y}{dx^{n-1}}\right)$$

When this method is used, the last two equations are usually combined into the single operation

$$\frac{d^{n-1}y}{dx^{n-1}} = \int f\left(x,\ y,\ \frac{dy}{dx}, --- \frac{d^{n-1}y}{dx^{n-1}}\right) dx$$

If the function, f(x, y, -------) can be produced automatically in any way whatever, the machine will, alone and without attention, grind out the desired answer.

The above is a more or less idealized case to illustrate the worst possible function that might arise. The system of connections actually used may be entirely different, being usually varied with the particular problem in hand to facilitate the solution.

It is with a function, $f(x, y, \frac{dy}{dx}, -----)$, then, that this thesis is concerned. If it is of a formal nature, the machine has within itself the means of producing it; if it is empirical, it may possibly be fitted by formal functions either as a whole or in sections; or, finally, it may be plotted as a series of curves, and some form of automatic curve tracer used to introduce them. The last method is a subject of a separate investigation.

The simplest operations that can occur in formal functions are addition and subtraction. Little comment need be made concerning them; the worst that can be said of the differential gears used for this purpose is that they are not easily corrected for backlash. Three front-lash units are required for complete correction, one for each of the three shafts connected to the unit.

Multiplication of two functions aan be done either with a multiplying board, or with integrators, but the former, not being automatic, is not a proper subject for discussion here. Furthermore, the latter is more accurate and is not subject to personal error. Under an integral sign the multiplication is done in the usual manner by successive integrations. That is, $\int f(x, y, \frac{dy}{dx}, --)\phi(x,y,--)dx$

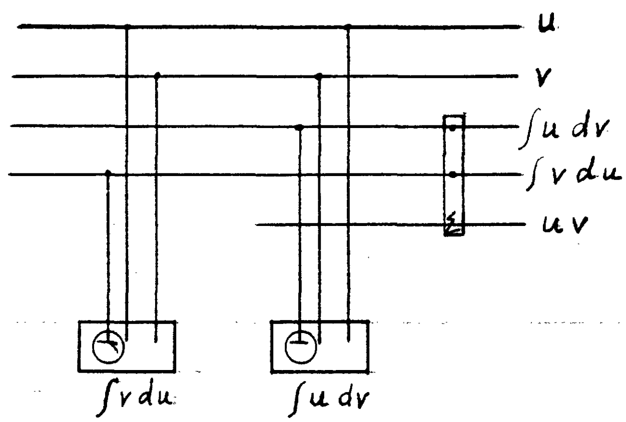

Fig. 1

Product of two functions

J. Guerrieri
June 1932

is the same as $\int f(x, y, \frac{dy}{dx}, ---)\ d(\int \phi(x, y, ---)\ dx)$. This method should be used whenever it is applicable. Otherwise the product can be obtained from (see fig. 1)

$$uv = \int u\ dv + \int v\ du$$

in which u and v are any functions. Quite frequently the two integrals of this last method already exist from their appearance elsewhere in the arrangement and need only to be added together in proper proportion to obtain the product.

To form a quotient four integrators are needed; of these, two go to make a reciprocal (by a method to be described presently) and the other two to multiply this reciprocal by the other function.

A few conventions to be used thruout the remainder of this discussion will now be given.

By the position of a shaft will be meant the value of the variable for that shaft, and the measure which determines this position is the angle thru which the shaft is moved from its assigned zero. Thus, if a shaft is labeled Ax (meaning that it makes A turns for unit change in x), then, by the position, x, of this shaft will be meant that it is Ax turns from its zero point.

By the displacement from zero of an integrator will be meant the position at that instant of the shaft driving the lead screw if the zero point for this shaft is chosen as that corresponding to a dead center position of the integrating wheel on its disk.

Unless otherwise noted the shaft labeled Ax will be the one independently driven.

The symbols n_1, n_2, n_3, ---- will be used for gear ratios. Thus a shaft labeled n Ax signifies that it is driven thru a gear ratio n from the Ax shaft.

By the origin for a shaft will be meant its zero position.

IV- Algebraic Polynomials

Now let it be required to obtain the square of some function, f(t). Let x be this function. Then the interconnection of the machine is wanted which will give x^2. Obviously, $x^2 = \int 2x\, dx$. The complete arrangement for doing this is shown diagrammatically in fig. 2. In the course of the solution Ax should make a large number of turns. A gear ratio, n_1, is chosen so that the maximum turns of the lead screw from zero does not exceed the allowable number of 40. The factor, $\frac{1}{32}$, comes from the design of the integrating unit. Thus is obtained

$$By = \int \frac{n_1 Ax}{32}\, d(Ax)$$

$$= \frac{n_1 A^2}{32} \int x\, dx$$

$$= \frac{n_1 A^2}{32} \left(\frac{x^2}{2} + c\right)$$

The constant of integration, c, is arbitrary. It can be made zero by defining the origin for the (By) shaft as its position when the integrator is at zero displacement.

A change of origins is all that is needed to produce

$$x^2 + ax + b$$

Thus let the origin for x be that at which the integra-

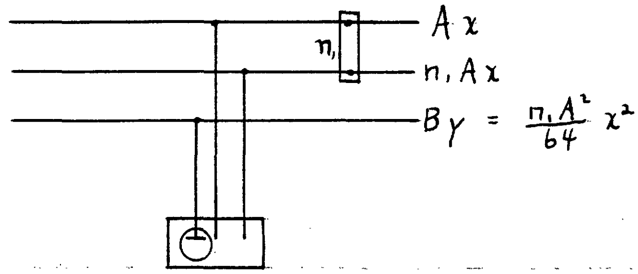

Fig 2

Arrangement for obtaining the square of a function x

Also for obtaining $x^2 + bx + c$

J. Guerrieri
June 1932

tor is displaced to x_0, and call the position of the By shaft at that instant, y_0. Then

$$By = \frac{1}{32}\int n_1 A(x + x_0)\, d(Ax)$$

$$= \frac{n_1 A^2}{32}\left(\frac{x^2}{2} + x_0 x + c\right)$$

$$= \frac{n_1 A^2}{64}\left(x^2 + 2x_0 x + y_0\right)$$

And the relations to be satisfied are

$$2x_0 = a$$

$$y_0 = b$$

$$n_1 A(x + x_0) < 40$$

If x^2 is integrated with respect to x, or if x is integrated with respect to x^2, x^3 results. The former arrangement will be used to obtain the general cubic

$$y = x^3 + bx^2 + cx + d$$

The first integration has already been shown to produce

$$\frac{n_1 A}{64}\left(x^2 + 2x_0 x + x_1\right)$$

The second integration is

$$By = \left(\frac{n_1 n_2 A^2}{(32)^2 (64)}\right)\int \left(x^2 + 2x_0 x + x_1\right) d(Ax)$$

$$= \left(\frac{n_1 n_2 A^3}{(32)^2 (64)}\right)\left(x^3 + 3x_0 x^2 + 3\, x x_0 + y_0\right)$$

in which x_0 and x_1, are the displacements of the two integrators when x is zero. The relations to be

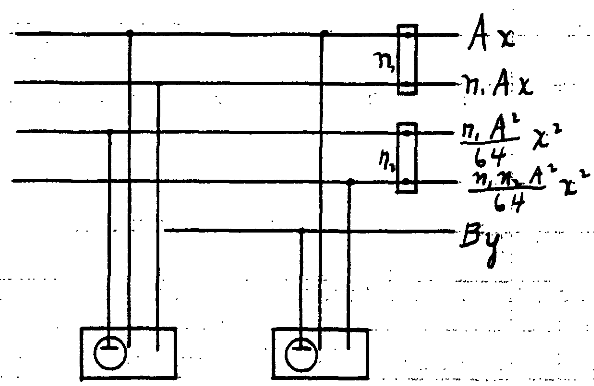

Fig. 3

Cubic $y = x^3 + bx^2 + cx + d$

J. Guerrieri
June 1932

satisfied are

$$3x_0 = b$$

$$3x_1 = c$$

$$y_0 = d$$

n $$n_1 A(x + x_0) < 40$$

$$\frac{n_1 n_2 A^2}{(64)}(x^2 + 2x_0 x + x_1) < 40$$

At first thought it would seem that the quartic could be formed with two integrators alone, but when the final relations are worked out, one finds that the coefficients of different terms are not independent. Consequently either an adder or another integrator is needed. The former will be used for the illustration.

The first integration is the same as for the preceding examples; the second is a repetition of the first. (fig. 4)

$$By = \int \frac{n_1 n_2 A^2}{(32)(64)}(x^2 + 2x_0 x + x_1) \, d\left(\frac{n_1 A^2}{64}(x^2 + 2x_0 x)\right)$$

$$= \frac{n_1^2 n_2 A^4}{(32)^2 (64)} \int (x^3 + 3x_0 x^2 + (x_1 + 2x_0^2)x + x_1 x_0) \, dx$$

$$= \frac{n_1^2 n_2 A^4}{(64)^3}(x^4 + 4x_0 x^3 + (2x_1 + 4x_0^2)x^2 + 4x_0 x_1 x + y_0)$$

To make the x term independent, add on $n_3 Ax$

$$By = \frac{n_1^2 n_2 A^4}{(64)^3}\left(x^4 + 4x_0 x^3 + (2x_1 + 4x_0^2)x^2 + \left(n_3 \frac{(64)^3}{n_1^2 n_2 A^3} + 4x_0 x_1\right)x + y_0\right)$$

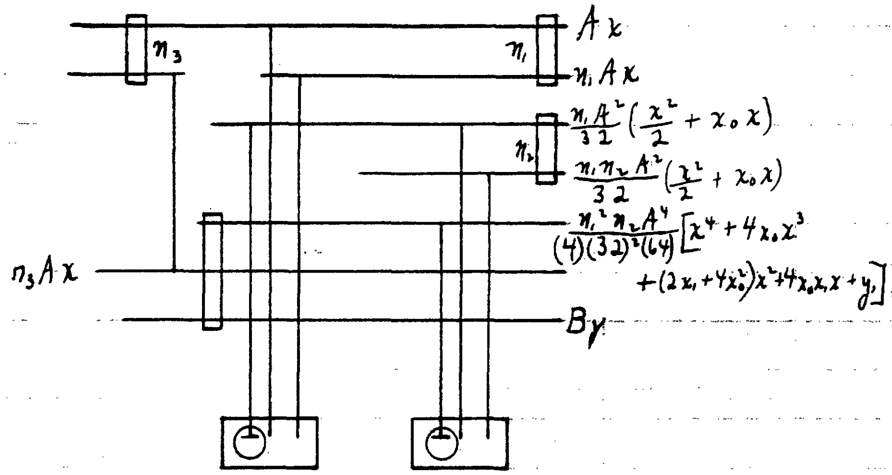

Fig. 4

Fourth power $y = x^4 + ax^3 + bx^2 + cx + d$

J. Guerrieri

June 1932

$$= k(x^4 + ax^3 + bx^2 + cx + d)$$

if

$$4x_0 = a$$

$$(2x_1 + 4x_0^2) = b$$

$$n_3 \frac{(64)^3}{n_1^2 n_2 A^3} + 4x_0 x_1 = c$$

$$y_0 = d$$

Gear ratios must conform to

$$n_1 A(x + x_0) < 40$$

$$\frac{n_1 n_2 A^2}{64} (x^2 + 2x_0 x + x_1) < 40$$

Continuing in this manner it is possible to obtain higher powers. In the appendix will be found derivations for the fifth and the sixth. In general it requires either an integrator or an adder to introduce one arbitrary constant, so that the least number of units is one less than the degree of the polynomial. Thus the fifth power would employ either three integrators and one adder or else four integrators.

V- Functions Convertible to Differential Equations

The functions sometimes encountered may be reducible to differential equations involving known operations. By reversing the procedure and solving the differential equation on the analyser, the required function may be obtained. The reciprocal will be used as an example, not only because of its simplicity, but also because of its importance from the standpoint of being the basis for division. A number of interesting points arise in connection with its set-up.

Let x be the function of which the reciprocal, y, is desired.

$$y = \frac{1}{x}$$

Differentiate with respect to x

$$\frac{dy}{dx} = -\frac{1}{x^2} = -y^2$$

This is a differential equation which involves the known operation of squaring a function. The solution, then, is

$$y = \int \frac{dy}{dx}\, dx$$

$$\frac{dy}{dx} = -y^2 = -\int 2y\, dy$$

The arrangement for these two equations is shown in fig. 5. As usual, one shaft is labeled Ax, and another is called

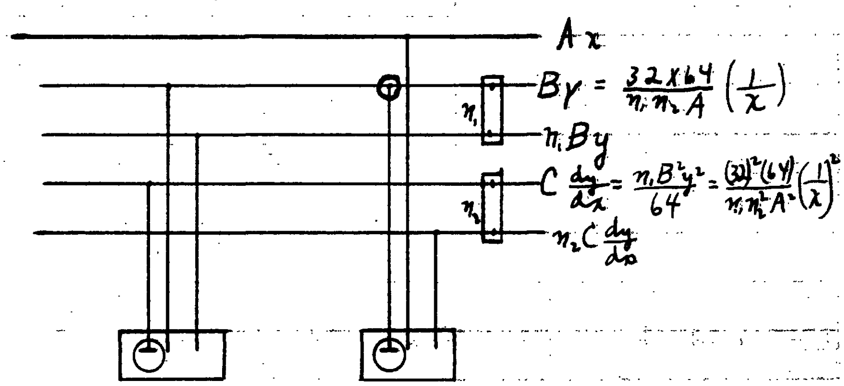

Fig 5

The reciprocal and the reciprocal squared

Also (with modifications) tanh
$(\text{sech})^2$

coth
$(\text{csch})^2$

cot
$(\text{cosec})^2$

tan
$(\text{sec})^2$

J. Guerrieri
June 1932

By. The second equation indicates that the square of y is $\frac{dy}{dx}$. Introduce the gear ratio, n_1, to allow for maximum travel of the lead screw.

$$C\frac{dy}{dx} = -2\frac{1}{32}\int(n_1 By)\ d(By)$$
$$= -2\frac{n_1 B^2}{32}y^2$$

A second gear ratio, n_2, is needed for the other integration.

$$By = \frac{1}{32}\int(n_2 C\frac{dy}{dx})\ d(Ax)$$
$$= -\frac{n_1 n_2 B^2 A}{(32)(64)}\int\frac{dy}{dx}\ dx$$

A negative connection is indicated altho in this case it turns out that a positive one would lead to the same result, for, if $y = -\frac{1}{x}$, then $\frac{dy}{dx} = \frac{1}{x^2} = y^2$.

Every solution of a differential equation has a closing equation for the constants, such as, in this example

$$B = \frac{n_1 n_2 B^2 A}{(32)(64)}$$

corresponding to the expression for the (n-1)st ~~nth~~ derivative

$$\frac{d^{n-1}y}{dx^{n-1}} = \int f(x,\ y,\ -----\ )dx$$

But this does not mean that B is always a function of A, as it sometimes happens that B drops out, for instance, when the equation is linear and has no term involving x alone, leaving to the initial setting the determination of this constant. When, however, as here, a relation does exist, B must be given its proper value in terms of A. Thus

$$B = \frac{(32)(64)}{n_1 n_2 A}$$

The limits of travel of the integrator impose the conditions

$$\frac{(32)(64)}{n_2 A x} < 40$$

$$\frac{(32)^2(64)}{n_1 n_2 A^2 x^2} < 40$$

The problem is now complete except for one detail; the initial setting must be made to conform to the equation of the desired function. Thus in the process of forming a squared term it was shown that the origin for y had to be that point at which the integrating wheel was at the center of the disk. But y and y^2 must pass thru zero simultaneously, so that both integrators in the present problem should start from zero displacement at the same instant. Of course when y is zero, x is infinite, and some other value must be used for determining x. A numerical example should make the method clear. Suppose

$y = 5$ is chosen for the setting; then, starting with the wheels at the centers of the disk, turn y to 5, turn y^2 to 25, interlock the machine and call the position of x $(=\frac{1}{y})$, $\frac{1}{5}$.

Unless these precautions are taken, some other function results, such as, for instance, the hyperbolic tangent or cotangent. The similarity of arrangements is very interesting and is well worth a word or two here. First note the following group of equations:

$$y = \pm a/x \qquad \frac{dy}{dx} = \pm ay^2$$

$$y = \tanh ax \qquad \frac{dy}{dx} = -a(y^2 - 1)$$

$$y = \coth ax \qquad \frac{dy}{dx} = -a(y^2 - 1)$$

$$y = \tan ax \qquad \frac{dy}{dx} = +a(y^2 + 1)$$

$$y = \cot ax \qquad \frac{dy}{dx} = -a(y^2 + 1)$$

Also compare the graphs (fig. 6).

Without going thru the derivation it will be said that the set-up, including the closing equations, are the same, differing only in the positions given to the units at the start and in the maximum travel allowed.

$$B = \frac{(32)(64)a}{n_1 n_2 A}$$

$$\frac{(32)(64)ay}{n_2 A} < 40$$

$$\frac{(32)^2(64)a^2}{n_1 n_2 A^2}(y^2 \pm 1) < 40$$

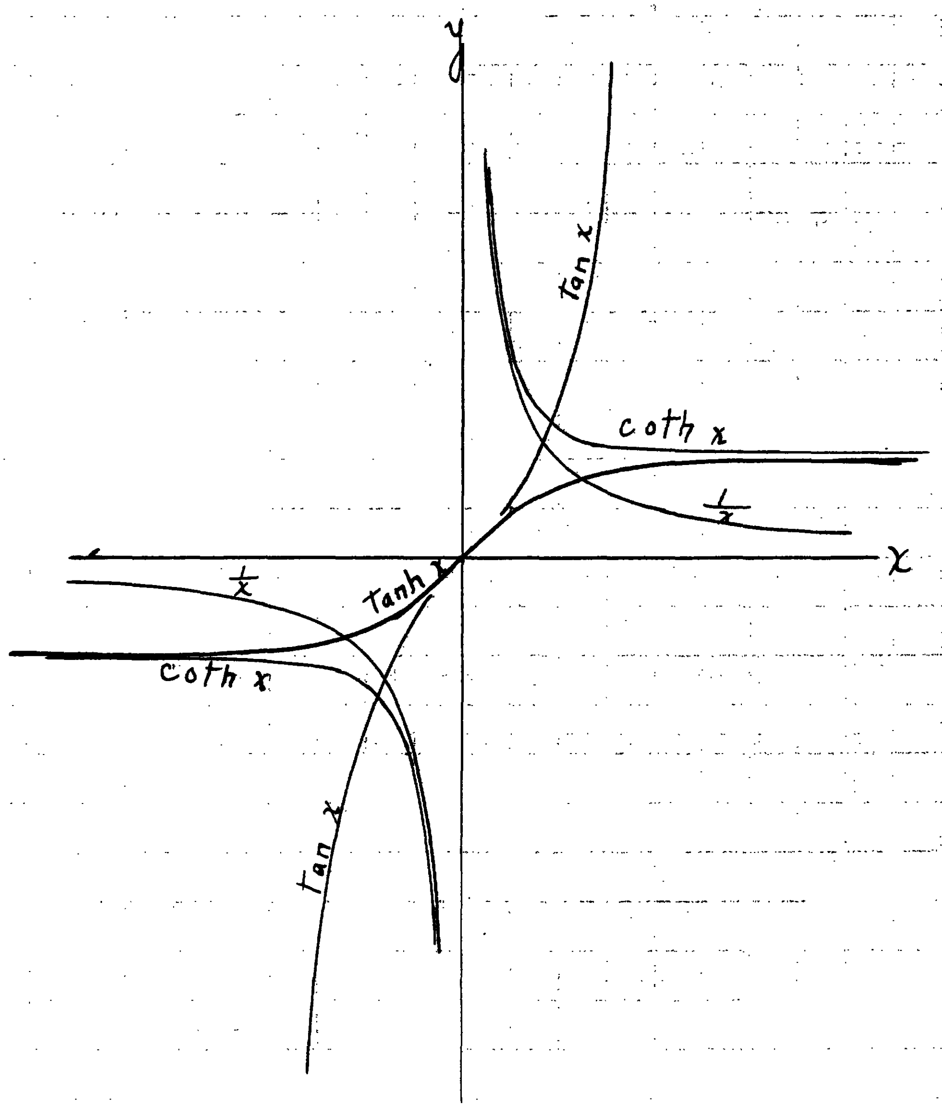

Fig. 6

J. Guerrieri
June 1932

But consider what happens when y is started at some value such that $1 > y > 0$, and $(y^2 - 1)$ is made the setting for the second integrator. Begin with x some positive value and increasing. $-\int(y^2 - 1)\,dx$ tends to increase y, but it cannot increase it above 1, because when $y = 1$, $d(\int(y^2 - 1)\,dx = 0$. The same applies when y approaches -1 . The function is tanh x .

If y is greater than one at the start, the result is the coth x, for, increasing x causes $-\int(y^2 - 1)\,dx$ to be negative, thereby decreasing y, but it cannot decrease below 1 because there $d(-\int(y^2 - 1)\,dx) = 0$.

A similar action occurs if y is less than (-1).

In a like manner the catangent is obtained if $(y^2 + 1)$ is the initial displacement of the second integrator.

The tangent is obtained if the connection is reversed at the circle (fig, 5), and $(y^2 + 1)$ is kept as the inital setting.

In the above four cases the $(y^2 \pm 1)$ shaft turns in accordance with the (hyperbolic secant)^2, the(hyperbolic cosecant)^2, the(cosecant)^2, and the (secant)^2, respectively.

To illustrate the manner of finding B when it is independent of A, consider the exponential,

$$y = a\, e^{\pm kx}$$

$$\frac{dy}{dx} = \pm ka\, e^{\pm kx} = \pm ky$$

By labeling shafts in the usual manner and introducing proper gear ratios, there results (see fig. 7)

$$By = \pm \frac{n_1 AB}{32} \int y\, dx$$

$$y = c_1 e^{\pm \frac{n_1 A}{32} x}$$

so that

$$\frac{n_1 A}{32} = k$$

Now if B is chosen arbitrarily, the initial setting must be made to correspond to it by giving the integrating wheel its proper position on the disk. The determining condition is that as the exponent of e goes from $-\infty$ to any value, x_1, y goes from zero to ae^{kx_1}. Hence $Bae^{\frac{n_1 A}{32}x_1}$ is the number of turns from zero (that is, from the position at which the wheel is in the center of the disk) of the By shaft. For any given starting position the number of turns which By makes during the solution is dependent on n_1, which in turn is related to A. For any given k, raising the value of A requires that n_1 be decreased, so that the turns of y is increased.

As an example let $k = 1$, $A = 1024$, then $n_1 = \frac{1}{32}$. Let the lead screw be 32 turns from zero when $x = 0$. Then, for $a = 1$, $B = 32(\frac{1}{n_1}) = 1024$. If a is to be

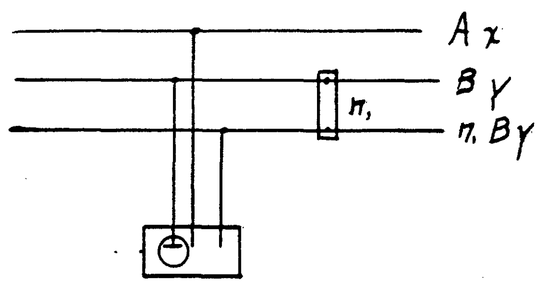

Fig 7

$$y = a\epsilon^{\pm kx}$$

J. Guerrieri
June 1932

changed to 2, B becomes 512 .

Notice that the x can be any function whatever, say, for instance, $-x^2$ in which case an integration gives the error function. (See also appendix, page 47)

A few other functions which were derived by converting them into differential equations will be found in the appendix. The list comprises

$y = k \ln x$	$\frac{dy}{dx} = k \frac{1}{x}$
$y = \begin{cases} \sin kx \\ \cos kx \end{cases}$	$\frac{d^2y}{dx^2} = - k^2 y$
$y = \begin{cases} \sinh kx \\ \cosh kx \end{cases}$	$\frac{d^2y}{dx^2} = + k^2 y$
$y = \sqrt{x}$	$\frac{dy}{dx} = \frac{1}{2y}$
$y = \sqrt[3]{x}$	$\frac{dy}{dx} = \frac{1}{3y^2}$
$y = \frac{1}{\sqrt{x}}$	$\frac{dy}{dx} = -\frac{y^3}{2}$
$y = \frac{1}{\sqrt[3]{x}}$	$\frac{dy}{dx} = -\frac{y^4}{3}$

VI- Functions from Certain Simple Integrations

In all the examples so far studied, the functions were given in advance, from which, by reasoning in terms of differential equations, the proper interconnections for the solutions were deduced. This method has the disadvantage that it does not always indicate the simplest way immediatdy. In this section a different plan will be followed; an equation involving a single integration will be written and solved, and an investigation will be made of the machine set-up only if the result is an interesting function. Actually, ane that has already proved to give a useful answer will be used, as there would be no point in burdening this thesis with the many attempts that failed to produce anything of interest. The particular operation to be considered is

$$y = \int x \, d(xy)$$

which leads to

$$y = \left(\frac{1}{1 - x^2}\right)^{\frac{1}{2}}$$

and by integration to $\sin^{-1}x$.

Let the shaft constants be introduced as before (fig. 8). Then

$$C\,xy = \frac{1}{32}\int(n_1 Ax)\, d(by) + \frac{1}{32}\int(n_2 By)\, d(Ax)$$

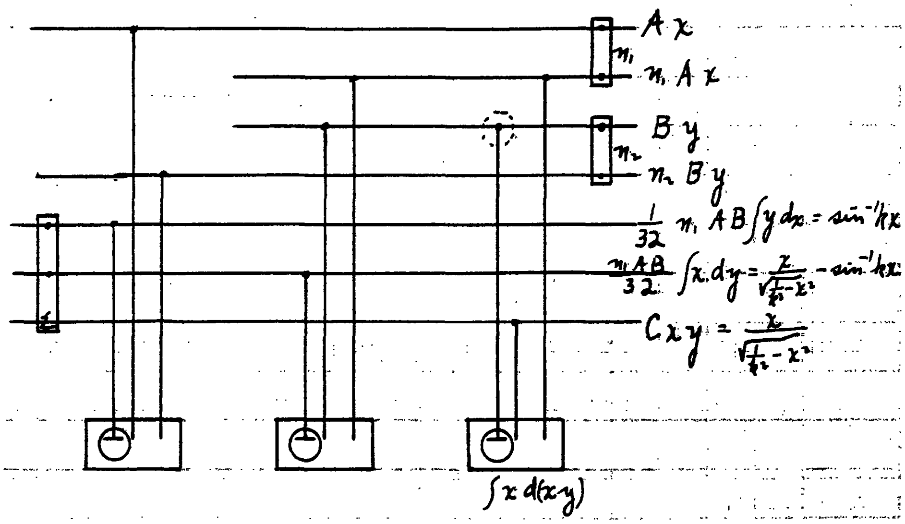

Fig. 8

$$y = \int x\, d(xy) \quad \text{producing} \quad \sin^{-1} kx$$

$$\frac{x}{\sqrt{\frac{1}{k^2} - x^2}}$$

$$\frac{1}{\sqrt{\frac{1}{k^2} - x^2}}$$

Also (reversed at circle) $\sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right)$

$$\frac{x}{\sqrt{1 + x^2}}$$

$$\frac{1}{\sqrt{1 + x^2}}$$

J. Guerrieri
June 1932

Make

$$n_1 = n_2$$

$$C = \frac{n_1 AB}{32}$$

and substituting these constants in their proper places

$$By = \frac{n_1^2 A^2 B}{(32)^2}\int x\ d(xy)$$

$$y = k^2\int x\ d(xy)$$

$$dy = k^2 x^2 dy + k^2 xy\ dx$$

$$y = \frac{1}{(1/k^2 - x^2)^{\frac{1}{2}}}$$

The inverse sine is already formed because one of the integrators is connected to produce $\int y\ dx$; likewise, from the other, $\int x\ dy = \frac{x}{(1/k^2 - x^2)^{\frac{1}{2}}} - \sin^{-1}kx$; the sum, (= xy) is $\frac{x}{(1/k^2 - x^2)^{\frac{1}{2}}}$. Multiplying the differential of this last function by x and integrating gives us $\frac{1}{(1/k^2 - x^2)^{\frac{1}{2}}}$ as it should.

As usual

$$n_1 Ax < 40$$

$$n_2 By < 40$$

But see how many integrators would be needed by the previous method. If $\frac{1}{(1 - x^2)^{\frac{1}{2}}}$ were made the starting point, four integrators would be required, plus one more for obtaining the inverse sin; if $y = \sin^{-1}x$, ($\frac{d^2y}{dx^2} = \frac{\sin y}{\cos^3 y}$ $= \sec^2 y \tan y$), again five integrators. Hence the new process is better from the standpoint of economy of units. However, it may happen that the intermediate functions are wanted at the same time, in which case the longer method may be preferred.

All that was said of the reciprocal concerning the necessity for proper initial setting applies here also. Thus one improper setting might result in

$$y = k\int (x + c)\, d(xy)$$

and in that way the following functions would be obtained

if $-(c^2/4) < k$

and $(x + c/2)^2 > k + c^2/4$

$$y = \left\{\frac{1}{(x + c/2)^2 - (k + c^2/4)}\right\}^{\frac{1}{2}} \left\{\frac{(x + c/2) + \sqrt{k + c^2/4}}{(x + c/2) - \sqrt{k + c^2/4}}\right\}^{\frac{c}{4}\frac{1}{\sqrt{k + c^2/4}}}$$

$(x + c/2)^2 < k + c^2/4$

$$y = \left\{\frac{1}{k + c^2/4 - (x + c/2)^2}\right\}^{\frac{1}{2}} \left\{\frac{\sqrt{k + c^2/4} + (x + c/2)}{\sqrt{k + c^2/4} - (x + c/2)}\right\}^{\frac{c}{4}\frac{1}{\sqrt{k + c^2/4}}}$$

if $k < -(c^2/4)$

$$y = \left(\frac{1}{(x + c/2)^2 + k + c^2/4}\right)^{\frac{1}{2}} e^{\frac{c}{2}\frac{1}{\sqrt{-(k + c^2/4)}} \tan^{-1}\frac{x + c/2}{\sqrt{-k - c^2/4}}}$$

if $k = -(c^2/4)$

$$y + \frac{1}{(x + c/2)}\, e^{\frac{c}{2}\frac{1}{x + c/2}}$$

But there are some new peculiarities arising in the correct arrangement which have not yet been discussed. Altho all the functions are continuous thru zero to - 1 , the machine is not capable of passing thru to the other side because of the loss of drive when the wheels are at the centers of the disks.

At $x = 1$ there is the singularity $\frac{dy}{dx} = \infty$, that can be approached only insofar as the limits of speed of the shafts permit. The back coupling of an integrator on itself thru an adder (which is essentially what we have here) seems to be a connection that must be treated with caution.

Fig. 9 is a skeletonized diagram of a part of fig. 8, which, by the elimination of one of the integrators, reduces to fig. 10 . If the rationof output to input of the remaining integrator is small enough, no trouble is experienced because the direction of motion of u is the same as the direction of torque applied to it by x . If the ratio exceeds a certain amount, however, the calculated direction of u is negative. If the integrator were reversible, u would be driven by it thru v, but the torque amplifiers do not permit this. It was thought advisable to actually set up fig. 10 in the laboratory,

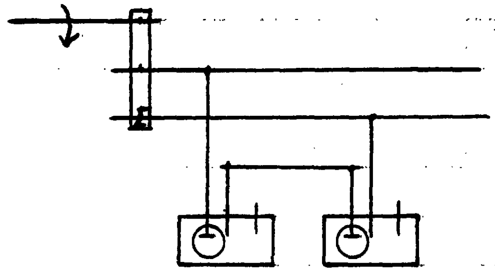

Fig. 9

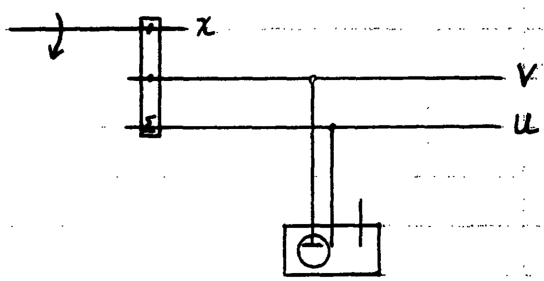

Fig. 10

Integrating wheel backcoupled thru an adder to its own driving disk

J. Guerrieri
June 1932

leaving x free to be turned by hand lest trouble should arise, to see exactly what would happen. The surprising result was that a positive torque applied to x caused it, (x), to move in a negative direction.

Thus we see that if y were given a position corresponding to $y = \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ (fig. 8), it would be necessary to devise a means of placing a negative torque on the xy shaft, or else return to the five integrator method of obtaining this function.

$\sinh^{-1} x = \ln(x + \sqrt{1 + x^2})$, however, does not cause trouble, requiring only that the connections be reversed at the dotted circle.

VII - Combinations Possible with a Single Integrator

A systematic study can be made of one- and two-integrator combinations, But with three integrators the number of arrangements possible is too large to be treated completely. Even with two integrators the number is exceedingly great, so much so that the writer has had time to investigate only a few of them.

With one integrator and one adder there are only six possible ways of interconnecting the shafts to produce functions, for, one shaft must be driven by the independent source of power and can be driven in no other way, a second must be driven by the output of the integrator, and a third may be driven by the first two thru an adder, but the lead screw can be driven from any one of these shafts while the disk can be driven from either the first or the third, thus allowing for six arrangements. They are

(1) x driving both disk and lead screw (fig. 2)

$$y = x^2 + bx + c$$

(2) x driving the disk, y driving the lead screw

$$y = a e^{\pm kx} \qquad \text{(fig. 7)}$$

(3) x driving the disk

(x + y) driving the lead screw (fig. 11)

$$y = \int (ax + by + c)\, dx$$

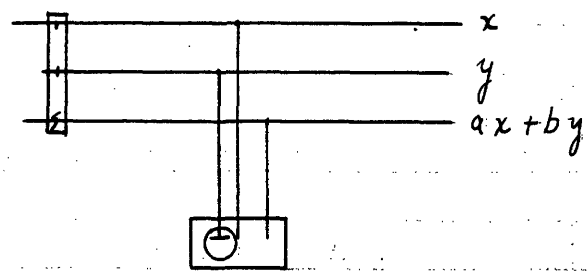

Fig 11

Skeletonized diagram of

$$y = \int (ax + by + c)\,dx$$

J. Guerrieri

June 1932

$$y = c_1 e^{bx} - (a/b)x - (a/b^2 + c/b)$$

$$(ax + by) = bc_1 e^{bx} - (a/b)$$

(4) (ax + by) driving the disk

x driving the lead screw (fig. 12)

$$y = \int (x + k)\, d(ax + by)$$

$$y = -(a/b)x - (a/b^2)\ln(x + k - 1/b) c_1 \qquad (b < 0)$$

Notice that this, like the two that follow, is one of the troublesome backcouplings, workable only for limited ranges of constants, unless the drive is shifted to the y shaft. An example in which this last is done will be found in a coming section.

(5) (ax + by) driving the disk

y driving the lead screw (diagram not given)

$$y = \int (y + k)\, d(ax + by)$$

$$x = c_1 + (1/a)\ln(y + k) - (b/a)y \qquad (b<0)$$

An explicit function of x cannot be obtained. Also see note under (4).

(6) (ax + by) driving the disk

(ax + by) driving the lead screw (fig. 13)

$$y = \int (ax + by)\, d(ax + by) \qquad (b<0)$$

y is of the form $a_1 x \pm \sqrt{b_1 + c_1 x}$

Also see note under (4)

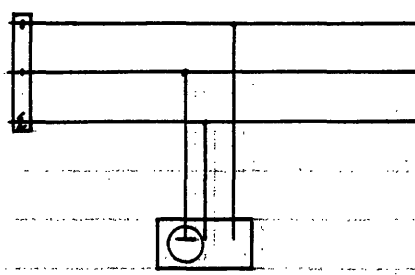

Fig 12

Skeletonized diagram

$$y = \int (x + k)\, d(ax + by)$$

$$= -\frac{a}{b} x - \frac{a}{b^2} \ln\left(x + k - \frac{1}{b}\right)$$

$$ax + by = -\frac{a}{b} \ln\left(x + k - \frac{1}{b}\right)$$

J. Guerrieri

June 1932

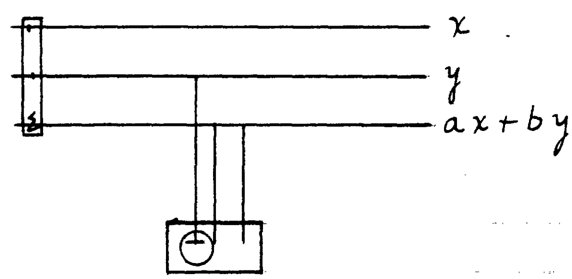

Fig. 13

Skeletonized diagram

$$y = \int (ax + by)\, d(ax + by)$$

J. Guerrieri
June 1932

Some of the results of a similar study of two-integrator arrahgements will be found in the appendix. The writer has not covered the fbeld completely, and it was deemed best, because of the limited time available, to concentrate on that portion of it which was most likely to prove fruitful, the part invalving backcoupling of integrators either directly or indirectly thru adders.

VIII- Combining of Functions - Artifices

It should be apparent by now that a wide variety of functions can be formed even with a small number of integrating units. Furthermore, any two of the arrangements already given can be combined to produce, tho possibly not in the simplest way, other mathematical expressions of a useful nature, and these in turn can go to form others. Exactly what the limitation of this process is remains for experience to tell; it may be a loss of accuracy due to multiplication of error in the individual units: it may be mechanical difficulties: it may be a practical consideration, such as the requiring of more time to arrange than would the plotting and manual introduction of the function; or it may be some other reason not yet apparent. Whatever it is, it will not be effective in the simpler cases.

One suggested example deserves special mention; it is the formation of x^k from e^{kx} and ln x, thus

$$x^k = e^{k \ln x} \quad \text{(fig. 14)}$$

An artifice that may at times be used to advantage is to have the independent variable shaft itself driven according to some function of another variable. For instance, suppose the $\sqrt{x + c}$ is needed as a factor.

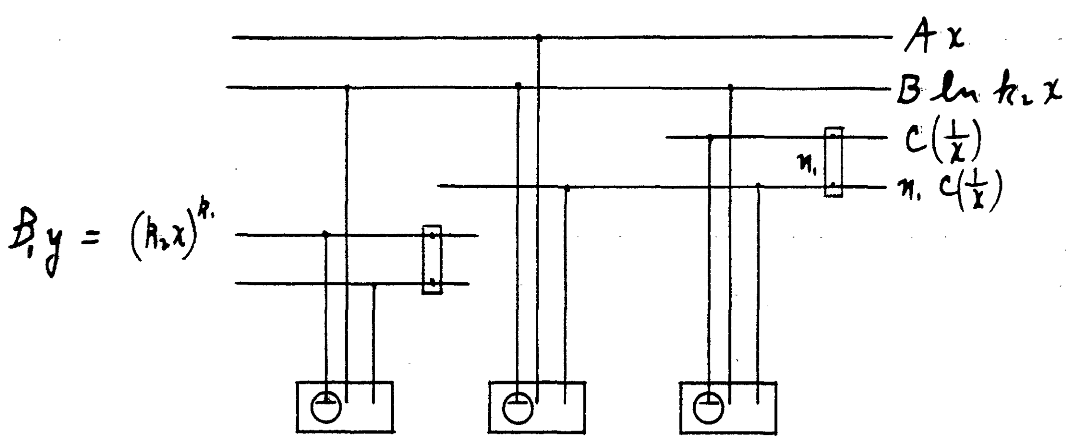

Fig. 14

$$y = e^{k_1 \ln k_2 x} = (k_2 x)^{k_1}$$

J. Guerrieri

June 1932

By ordinary methods three integrators would have to be used, but if x is driven by another shaft, u, so that $(x + c) = u^2$, the u shaft may be used for $\sqrt{x + c}$. A serious limitation is the power that the shafting requires, making much amplification of torque necessary, for, the entire machine will have to be driven by the very small force that the integrating wheel exerts.

Finally it must not be forgotten that under an integral sign it is easier to introduce a function inthe form of a differential, if the transformation can be readily made, rather than go thru the process of multiplication that would otherwise be necessary. As a last example the interconnections for the equation

$$\frac{d^2y}{dx^2} = -\frac{x+2}{x}\,y$$

will be worked out.

Rewrite this in the form

$$y = \int dx \int -(1 + 2/x)\,y\,dx$$
$$= \int dx \int y\,d(-x - 2\ln x)$$

In the index of functions will be found $-(a/b)x - (a/b^2)\ln(x + k - 1/b)$ in which b is negative if x is the independent drive. However, it is possible to run the machine from the $-(a/b)x - (a/b^2)\ln(x + k - 1/b)$ shaft, in which case positive values of b are allowable. It remains to be seen whether a, b and k can be chosen to fit our problem. If so

$$k - (1/b^2) = 0$$

$$(a/b^2) = 2$$

$$(a/b) = 1$$

and obviously, $a = 1/2$, $b = 1/2$, $k = 2$

The range over which the answer is wanted is

$$2/65 < x < 4$$

and the boundary conditions are (being in this case maximum values also)

$$x = 4$$

$$y = 0.4$$

$$\frac{dy}{dx} = 1$$

The problem, then, is to determine the shaft constants A, B, C, D, and E (fig. 14), the gears, n_1, n_2, n_3, and n_4, and the settings for each of the three integrators. Since the available ratios are 1:2 , 2:3 , and 1:4, a choice of A some multiple of these factors, and such as to have Ax make several thousands turns in the course of the solution is preferred. As a tentative value let it be A = 1024. The upper half of fig. 15 represents the part of the machine that is introducing the function (-x -2ln x). From the adder is obtained

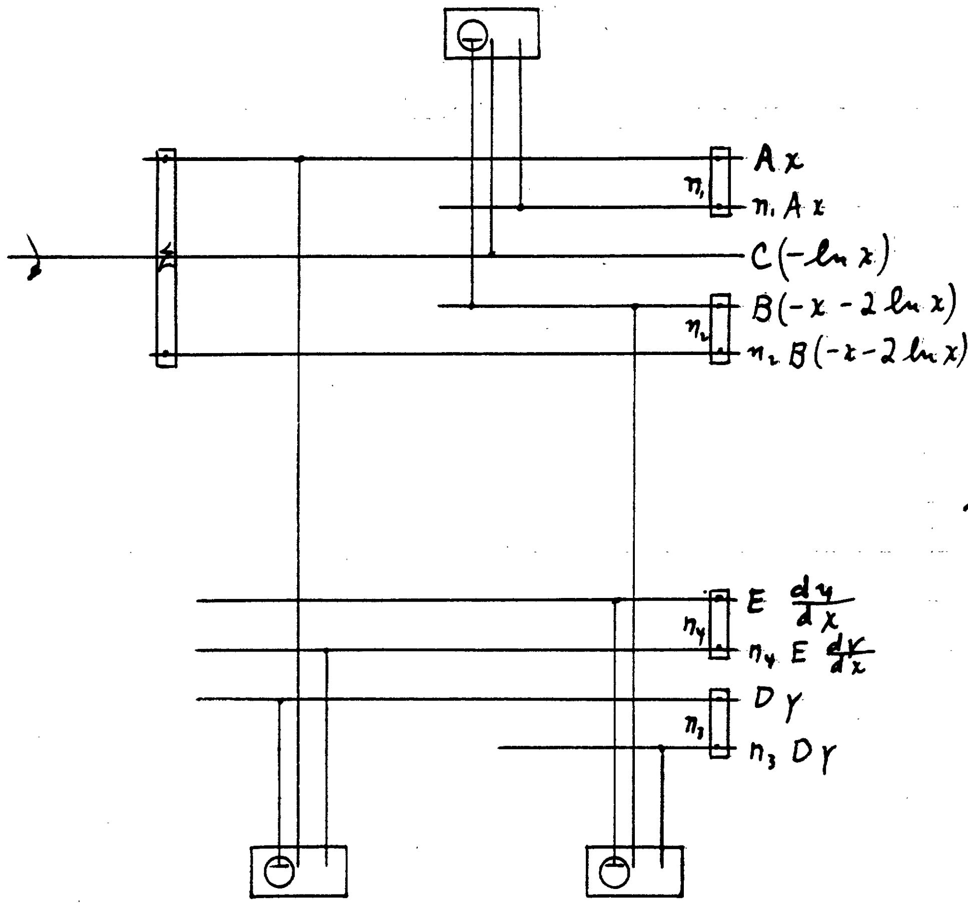

Fig. 15

Diagram for $\frac{d^2y}{dx^2} = -\left(1 + \frac{2}{x}\right) y$

J. Guerrieri
June 1932

$$Ax + n_1 B(-x - 2 \ln x) = C(-\ln x)$$

which can be so if

$$A = n_1 B = 1024,$$

$$2 n_1 B = C = 2048$$

The equation of the integrator is

$$B(-x - 2 \ln x) = \frac{1}{32} \int n_1 A(x + k)\, d(-C \ln x)$$

$$= \frac{n_1 (1024)(2048)}{32} \int (x + 2)\, d(-\ln x)$$

$$= n_1 (1024)(64) \int (-1 - 2/x)\, dx$$

$$= n_1 (1024)(64) \left((-x - 2 \ln x) \right)$$

so that

$$B = n_1 (1024)(64)$$

The third relation needed to determine these constants is

$$n_1 A(x + k) < 40$$

Then, $n_1 = \text{say, } \frac{1}{162} = \frac{1}{(9)(9)(2)}$

$$B = \frac{(1024)(32)}{81}$$

$$n_1 = \frac{81}{32}$$

For the determination of the origins, some point other than zero must be used. If it is $x = 1$, then, $\ln x = 0$, and $(-x - 2\ln x) = -1$. But notice that the integrator is started from $(x + 2) = 3$

There still remains to be found the constants and settings for the lower half of the figure, the method being not much different from that already used. From

$$E \frac{dy}{dx} = \frac{1}{32}\int n_3 Dy \; d(B(-x - 2 \ln x))$$

and

$$Dy = \frac{n_3 DB}{(32)}\frac{n_4 A}{(32)}\int \frac{dy}{dx}\, dx$$

comes the closing equation

$$D = \frac{n_3 n_4 A \; B}{(32)^2} D$$

$$n_3 n_4 B = 1$$

As before

$$n_3 Dy_m < 40$$

$$\frac{n_3 n_4 BD}{32}\left(\frac{dy}{dx}\right)_{max} < 40$$

Therefore

$$\frac{D}{32} < 40$$

$$D = \text{say, } (32)(36) = 1152$$

$$n_3 = \text{say, } \frac{9}{128}$$

$$n_4 = \frac{9}{256}$$

This completes the problem.

IX- Conclusion

The foregoing sections have attempted to set forth methods of attack rather than a wide variety of examples, but it is believed that a sufficient number of the latter have been obtained to be of considerable use in actual practise. Consequently an index has been compiled which includes most of the functions in this thesis, excluding only those arising from a change of drive to some shaft other than the independent variable. In the general equation

$$\frac{d^{n-1}y}{dx^{n-1}} = \int f(x,\ y,\ \frac{dy}{dx},\ \text{-----}\ \frac{d^{n-1}y}{dx^{n-1}})$$

if, say, $\frac{dy}{dx}$ is the variable of which some function is wanted, the drive cannot be changed, and thus is seen the limited applicability of such an artifice.

In general the following conclusions may be drawn :

(1) That many functions which are the solutions of the simpler differential equations can be introduced automatically by proper interconnection of the machine.

(2) That due to certain peculiarities of the mechanisms a few of these arrangements must be treated with caution.

(3) That the limitations of this process are mechanical in nature and should diminish with improve-

ments in technique.

(4) And finally, that in the simpler cases at least, a great advantage results if the functions are introduced automatically rather than manually.

APPENDIX I

Fifth and Sixth Degree Polynomials of a Single Variable

Fifth degree:

From the quadratic and the cubic (page 11) are obtained

$$\frac{n_1 A^2}{64}(x^2 + 2x_0 x + x_1)$$

and

$$\frac{n_1 n_2 A^3}{(32)(64)}(x^3 + 3x_0 x^2 + 3x_1 x + x_2)$$

Integrate the cubic with respect to the quadratic,

$$\frac{n_1^2 n_2 A^5}{(32)^2(64)}\frac{n_3}{(32)}\int\left(x^3 + 3x_0 x^2 + 3x_1 x + x_2\right)\left(x + x_0\right)dx$$

$$\frac{n_1^2 n_2 n_3 A^5}{(32)^3(64)}\left(\frac{x^5}{5} + \frac{4x_0 x^4}{4} + \frac{(3x_0^2 + 3x_1)x^3}{3} + \frac{(3x_1 x_0 + x_2)x^2}{2} + (x_2 x_0)x + \frac{y_0}{5}\right)$$

The x term is still dependent, so add $n_4 Ax$ to it. Then

$$By = \frac{n_1^2 n_2 n_3 A^5}{(5)(32)^3(64)}(x^5 + 5x_0 x^4 + 5(x_0^2 + x_1)x^3 + 2.5(x_0 x_1 + x_2)x^2 + (5x_0 x_2 + n_4 k^{-1})x + y_0)$$

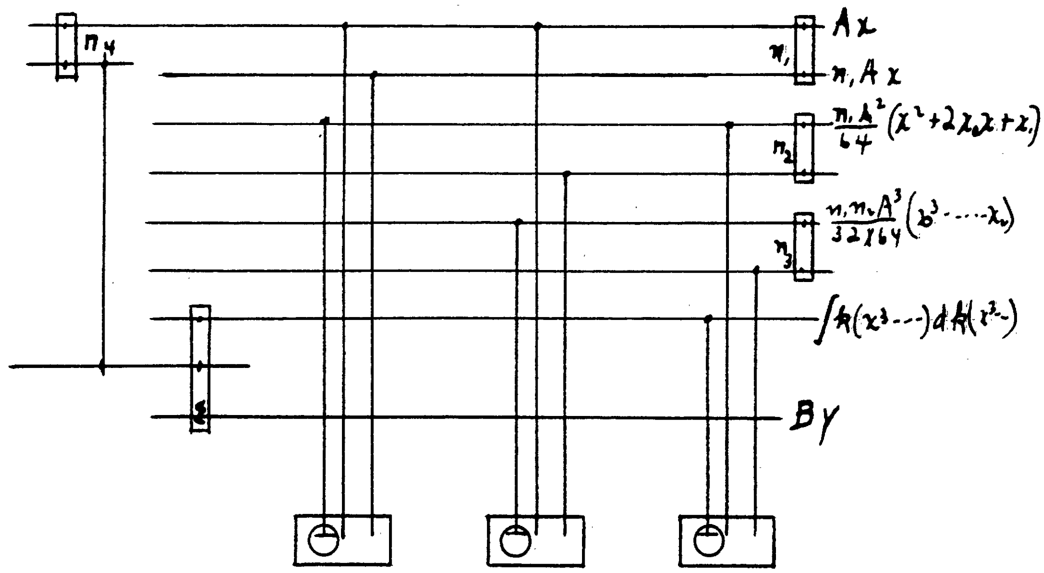

Fig. 16

Fifth Power $y = (x^5 + ax^4 + bx^3 + cx^2 + dx + e)$

J. Guerrieri

June 1932

$$= k(x^5 + ax^4 + bx^3 + cx^2 + dx + e)$$

if

$$5x_0 = a$$

$$5(x_0^2 + x_1) = b$$

$$2.5(3x_0x_1 + x_2) = c$$

$$5(x_0 x_2) + n_4 \frac{(32)^3 (64)(5)}{n_1^2 n_2 n_3 A^4} = d$$

$$y_0 = e$$

Also

$$n_1 A(x + x_0) < 40$$

$$\frac{n_1 n_2 A^2}{64}(x^2 + 2x_0 x + x_1) < 40$$

$$\frac{n_1 n_2 n_3 A^3}{(32)(64)} (x^3 + 3x_0 x^2 + 3x_1 x + x) < 40$$

Sixth degree:

Integrate the cubic with respect to itself,

$$\frac{n_1^2 n_2^2 n_3 A^6}{(32)^2(64)^2}\int(x^3 + 3x_0x^2 + 3x_1x + x_2)(3x^2 + 6x_0x + 3x_1)dx$$

$$= K\int(3x^5 + 15x_0x^4 + (18x_0^2 + 12x_1)x^3 + (27x_0x_1 + 3x_2)x^2 + (6x_0x_2 + 9x_1^2)x + 3x_1x_2)$$

$$= 0.5K(x^6 + 1.2x_0x^5 + (9x_0^2 + 6x_1)x^4 + (18x_0x_1 + 2x_2)x^3 + (6x_0x_2 + 9x_1^2)x + 6x_1x_2x + y_0)$$

The x^2 and x coefficients are still dependent, so add on

$$n_4\frac{n_1A^2}{64}(x^2 + 2x_0x)$$

and

$$n_5Ax$$

Then

$$1.2x_0 = a$$

$$9x_0^2 + 6x_1 = b$$

$$(18x_0x_1 + 2x_2) = c$$

$$6x_0x_2 + 9x_1^2 + n_4\frac{(32)^3(64)^2}{n_1^2n_2^2n_3A^5} = d$$

$$6x_1x_2 + n_4\frac{(32)^2(64)^2}{n_1n_2^2n_3A^4} + n_5\frac{(32)^3(64)^2}{n_1^2n_2^2n_3A^5} = e$$

$$y_0 = f$$

Also

$$n_1A(x + x_0) < 40$$

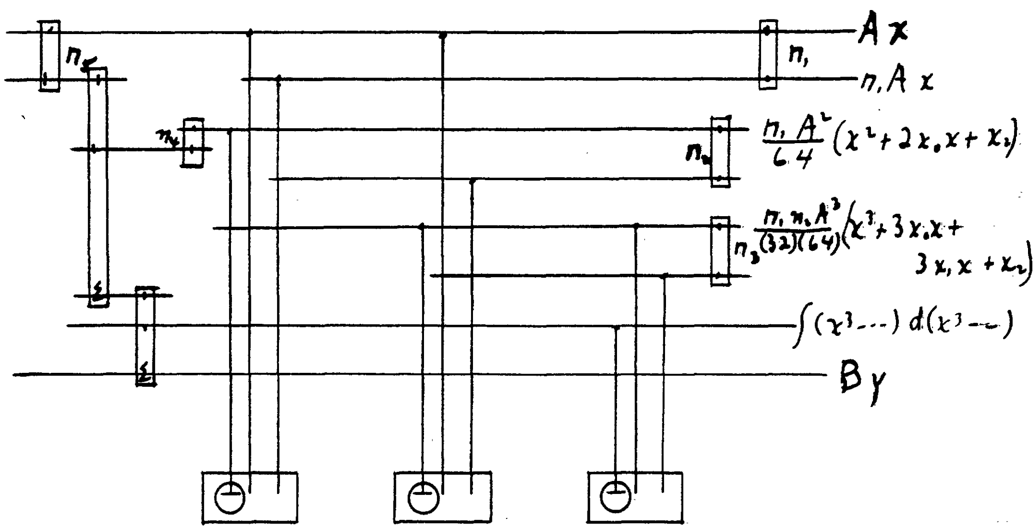

Fig. 17

Sixth Power $y = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

J Guerrieri
June 1932

$$\frac{n_1 n_2 A^2}{64}(x^2 + 2x_0 x + x_2) < 40$$

$$\frac{n_1 n_2 n_3 A^3}{(32)(64)}(x^3 + 3x_0 x^2 + 3x_1 x + x_2) < 40$$

APPENDIX II

A Few Functions Obtained by Means of Differential Equations

(1) The logarithm:

$$y = \ln x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Apparently an integration of the reciprocal gives ln x. (For ather methods using fewer units see index)

(2) $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{y}$$

Form the recproical of y and integrate it with respect to x. (fig. 18)

$$By = \frac{1}{32} \int \frac{(32)(64)A}{n_1 B} \frac{dx}{y}$$

$$B^2 = \frac{(64)A}{n_1}$$

$$B = 8\sqrt{\frac{A}{n_1}}$$

At the same time is formed

$$\frac{1}{y} = \frac{1}{\sqrt{x}}$$

$$\frac{1}{y^2} = \frac{1}{x}$$

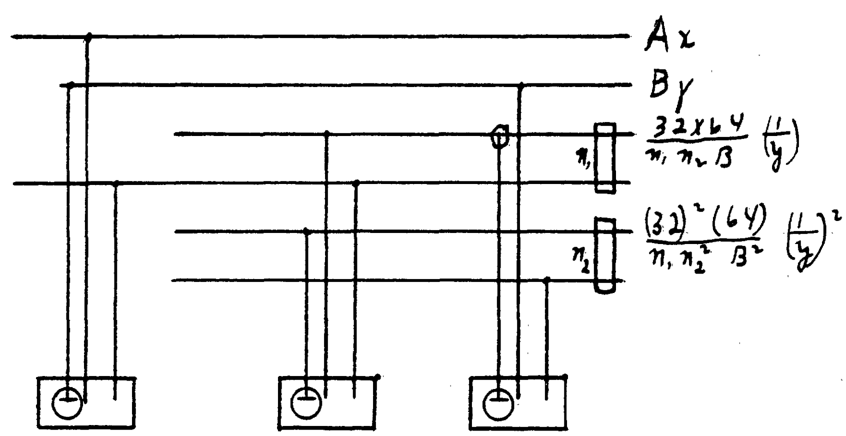

Fig. 18

$$y = \sqrt{x}$$

$$\frac{1}{y} = \frac{1}{\sqrt{x}}$$

$$\frac{1}{y^2} = \frac{1}{x}$$

J. Guerrieri

June 1932

(3) $y = \sqrt[3]{x}$

$$\frac{dy}{dx} = \frac{1}{3}\ \frac{1}{y^2}$$

In fig. 18 integrate $\frac{1}{y^2}$ instead of $\frac{1}{y}$

$$By = \frac{1}{32}\int \frac{(32)^2(64)\ A}{n_1 n_2 B^2}\ \frac{dx}{y^2}$$

$$B^3 = \frac{(32)(64)(3)\ A}{n_1 n_2}$$

$$B = 8\sqrt[3]{\frac{12A}{n_1 n_2}}$$

The other functions are

$$\frac{1}{y} = \frac{1}{\sqrt[3]{x}}$$

$$\frac{1}{y^2} = \frac{1}{\sqrt[3]{x^2}}$$

(4) $y = \frac{1}{\sqrt{x}}$. Of course this could be obtained from (2), but this other method gives different intermediate functions.

$$\frac{dy}{dx} = -\ \frac{y^3}{2}$$

Form y^3 and integrate with respect to x (fig. 19)

$$By = -\frac{1}{32}\int \frac{n_1 n_2 n_3 B^3 A}{(32)^3}\ \frac{y^3 dx}{(2)}$$

$$B = \frac{(32)^2}{\sqrt{n_1 n_2 n_3 A}}$$

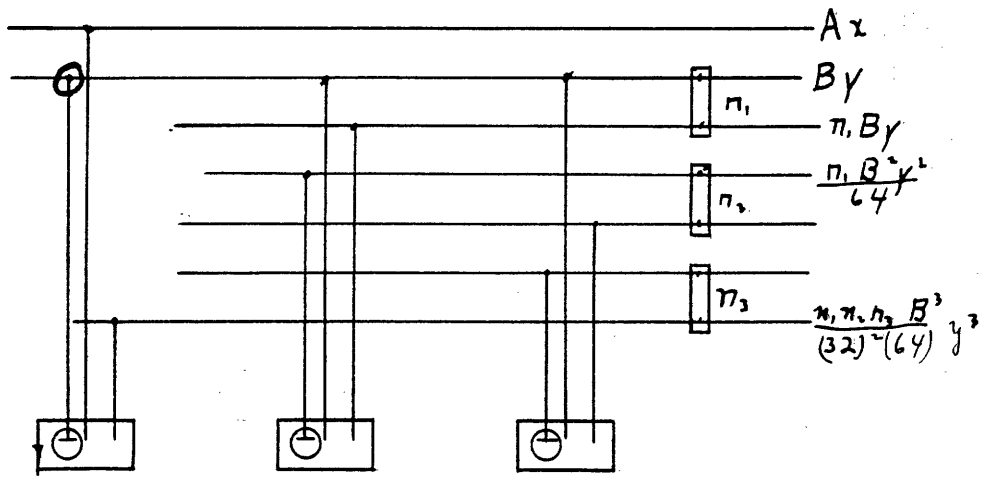

Fig. 19

$$y = \frac{1}{\sqrt{x}}$$

$$y^2 = \frac{1}{x}$$

$$y^3 = \frac{1}{\sqrt{x^3}}$$

J. Guerrieri

June 1932

And the other functions are

$$y^2 = \frac{1}{x}$$

$$y^3 = \frac{1}{\sqrt{x^3}}$$

(5) $y = \frac{1}{\sqrt[3]{x}}$

$$\frac{dy}{dx} = -\frac{1}{3}y^4$$

The method is similar to that of the preceding example (diagram not given)

$$By = -\frac{n_1^2 n_2 n_3 B^4 A}{(32)^3}\int \frac{y^4 dx}{(8)}$$

$$B = \frac{(64)}{\sqrt[3]{n_1^2 n_2 n_3 (3) A}}$$

The other functions are

$$y^2 = \frac{1}{\sqrt[3]{x^2}}$$

$$y^4 = \frac{1}{\sqrt[3]{x^4}}$$

(6) $y = kx \ln x$

$$\frac{dy}{dx} = k(1 + \ln x)$$

A simple integration with proper initial setting will produce this function. Any convenient method of forming ln x may be used. Then

$$y = k\int(\ln x + 1)dx$$

(7) $y = a \sin kx$, or

$y = b \cos kx$

$$\frac{d^2y}{dx^2} = - k^2 y$$

$$By = \frac{n^2 CA}{32} \int \frac{dy}{dx} dx$$

$$= - \frac{n_2 A}{32} \int \frac{dx}{32} \int n_1 BA\ y\ dx$$

Therefore

$$\frac{n_1 n_2 A^2}{(32)^2} = k^2$$

If $a = 1$, B is defined as the number of turns of the By shaft from zero when it is at its maximum. For any other value of a, change B to $\frac{1}{a}$ as much.

8) $y = a \cosh kx$, or

$y = b \sinh kx$

This is the same as for sin and cos except that the connection is reversed at the circle. For $a = 1$, B is the number of turns of the integrator from zero at its minimum position. For any other a, B is $\frac{1}{a}$ as much.

If n_1 and n_2 are equal, and $a = b$, then $C = B$. If $a \neq b$, $C = \frac{a}{b} B$

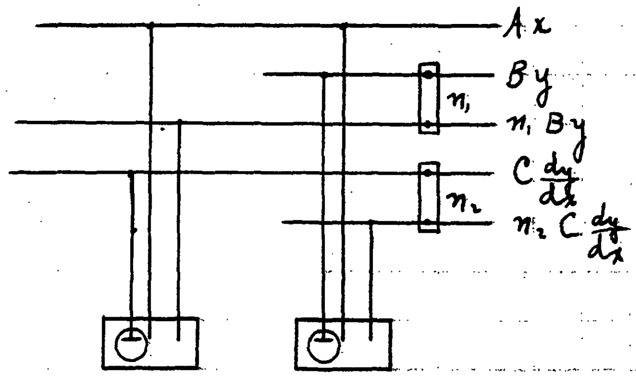

Fig 20

$$y = \begin{cases} a \sin kx \\ b \cos kx \end{cases}$$

J. Guerrieri
June 1932

(9) $y = \tan^{-1} x$

$$\frac{dy}{dx} = \frac{1}{x^2+1}$$

Form x^2 and then the reciprocal of (x^2+1) (fig. 21) (A simpler method **has** not been encountered alltho there may be one)

$$By = \frac{1}{32}\int \frac{(32)(64)^2 A}{n_1 n_3 A^2} \frac{dx}{(x^2+1)}$$

$$= B \tan^{-1} x$$

$$B = \frac{(64)^2}{n_1 n_3 A}$$

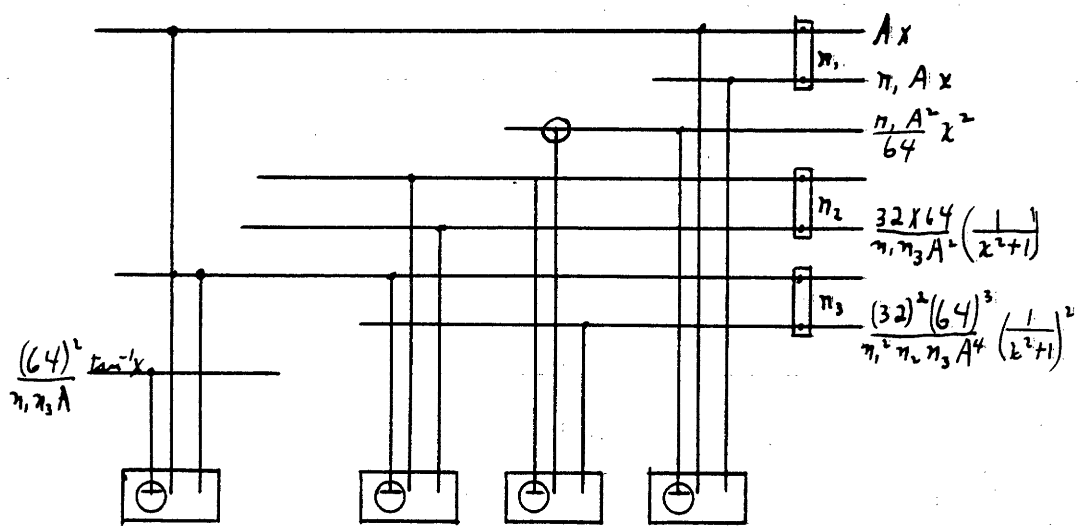

Fig. 21

$$y = \tan^{-1} x$$

J. Guerrieri
June 1932

APPENDIX III

A Few Functions Obtained from a Study of Combinations of Two Integrators

(The manner of obtaining these functions from their differential equations becomes apparent after they are discovered, but since they were originally encountered in the study of two-integrator combinations that method will be used for this exposition)

(1a) $y = (x + c)e^{kx}$

Ax driving the first disk

$n_1 By_1$ driving the first lead screw

By_1 driving the second disk

$n_2 Ax$ driving the second lead screw (fig. 22)

$$y_1 = e^{\frac{n_1 A}{32}x}$$

$$y_2 = \frac{1}{32}\int n_2 A(x + c)\,d\left(e^{\frac{n_1 A}{32}x}\right)$$

$$= \frac{n_1 A}{32}(x + c_1)e^{\frac{n_1 A}{32}x}$$

In a similar manner, with three integrators,

$$y = k\int (x^2 + bx + c)\,d(e^{kx})$$

$$= k(x^2 + b_1 x + c_1)e^{kx}$$

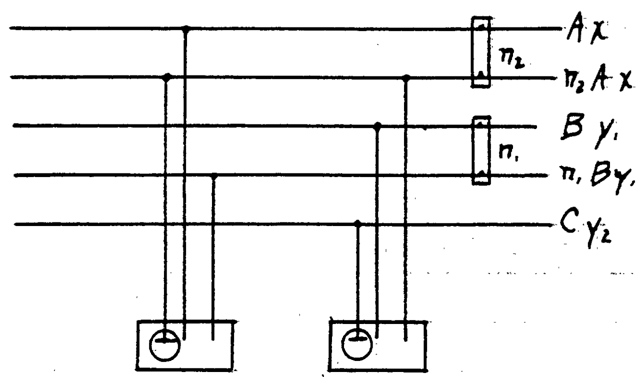

Fig. 22

$$y_2 = (x + c)\,\epsilon^{kx}$$

$$y_1 = \epsilon^{kx}$$

J. Guerrieri

June 1932

(11) $y_1 = \int (e^{-x^2}) dx$

$y = e^{-x^2}$

Ax driving the first disk

$n_2 C y_2$ driving the first lead screw

$B y_1$ driving the second disk

$n_1 A x$ driving the second lead screw (fig. 23)

$$By_1 = \frac{n_2 CA}{32} \int y_2 \, dx$$

$$Cy_2 = - \frac{n_1 AB}{32} \int x \, dy_1$$

$$dy_2 = - \frac{n_1 AB}{32C} x \, dy_1 = - \frac{n_1 A\not{B}}{(32)\not{C}} \frac{n_2 \not{C} A}{(32)\not{B}} x y_2 \, dx$$

$$y_2 = e^{-\frac{n_1 n_2}{2}\left(\frac{A}{32}\right)^2 x^2}$$

$$y_1 = \frac{n_2 CA}{32} \int y_2 \, dx$$

$$= \frac{n_2 CA}{32} \int \left(e^{-\frac{n_1 n_2}{2}\left(\frac{A}{32}\right)^2 x^2} \right) dx$$

It is now seen that this is the same as reversing the order of integration in the formation of the exponential of the square. Thus

$$y = \int y_2 \, d(x^2)$$

$$= \int y_2 \, d\left(\int 2x \, dx\right)$$

$$= \int y_2 (2x) \, dx$$

$$= \int (2x) \, d\left(\int y_2 \, dx\right)$$

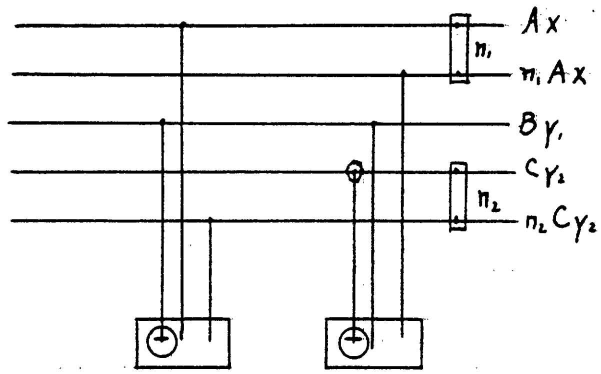

Fig. 23

$$y_1 = \int \epsilon^{-x^2}\, dx$$

$$y_2 = \epsilon^{-x^2}$$

J. Guerrieri

June 1932

(12) $y_1 = \ln kx$

$y_2 = \frac{1}{x}$

Ax driving the first disk

$n_1 C y_2$ driving the first lead screw

By driving the second disk

$n_1 C y_2$ driving the second lead screw (fig. 24)

By inspection it is seen that the second integrator forms the exponential of y_1.

$$y_2 = e^{-ky_1}$$

Then

$$By_1 = \frac{1}{32} \int n_1 C \, y_2 \, d(Ax)$$

$$B \, dy_1 = \frac{n_1 AC}{32} e^{-ky_1} dx$$

$$\frac{Be^{ky_1}}{k} = \frac{n_1 AC \, x}{32}$$

$$y_1 = \frac{1}{k} \ln\left(\frac{n_1 AC \, kx}{32 \, B}\right)$$

$$y_2 = e^{-\ln \frac{n_1 AC}{32 B} kx}$$

$$= \frac{32 \, B}{n_1 AC \, k} \frac{1}{x}$$

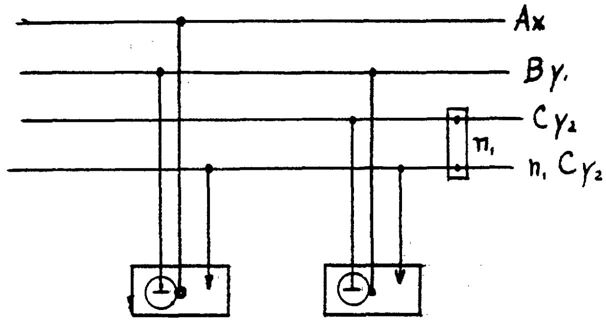

Fig. 24

$$y_1 = \ln k(x+c)$$

$$y_2 = \frac{1}{x+c}$$

J. Guerrieri
June 1932

(13) $y = a e^{-kx} \sin(bx + c)$

Ax driving the first disk

$n_1 Cy_2$ driving the first lead screw

Ax driving the second disk

$n_2(By_1 + Cy_2)$ driving the second lead screw

$$By_1 = \frac{n_1 CA}{32} \int y_2 \, dx$$

$$Cy_2 = - \frac{n_2 A}{32} \int (By_1 + Cy_2) \, dx$$

$$\frac{dy_2}{dx} = - \frac{n_2 A (By_1 + Cy_2)}{32C}$$

$$\frac{d^2y_2}{dx^2} = - \frac{n_2 A}{32C} \left(\frac{n_1 AC \, y_2}{32} + (C) \frac{dy_2}{dx} \right)$$

$$\frac{d^2y_2}{dx^2} + \frac{n_2 A}{32} \frac{dy_2}{dx} + \frac{n_1 n_2 A^2}{(32)^2} y_2 = 0$$

A very familiar formula. The imaginary roots are the only ones that are wanted since the real roots lead to two exponentials which are preferably made separately.

$$y_2 = a \, e^{-\frac{n_2 A}{2 \times 32} x} \sin\left(\sqrt{\frac{n_1 n_2 A}{(32)^2} - \left(\frac{n_2 A}{64}\right)^2} \; x + c \right)$$

y_1 = same with different a and c.

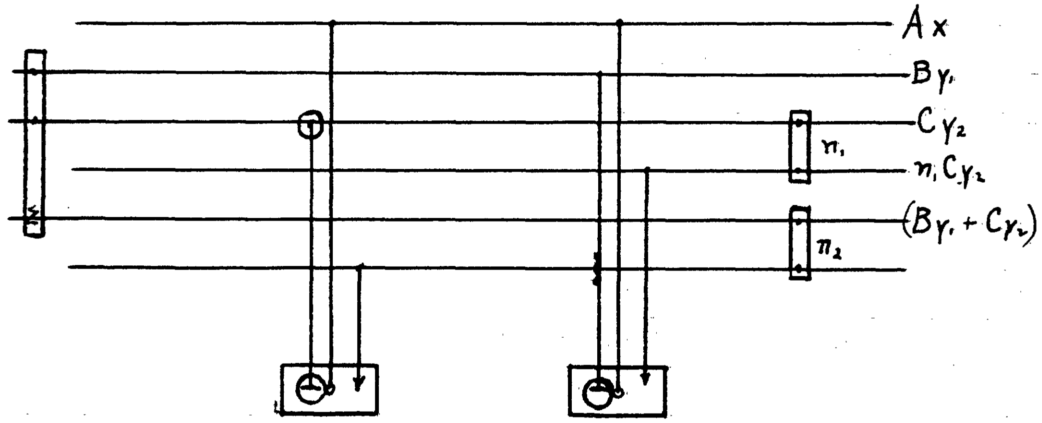

Fig. 25

$$y = a e^{-kx} \sin(bx + c)$$

J. Guerrieri

June 1932

Bibliography

(1) Publications from the Massachusetts Institute of Technology, No. 75 , The Differential Analyser, V. Bush or, The Journal of the Franklin Institute, Vol. 212, No. 4, October 1931 .

(2) "On an Integrating Machine having a New Kinematic Principle" Sir William Thomson, Proceedings of the Royal Society of London, February 3, 1876, Vol. XXIV, Page 262

(3) "Mechanical Calculating Devices", J. Russell, M.I.T.E.E. Seminar, 1929
"Mechanical Calculating Devices," N. Howitt, M.I.T.E.E. Seminar, 1929
and others.

Index of Functions

(Note: $f(x)$ can be changed to $f(x+c)$ by changing the origin for the x shaft. $f(x)$ can be changed to $(c + f(x))$ by changing the origin for the $f(x)$ shaft. Functions grouped together come from a single arrangement. Where functions are given more than once, they come from different methods.)

Function	No. of Integrators	No. of Adders	Page
$\sqrt[3]{x}$, $1/\sqrt[3]{x}$, $1/\sqrt[3]{x^2}$	3	0	42
$1/\sqrt[3]{x}$, $1/\sqrt[3]{x^2}$, $1/\sqrt[3]{x^4}$	3	0	43
$1/(1 - x^2)^{\frac{1}{2}}$, $x/(1 - x^2)^{\frac{1}{2}}$, $\sin^{-1}x$, $x/(1 - x^2)^{\frac{1}{2}} - \sin^{-1}x$	3	1	22- 25
$1/(1 + x^2)^{\frac{1}{2}}$, $x/(1 + x^2)^{\frac{1}{2}}$, $\sinh^{-1}x$ or $\ln(x + (1 + x^2)^{\frac{1}{2}})$, $x/(1 + x^2)^{\frac{1}{2}} - \sinh^{-1}x$	3	1	26
$\sin x$, $\cos x$	2	0	44
$\tan x$, $\sec^2 x$	2	0	18
$\cot x$, $\csc^2 x$	2	0	18
$\sinh x$, $\cosh x$	2	0	44

Function	No. of Integrators	No. of Adders	Page
$\tanh x$ $\operatorname{sech}^2 x$	2	0	18
$\coth x$ $\operatorname{csch}^2 x$	2	0	18
$\tan^{-1} x$	4	0	45
$\ln x$ $x - \ln x$	1	1	28
x^k $\ln x$	3	0	30
$\ln x$ $1/x$ $1/x^2$	3	0	41
$x \ln x$	-	-	43
e^x	1	0	20
$(x + c)e^x$	2	0	46
$(x^2 + bx + c)e^x$	3	0	46
e^{-x^2} $\int e^{-x^2} dx$	2	0	47
$e^{-x}\sin(bx + c)$	2	1	49