Soft - Collinear Effective Theory (SCET)

(5 classes)

For this part we'll switch our sign convention for $\xi = i g T^A \gamma^\mu$ to agree with Refs.

Outline

Class 1: Intro, Degrees of Freedom, Scales
Expansion of Spinors, Propagators
Power Counting, see 2, 3

Class 2: Gauge Symmetry, Reparameterization Inv.
Construction of currents, Lagrangian
Multipole Expansion, One-loop Calculation
see 4, 6, and for One-loop 2

Class 3: Matching & Running (Summing Sudakov Logs)
SCETI $\to$ SCETII, Factorization type
(I: Hard-collinear, II: Soft-collinear, III: soft-collinear)
see 2, 5 and for I $\to$ II see 7

Class 4: D=5, $B \to X_i \gamma$ examples
see 5, 10 respectively

Class 5: Pion form factor, $B \to D\pi$ example
see 5, (1, 3) respectively

(Power Corrections & $B \to \pi \ell \nu$)
see 10, 11
Want an EFT for energetic hadrons, $E_H \gg Q \gg \Lambda_{QCD}$

Why? Many processes have large regions of phase space where the hadrons are energetic, $E_H \gg M_{\pi}$

- $B$-decays: $B \to \pi \nu$, $B \to K^* \gamma$, $B \to \pi \pi$, $B \to X \nu e\nu$
- $B \to X s \gamma$, $B \to D^* \pi$, ...

$M_{\pi} = 5.279 \text{ GeV} \gg \Lambda_{QCD}$

- Hard Scattering:
  - $e^- p \to e^- X (Drell Yan)$
  - $p F \to X l^+ l^-$
  - $\gamma^* \gamma \to \pi^0$
  - $\gamma^* p \to \gamma^{(*)} p'$ (Deeply Virtual Compton Scattering)

- Need to separate perturbative, $dS(Q)$, and non-perturbative "$dS(\Lambda_{QCD})"$ effects → factorization

What are the low energy degrees of freedom?

1. $B \to D \pi$

\[ p_{\pi}^- = (2.310 \text{ GeV}, 0, 0, -2.304 \text{ GeV}) = Q \ n^m \text{ to good approx.} \]

$Q \gg \Lambda$, $n^m = (1, 0, 0, -1)$, $n^2 = 0$ light-like

In $0,1,2,3$ basis
Basis vectors $\vec{n}^\mu, \vec{\bar{n}}^\mu$ 

Use light-cone coordinates: $n^2 = 0$, $\bar{n}^2 = 0$, $n \cdot \bar{n} = 2$.

Vectors $P^\mu = \frac{\vec{n}^\mu}{2} \cdot \bar{n} \cdot P + \frac{\vec{\bar{n}}^\mu}{2} \cdot n \cdot P + P_\perp$

Orthogonal $\vec{n}^\mu, \vec{\bar{n}}^\mu$

Metric $g_{\mu\nu} = \frac{n^\mu n^\nu + \bar{n}^\mu \bar{n}^\nu}{2} + g_{\mu\nu}^{\perp}$

Define $P^+ = n \cdot P$, $P^- = \bar{n} \cdot P$

Since $n^2 = 0$ we needed to define complementary vector $\bar{n}$.

Choice $n^\mu = (1, 0, 0, -1)$, $\bar{n}^\mu = (1, 0, 0, 1)$ is possible, but other choices also work, e.g., $n^\mu = (1, 0, 0, -1)$, $\bar{n}^\mu = (3, 2, 2, 1)$ (more on this later)

In $B \to 0\pi$ the $B, D$ are soft, $E_H \sim M_H$

We can use HQET for their constituents, e.g., quarks & gluons with $p^\mu \sim \Lambda$

But pion is "collinear", $E_H \gg M_H$

In rest frame $\pi$ has quark & gluon constituents $p^\mu \sim (\Lambda, \Lambda, \Lambda, \Lambda)$

Boosting $\pi \to$ has constituents $p^\mu \sim (\frac{\Lambda^2}{a}, Q, \Lambda, \Lambda)$

Collinear fluctuations around $(0, Q, 0) = P_\perp$

Note: Boost in direction orthogonal to $\perp$ directions changes $p^+, p^-$ multiplicatively $p^+ \to a p^+$, $p^- \to \frac{1}{a} p^-$.
(p^+, p^-, p^z) \sim Q (x^2, 1, \lambda) \text{ is collinear}

where \( \lambda \ll 1 \) is a small parameter. (above eq. \( \lambda = \frac{A}{Q} \))

What makes this EFT different?

Usually, we separate scales \( M_1 \gg M_2 \) and have

\[
\sum_{i=1}^{N} C_i (\mu, M_1) \, O_i (\mu, M_2)
\]

short distance \quad long distance

Wilson coeffs \quad operators

\( \sum \) in HQET

the B-meson

\( M_B \gg \Lambda \)

\( p_a^\mu \sim M_B \)

\( p_b^\mu \sim \Lambda \)

picture momenta

now we have overlap between perturbative & non-perturbative momenta in \( p^- \) component

\( E_{\pi} \sim M_B \)

\( p_a \sim (\frac{M_B}{m_b}, M_B, \lambda) \)

\( p_a^2 - \Lambda^2 \)

\( p_a^\mu \sim M_B^2 \)

some

\( p_a^2 - \Lambda^2 \)

overlap in \( p^- \), but

\( p_b^2 \ll p_a^2 \) still
2. inclusive decay \( B \to X_s \gamma \) from \( b \to s \gamma \)

\[ E_\gamma = \frac{m_B^2 - m_{X_s}^2}{2m_B} \left( 0 \to \frac{m_B^2 - m_{X_s}^2}{2m_B} \right) \]

for \( m_X \in \left[ m_B, m_{X_s}^* \right] \)

For \( m_X^2 \sim m_B^2 \)

\[ \text{standard OPE} \]

\[ \text{just like we} \]

\( X \) has hadrons in all directions

did for \( B \to X\gamma \)

For \( m_X^2 \sim \Lambda^2 \)

\[ \text{exclusive decay} \]

(not inclusive)

For \( m_X^2 \sim m_B \Lambda \)

\[ \text{jet of hadrons in } X \]

\[ (p^+, p^-, p_{\perp}) \sim (\Lambda, Q, \sqrt{\Lambda Q}) \sim Q (\lambda^2, 1, \lambda) \]

not transverse

collinear again

\[ \text{this time } \gamma = \frac{\sqrt{\Lambda}}{Q} \ll 1 \]
Infrared Degrees of Freedom have $p^2 \ll Q^2 \lambda^2$

<table>
<thead>
<tr>
<th>Modes</th>
<th>$p^\mu$</th>
<th>$p^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collinear</td>
<td>$Q(x^2,1,1)$</td>
<td>$Q^2 \lambda^2$</td>
</tr>
<tr>
<td>Soft</td>
<td>$Q(x^2,1,1)$</td>
<td>$Q^2 \lambda^2$</td>
</tr>
<tr>
<td>Ultrasoft (uSoft)</td>
<td>$Q(x^2,x^2,1)$</td>
<td>$Q^2 \lambda^4$</td>
</tr>
</tbody>
</table>

Off shell modes have $p^2 \gg Q^2 \lambda^2$ and are integrated out into Wilson coefficients $C(\mu)$

\[ p^\mu \sim Q(1,1,1) \]

Useful Cases

**SCET**

\[ \alpha = \sqrt{\frac{\Lambda}{Q}} \]

\[
\begin{bmatrix}
\text{Collinear} & p_c^2 \sim Q\lambda \\
\text{Soft} & p_s^2 \sim \lambda^2 \\
\text{uSoft} & p_{u}^2 \sim \lambda^2
\end{bmatrix}
\]

Examples

- $B \to Xs \gamma$
- $D \to S \gamma$

**SCET**

\[ \alpha = \frac{\Lambda}{Q} \]

\[
\begin{bmatrix}
\text{Collinear} & p_c^2 \sim \Lambda^2 \\
\text{Soft} & p_s^2 \sim \Lambda^2 \\
\end{bmatrix}
\]

Examples

- $B \to D \pi$
- $\gamma^* \gamma \to \pi^0$

The theory $\text{SCET}^\Pi$ can be derived from $\text{SCET}^*$ so we'll study it first

Quantization: $\exists C(\ell, Q_i) \text{ becomes continuous}$

\[ \int d^3 \ell \ C(\ell) \phi(\ell) \]

Since $p^-$ were same size
Collinear Spinsors U_n labelled by direction \( n \) (recall HQET spinsors \( U_n \))

\[
U(p) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} U \\ \sqrt{2} \frac{\vec{p} \cdot \vec{n}}{p^0} \end{array} \right), \quad \bar{U}(p) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \frac{\vec{p}}{p^0} \cdot \bar{U} \\ \bar{n} \end{array} \right)
\]

Massless QCD spinsors \( U(p) \) and expand \( \Pi \cdot p = p^0 + p^3 = Q + i \frac{Q}{2} \) \( \bar{n} = (1, 0, 0, -i) \) \( n \cdot p \ll Q \), \( p_i \ll Q \)

\[
\frac{\vec{p} \cdot \vec{n}}{p^0} = \sigma^3
\]

\[
U_n = \frac{1}{\sqrt{2}} \left( \begin{array}{c} U \\ \sigma^3 U \end{array} \right) = \left\{ \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \right\} \text{ particles}
\]

\[
\bar{U}_n = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sigma^3 \bar{U} \\ \bar{U} \end{array} \right) = \left\{ \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\} \text{ antiparticles}
\]

\[
\alpha = \left( \begin{array}{cc} 1 & -\sigma^3 \\ \sigma^3 & -1 \end{array} \right)
\]

\( \alpha U_n = \alpha \bar{U}_n = 0 \)

\[
\frac{\alpha \bar{\alpha}}{4} = \frac{1}{2} \left( \begin{array}{cc} 1 & \sigma^3 \\ \sigma^3 & 1 \end{array} \right)
\]

\[\frac{\alpha \bar{\alpha}}{4} U_n = U_n, \quad \frac{\alpha \bar{\alpha}}{4} \bar{U}_n = \bar{U}_n \]

Projection Operator \( 1 = \frac{\alpha \bar{\alpha}}{4} + \frac{\bar{\alpha} \alpha}{4} \)

Field \( \bar{\chi} \sigma \chi = \bar{v}_n + \bar{v} \bar{n} \)

We'll integrate out "small" component \( \bar{v} \bar{n} \)
Collinear Propagators

\[ p^2 + i\epsilon = \frac{\Lambda^2}{p^2 + i\epsilon} \sim \frac{\Lambda^2}{2\Lambda^2 + \Lambda^2 \lambda} \text{ same scale} \]

Fermions

\[
\frac{\not{p} \not{q}}{p^2 + i\epsilon} = \frac{i\alpha}{2} \frac{\not{p} \not{q}}{p^2 + i\epsilon} + \ldots
\]

\[
= \frac{i\alpha}{2} \frac{\not{p} + \not{q} + i\epsilon \text{ sign}(\not{p} \not{q})}{\not{p} \not{q}} + \ldots
\]

\[ \text{from } T \in \mathfrak{T}_\alpha(x), \mathfrak{T}_\alpha(0) \}

Gluons

\[-ig^a \eta^{\mu\nu} \text{ stays same as QCD} \]

\[ g^{\mu\nu} \eta^a \]

\[ \text{true in any gauge} \]

\[ \text{(e.g. Feynman Gauge)} \]

Power counting for collinear fields

\[ L = \int d^4x \int_0^\Lambda \frac{d^2}{\Lambda^2} \left[ \frac{i\alpha^2}{2} + \ldots \right] \eta_0 \]

\[ a_\Lambda^2 \lambda^2 \eta_0 \]

\[ a_\Lambda^2 \lambda^2 \eta_0 = \Lambda^{2-2} \]

set \( \lambda \sim \Lambda \) ie normalize kinetic term so no Lambda's

then \[ \eta_0 \sim \Lambda \]

For gluons find \( A_\mu^a = (A_\mu^+, A_\mu^-, A_\mu^+) \sim (\lambda^2, 1, 1) \)

just like collinear momenta

ie have \( q_\mu + A_\mu^a = \partial_\mu^a \) homogeneous covariant derivative
Currents

\[ \text{QCD} \quad b \to u e^c \quad J = \bar{u} \Gamma b \quad p = \gamma^\mu (1 - \gamma_5) \]

if \( u \) energetic match onto SCET ( \& HET for \( b \))

\[ J^{\text{off}} = \bar{u}_n \Gamma h_r \]

\[ \begin{array}{c}
\downarrow \\
\text{dashed} \\
\text{for collinear}
\end{array} \]

\[ \text{Consider} \]

\[ \begin{array}{c}
\downarrow \\
\text{no power} \\
\text{Suppression} \\
\text{for these gluons}
\end{array} \]

\[ \bar{q}_n \Gamma \left( \frac{i (k^+ m_b)}{k^2 - m_b^2} \right) i g T^A \gamma^\mu h_r = -g \bar{q}_n \Gamma \left( \gamma^\mu (1 + \sigma) + \frac{\pi \eta}{2} \gamma^5 \right) h_r \\
= -g \frac{\bar{q}_n}{\pi^\mu} \Gamma T^A \left( -\frac{\eta}{2} (1 - \sigma) + 2 \gamma^5 \right) h_r \\
\text{(add more gluons later)}
\]
Which fields can interact in a local way?

\[ p + k = \frac{n^\mu}{2} \bar{n} \cdot p + \frac{n^\mu}{2} n \cdot (p + k) + P_k + \ldots \]

Still collinear: local

\[ p + k = \frac{n^\mu}{2} \bar{n} \cdot (p + k) + \frac{n^\mu}{2} n \cdot (p + k) + P_k + k \]

Still collinear: local

offshell: integrate it out (prev. eq.)

\[ m^2 \uparrow \]

\[ k \] offshell

local

\[ p \]

\[ n \]

in SCET-II

\[ p + k = \frac{n^\mu}{2} \bar{n} \cdot p + \frac{n^\mu}{2} n \cdot k + \ldots \]

\[ (p + k)^2 = \bar{n} \cdot p \cdot n \cdot k \sim Q^2 \lambda \gg Q^2 a^2 \]

IR d.o.f.

Fields which radiate interactions in SCET-II are offshell making it more complicated so we postpone further discussion to after developing SCET-II.
More on Power Counting

Separate \( Q, Q^2, Q^2 \) moments

Analogy

- HQET: \[ P^\mu = m \nu v^\mu + k^\mu \]
- SCET: \[ P^\mu = p^\mu + k^\mu \]

Node Expr

\[ \Psi(x) = \left[ \sum \frac{1}{S(P^3)} \sigma(P^0) \left[ U(P) a(p) e^{-iP \cdot x} + U(P) b^*(p) e^{iP \cdot x} \right] \right] \]

\[ = \Psi^+ + \Psi^- \]

Expand \&

Write \[ \Psi^+(x) = \sum_p e^{-iP \cdot x} \Psi^+_{n, p}(x) \quad \text{a } \Psi^+_{\bar{n}, p} = 0 \]

\[ \Psi^-(x) = \sum_p e^{iP \cdot x} \Psi^+_{\bar{n}, p}(x) \]

\[ \uparrow \text{ both have } \Theta(\bar{n}, p) \]

Now define \[ \Psi_{n, p}(x) = \Psi^+_{n, p}(x) + \Psi^-_{\bar{n}, p}(x) \]

\[ \bar{n}, p > 0 \text{ particles } \quad E = \frac{p^2}{2} > 0 \]

\[ \bar{n}, p < 0 \text{ anti-particles } \quad E = -\frac{\bar{n}, p^2}{2} > 0 \]

Similar for Gluons \[ A_{n, q}^\mu \]

\[ \text{destroy} \quad A_{n, q}^\mu = A_{\bar{n}, \bar{q}}^\mu \text{ create} \]

In HQET label \( \nu^\mu \) was conserved by gluons

In SCET labels are changed by collinear gluons

Collinear \[ p' = a + p \]

Soft \[ \bar{q} \]

\[ (p, k) \quad (\bar{p}, \bar{k}) \]
Introduce Label Operator for $p^\mu$ momenta

$$p^\mu \left( \phi_{\alpha_1}^+ \phi_{\beta_2}^+ \ldots \phi_{\alpha_l}^+ \phi_{\beta_2}^- \right) = (p_1^+ + p_2^+ + \ldots - q_{\alpha_1}^+ - q_{\beta_2}^-) \right)$$

eigenvalue $\phi_{\alpha_1} \ldots$

"derivative" for labels $p^\mu$

derivative for residual $\partial x^\mu$

$$i \partial x^\mu \equiv e^{-i p^\mu x} \phi_{\alpha \rho} (x) = \sum_p e^{-i x^\mu p^\mu} (p^\mu + i \partial x^\mu) \phi_{\alpha \rho} (x)$$

in products of fields this makes labels conserved

Summary

<table>
<thead>
<tr>
<th>Type</th>
<th>$(p^+, p^-, p^\perp)$</th>
<th>Fields</th>
<th>Field Scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>collinear</td>
<td>$(x^2, 1, \lambda)$</td>
<td>$\phi_{\alpha \rho} (x)$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(A_{\alpha}^+, A_{\beta}^-, A_{\gamma}^\perp)$</td>
<td>$(x^2, 1, \lambda)$</td>
</tr>
<tr>
<td>soft</td>
<td>$(x, 1, 1)$</td>
<td>$q_{\alpha \rho}$</td>
<td>$\lambda^2 x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_{\alpha \rho}$</td>
<td>$\lambda^2 x$</td>
</tr>
<tr>
<td>USoft</td>
<td>$(x^2, x^2, x^2)$</td>
<td>$q_{\alpha \rho}$</td>
<td>$\lambda^3 x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A_{\alpha \rho}$</td>
<td>$\lambda^2$</td>
</tr>
</tbody>
</table>
Collinear Lagrangian

Write $\mathcal{L} = \mathcal{L}_n + \mathcal{L}_\pi$, $\mathcal{L}_n = \rho_n \mathcal{L}$, $\mathcal{L}_\pi = \rho_\pi \mathcal{L}$

\[ \rho_n = \frac{\sigma \alpha}{4}, \quad \rho_\pi = \frac{\bar{\sigma} \alpha}{4} \]

integrate out $\mathcal{L}_\pi$

(start by keeping single gluon field, and no phase redefinition)

\[ \mathcal{L} = \frac{i}{\sqrt{2}} \partial \bar{\mathcal{L}} = (\bar{\mathcal{L}} + \bar{\mathcal{L}}_\pi) \left( i \frac{\mathcal{L}}{\sqrt{2}} \pi \cdot 0 + i \frac{\mathcal{L}_\pi}{\sqrt{2}} \pi \cdot 0 + i \mathcal{L}_\pi \right) (\bar{\mathcal{L}} + \bar{\mathcal{L}}_\pi) \]

\[ = \bar{\mathcal{L}} \frac{\mathcal{L}}{2} \pi \cdot 0 \mathcal{L} + \bar{\mathcal{L}}_\pi \frac{\mathcal{L}_\pi}{2} \pi \cdot 0 \mathcal{L} + \bar{\mathcal{L}}_0 \mathcal{L}_\pi + \bar{\mathcal{L}}_\pi \mathcal{L}_0 \mathcal{L} \]

\[ \frac{\delta}{\delta \mathcal{L}} : \quad c.o.m.
\]
\[ \frac{\mathcal{L}}{2} \pi \cdot 0 \mathcal{L} + i \mathcal{L}_\pi \mathcal{L} = 0 \]
\[ i \mathcal{L}_0 \mathcal{L} + \frac{\mathcal{L}_\pi}{2} \pi \cdot 0 \mathcal{L} = 0 \]
\[ \mathcal{L}_\pi = \frac{1}{i \pi \cdot 0} \mathcal{L}_0 \mathcal{L} \]

Think of $\frac{1}{i \pi \cdot 0} f(x) = \int d^4 \rho \frac{e^{-i \rho \cdot x}}{\pi \cdot \rho} f(\rho)$ for inverse derivative

Now

\[ \mathcal{L} = \bar{\mathcal{L}} \left( i \pi \cdot 0 + i \mathcal{L}_\pi \frac{1}{i \pi \cdot 0} i \mathcal{L}_\pi \right) \frac{\mathcal{L}}{2} \mathcal{L} \]

Next: introduce collinear & soft gluon fields & phases $e^{-i \mathcal{L} \cdot \mathcal{X}}$

- recall $A_\mu^s$ has $p^2 \sim a^2 \lambda^2 \ll P^2 \sim a^2 \lambda^2$

ie long wavelength, its like a classical background field

so for $A_\mu^s$ and $\pi_0$ are concerned

write $A_\mu = A_\mu^s + A_\mu^s$

- make phase redefinition: $i \mathcal{L}_\mu \rightarrow p_\mu + i \mathcal{L}_\mu$

and we get a $e^{-i \mathcal{L} \cdot \mathcal{X}}$ out front irrespective of number of fields we have $\frac{1}{i \pi \cdot 0}$ mean we have Feyn rules with $0, 1, 2, 3, 4, 5, \ldots$ gluons)
\[ \text{i} \cdot D = \text{i} \cdot D^e + \gamma \cdot \text{An}_b + \gamma \cdot \text{An}_b \\
\text{we will suppress the } \gamma \text{ from now on} \]

\[ \text{i} \cdot D = \gamma n \]

\[ \text{we drop it} \]

\[ \text{i} \cdot D = (\gamma \cdot \text{An}_b + \gamma \cdot \text{An}_b) \]

\[ \text{we drop it} \]

So Leading Order Action is \( \mathcal{L}^{(0)} \) \( \mathcal{L}^{(0)} = \mathcal{L}^{(0)} \)

\[ \gamma \cdot \gamma \cdot \gamma \]

- all fields are at \( x \) & derivatives \( \frac{\text{D}}{\gamma} \)
- action is explicitly local at \( Q^2 \) scale
- action is essentially local at \( Q \) (D\gamma in numerator)
- only non-local at \( \sim Q \) scale
- terms are same size in power counting!

Repeating for Gluons: we find

\[ \mathcal{L}^{(0)} = \frac{1}{4} G_{\mu \nu} G^{\mu \nu} = -\frac{1}{2} + \left[ G_{\mu \nu} G^{\mu \nu} \right] , \quad G^{\mu \nu} = \frac{i}{2} \left[ D^\mu , D^\nu \right] \]

\[ \cdots \]

\[ \mathcal{L}^{(0)} = \frac{1}{2} g^2 + \frac{1}{2} \left( \left[ i D^\mu + g \text{An}_b , i D^\nu + g \text{An}_b \right] \right)^2 \]

\[ \text{where } i D^\mu = i \frac{\text{D}}{2} n \cdot O + \text{D}^\mu + \frac{\text{D}^\mu}{2} \]

\[ \text{see hep-ph/0109045} \]
we drop some terms in constructing $x^{(0)}$ $x^{(0)}$

Argument so far was \textit{tree level}. To go further we need symmetries
(plus power counting still)

1. Spin structure
2. Gauge Symmetry
3. Reparameterization Invariance

1. Spin. Easiest in two-component form (rather than 4 component $\gamma_n$ with $\alpha^\mu \gamma_n = \gamma_n$)

\[ \gamma_n = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \gamma_0 \\ \sigma_3 \gamma_n \end{array} \right) \]

\[ x = \gamma_{\mu \nu} \left\{ m \cdot 0 + \frac{i D_\mu^\nu}{\sqrt{2}} \right\} \gamma_{\nu \rho} \]

not $SU(2)$

just $U(1)$: helicity $h = \frac{i \epsilon_{\mu \nu}}{\sqrt{2}} [\gamma_\mu, \gamma_\nu]$ generator

$h \sim \sigma_3$ spin along direction of motion

2. Gauge Symmetry $U(x) = \exp \left[ i d^\mu(x) T^\mu \right]$

Need to consider $U's$ which leave us within EFT

\[ e.g. \quad i d^\mu \phi = 0 \Rightarrow \phi \text{ then } \gamma_{\mu \nu} = U(x) \gamma_{\mu \nu} \text{ would no longer have } p^2 \lesssim A^2 \lambda^2 \]
- Two classes of gauge transformation for two gauge fields.

- In momentum space we have convolutions for \( \mathcal{U}_c \)

\[ q_{n_1 \rightarrow q_{n_2}} \rightarrow \mathcal{U}_c \]

We'll write shorthand \( q_{n_1} \rightarrow q_{n_2} \rightarrow \mathcal{U}_c \)

Now \( q_{n_2} \mathcal{U}_c \rightarrow q_{n_2} \mathcal{U}_c \) since otherwise we give large momentum to an usoft field.

Aside: Wait! recall our heavy-to-light current

\[ q_{n_1} \Gamma_{h_{n_2}} \rightarrow q_{n_1} \mathcal{U}_c^+ \Gamma_{h_{n_2}} \] it's not gauge invariant.

But we had to integrate out off-shell propagators.

\[
\Gamma \equiv \Gamma_{W} = \prod_{m=0}^{\infty} \sum_{\text{perms}} \left( \frac{-g}{\bar{p} \cdot A_{n_1} \cdots A_{n_m}} \right) \times \prod_{b_1} \Gamma_{a_1} \cdots \Gamma_{a_1}
\]

Here \( W \) is a Wilson line.
Short form \( W = \left[ \sum_{p,m} \exp \left( -\frac{g}{\rho} \hat{n}_A \phi_\rho \right) \right] \)

If we set residual coordinate \( x = 0 \)

Fourier transform \( W = W(y, -\infty) \)

\[ = \rho \exp \left( i g \int_{-\infty}^{y} ds \, \hat{n}_A \phi_\rho (s) \right) \]

\[ \hat{W}_n(y) W(y, -\infty) \hat{w}_I (-\infty) \]

\[ \text{up} \quad \text{up} \quad \text{soft field is at "long" distance} \]

\[ \text{short distance} \quad \text{and doesn't see short distance} \]

\[ \text{interactions in this direction} \]

Now \( W \to U_{s} W \) and \( \hat{W} \) \( U_{s} \) is invariant

\[ \text{end Aside} \]

**Gauge Transformations**

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( U_{c} )</th>
<th>( U_{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_{c} )</td>
<td>( U_{c}, \gamma_{\mu p} )</td>
<td>( U_{s}, \gamma_{\mu p} )</td>
</tr>
<tr>
<td>( A_{\mu} )</td>
<td>( U_{c} A_{\mu} U_{s}^{+} + \frac{i}{g} U_{c} \left[ i D_{\mu}, U_{c} \right] )</td>
<td>( U_{s} A_{\mu} U_{s}^{+} )</td>
</tr>
<tr>
<td>( W )</td>
<td>( U_{c} W )</td>
<td>( U_{s} W U_{s}^{+} )</td>
</tr>
<tr>
<td>( \sigma_{s} )</td>
<td>( \sigma_{s} )</td>
<td>( \sigma_{s} )</td>
</tr>
<tr>
<td>( A_{s} )</td>
<td>( A_{s} )</td>
<td>( A_{s} )</td>
</tr>
<tr>
<td>( Y )</td>
<td>( Y )</td>
<td>( Y )</td>
</tr>
</tbody>
</table>

- homogeneous in \( \sigma \), recall \( i D_{\mu} \) \( \eta_{\mu} \) in \( D \) in it
- \( U_{s} A_{\mu} U_{s}^{+} \) is like background field transformation of quantum field \( A_{\mu} \)
Case Symmetry ties together $\ln D = \ln \mathcal{O} + \gamma \ln A_0 + \gamma \ln A_1$

Mass Dimension + lower counting means either $\ln D \sim \mathcal{O}_2$
or $\frac{1}{\mathcal{O}} (\ln \mathcal{O})^2 \sim \mathcal{O}_2^2$, no other $\mathcal{O}_2$ operators

What about a coefficient $c$ between $\ln D \sim \mathcal{O}_2$ terms?

What about other operators like

$\frac{1}{\ln \mathcal{O}_2} c \ln \mathcal{O}_2 \frac{\mathcal{O}_2}{2} \ln \mathcal{O}_2$

see hep-ph/0204229

(3) Reparameterization Invariance (RPI)

$n, \bar{n}$ break Lorentz invariance $n^\mu M_{\mu\nu}, \bar{n}^\nu M_{\mu\nu}$

(only $E_1^{\mu\nu} M_{\mu\nu}$ preserved)

3 types of RPI which keep $n^2 = \bar{n}^2 = 0, n \cdot \bar{n} = 2$

I $n \rightarrow n + A_1$
$\bar{n} \rightarrow \bar{n}$

II $n \rightarrow n$
$\bar{n} \rightarrow \bar{n} + E_1$

III $n \rightarrow e^{x}n$
$\bar{n} \rightarrow e^{-x}\bar{n}$

Type III is simple: implies that for any operator with an $n^\mu$ we have corresponding $A$ in denominator or a corresponding $\bar{n}$ in numerator.

e.g. $L_{99}$ had $\bar{n} \frac{1}{\ln \mathcal{O}}$, $\bar{n} \cdot \mathcal{O}$

Can't have $\bar{n} \cdot \mathcal{O}$ for example.
Power Counting: \[ \Delta T \sim \lambda^2 \]
\[ \xi \sim \lambda^0, \quad \Delta \sim \lambda^0 \]

\( \max \) power that leaves scale of collinear momenta intact

i.e. we only care about restoring Lorentz invariance for the set of fluctuations described by SCET

\[ \text{Find } \]
\[ n \cdot D \rightarrow n \cdot D + \Delta \cdot D \]

\[ D_{\mu} \rightarrow D_{\mu} + \frac{\Delta_{\mu}}{2} \frac{n \cdot D - \bar{n} \cdot D}{2} \]

\[ \bar{n} \cdot D \rightarrow \bar{n} \cdot D \]

\[ \Upsilon_n \rightarrow (1 + \frac{1}{4} \frac{\Delta \cdot D}{n \cdot D}) \Upsilon_n \]

\[ W \rightarrow W \]

\[ \text{Under II} \]

\[ n \cdot D \rightarrow n \cdot D \]

\[ D_{\mu} \rightarrow D_{\mu} + \frac{\xi_{\mu}}{2} \frac{n \cdot D - \bar{n} \cdot D}{2} \]

\[ \bar{n} \cdot D \rightarrow \bar{n} \cdot D + \xi \cdot D \]

\[ \Upsilon_n \rightarrow (1 + \frac{1}{4} \frac{\xi \cdot D}{n \cdot D}) \Upsilon_n \]

\[ W \rightarrow \left[ (1 - \frac{1}{2 \bar{n} \cdot D}) e^{\pm D} \right] W \]
\[ S^{(2)} \left( \frac{\gamma_\nu \cdot i e \gamma_\tau}{e \cdot p} \frac{i D^\nu}{2} \bar{\tau}_n \right) = -\bar{\tau}_n i \Delta^\nu D^\nu \frac{\gamma_\nu}{2} \bar{\tau}_n \]

\[ S^{(2)} \left( \frac{\gamma_\nu \cdot i e \gamma_\tau}{e \cdot p} \frac{i D^\nu}{2} \bar{\tau}_n \right) = \bar{\tau}_n i \Delta^\nu D^\nu \frac{\gamma_\nu}{2} \bar{\tau}_n \]

These terms are connected.

Type II rules out the \( D_\mu^L \) \( D_\mu^R \) operator in \( \bar{\ell}_R \).

Just like HQET, RPI also relates Wilson coefficients of leading and subleading currents. See hep-ph/0211251.

What about the Wilson Coefficients? They have \( C(\bar{\ell}, \mu) \) depend on large momenta picked out by \( \bar{\ell} \sim \tau_0 \).

\[ C(-\bar{\ell}, \mu) \left( \bar{\tau}_{n, \rho} W \right) \Gamma h_v = \left( \bar{\tau}_{n, \rho} W \right) \Gamma h_v C(\bar{\ell}^+) \]

We must act on product \( \left( \bar{\tau}_{n, \rho} W \right) \) since only the momentum of this combination is gauge invariant.

Can write

\[ \int d\omega \, C(\omega, \mu) \left[ \left( \bar{\tau}_{n, \rho} W \right) \delta(\omega - \bar{\ell}^+) \Gamma h_v \right] = \int d\omega \, C(\omega, \mu) U(\omega) \]

Factorization of hard "\( C \)" and collinear "\( U \)" involves a convolution.
More collinear fields: if we have $>1$ energetic hadron or $>1$ jet we need more $n$'s (or $\tilde{n}$'s)

In general just odd $\sum_n \bar{\xi}_n^{(0)}$ etc.

If we have $n_1, n_2, n_3, \ldots$ then these directions are distinct only if

$$n_i \cdot n_j \gg \Lambda^2 \quad i \neq j$$

for

$$p_2 = Q n_2$$

if

$$n_1 \cdot p_2 = Q n_1 \cdot n_2 \sim Q \Lambda^2$$

then $p_2$ is collinear in $n_1$ direction.

Discrete Symmetries

$$C^{-1} \gamma_{\mu,\rho} C = - \left[ \gamma_\mu, -\gamma_\rho \right]^T$$

$$P^{-1} \gamma_{\mu,\rho}(x) P = \gamma_0 \gamma_{\mu,\rho} (x_P)$$

$$T^{-1} \gamma_{\mu,\rho}(x) T = \gamma_\tau \gamma_{\mu,\rho} (x_T)$$

$P$ and $T$ relate $n$ to $\tilde{n}$ direction.
Using Gauge INV, RPI, Power Counting:

\[ \mathcal{L}^{(0)}_{q\bar{q}} = \mathcal{L}_{\pi\rho} \left( i \not{n} \cdot \not{D} + i \alpha_2 \frac{1}{i \not{n} \cdot \not{D}} i \theta \not{D} \right) \frac{\alpha}{2} \mathcal{L}_{\rho,\rho} \]

is unique at LO

Propergator

\[ \frac{i}{\pi^2} \frac{\Theta(-\pi \rho)}{\pi \rho + \frac{m^2}{\pi \rho} + i \epsilon} - \frac{i \alpha}{2} \frac{\Theta(-\pi \rho)}{-n \cdot p - \frac{m^2}{\pi \rho} + i \epsilon} = \frac{i \alpha}{2} \frac{n \cdot \rho}{n \cdot p \pi \rho + m^2 + i \epsilon} \]

has two poles

Multipole Expansion

- action only has uoqf n.Dus gluons

- action only sees n.k uoqf momenta; equiv to multipole expansion

\[ \mathcal{L} \]

\[ = \frac{n \cdot \rho}{n \cdot p \cdot (p+k)^2 + m^2 + i \epsilon} \]

on-shell

= \[ \frac{n \cdot \rho}{n \cdot p \cdot n \cdot k + m^2 + i \epsilon} \]

propagators reduce to eikonal approximation when appropriate

- for collinear gluons attaching to the quark

\[ \frac{n \cdot (p+k)}{n \cdot (p+k) + (p+k)^2 + i \epsilon} = \frac{n \cdot (p+k)}{(p+k)^2 + i \epsilon} \]
Back to Wilson line, \( W = \left[ \xi \exp \left[ -\frac{g}{\rho} \bar{\Lambda} A_{\alpha \beta} \right] \right] \)

Definition is \( i\bar{\Pi} D_\alpha W = 0 \) \( \therefore \bar{\Pi} W = 1 \)

As an operator
\[
\begin{align*}
    i\bar{\Pi} D_\alpha W &= W \bar{\Pi} \\
    i\bar{\Pi} D_\alpha &= W \bar{\Pi} W^+ \\
    (i\bar{\Pi} D_\alpha)^k &= W \bar{\Pi}^k W^+
\end{align*}
\]

\[
\begin{align*}
    f(i\bar{\Pi} D_\alpha) &= W f(\bar{\Pi}) W^+ \\
    \text{part of collinear operator, } p^2 \Lambda^2 \\
    \text{hard Wilson coefficient, } p^2 \Lambda^2 \\
    &= \int d\omega f(\omega) W S(\omega - \bar{\Pi}) W^+
\end{align*}
\]

This is \textbf{Hard-Collinear Factorization}!

IR divergences, Matching, & Running

Consider Heavy-to-Light Current for \( b \rightarrow s \gamma \)

\[
\begin{align*}
    J^{Q=0} &= \bar{s} \gamma b \\
    J^{scet} &= (\bar{t} \gamma W) \Gamma_{hr} C(\bar{f}^+) \\
    (\Gamma = \sigma^{\mu \nu} P^\mu F_{\nu 0}) \text{ ie } U_{78}
\end{align*}
\]
QCD graphs at one-loop regulate IR with \( p^2 \neq 0 \)

\[
\begin{align*}
\bar{b} & \rightarrow p_0 \rightarrow p \\
\end{align*}
\]

\[
\sum b \frac{d\sigma_{CF}}{4\pi} \left[ \ln^2 \frac{p^2}{m_b^2} + 2 \ln \frac{p^2}{m_b^2} + \ldots \right]
\]

\[
\gamma = 1 - \frac{d\sigma_{CF}}{4\pi} \left[ \frac{1}{E_{\text{UV}}} + \frac{2}{E_{\text{IR}}} + 3 \ln \frac{\mu^2}{m_b^2} + \ldots \right] \quad \text{IR regulated by dim reg here}
\]

\[
\gamma_5 = 1 - \frac{d\sigma_{CF}}{4\pi} \left[ \frac{1}{E_{\text{UV}}} - \ln \frac{p^2}{\mu^2} \right]
\]

\[
\gamma_{\text{tensor}} = 1 + \frac{d\sigma_{CF}}{4\pi} \frac{1}{\epsilon} \quad \text{(tensor current not conserved)}
\]

\[
\sum = \sum b \left[ 1 - \frac{d\sigma_{CF}}{4\pi} \left( \ln^2 \frac{p^2}{m_b^2} + \frac{3}{2} \ln \frac{p^2}{m_b^2} + \frac{1}{E_{\text{IR}}} + 2 \ln \frac{\mu^2}{m_b^2} + \ldots \right) \right]
\]

SCET

\[
\left( \frac{d^4 h}{(n \cdot k + \epsilon)(k^2 + \epsilon)(n \cdot h + p^n_{\mu} + \epsilon)} \right)
\]

\[
- \sum b \frac{d\sigma_{CF}}{4\pi} \left[ \frac{1}{E^2} + \frac{2}{E} \ln \left( \frac{\mu \cdot p}{-p^2 + \epsilon} \right) + 2 \ln^2 \left( \frac{\mu \cdot p}{-p^2 + \epsilon} \right) + \frac{3\pi^2}{4} \right]
\]

\[
\gamma_{n^\mu n^\nu} = 0 \quad \text{in Feyn. Gauge}
\]

\[
\gamma_{\text{SCET}} = 1 + \frac{d\sigma_{CF}}{4\pi} \left[ \frac{Z_{\text{UV}}}{E_{\text{UV}}} - \frac{Z_{\text{IR}}}{E_{\text{IR}}} \right]
\]
Collinear Graphs

\[ \int \frac{d^4k}{(2\pi)^4} \frac{n \cdot \bar{n}}{k \cdot k} \frac{\bar{n} \cdot (p+k)}{(k+p)^2} \]

\[ = -g^2 \Gamma \frac{\alpha s (\alpha s)}{4\pi} \left[ \frac{-2}{E^2} \left( \frac{\mu^2}{-p^2+i\epsilon} \right) - \ln \left( \frac{\mu^2}{-p^2+i\epsilon} \right) \right] \]

\[ - 2 \ln \left( \frac{\mu^2}{-p^2+i\epsilon} \right) - 4 + \frac{\pi^2}{6} \]

\[ \begin{array}{c}
\alpha s \cdot \bar{n} \cdot \bar{n} = 0
\end{array} \]

\[ \begin{array}{c}
E = 1 - \frac{\alpha s (\alpha s)}{4\pi} \left[ \frac{1}{G_W} + \ln \left( \frac{\mu^2}{p^2} \right) \right]
\end{array} \]

IR matches:
- \( \ln^2 (p^2) \)
- \( \ln (p^2) \)
- \( \ln (p^2) \)
- \( \sqrt{E IR} \)

If we had neglected collinear graphs, this would not be true.

(historically, only usoft graphs were included for a while, an "EFT" called LEET)

This is a check that we have correct degrees of freedom.
UV divergences in SCET need a counterterm (a $p^2$ term in $\frac{1}{\epsilon}$ term)

$$Z = 1 + \frac{ds(C_F)}{4\pi} \left[ \frac{1}{\epsilon^2} + \frac{2}{\epsilon} \ln \left( \frac{\mu}{\pi \cdot p} \right) + \frac{5}{2\epsilon} \right]$$

For $\frac{3}{2}$ in $\Gamma$ hr current \[ \uparrow \quad \uparrow \quad \text{L} \quad \text{L} \quad \text{port of NLL} \]

---

**Running**

in general we must be careful with coefficients $C(\mu, \bar{p})$ since they act like operators.

---

In our example $\bar{p} \rightarrow \pi \cdot p$ of external field always

the non-trivial case

$$C(\mu, \pi \cdot (p + k) + \pi \cdot (-k))$$

$$= C(\mu, \pi \cdot p)$$

So

$$\frac{\mu}{2 \epsilon} C(\mu) = -\frac{ds(\mu)}{\pi} \ln \left( \frac{\mu}{\bar{p}} \right) C(\mu)$$

LO anom dim

---

**Solution:**

QED $\quad ds = \text{fixed } \quad C_F = 1$

$$C(\mu) = \exp \left[ -\frac{\alpha}{2\pi} \ln^2 \left( \frac{\mu}{\bar{p}} \right) \right]$$

$\uparrow$ Sums $[\alpha \ln^2]^k$ terms, Sudakov double logs

QCD $\quad C(\mu) = \exp \left[ -\frac{4\pi C_F}{3 \alpha_s (m_b)} \left( \frac{1}{2} - 1 + \ln z \right) \right]$

$z = \frac{ds(\mu)}{ds(m_b)}$

$M_b = \text{matching scale}$
In more complicated cases $C(\bar{\mathcal{F}}, \mathcal{F}^+)$ will be sensitive to \( n \) th loop momentum and we'll get
\[
\frac{\mu^2}{2\pi} C(\mu, \omega) = \int d\omega' \gamma(\omega, \omega') C(\mu, \omega')
\]

Examples:
- DIS
- Alterelli-Parisi evolution
- $\gamma^* \pi^0 \rightarrow \pi^0$
- Brodsky-Lepage evolution
- $\gamma^* \rho \rightarrow \gamma \rho'$
- Deeply-Virtual Compton-Scatter evolution

(actually these are all the evolution of a single type of operator \((\text{isotriplet})\) or operators \((\text{isosinglet})\))

Note: series in $\ln C(\mu)$

\[
\begin{array}{ccc}
\text{LL} & ds^n \ln^{n+1} & \frac{1}{\epsilon} \quad - \\
\text{NLL} & ds^n \ln^n & \frac{1}{\epsilon} \quad \frac{1}{\epsilon^2} \\
\text{NNLL} & ds^n \ln^{n-1} & \text{matching} \quad \frac{1}{\epsilon} \quad \frac{1}{\epsilon^2} \\
\end{array}
\]

differ from single log case somewhat
Recall we had multipole expansion, if we consider

$$\rightarrow T = \sum_{\text{perms}} \frac{(g)^m}{n^k_1 n^k_2 \cdots n^k_r} a^1 a^2 \cdots a^m$$

on-shell so

$$\frac{1}{n^k + p^2_{\text{exp}}} \rightarrow \frac{1}{n^k}$$

Motivates us to consider a Field Redefinition

$$\gamma_{\bar{1}1} (x) = Y(x) \gamma_{\bar{1}1}^{(0)} \quad \gamma_{\bar{1}1} = Y A_{\bar{1}1}^{(0)} Y^+$$

$$Y(x) = \text{Pexp} \left( i g \int_0^x ds \ A_{\bar{1}1}^{(0)} (x + s) T^a \right)$$

$$n_1 D Y = 0 \quad Y^+ Y = 1 \quad \text{find} \quad W = Y W^{(0)} Y^+$$

$$\gamma_{\bar{1}1}^{(0)} = \bar{\gamma}_{\bar{1}1}^{(0)} \gamma_{\bar{1}1}^{(0)}$$

$$= \bar{\gamma}_{\bar{1}1}^{(0)} \gamma_{\bar{1}1}^{(0)}$$

$$= \bar{\gamma}_{\bar{1}1}^{(0)} \gamma_{\bar{1}1}^{(0)}$$

For all $n_1 A_{\bar{1}1}$'s disappear!

Formally

$$X(\gamma_{\bar{1}1}, A_{\bar{1}1}, n_1 A_{\bar{1}1}) = X(\gamma_{\bar{1}1}^{(0)}, A_{\bar{1}1}^{(0)}, 0)$$
These interactions don't disappear, but are moved out of Lagrangian and into currents.

\[ J = \bar{\tau}_n N \Gamma h \nu = (\bar{\tau}_n^{(0)} W^{(0)} - \Gamma (\gamma^+ h \nu)) \]

If our current was a collinear singlet,

\[ J = (\bar{\tau}_n W) \Gamma (W^+ \tau_n) = \bar{\tau}_n^{(0)} W^{(0)} \gamma^+ \Gamma (W^{(1)} \tau_n^{(1)}) \]

This is quite powerful. It sums up an infinite class of diagrams in eq 1:

\[ = \]

\[ \begin{array}{c}
\text{usoft} \\
\text{collinear} \\
\text{part} \\
\text{part}
\end{array} \]

In eq 2, the usoft gluons decouple at lowest order (from any diagram).

This is color transparency:

- the usoft gluons decouple from the energetic partons in a color singlet state
- they just "see" overall color singlet due to multipole expn.
\[ q = q_s + q_c = 2(\alpha_s \lambda), \quad q^2 = \alpha^2 \lambda^2 \gg (\alpha_s^2) \text{ offshell} \]

Integrating out these fluctuations builds up a soft Wilson line \( S_0 \) (analogous to \( Y_{\Sigma n.AW} \) but with soft fields).

Consider heavy-light soft-collinear current \( \bar{c} \gamma \mu v u \).

\( s = \text{soft}, \quad c = \text{collinear} \)
\( 0 = \text{offshell} \)

Adding more gives \( \bar{c} \gamma S_0 \gamma W_{\mu
u} \).

Need 3-gluon, 4-gluon vertices too.

These flip the order of \( S_0^+ \) and \( W \).

\[ (\bar{c} \gamma W) \gamma (S_0^+ \gamma W) \]
\[ \text{collinear} \quad \text{soft} \]
\[ \text{gauge} \quad \text{gauge} \]
\[ \text{invariant} \quad \text{invariant} \]

This is soft-collinear factorization.
Better Method

1) Match OPE onto SCET\textsubscript{I} \hspace{1cm} \text{usoft} \hspace{1cm} P_0^2 \sim \Lambda^2 \\
    \text{collinear} \hspace{1cm} P_c^2 \sim Q \Lambda

2) Factorize with field redefinition

3) Match SCET\textsubscript{I} onto SCET\textsubscript{II} \hspace{1cm} \text{soft} \hspace{1cm} P_s^2 \sim \Lambda^2 \\
    \text{collinear} \hspace{1cm} P_c^2 \sim \Lambda^2

Notes

- This gives us a simple procedure to construct SCET\textsubscript{II}

- usoft fields in I are simply renamed soft in SCET\textsubscript{II}

- In cases where we have time-ordered products in SCET\textsubscript{I}, with \( \geq 2 \) operators involving both usoft & collinear fields, we can generate a non-trivial coefficient in SCET\textsubscript{II} (jet function \( J \))

\[
\frac{P_0^2}{P_c^2 \sim Q \Lambda} \left( \frac{P_s^2 \sim \Lambda^2}{P_c^2 \sim \Lambda^2} \right)
\]

*\( \text{e.g. general structure is:} \)

\[
\int \text{d} q_1 \cdot \text{d} k_2 \cdot \text{d} k_3 \cdot C(q) \cdot J(p, p_c, k) \cdot (\bar{\nu}_n W)_{p_c} \Gamma \left( \gamma^+ \bar{\epsilon}_s \right) k^+ \]

*\( \text{in g.s. in SCET, allow it to depend on } k^+ \)

*\( \text{e.g.)}

1) \( J^I = \bar{\nu}_n W \Gamma h \nu \)

2) \( J^I = \bar{\nu}_n W^0 \Gamma \gamma_5 h \nu \)

3) \( J^I = (\bar{\nu}_n W) \Gamma (S^+ h \nu) \hspace{1cm} \text{same as before} \)

* In this case all T-products in SCET\textsubscript{II} & SCET\textsubscript{II} match up, so matching is easy.
Consider two operators

\[ p^2 = k^2 - \Lambda^2 \]

\[ U \rightarrow S \]

When we leave off-shellness of external collision fields the intermediate line still has \( p^2 \sim \Lambda^2 \) and must really be integrated out.

Note: \( T^+ \sim \Lambda^{2k} \Rightarrow U^+ \sim \eta^{k+E} \)

where \( \eta = \lambda^2 = \frac{\Lambda}{\Lambda} \), \( E > 0 \)

The extra factor of \( E \) comes from changing the scaling of the external fields.

\[ Y^I \sim \lambda \]

\[ Y^I \sim \eta = \lambda^2 \]

This implies there are no mixed soft-collinear Lagrangian terms at leading order

- after field redefinition there are no mixed \( Y^I \) operators at LO
- mixed \( Y^I(1) \) give \( T \Sigma Y^I(1), Y^I(1) \sim \lambda^2 \)

⇒ matches to \( U^+ \sim \eta \) ie \( U^+(1) \) at worst NLO
Rule of thumb:

- Factorization is simplest to analyze in a frame where the hard physics is "hard" \( p^2 \sim (Q, Q, 0) \) or \( p^2 \sim (Q, Q, Q) \).

- Breit frame, \( B \)-meson rest frame.

Processes:

\( \gamma^* \gamma \rightarrow \pi^* \)

\( \Pi \)-\( \gamma \) form factor at \( Q^2 \gg \Lambda^2 \) for \( \gamma^* \)

Breit frame \( q^\mu = Q^2 (n^\mu - \bar{n}^\mu) \), \( p_\gamma^\mu = E \bar{n}^\mu \)

Pion = collinear in \( n \)-direction

\( \gamma^* M \rightarrow M' \)

\( M-M' \) (meson) form factor \( Q^2 \gg \Lambda^2 \) for \( \gamma^* \)

\( M = \) collinear in \( n \)

\( M' = \) collinear in \( \bar{n} \) (say)

\( B \rightarrow D\pi \)

Matrix \( E_{1+} \otimes 4-\)quark Operators

\( Q = (m_b, m_c, E_{\pi}) \gg \Lambda \)

\( B, D \) are soft \( p^2 \ll \Lambda^2 \), \( \pi \) - collinear \( p_{\pi}^2 \gg \Lambda^2 \)

DIS

\( e^- p \rightarrow e^- X \)

Structure Functions at \( Q^2 \gg \Lambda^2 \)

\( 1-x \gg \Lambda^2/Q \) (ie not near endpts \( \frac{1}{x} \) in Bjorken \( x \))

Breit frame: Proton - collinear, \( X \)-hard

Drell-Yan

\( e^- e^+ \rightarrow X \)

\( d\sigma/dQ^2 \)

\( Q^2 = \text{inv mass of } l^+l^- \gg \Lambda^2 \)

... many more
QCD observable

\[ \langle \pi^0(p_{\pi}) | J_\mu(0) | \gamma(p, q) \rangle = \frac{i e e^3}{2} \int d^4z \ e^{-ip_{\pi} \cdot z} \langle \pi^0(p_{\pi}) | T J_\mu(0) J_{\bar{\nu}}(z) | 0 \rangle \]

\[ = -i e F_{\pi \gamma}(Q^2) \varepsilon_{\mu \nu \rho \sigma} \frac{P_{\pi}^\rho \varepsilon^\sigma}{Q} \]

c.m. current \( J^\mu = \bar{u} \gamma^\mu u \)
\( \hat{Q} = \frac{m}{2} + \frac{1}{6} = \left( \frac{2}{3} \right) \)

For \( Q^2 \gg \Lambda^2 \), \( F_{\pi \gamma} \) simplifies (ala Brodsky-Lepage)

Frame \( \tilde{q}^\mu = \frac{Q}{2} (\eta^\mu - \bar{\eta}^\mu) \)
\( P_{\gamma}^\mu = E \bar{\eta}^\mu \)

\( P_{\pi}^\mu = P_{\gamma} + P_{\pi} = \frac{Q}{2} \eta^\mu + \left( E - \frac{Q}{2} \right) \bar{\eta}^\mu \)
\( E = \frac{Q}{2} + \frac{m_{\pi}^2}{2Q} \)

Use collinear d.o.f. for pion, \( p_{\pi}^2 \sim \Lambda^2 \)

Intermediate propagator is hard, gets integrated out

SCET operator at leading order (for time-ordered product) is

\[ \mathcal{O} = \frac{i e e^3}{Q} \left[ \bar{\eta}_{\rho} W \right] \Gamma C(\bar{p}, \bar{p}^\prime, \mu) \left[ W^\dagger \eta_{\rho} \right] \]

- obeys current conservation
- \( \frac{1}{Q} \) factor
- for \( C \) dimensionless
charge conjugation, \( T \Sigma, J \) was even so \( \xi \) is even need \( C(\mu, \bar{r}, r) = C(\mu, -\bar{r}^+, -\bar{r}^-) \)

flavor structure \( F = \bar{Q}Y_5 \quad 3J_{\Sigma} \quad \hat{Q}^2 \)
spin structure \( \Gamma = \bar{r}Y_5 \quad 3J_{\pi} \quad \hat{Q}^2 \)
for pion 2nd order in e.m.

color singlet, purely collinear, soft gluon decouple
( ignore SCET-I & soft gluon in SCET-II )

\[
\frac{Q^2}{\pi} F_{\pi \gamma} = \frac{i}{\alpha} \langle \pi^0 | \bar{\tau}_{\eta, \rho}^{(\bar{r})} W^{(\bar{r})} \Gamma C W^{(\bar{r})} | \eta_{\rho}^{(\bar{r})} \rangle_{10} \]
\[
\text{write} \quad \bar{r}_{\pm} = \bar{r}_{\pm}^+ \pm \bar{r}^- \quad \text{in} \quad C(\bar{r}_+, \bar{r}_-, \mu) \]

Now \( \bar{r}_- \) gives total momentum of \( \bar{r}_W \) \( \bar{r}_W^+ \) operator
\( \bar{r}_- \) momentum of pion
\[
\bar{r}_- = \bar{r}_\pi = Q \]
\[
\rightarrow \text{total momentum} \\
F_{\pi \gamma}(Q^2) = \frac{2i}{Q^2} \left( \int dw \ C(w, \mu) \langle \pi^0 | \bar{\tau}_{\eta, \rho}^{(\bar{r})} W^{(\bar{r})} \Gamma S(\omega - \bar{r}_\pi) W^{(\bar{r})} | \eta_{\rho}^{(\bar{r})} \rangle_{10} \right) \]

Non-perturbative matrix element:

position space
\[
\langle \pi_{n, \rho}^{(\bar{r})} | \eta_{\rho}^{(\bar{r})} \bar{Q} Y_5 \bar{T}\bar{r}^2 \ W(y, x) \ | \eta_{n, \mu} \rangle_{10} = -i f_{\pi} \eta_{\rho} S_{\pi}(z + \frac{1}{\lambda(x)} \bar{r}_\pi) \int_0^1 dz e^{i \bar{r}_\pi(x)} \rho_{\pi}(z, z) \]
\[
S_{\pi}(z) = 1 \quad \eta_{n, \mu} = \text{F.T} \ \bar{Q} Y_{n, \rho} \ \text{wrt} \ \rho \]
\[ \langle \tilde{T}_{\mu \nu} | \overline{T}_{\mu \nu}, W \sigma \gamma_5 \gamma^3 \gamma^8 \gamma^8 (\omega - \tilde{p}_+ \tilde{p}_- \tilde{p}_+ \tilde{p}_- | 0 \rangle \]

\[ = -i \int_T \tilde{p}_- \int_0^1 dz \frac{1}{z} \lambda \epsilon (\omega - (2z - 1) \tilde{p}_- \tilde{p}_+) \phi_T (x, z) \]

Plug it into \( F_{\pi \gamma} (Q^2) \) and do integral over \( \omega \)

\[ F_{\pi \gamma} (Q^2) = \frac{2 \pi \mu}{Q^2} \int_0^1 dz \frac{1}{z} \lambda \epsilon \left( (2z - 1) Q, Q, \mu \right) \phi_T (x, \mu) \]

- \( \phi_T \) is universal distribution function (light-cone w.f.n.) for pions.
- \( C \) is process dependent, this is all order factorization in \( \lambda \epsilon \), \( \lambda \epsilon \) in \( \lambda \epsilon \).
- One-dim convolution between variables of comparable size

\[ \text{hard} \sim (Q, Q, Q) \]
\[ \text{collinear} \sim \left( \frac{1}{Q^2}, Q, Q \right) \]

Convolution

Tree level calculation of matching:

\[ \exp ( \mu \gamma^8 + \nu \gamma^8 ) = i e \frac{1}{2} \lambda \epsilon \rho \rho' \rho \nu \nu' \nu (\frac{Q}{2} \gamma^8) \hat{Q}^2 \]

\[ \times \left( \frac{1}{\tilde{p}_- \tilde{p}_+} - \frac{1}{\tilde{p}_- \tilde{p}_+'} \right) \]

So \( C = \frac{1}{6} \sqrt{2} \left( \frac{Q}{\tilde{p}_+} - \frac{Q}{\tilde{p}_-} \right) \)

\( C (\omega = (2x - 1) 0) = \frac{1}{6} \sqrt{2} \left( \frac{1}{x} + \frac{1}{1-x} \right) \)

Matched with \( W = 1 \)
Using Charge Conjugation +1 for $|\pi^0>$ we can prove
\[ \Theta_\pi(x) = \Theta_\pi(1-x) \]

So only \[ \frac{\Theta_\pi(x)}{x} \] appears in our prediction.

**Interpretation:** Naively

Really, momentum fractions at point where quarks are produced.

Hadronization process can change $x$ but is entirely encoded in $\Theta_\pi(x)$.

**Matching Beyond Tree Level**

**Full Theory**

**SCET**

Difference will be IR finite (with same regulator used) and give $C$ at one-loop.
Another Exclusive Example

\[ B \to D \pi \]
take \( m_b, m_c, E_\pi \gg \Lambda_{QCD} \),
\( (\bar{Q} \to \Lambda_{QCD}) \)

QCD operators at \( \mu = m_b \)

\[ H_0 = \frac{4G_F}{\sqrt{2}} \, V_{ud} \, V_{ub} \, \left[ C_6 \, O_6 + C_9 \, O_8 \right] \]

where

\[ O_6 = \left[ \bar{e} \gamma^\mu P_L b \right] \left[ \bar{d} \gamma^\nu P_L u \right] \]
\[ O_8 = \left[ \bar{e} \gamma^\mu P_L T^a b \right] \left[ \bar{d} \gamma^\nu P_L T^a u \right] \]

Want to factorize the matrix element \( \langle 0 | O_0,8 | B \rangle \)

History

\[ \langle O_0 \rangle = N \, F_{B \to D} \pi \]

\[ (87) \text{ Bjorken, Color Transparency} \]

\[ (91) \text{ Dugan, Grinstein} \]

\[ \text{LEET} \quad \mathcal{K}_{BF} = \mathcal{F}_B \cdot \mathcal{D} \mathcal{F}_D \]

\[ \langle O_0,8 \rangle = N \, \gamma(w_0, \mu) \int_0^1 d\alpha \, T(x, \mu, \frac{m_c}{m_b}) \, \pi(x, \mu) \]

\[ \uparrow \]

Generalized Factorization

Result should hold to all orders in \( \alpha_s \)

but LO in \( \sqrt{Q} \)

Idea, show

\[ \begin{align*}
\text{Conjecture} & \quad \text{Politzer, Wise (91)} \\
\text{Two-Loop Results} & \quad \text{Beneke et al. (00)} \\
\text{All-Orders Results} & \quad \text{Boyer, Pinjol, I.S.} \\
\end{align*} \]

\[ \text{hep-ph/0107002} \]

\[ \begin{align*}
\text{ie no gluons between quarks in } B, D \\
\text{and quarks in pion at LO in } \sqrt{Q} \\
\end{align*} \]
B, D are soft \[ p^2 \ll \Lambda^2 \] \quad \implies \quad SCET_{\text{II}} \quad \text{degree of freedom}

\[ \Pi \text{ is collinear} \quad p^2 \ll \Lambda^2 \]

Let's use SCET_{\text{II}} as an intermediate step

[1]

\[ \begin{align*}
\mathcal{O}_0 & \longrightarrow \mathcal{O}_0^{1/5} = \left[ \bar{f}_n \Gamma^{h, h \gamma} W f_{\Lambda, \text{soft}} \right] \left[ \bar{f}_{n, \text{soft}} W f_{\Lambda, \text{soft}} \Gamma_{\text{Co}(P_+)} W^+ \gamma_{\Lambda, \text{soft}}^{(5)} \right] \\
\mathcal{O}_8 & \longrightarrow \mathcal{O}_8^{1/5} = \left[ \bar{f}_n \Gamma^{h, h \gamma} W f_{\Lambda, \text{soft}} \right] \left[ \bar{f}_{n, \text{soft}} W f_{\Lambda, \text{soft}} \Gamma_{\text{Co}(P_+)} W^+ \gamma_{\Lambda, \text{soft}}^{(5)} \right]
\end{align*} \]

Mix in general

\[ \uparrow \quad \text{SCET_{\text{II}} operators} \quad \uparrow \quad \text{collinear} \quad p^2 \ll \Lambda^2 \]

[2]

Make field redefinitions \[ Y_{n, \Lambda} = Y_{n, \Lambda}^{(\text{soft})} \]

in \[ \mathcal{O}_0^{1/5} \]

\[ \bar{f}_n^{(5)} W f_{\Lambda, \text{soft}} \]

in \[ \mathcal{O}_8^{1/5} \]

\[ \bar{f}_n^{(5)} W f_{\Lambda, \text{soft}} \]

\[ T^a \otimes Y \otimes Y^+ = Y T^a Y^+ \]

\[ Y^+ T^a Y = Y^{ab} T^b \]

\[ Y \quad \text{adjoint Wilson line} \]

\[ T^a \otimes Y \otimes Y \otimes T^a \]

\[ \Lambda \text{ moves \text{soft} Wilson lines next to } h \text{ fields} \]

in \[ \mathcal{O}_8^{1/5} \]

at some \( \mu^2 \ll \Lambda^2 \).

[3]

Match SCET_{\text{II}} onto SCET_{\text{III}}; trivial here \( Y \rightarrow S \) etc again \( Y_{n, \Lambda}^{(\text{soft})} \rightarrow Y_{n, \Lambda} \) in \( \Pi \)

\[ \begin{align*}
\mathcal{O}_0^{1/5} & = \left[ \bar{f}_n^{(5)} \Gamma^{h, h \gamma} W f_{\Lambda, \text{soft}} \right] \left[ \bar{f}_{n, \text{soft}} W f_{\Lambda, \text{soft}} \Gamma_{\text{Co}(P_+)} W^+ \gamma_{\Lambda, \text{soft}}^{(5)} \right] \\
\mathcal{O}_8^{1/5} & = \left[ \bar{f}_n^{(5)} \Gamma^{h, h \gamma} W f_{\Lambda, \text{soft}} \right] \left[ \bar{f}_{n, \text{soft}} W f_{\Lambda, \text{soft}} \Gamma_{\text{Co}(P_+)} W^+ \gamma_{\Lambda, \text{soft}}^{(5)} \right]
\end{align*} \]
Take Matrix Elements

\[ \langle \bar{\Phi}_m | \mathcal{H} | \Phi_0 \rangle = \frac{i}{2} \int d^3 \mathbf{p} E_p \int dx \, C(2E_p(2x_1)) \phi(x) \]

\[ \langle D_\nu | \mathcal{H} | B \rangle = N' \, \Phi(\omega, \mu) \]

(\text{Isgur-Wise function})

Combining we get final result \( \Rightarrow \) SCET made it easy!!

Here we took \( B, D \) to be purely soft so they have no contractions with collinear fields.

\& we took \( \Phi_0 \) to be purely collinear so it has no contractions with soft fields (\( \Phi \) factors into two matrix elements).

For \( 0 \): \[ \langle D_\nu | \bar{\Phi}_m \mathcal{Y} T^a \mathcal{Y}^+ \Phi_0 \rangle = 0 \]

\text{color act as between color singlet states operator}
Take Matrix Elements

\[ \langle \pi^{-} \mid q_{n} W^{+} C_{o} (P_{e}) \pi^{-} q_{n} \mid 0 \rangle = \frac{i}{2} \int_{\pi} \, E_{n} \, \int_{0}^{1} \, C(z E_{\pi}(z x_{1})) \, \phi(x) \]

\[ \langle D_{e} \mid \pi^{-} \mid h_{o} B \rangle = N' \, \gamma(w_{o}, \mu) \]

*Isigur-Wise function

Combining we get final result → SCET made it easy!!

Here we took \( B, D \) to be purely soft so they have no contractions with collinear fields.

& we took \( \pi^{-} \) to be purely collinear so it has no contractions with soft fields (i.e., factors into two matrix elements)

For \( U_{8} \):

\[ \langle D_{e} \mid \pi^{-} \gamma^{\prime} T^{\alpha} \gamma^{\dagger} h_{o} \mid B_{e} \rangle = 0 \]

*Color octet between color singlet states

*Operator

Find

Factorization Formula

\[ \langle \pi^{-} D_{e} \mid U_{0} \mid B \rangle = i N \, \gamma(w_{o}, \mu) \int_{0}^{1} \, C(z E_{\pi}(z x_{1})) \, \phi(x, \mu) \]

\[ + \, O(1/a_{Q}) \]

*Here \( \gamma(w_{o}, \mu) \) is Isigur-Wise function at max. recoil

\[ w_{o} = \frac{m_{b}^{2} - m_{D}^{2}}{2 m_{b}} \]

*Gives Factorization for so called Type I (\& III) decays

\( B^{-} \to D^{0} \pi^{-}, \quad B^{0} \to D^{+} \pi^{-}, \quad B^{0} \to D^{+} \pi^{-}, \quad B^{0} \to D^{*+} \pi^{-}, \quad \cdots \)

\( B^{-} \to D^{0} \pi^{-}, \quad B^{0} \to D^{0} \pi^{0}, \quad B^{0} \to D^{*+} \pi^{-}, \quad B^{-} \to D^{*+} \pi^{-}, \quad \cdots \)

predicts Type - II decays are suppressed \( B^{0} \to D^{0} \pi^{0} \).....
Inclusive Examples

**DIS**

A rich subject, only the aspects related to QCD factorization are covered here using SCET

References: § 1.8 of text

Aneesh M.'s review: hep-ph/9204208

Bob J.'s review: hep-ph/9602236

Our paper: hep-ph/0202088 (for material below)

$$e^- p \to e^- X$$

$$Q^2 \gg \Lambda^2$$

$$q^2 = -Q^2, \quad x = \frac{Q^2}{2 p \cdot q}$$

$$p^x = p + q^x$$

\[
P_x^2 = \frac{Q^2}{x} (1-x) + m_p^2
\]

Regions

\[
\begin{align*}
p_x^2 & \sim \frac{1}{x} \quad \text{inclusive OPE} \\
p_x^2 & \sim \frac{1}{x} \quad \text{endpoint region} \\
p_x^2 & \sim \frac{\Lambda^2}{Q^2} \quad \text{resonance region}
\end{align*}
\]

Parton Variables

struck quark carries some fraction $\xi$ of proton momentum

$$\vec{\pi} \cdot p = \xi \vec{\pi} \cdot \vec{P}$$

$$p'^2 = Q^2 (\frac{1}{x} - 1)$$

We'll see where the variable $\xi$ comes from in QCD
Frames

Breit Frame

\[ q_\mu^B = \frac{Q}{2} (\vec{p}_r^\mu - \vec{p}_l^\mu) \]

\[ p_\mu^B = \frac{p^\mu}{2} \vec{e}_r \cdot \vec{p} + \frac{p^\mu}{2} \vec{e}_l \cdot \vec{p} \]

\[ = \frac{p^\mu}{2} \frac{Q}{x} + \ldots \quad \text{collinear} \]

\[ p_{x,\mu}^B = \frac{p^\mu}{2} Q + \frac{p^\mu}{2} \frac{Q}{x} (1-x) + \ldots \quad \text{hard} \]

Proton is made of collinear quarks and gluons

Rest Frame

\[ p_\mu^R = \frac{m_p}{2} (\vec{n}_R^\mu + \vec{n}_L^\mu) \quad \text{soft} \]

\[ q_\mu^R = \frac{\vec{n}_R^\mu}{2} \frac{Q^2}{m_p x} - \frac{\vec{n}_L^\mu}{2} \frac{Q}{m_p x} + \ldots \]

\[ p_{x,\mu}^R = \text{sum} \quad \text{"collinear"} \]

\[ p_{x,\mu}^R \sim Q^2 \]

Much like \( B \rightarrow X e \nu \), we can write cross section in terms of leptonic & hadronic tensors

\[ d\sigma = \frac{d^3k'}{2|k'|} \frac{e^4}{sQ^4} L^{\mu \nu} (k, k') W_{\mu \nu} (P, q) \]

We'll only look at spin-averaged case for DIS

\[ W_{\mu \nu} = \frac{1}{\pi} \text{Im} \ T_{\mu \nu} \]

\[ T_{\mu \nu} = \frac{1}{2} \sum_{\text{spin}} \langle p | \hat{T}_{\mu \nu} (q) | p \rangle \]

\[ \hat{T}_{\mu \nu} = i \int d^4x \ e^{i q \cdot x} \ T \left[ J_\mu (x) J_\nu (0) \right] \]

\[ \\text{e.m. currents} \]
$$T_{\mu \nu} = \left( - g_{\mu \nu} + \frac{g_{\mu \nu} b_\nu}{q^2} \right) T_1 (x, Q^2) + \left( P_\mu + \frac{b_\mu}{2x} \right) \left( P_\nu + \frac{b_\nu}{2x} \right) T_2 (x, Q^2)$$

satisfies current conservation, and so on.

Wanted imaginary part of forward scattering.

First match onto SCET operators at leading order:

$$\hat{T}^{\mu \nu} = \frac{g^{\mu \nu}}{Q} \left( O_1^{(i)} + \frac{O_3}{Q} \right) + \frac{(n^\nu + \tilde{n}^\nu)(n^\mu + \tilde{n}^\mu)}{Q} \left( O_2^{(i)} + \frac{O_3}{Q} \right)$$

$O^{(i)}$ operators:

$$O_1^{(i)} = \tilde{f}_{n_{i\nu}, W} \frac{1}{2} C_{ij}^{(i)} (\bar{p}_+, \bar{p}_-) \, W^+ n_{i\nu}$$

$$O_3 = \text{tr} \left[ W^+ B_{\perp}^3 W \, C_{ij}^{(i)} (\bar{p}_+, \bar{p}_-) \, W^+ B_{\perp} \, W \right]$$

where $i, j \in \{u, d, \ldots\}$

$B_{\perp}^3 = [i \bar{c} \tilde{D}_c, i \bar{D}_c c]$

$O_1^{(i)}$ will lead to quark, anti-quark p.d.f.'s

$O_3 $ will lead to gluon p.d.f.'s

Work out quark contribution in detail:

$$O_3^{(i)} = \int \, dw_1, dw_2 \, C_{ij}^{(i)} (w_+, w_-) \left[ \left( \frac{\bar{f}_n}{w_1} \right) \omega_1 \frac{w_1}{w} \left( \omega'^{\nu} \right) \omega_2 \right]$$

$$\hat{T} = \omega_1 \pm \omega_2$$
Coord space: \( f_i/p (z) = \int d\tau \ e^{-i\tau \cdot \vec{r}} \langle p | \bar{\psi} (y) \ W(y, -y) \ \bar{\psi} (y) | p \rangle \)

Space proton distn for quark \( i \) in proton \( p \)

\( \bar{f}_i/p (z) = - \bar{f}_i/p (-z) \) for antiquark

Mom.

\(< p_1 | (\bar{\tau}_0 \omega)_{\theta_1} \ W (W \bar{\theta}_0)_{\theta_2} | p_0 \rangle = \imath \bar{\pi} \cdot \vec{p} \int \frac{d^3 \delta (\omega -)}{3!} \)

\[ \times \left[ \delta (\omega - 2 \pi \bar{\pi} \cdot \vec{p}) \ f_i/p (z) - \delta (\omega + 2 \pi \bar{\pi} \cdot \vec{p}) \ \bar{f}_i/p (z) \right] \]

↑ positive \( \omega_1 = \omega_2 \)

↑ negative \( \omega_1 = \omega_2 \)

give particles

give antiparticles

\((\bar{\tau}_0 \omega)_{\theta_1} \ W (W \bar{\theta}_0)_{\theta_2}\) is like a number operator

for collinear quarks with momentum \( \omega \)

Charge Conjugation:

\[ C_{{\tau}^i} (-\omega_+, \omega_-) = - C_{{\tau}^i} (\omega_+, \omega_-) \]

relates Wilson coefficients for quarks & antiquarks directly at operator level

Now matching

\[ T_1 = \frac{- C_1 (\omega)}{A} \langle \sigma^{i} (\omega) \rangle \]

\[ T_2 = \left( \frac{4 \pi}{A} \right)^2 \frac{1}{A} \left( C_0 (\omega) - \frac{C_1 (\omega)}{4} \right) \langle \sigma^{i} (\omega) \rangle \]

Delta functions set

\[ \omega_+ = \pm \frac{2 \pi \bar{\pi} \cdot \vec{p}}{x} \]

\[ \omega_- = 0 \]
Defining \( H_3 (z) = C_3 \left( 2Q, z, 0, Q, \mu \right) \)

\[
T_1 (x, Q^2) = - \frac{1}{x} \int_0^1 d \frac{T}{x} \ H_1^{(i)} \left( \frac{T}{x} \right) \left[ f_{i1} (T) + \bar{f}_{i1} (T) \right]
\]

\[
T_2 (x, Q^2) = \frac{4x}{Q^2} \int_0^1 d \frac{T}{x} \ (4 H_2^{(i)} (T) - H_1^{(i)} (T)) \left[ f_{i1} (T) + \bar{f}_{i1} (T) \right]
\]

This is factorization for DIS into computable coefficient \( H_i \)
universal non-perturbative function \( f_{i1} / p, \bar{f}_{i1} / p \)

(reshat up in many processes)

Coefficients \( C_i \) were dimensionless and can only have \( ds (\mu \ln (\mu / Q)) \) dependence on \( Q \)

\( \Rightarrow \) Bjorken scaling

Tree level matching:

\[
\begin{align*}
\text{find just } g_{\mu}^{\mu} \quad \text{ie } \quad C_2 &= 0 \quad \Rightarrow \quad \text{gives Callan-Gross relation } \quad \frac{\omega_1}{\omega_2} = \frac{Q^2}{4x^2} \\
C_1 (\omega_2) &= 2 e^2 Q^2 \left[ \frac{Q}{(\omega_2 - 2Q) + i\epsilon} - \frac{Q}{(\omega_2 + 2Q) + i\epsilon} \right] \\
\text{Im } \ H_1 &= - e^2 Q^2 \pi \left( \frac{y}{x} - 1 \right) \\
\end{align*}
\]

gives parton model interpretation that \( y = x \)
Another inclusive example \( B \to X_s \gamma \)

Here we will need both \( u \) soft and \( c \) collinear d.o.f. in SCET

\[
\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{\alpha_s}} V_{tb} V_{ts}^* \, C_7 \, U_7 , \quad U_7 = e \frac{m_b}{16\pi^2} \sigma^{\mu\nu} F_{\mu\nu} \, \bar{P}_b \, P_r b
\]

Photon \( g^\mu = E_{\gamma} \bar{\pi}^\mu \)

\[
\frac{1}{p_\gamma} \, \frac{dF}{dE_{\gamma}} = \frac{4E_{\gamma}}{m_b^3} \left( -\frac{i}{\pi} \right) \text{Im} \, T
\]

\[
T = \frac{i}{m_b} \int d^4x \, e^{-i p_{\gamma} x} \langle \bar{b} \mid T J_\mu^+(x) J_-^\mu(0) \mid b \rangle
\]

\[
J_\mu = s \, i \sigma^\nu \bar{q}_u \, P_L \, b
\]

Consider endpoint region

\[
\frac{m_b}{\Lambda} - E_{\gamma} \ll \Lambda \alpha_s \quad p_x^2 = m_b \Lambda
\]

\( B \) rest frame \( p_b = \frac{m_b}{2} (n^\mu + \bar{n}^\mu) = p_x + \bar{p} \)

\[
p_x = \frac{m_b}{2} n^\mu + \frac{\bar{n}^\mu}{2} (m_b - 2E_{\gamma})
\]

Collinear

so quark and gluons in \( X \) are collinear with \( p_c + m_b \Lambda \)

\( B \) has \( u \) soft \( \gamma \) light d.o.f.
match onto LO SCET operator

\[ J_{\mu} = - E \gamma_{\mu} e^{-i (p_{T} / 2 - m_{W}) \cdot x} \sqrt{s} W Y^{+} p_{L} h_{\nu} C(p^{+}, \nu) \]

\[ \frac{C^{2}(x, y)}{\sqrt{s}} \frac{C^{2}(x, y)}{\sqrt{s}} \]

our heavy-to-light correct from earlier \[ \nu \equiv \bar{\nu} \]

The coefficient \( C(p^{+}) \) has \( \bar{p}^{+} = M_{6} \) since this is total momentum of s-quark jet in \( \bar{p}^{+} p_{x} \)

Factor with Field redefinition

\[ J_{\alpha}^{\mu} = \gamma_{\mu}^{(a)} \gamma_{5}^{(a)} p_{L} Y^{+} h_{\nu} \]

\[ T_{\alpha} = i \int d^{4}x \ e^{i (m_{W} / 2 - \xi - \eta) \cdot x} \langle \bar{B} | T_{\alpha}^{+} \rangle \langle x | J_{\mu}^{\alpha}(x) J_{\alpha}^{\mu}(x) | \bar{B} \rangle \]

\[ \text{factored} \]

\[ = i \int d^{4}x \ e^{i \xi} \langle \bar{B} | T_{\alpha}^{+} \rangle \langle x | (\bar{W})^{(a)}(x) \gamma_{\nu}^{(a)}(x) \gamma_{\nu}(0) \rangle \langle \bar{B} | \gamma_{\nu}(0) \gamma_{\nu}(0) \rangle \]

\[ \text{spin and color indices suppressed} \]

\[ = \frac{1}{2} \int d^{4}x \int d^{4}k \ e^{i (m_{W} / 2 - \xi - \eta - k) \cdot x} \langle \bar{B} | T_{\alpha}^{+} \rangle \langle x | (\bar{W})^{(a)}(x) \gamma_{\nu}^{(a)}(x) \gamma_{\nu}(0) \rangle \langle \bar{B} | \gamma_{\nu}(0) \gamma_{\nu}(0) \rangle \]

\[ \langle \bar{B} | T_{\alpha}^{+} \rangle \langle x | (\bar{W})^{(a)}(x) \gamma_{\nu}^{(a)}(x) \gamma_{\nu}(0) \rangle \langle \bar{B} | \gamma_{\nu}(0) \gamma_{\nu}(0) \rangle \]

in \( T_{\alpha}^{+} \) we then get

\[ S(z) = \frac{1}{2} \int d^{4}x \ e^{-i \frac{1}{2} z + 1 \cdot x} \langle \bar{B} | T_{\alpha}^{+} \rangle \langle x | (\bar{W})^{(a)}(x) \gamma_{\nu}^{(a)}(x) \gamma_{\nu}(0) \rangle \langle \bar{B} | \gamma_{\nu}(0) \gamma_{\nu}(0) \rangle \]

\[ S(z) = \frac{1}{2} \langle \bar{B} | \gamma_{\nu}(0) \rangle \langle \bar{W} | (s \cdot \nu - k^{+}) h_{\nu} \rangle \langle \bar{B} | \gamma_{\nu}(0) \rangle \]

\[ \Rightarrow \]

only depends on \( k^{+} \) !

so do \( k^{+}, k^{+} \) integrals
Imaginary part is in jet function

\[ J(k^+) = -\frac{1}{\mu^2} \text{Im} \ J_\rho(k^+) \]

(Tree level)

\[ J(k^+) = S(k^+) \]

All order's factorization

\[ \frac{1}{\Gamma_0} \frac{d\Gamma}{dE_\gamma} = N \ C(m_b, \mu) \int d\ell^+ \ S(\ell^+) \ J(\ell^++m_b-2E_\gamma) \]

\[ \uparrow \]

\[ 2E_\gamma - m_b \]

\[ \quad \quad \quad \quad \quad \]

\[ \rho^2 - m_b^2 \]

\[ \rho^2 - m_b^2 \]

\[ \rho^2 - m_b^b \]

\[ \uparrow \]

Shape function is seen in the data