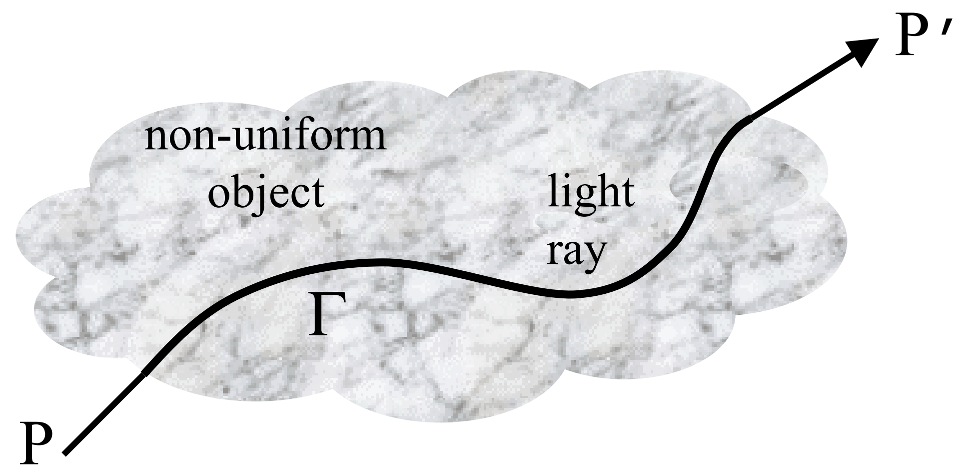
Lenses and imaging

- Huygens principle and why we need imaging instruments
- A simple imaging instrument: the pinhole camera
- Principle of image formation using lenses
- Quantifying lenses: paraxial approximation & matrix approach
- "Focusing" a lens: Imaging condition
- Magnification
- Analyzing more complicated (multi-element) optical systems:
  - Principal points/surfaces
  - Generalized imaging conditions from matrix formulae

The minimum path principle

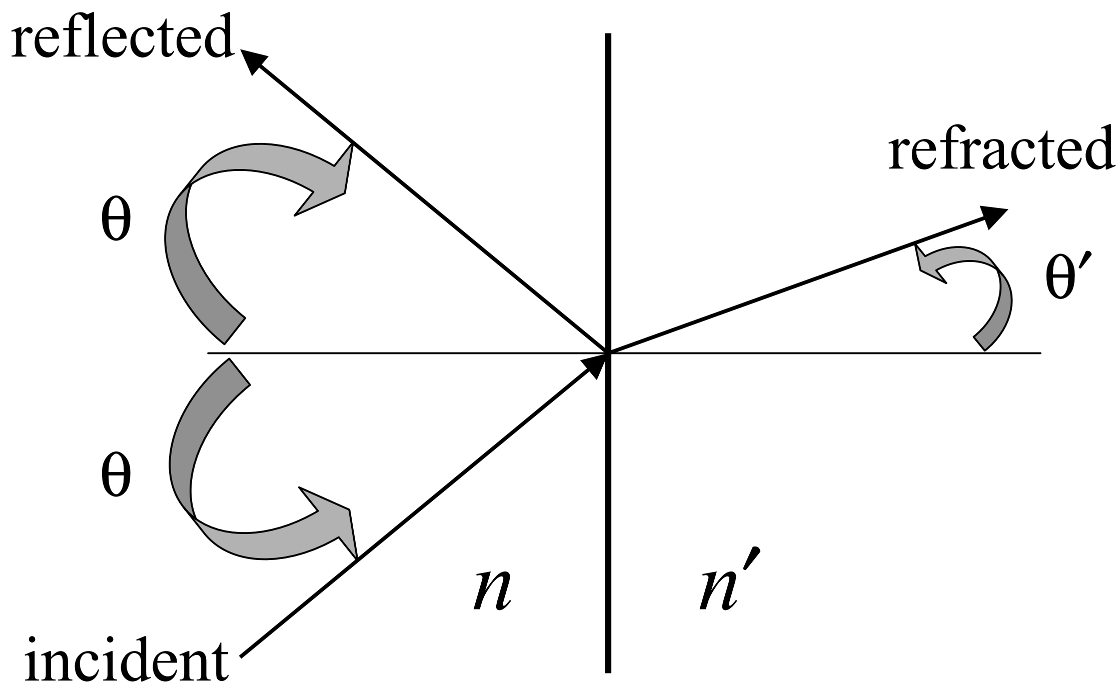


$$\int_{\Gamma} n(x, y, z)\,\mathrm{d}l$$

Γ is chosen to minimize this "path" integral, compared to alternative paths

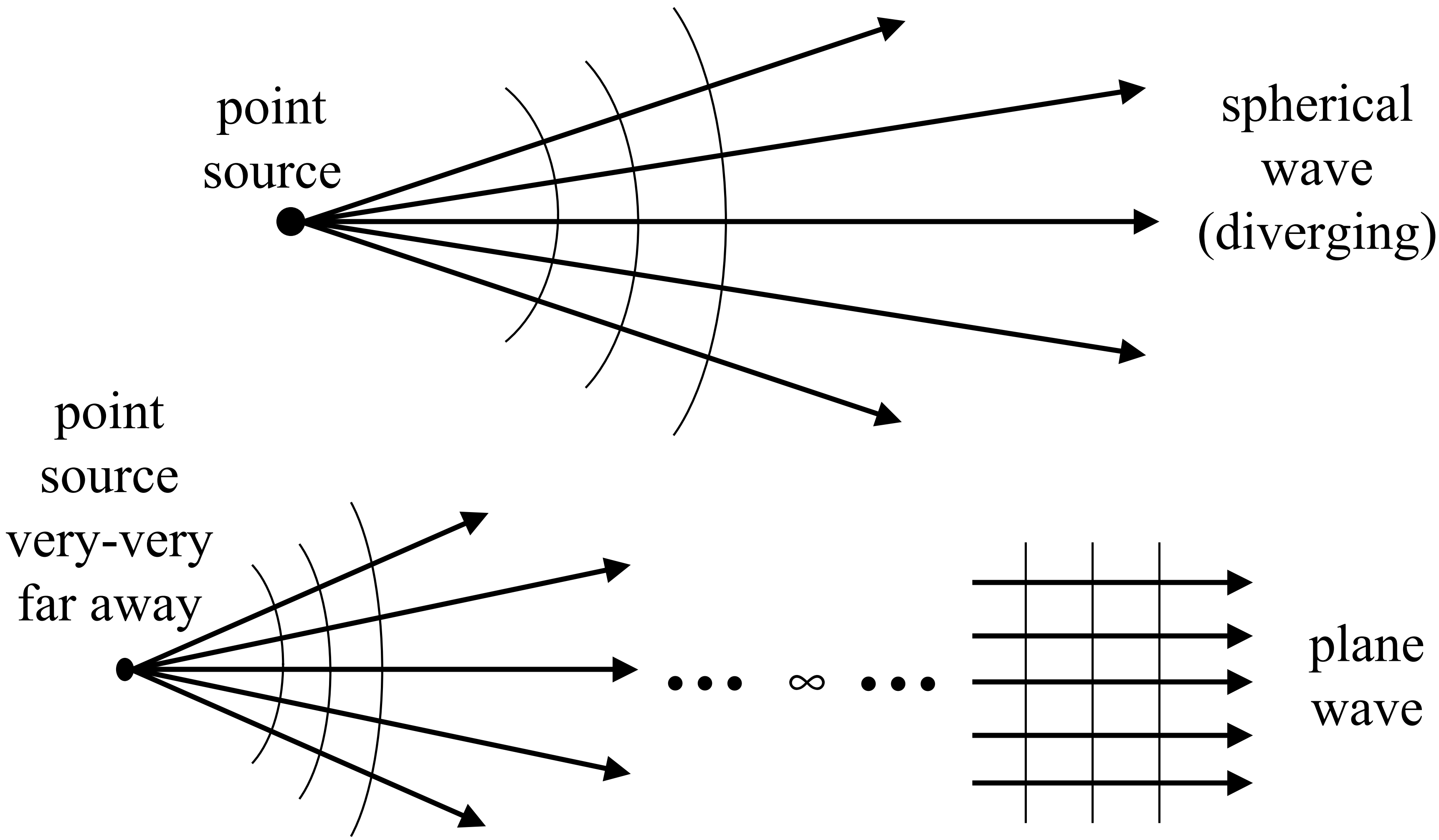
(aka **Fermat**'s principle)
Consequences: law of reflection, law of refraction

The law of refraction



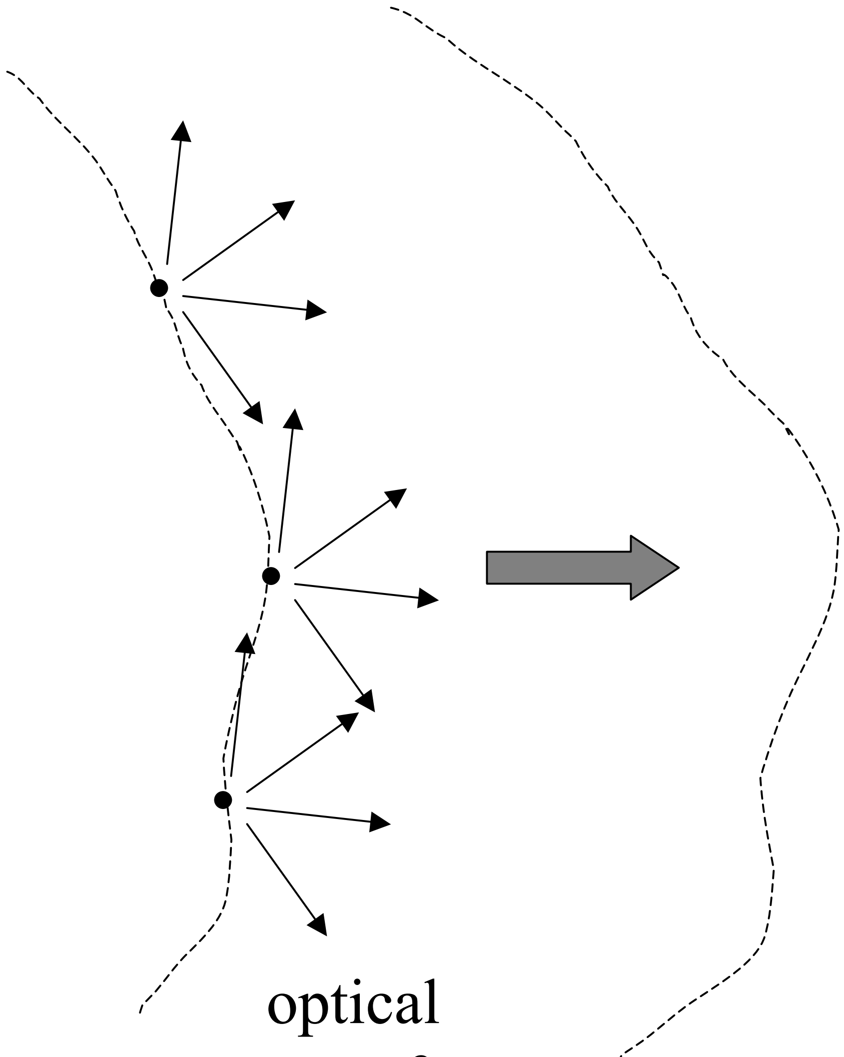
$$\boxed{n \sin\theta = n' \sin\theta'}$$ Snell's Law of Refraction

Ray bundles



point source
spherical wave (diverging)
point source very-very far away
• • • ∞ • • •
plane wave

Huygens principle

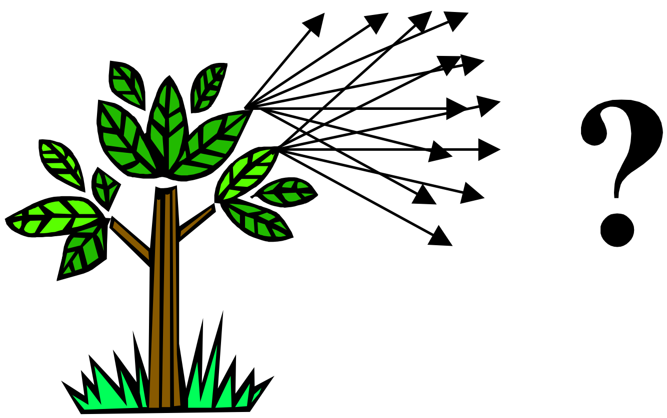


Each point on the wavefront acts as a secondary light source emitting a spherical wave

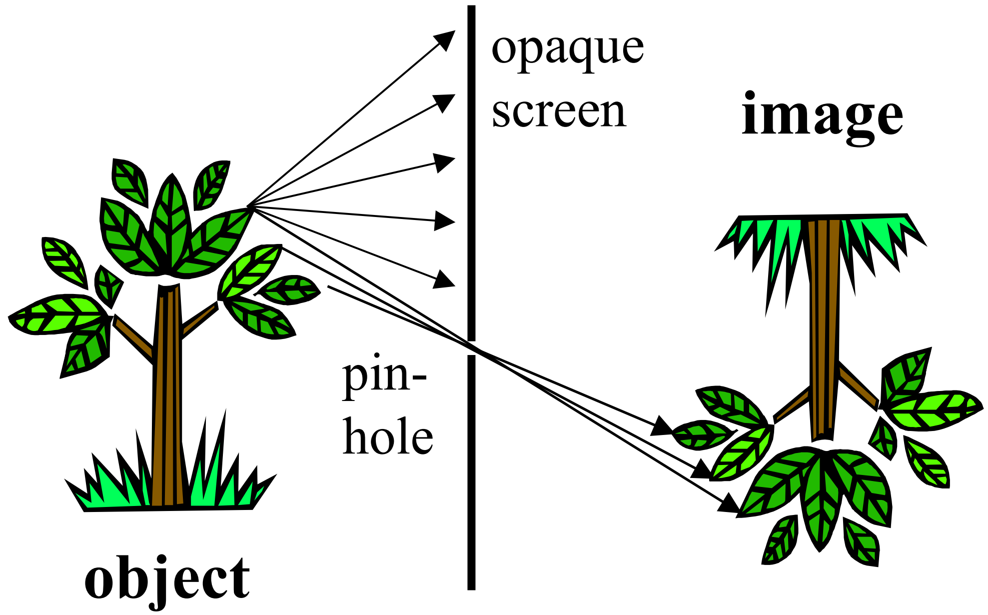
The wavefront after a short propagation distance is the result of superimposing all these spherical wavelets

Why imaging systems are needed

- Each point in an object scatters the incident illumination into a spherical wave, according to the Huygens principle.
- A few microns away from the object surface, the rays emanating from all object points become entangled, delocalizing object details.
- To relocalize object details, a method must be found to reassign ("focus") all the rays that emanated from a single point object into another point in space (the "image.")
- The latter function is the topic of the discipline of Optical Imaging.



The pinhole camera

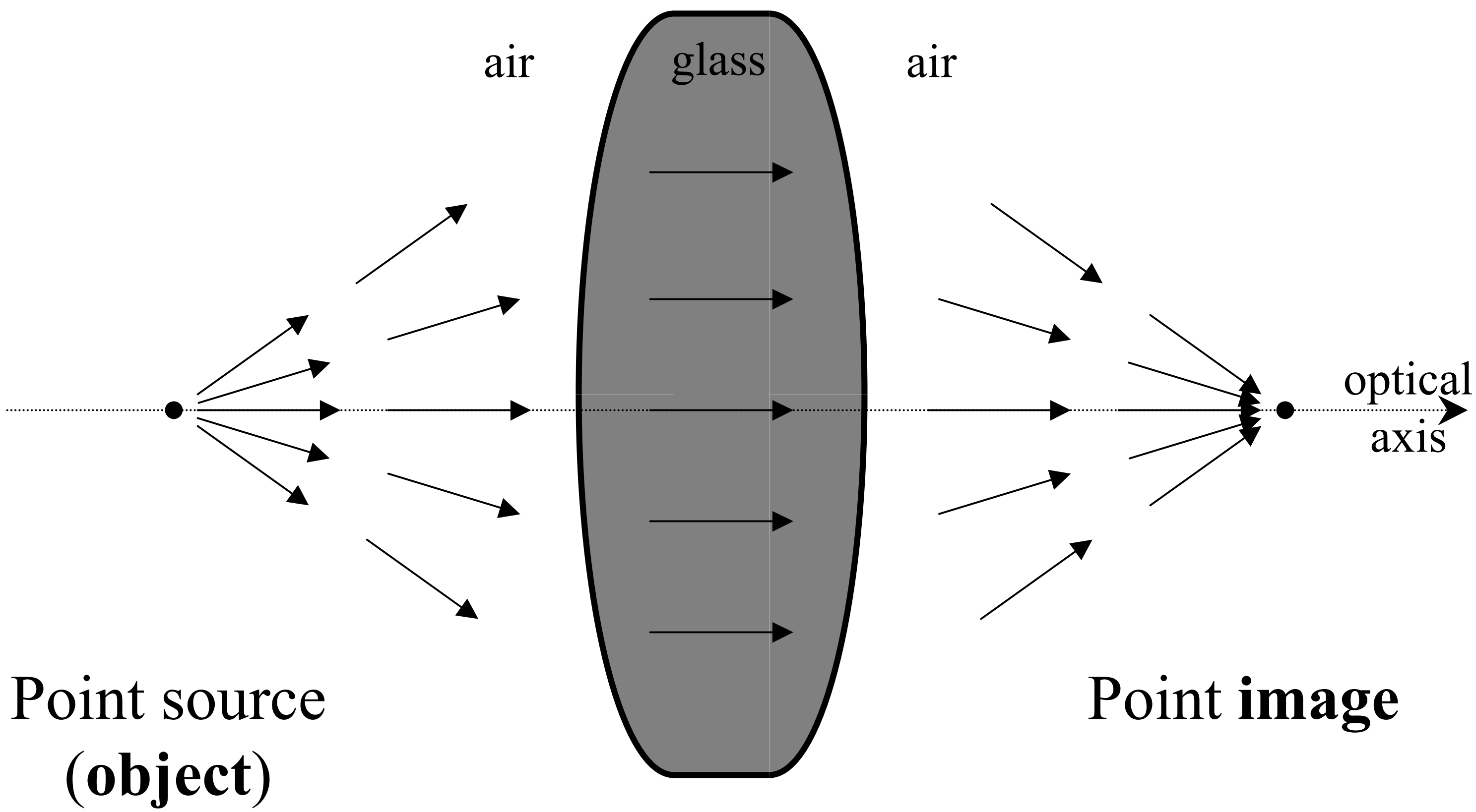


• The pinhole camera blocks all but one ray per object point from reaching the image space $\Rightarrow$ an image is formed (*i.e.*, each point in image space corresponds to a single point from the object space).

• Unfortunately, most of the light is wasted in this instrument.

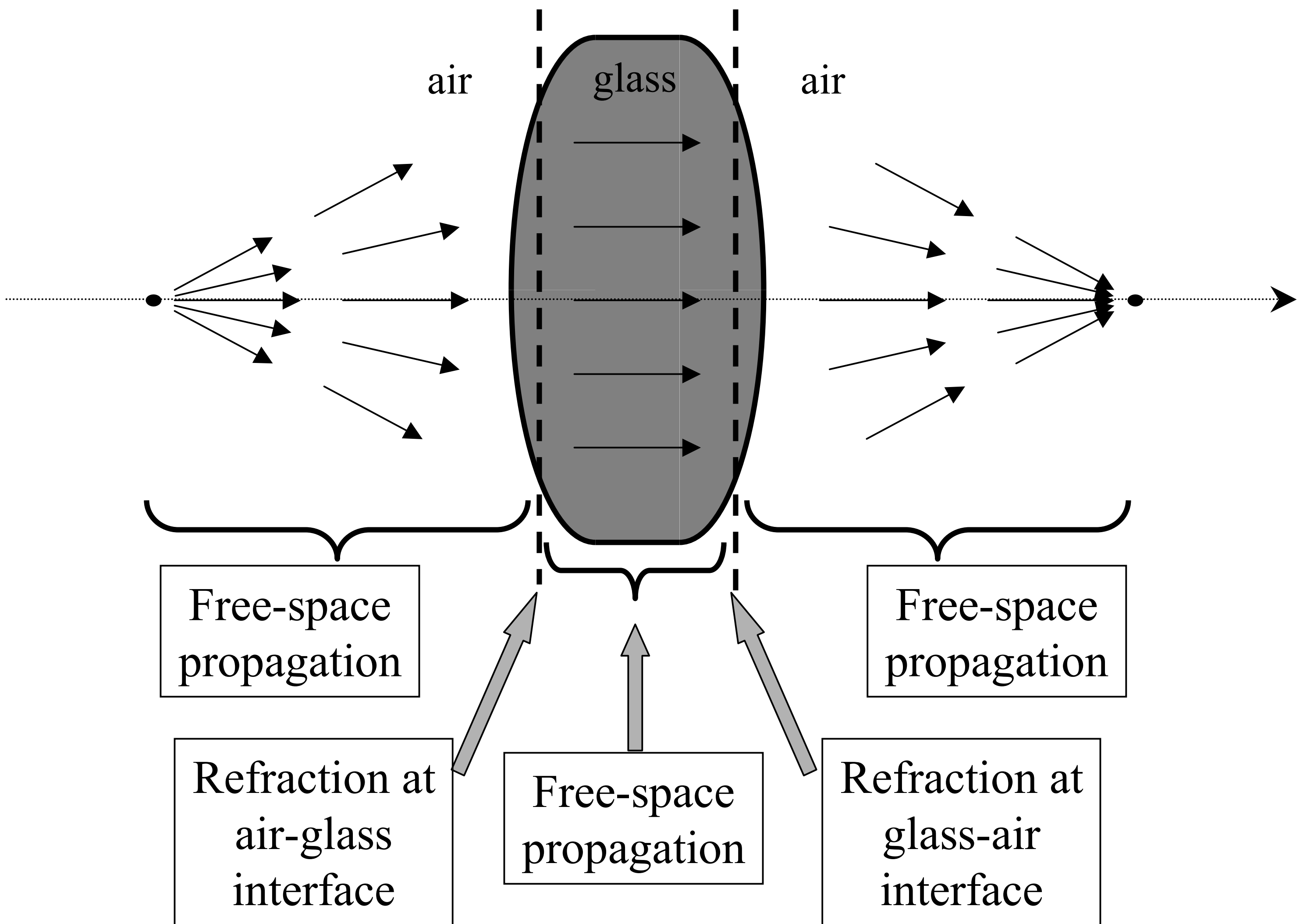
• Besides, light diffracts if it has to go through small pinholes as we will see later; diffraction introduces artifacts that we do not yet have the tools to quantify.

Lens: main instrument for image formation



The curved surface makes the rays bend proportionally to their distance from the “optical axis”, according to Snell’s law. Therefore, the divergent wavefront becomes convergent at the right-hand (output) side.

Analyzing lenses: paraxial ray-tracing



Paraxial approximation /1

- In paraxial optics, we make heavy use of the following approximate (1st order Taylor) expressions:

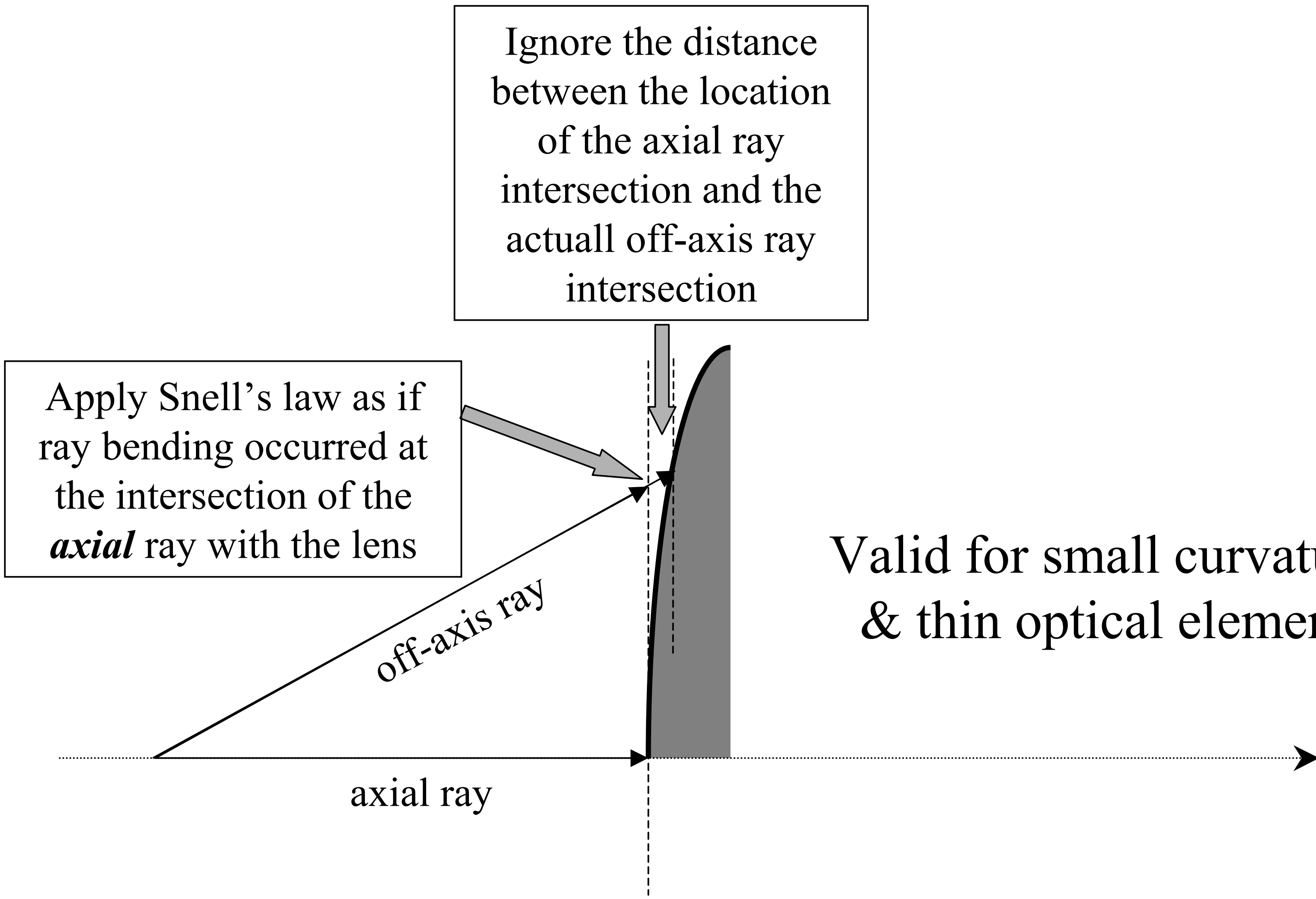
$$\sin\varepsilon \approx \varepsilon \approx \tan\varepsilon \qquad \cos\varepsilon \approx 1$$

$$\sqrt{1+\varepsilon} \approx 1 + \frac{1}{2}\varepsilon$$

where ε is the angle between a ray and the optical axis, and is a small number ($\varepsilon \ll 1$ rad). The range of validity of this approximation typically extends up to ~10-30 degrees, depending on the desired degree of accuracy. This regime is also known as "Gaussian optics."

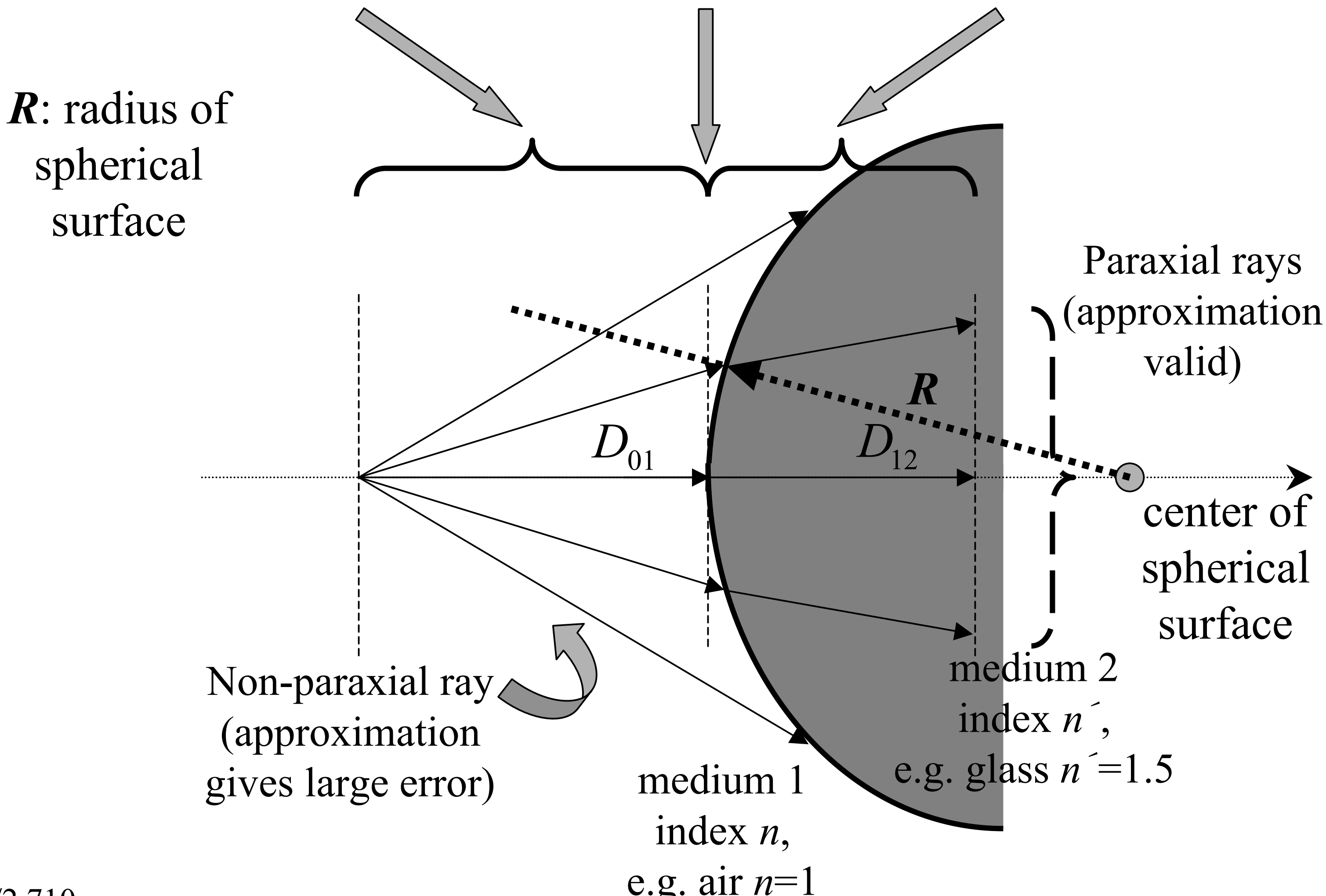
Note the assumption of existence of an optical axis (*i.e.*, perfect alignment!)

Paraxial approximation /2

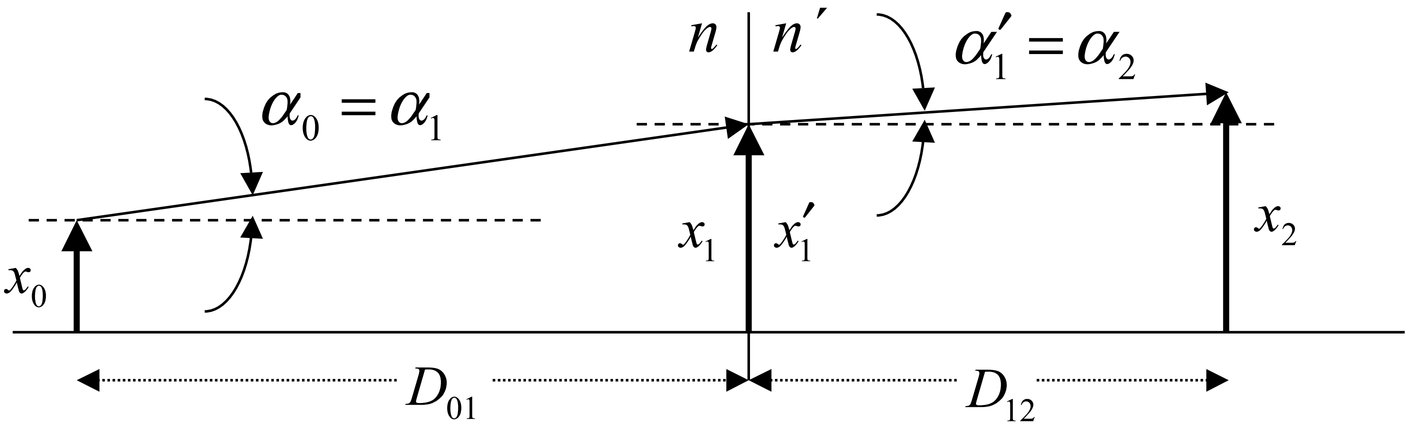


Valid for small curvatures & thin optical elements

Example: one spherical surface, translation+refraction+translation



Translation+refraction+translation /1



Starting ray: location x_0 direction α_0

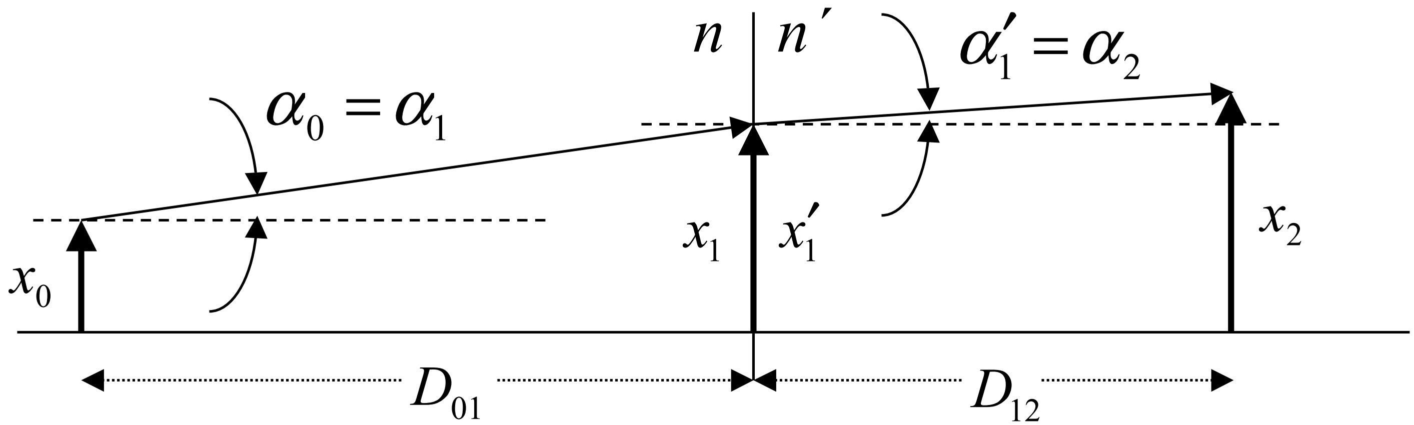
Translation through distance D_{01} (+ direction):

$$\begin{cases} x_1 = x_0 + D_{01}\alpha_0 \\ \alpha_1 = \alpha_0 \end{cases}$$

Refraction at positive spherical surface:

$$\begin{cases} x_1' = x_1 \\ \alpha_1' = \dfrac{n}{n'}\alpha_1 + \left[\dfrac{(n-n')}{n'R}\right]x_1 \end{cases}$$

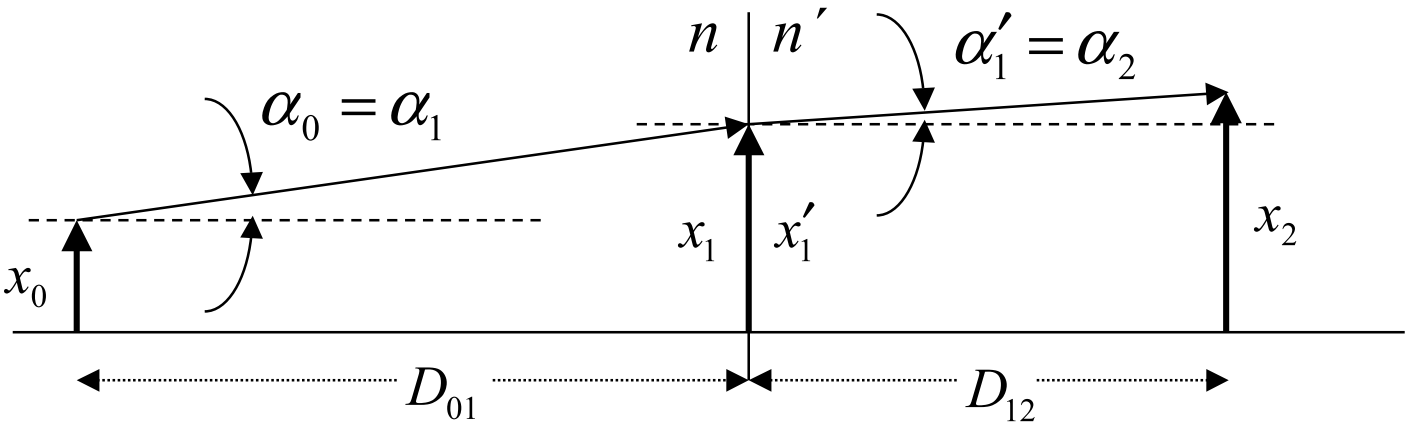
Translation+refraction+translation /2



Translation through distance D_{12} (+ direction): $\begin{cases} x_2 = x_1 + D_{12}\alpha_1' \\ \alpha_2 = \alpha_1' \end{cases}$

Put together:

Translation+refraction+translation /3

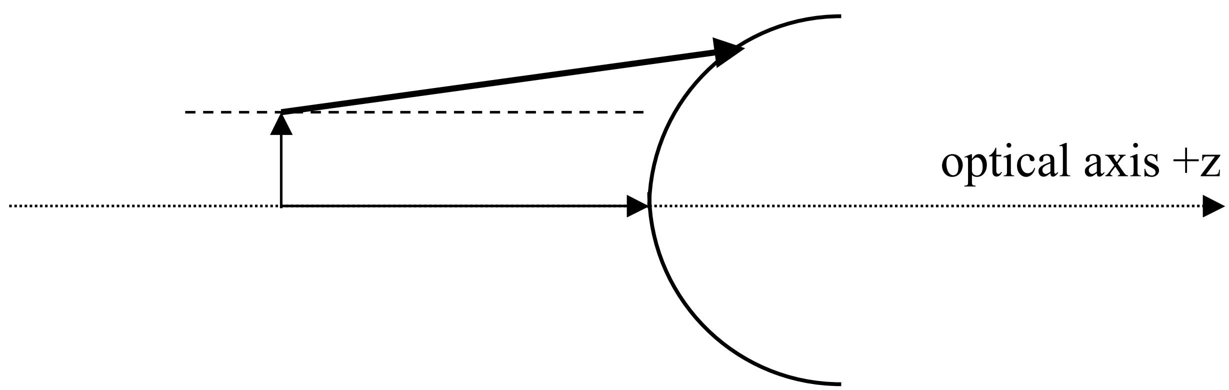


$$x_2 = \left[\frac{(n-n')D_{12}}{n'R} + 1\right]x_0 + \left[D_{01} + \frac{nD_{12}}{n'} + \frac{(n-n')D_{01}D_{12}}{n'R}\right]\alpha_0$$

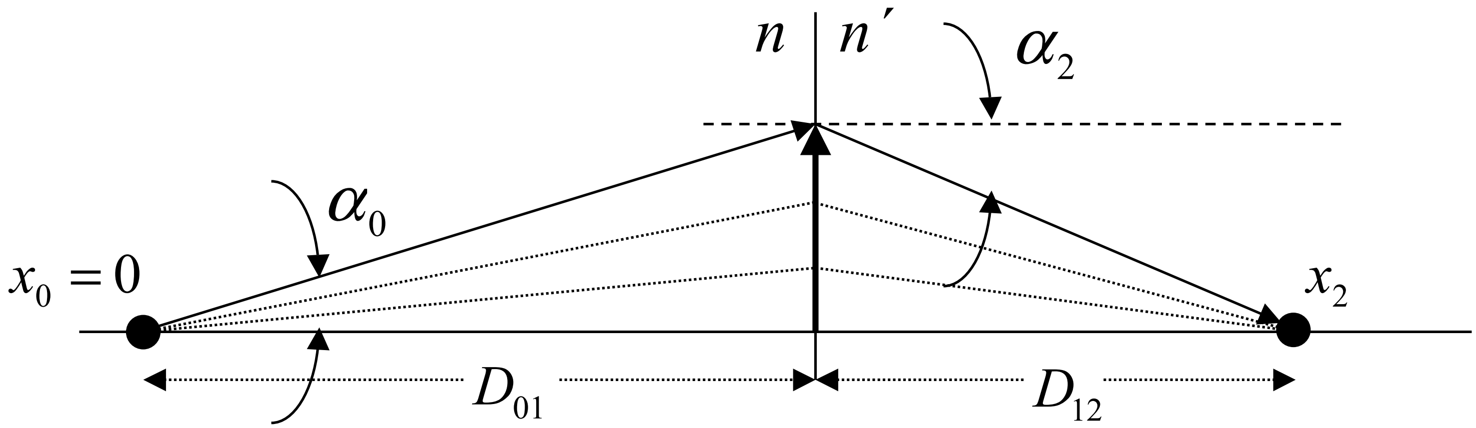
$$\alpha_2 = \left[\frac{n-n'}{n'R}\right]x_0 + \left[\frac{n}{n'} + \frac{(n-n')D_{01}}{n'R}\right]\alpha_0$$

Sign conventions for refraction

- Light travels from left to right
- A radius of curvature is positive if the surface is convex towards the left
- Longitudinal distances are positive if pointing to the right
- Lateral distances are positive if pointing up
- Ray angles are positive if the ray direction is obtained by rotating the +z axis counterclockwise through an acute angle



On-axis image formation

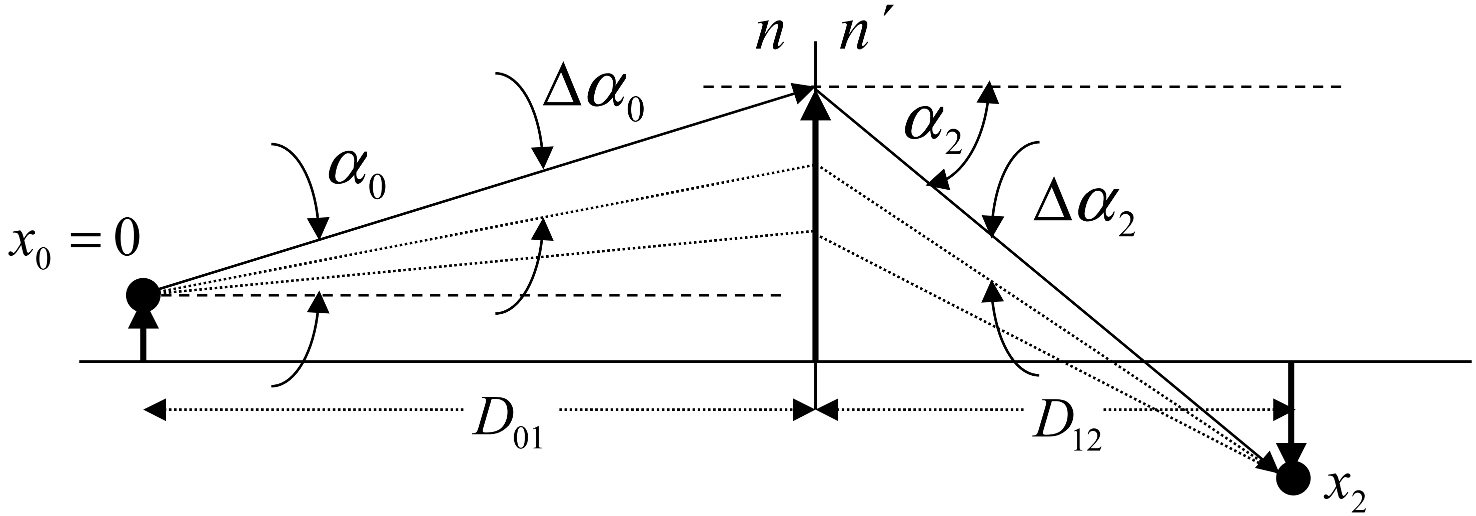


All rays emanating at x_0 arrive at x_2 irrespective of departure angle α_0 $\Rightarrow$ $\dfrac{\partial x_2}{\partial \alpha_0} = 0$ $\Rightarrow$

$$\frac{n'}{D_{12}} + \frac{n}{D_{01}} = \frac{n' - n}{R}$$

"Power" of the spherical surface [units: diopters, 1D=1m^{-1}]

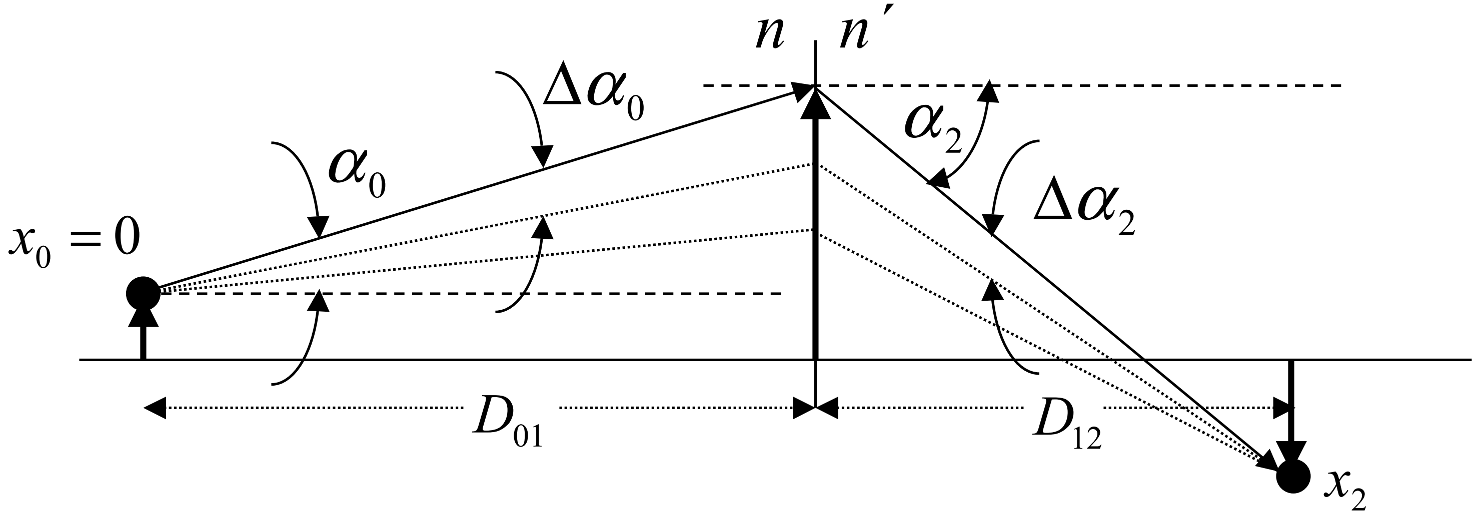
Magnification: lateral (off-axis), angle



Lateral $$m_x = \frac{x_2}{x_0} = \frac{n-n'}{R}\frac{D_{12}}{n'} + 1 = ... = -\frac{n}{n'}\frac{D_{12}}{D_{01}}$$

Angle $$m_\alpha = \frac{\Delta\alpha_2}{\Delta\alpha_0} = -\frac{D_{01}}{D_{12}}$$

Object-image transformation

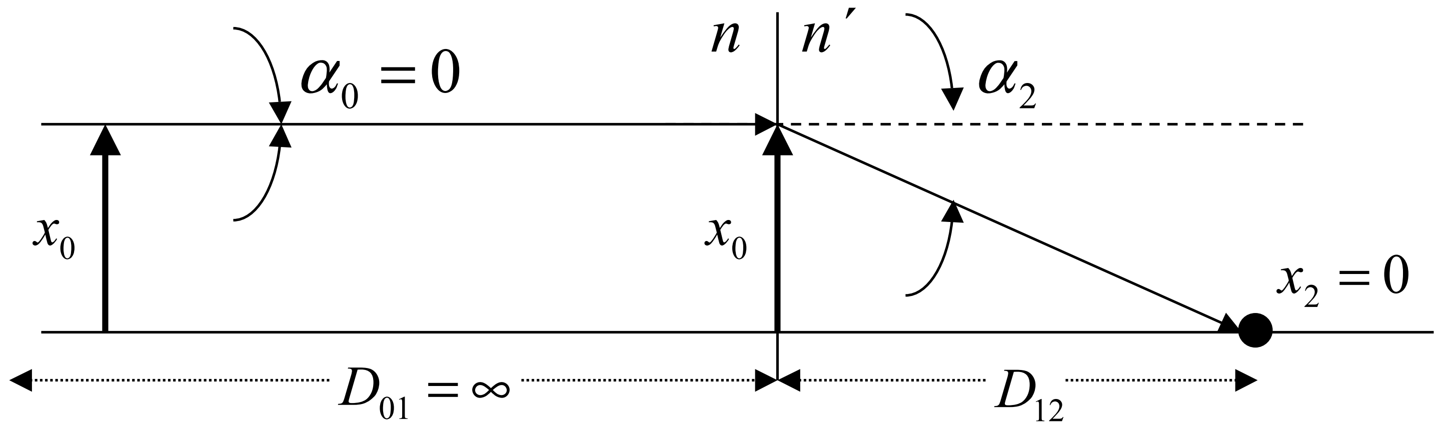


$$x_2 = m_x x_0$$

$$\alpha_2 = -\frac{1}{f'} x_0 + m_\alpha \alpha_0$$

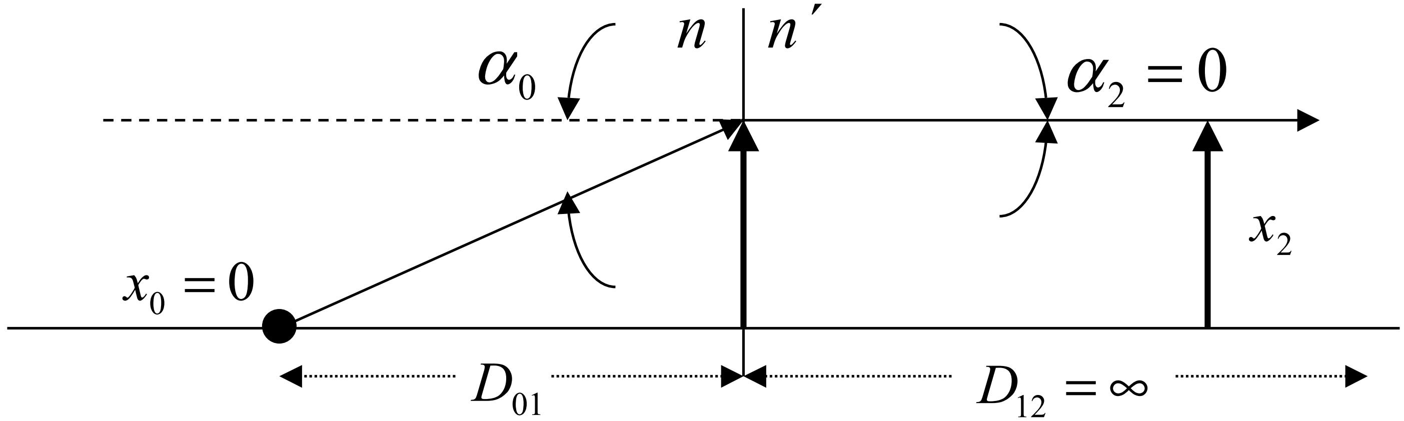
Ray-tracing transformation (paraxial) between object and image points

Image of point object at infinity



$$\frac{n'}{D_{12}} = \frac{n'-n}{R} \Rightarrow D_{12} = \frac{n'R}{n'-n} \equiv f' : \text{image focal length}$$

Point object imaged at infinity



$$\frac{n}{D_{01}} = \frac{n' - n}{R} \Rightarrow D_{01} = \frac{n'R}{n' - n} \equiv f : \text{object focal length}$$

Matrix formulation /1

$$x_1 = x_0 + D_{01}\alpha_0$$
$$\alpha_1 = \alpha_0$$

translation by distance D_{01}

$$x_1' = x_1$$
$$\alpha_1' = \frac{n}{n'}\alpha_1 + \left[\frac{(n-n')}{n'R}\right]x_1$$

refraction by surface with radius of curvature R

$$x_2 = m_x x_0$$
$$\alpha_2 = -\frac{1}{f'}x_0 + m_\alpha \alpha_0$$

ray-tracing object-image transformation

$$\alpha_{\text{out}} = M_{11}\alpha_{\text{in}} + M_{12}x_{\text{in}}$$
$$x_{\text{out}} = M_{21}\alpha_{\text{in}} + M_{22}x_{\text{in}}$$

form common to all

Matrix formulation /2

$$\alpha_{\text{out}} = M_{11}\alpha_{\text{in}} + M_{12}x_{\text{in}}$$
$$x_{\text{out}} = M_{21}\alpha_{\text{in}} + M_{22}x_{\text{in}}$$

$$\begin{pmatrix} n_{\text{out}}\alpha_{\text{out}} \\ x_{\text{out}} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} n\alpha_{\text{in}} \\ x_{\text{in}} \end{pmatrix}$$

<u>Refraction by spherical surface</u>

$$x_1' = x_1$$
$$\alpha_1' = \frac{n}{n'}\alpha_1 + \left[\frac{(n-n')}{n'R}\right]x_1$$

Power

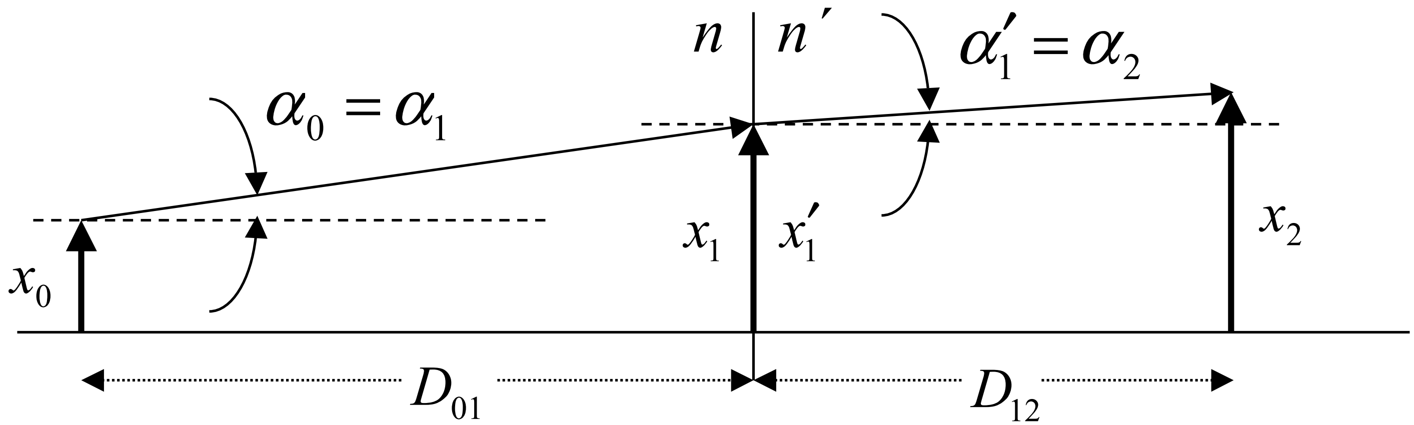
$$\begin{pmatrix} n'\alpha_1' \\ x_1' \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n'-n}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix}$$

<u>Translation through uniform medium</u>

$$x_1 = x_0 + D_{01}\alpha_0$$
$$\alpha_1 = \alpha_0$$

$$\begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{D_{01}}{n} & 1 \end{pmatrix} \begin{pmatrix} n\alpha_0 \\ x_0 \end{pmatrix}$$

Translation+refraction+translation



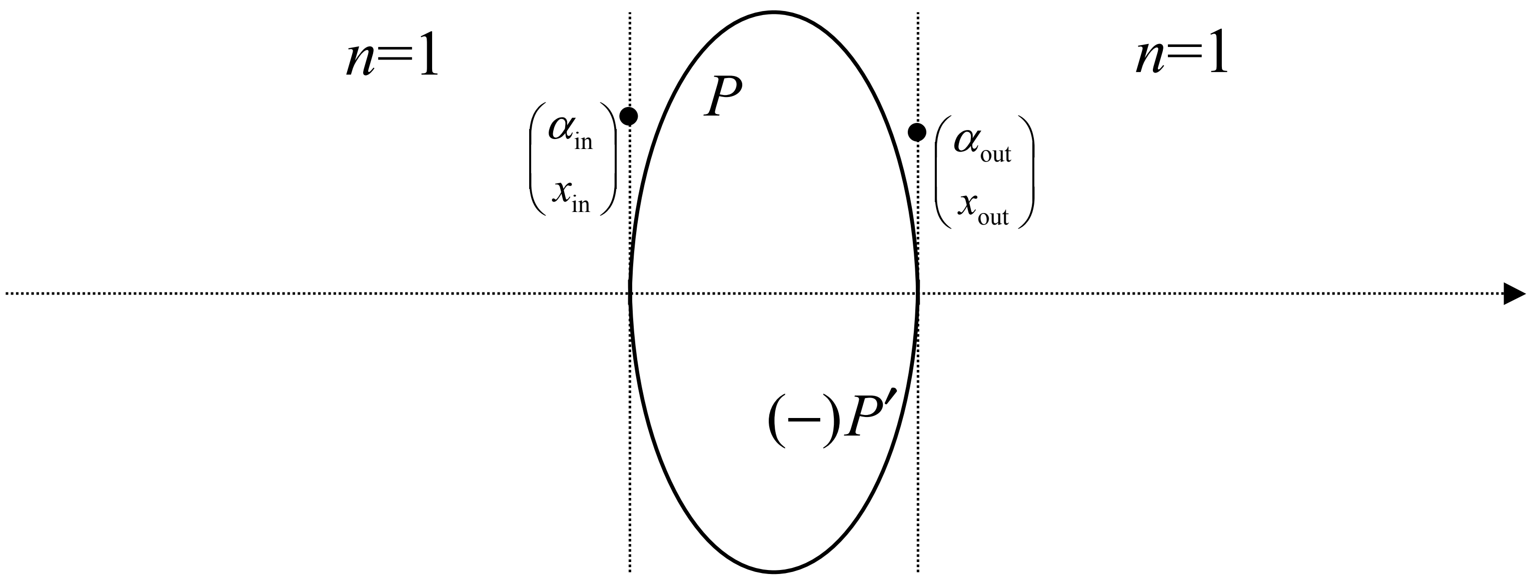
$$\begin{pmatrix} n'\alpha_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \text{translation} \\ \text{by } D_{12} \end{pmatrix} \begin{pmatrix} \text{refraction} \\ \text{by r.curv. } R \end{pmatrix} \begin{pmatrix} \text{translation} \\ \text{by } D_{01} \end{pmatrix} \begin{pmatrix} n\alpha_0 \\ x_0 \end{pmatrix}$$

result…

$$n'\alpha_2 = \left[\frac{n-n'}{R}\right]x_0 + \left[n + \frac{(n-n')D_{01}}{R}\right]\alpha_0$$

$$x_2 = \left[\frac{(n-n')D_{12}}{n'R} + 1\right]x_0 + \left[D_{01} + \frac{nD_{12}}{n'} + \frac{(n-n')D_{01}D_{12}}{n'R}\right]\alpha_0$$

Thin lens



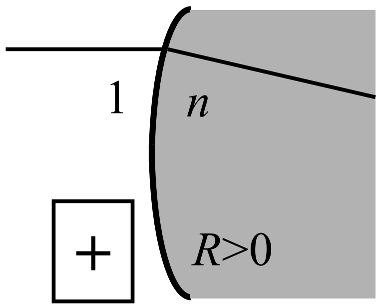
$$\begin{pmatrix}\alpha_{\text{out}}\\ x_{\text{out}}\end{pmatrix}=\begin{pmatrix}1 & -\left[P+P'\right]\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\text{in}}\\ x_{\text{in}}\end{pmatrix}$$

$$P_{\text{thin lens}}=\frac{n-1}{R}+\frac{1-n}{R'}=\boxed{(n-1)\left(\frac{1}{R}-\frac{1}{R'}\right)}$$

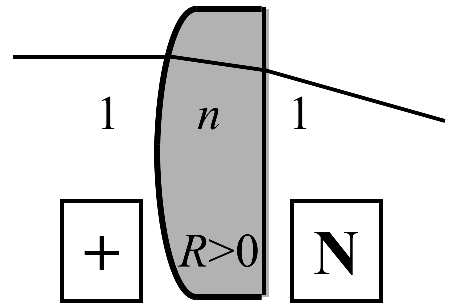
Lens-maker's formula

The power of surfaces

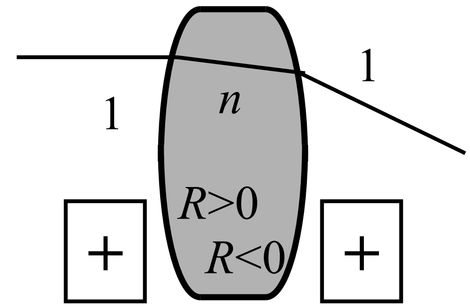
- Positive power bends rays "inwards"



Simple spherical
refractor (positive)

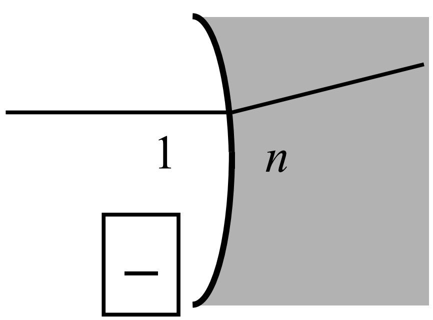


Plano-convex
lens

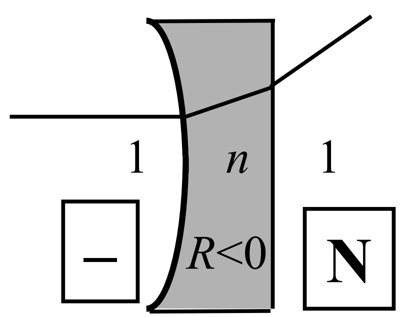


Bi-convex
lens

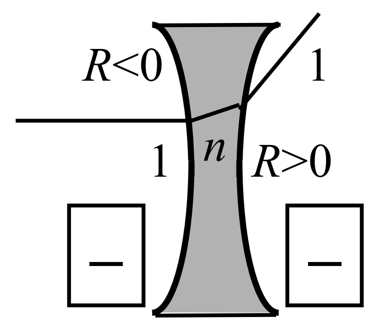
- Negative power bends rays "outwards"



Simple spherical
refractor (negative)



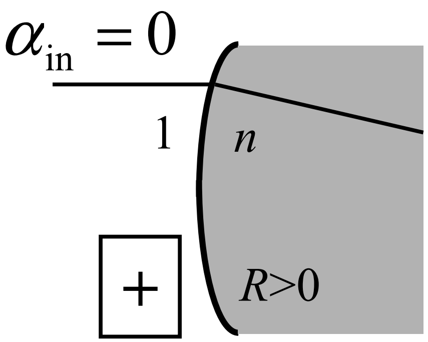
Plano-concave
lens



Bi-concave
lens

The power in matrix formulation

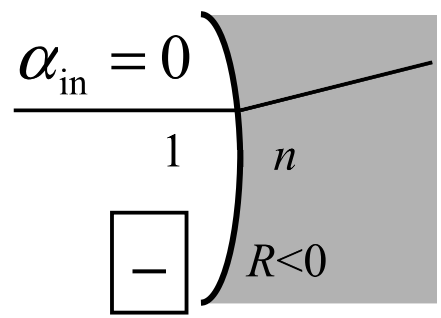
$$\begin{pmatrix} n_{\text{out}}\alpha_{\text{out}} \\ x_{\text{out}} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} n_{\text{in}}\alpha_{\text{in}} \\ x_{\text{in}} \end{pmatrix}$$



Simple spherical refractor (positive)

$$n\alpha_{\text{out}} \begin{cases} = M_{12} x_{\text{in}} \\ < 0 \end{cases}$$

$$M_{12} = -\frac{n-1}{R}$$



Simple spherical refractor (negative)

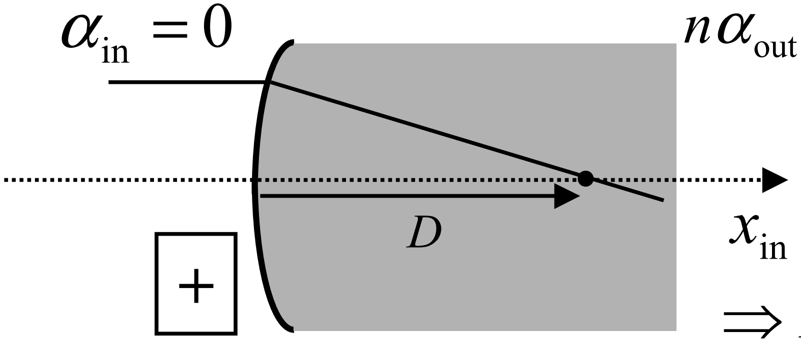
$$n\alpha_{\text{out}} \begin{cases} = M_{12} x_{\text{in}} \\ > 0 \end{cases}$$

$$M_{12} = -\frac{n-1}{R}$$

$$(\text{Ray bending}) = (\text{Power}) \times (\text{Lateral coordinate})$$

$$\boxed{\Rightarrow (\text{Power}) = -M_{12}}$$

Power and focal length



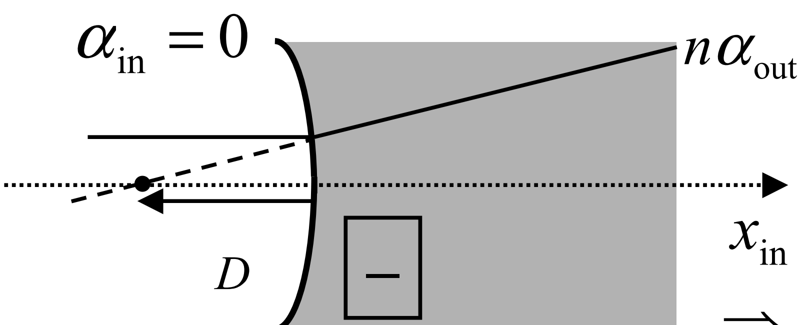
Simple spherical refractor (positive)

$$n\alpha_{\text{out}} \begin{cases} = M_{12} x_{\text{in}} \\ < 0 \end{cases}$$

$$x_{\text{in}} + D\alpha_{\text{out}} = 0$$

$$\Rightarrow D = -1/M_{12}$$

$$(D > 0)$$



Simple spherical refractor (negative)

$$n\alpha_{\text{out}} \begin{cases} = M_{12} x_{\text{in}} \\ > 0 \end{cases}$$

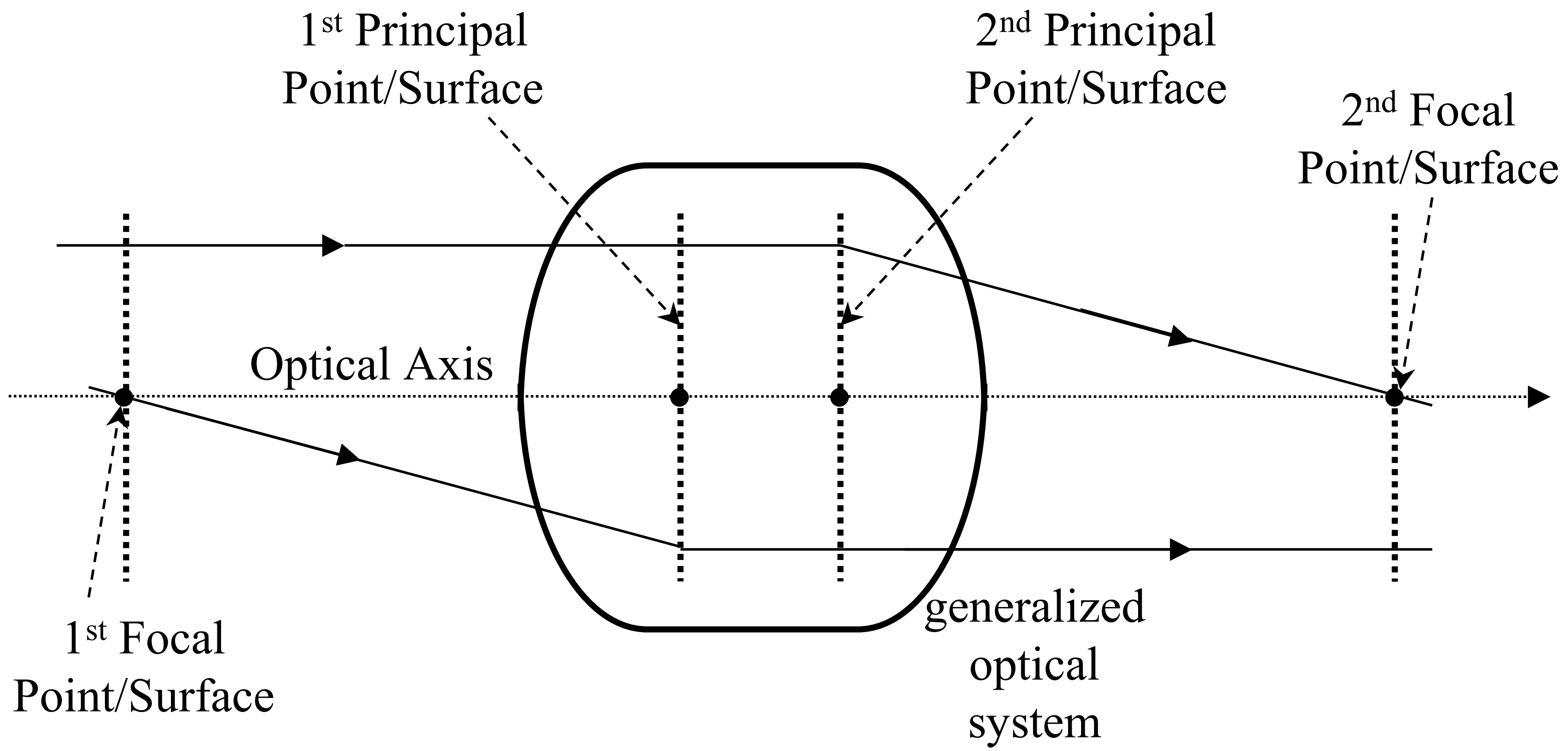
$$x_{\text{in}} + D\alpha_{\text{out}} = 0$$

$$\Rightarrow D = -1/M_{12}$$

$$(D < 0)$$

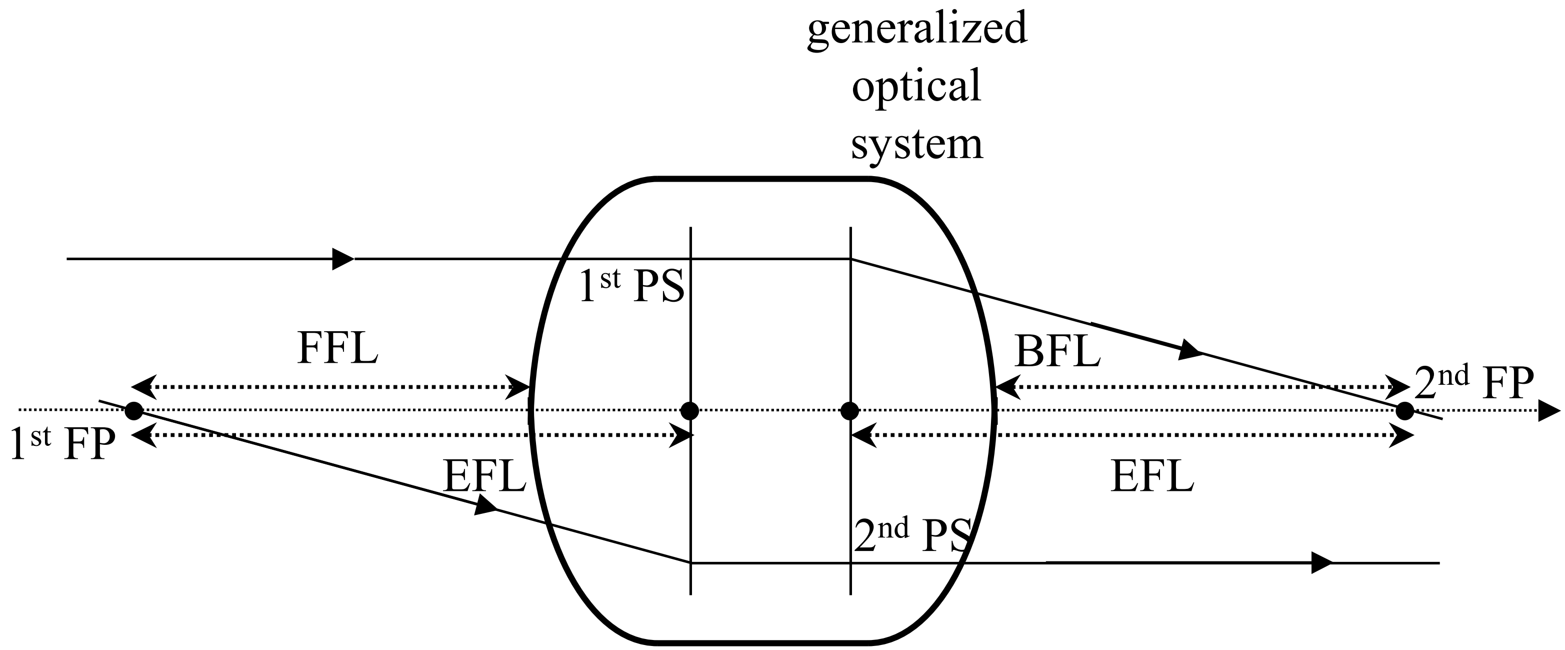
$$\boxed{(\text{Focal length}) = \; = 1/(\text{Power}) = -1/M_{12}}$$

Thick/compound elements: focal & principal points (surfaces)



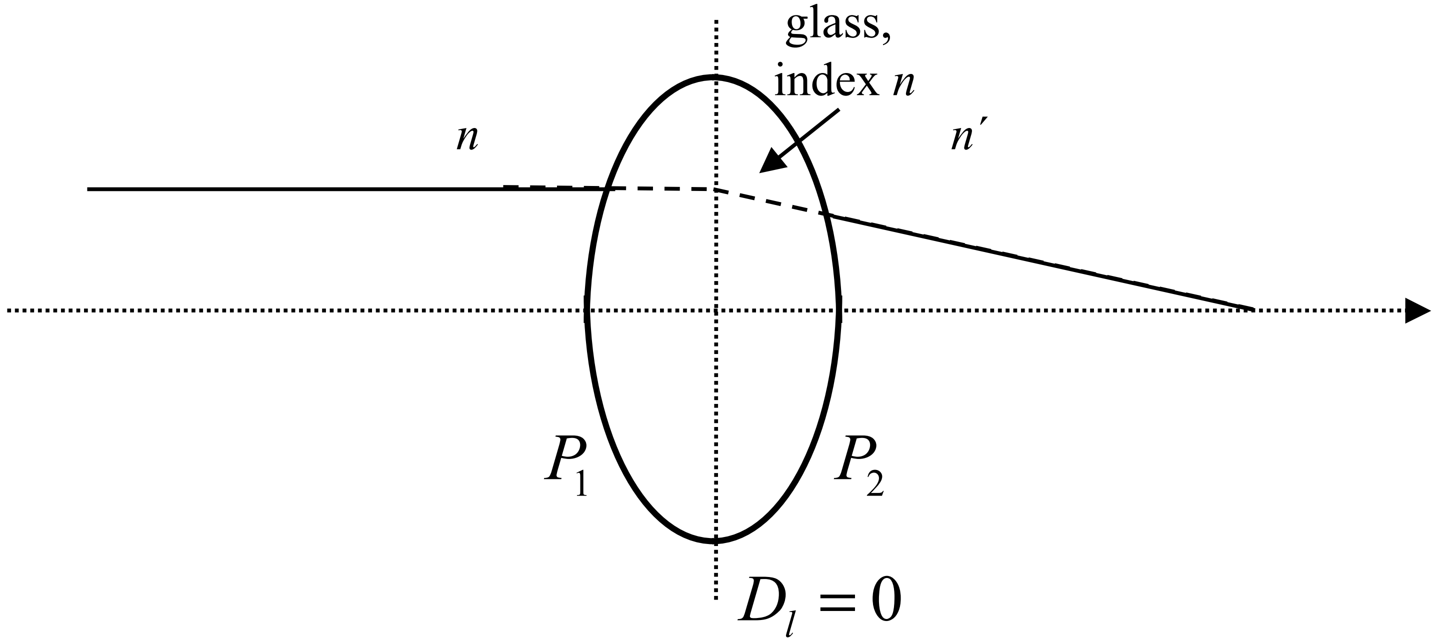
<u>Note</u>: in the paraxial approximation, the focal & principal surfaces are flat (*i.e.*, planar). In reality, they are curved (but not spherical!!).The exact calculation is very complicated.

Focal Lengths for thick/compound elements



EFL: Effective Focal Length (or simply "focal length")
FFL: Front Focal Length
BFL: Back Focal Length

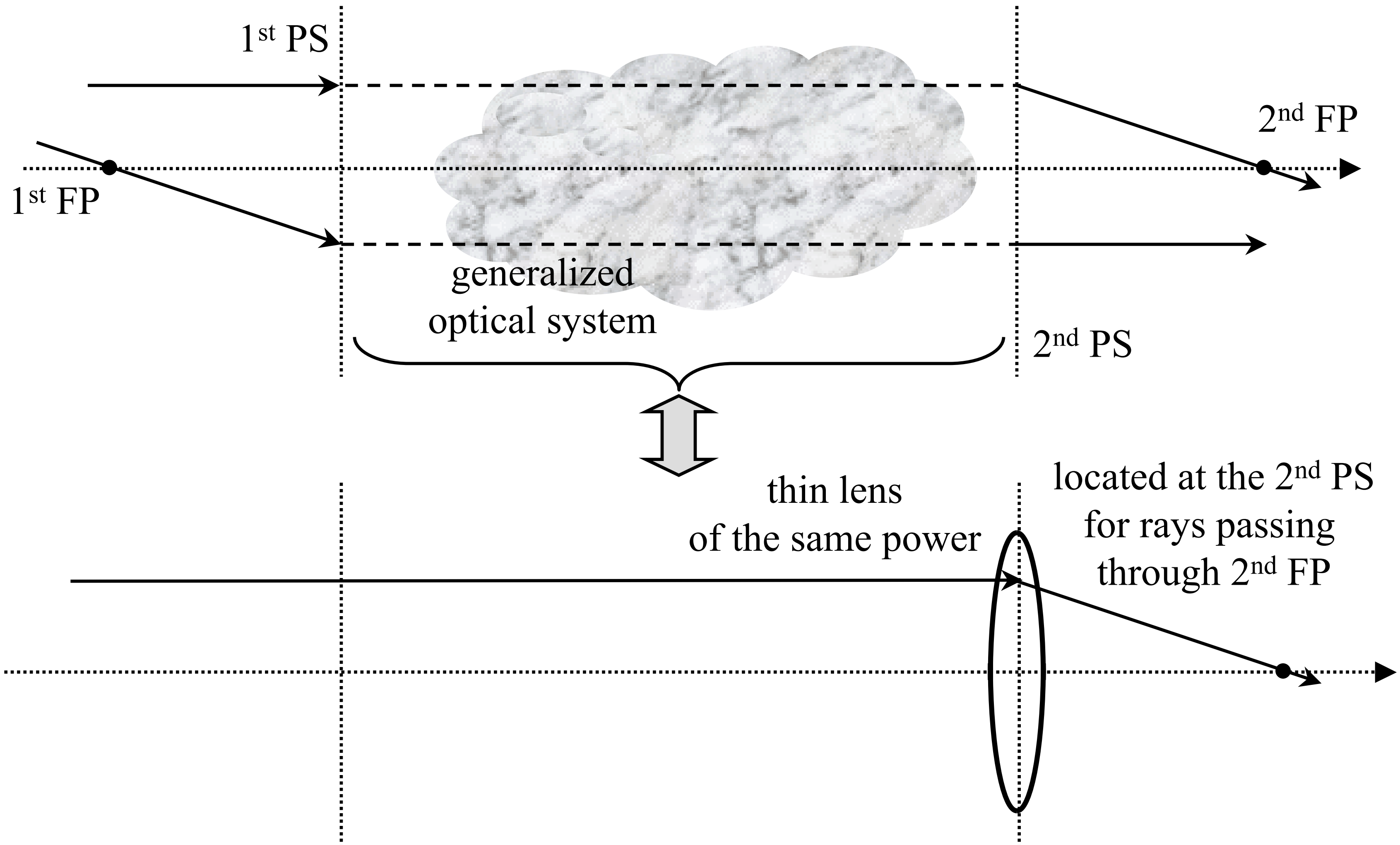
PSs and FLs for thin lenses



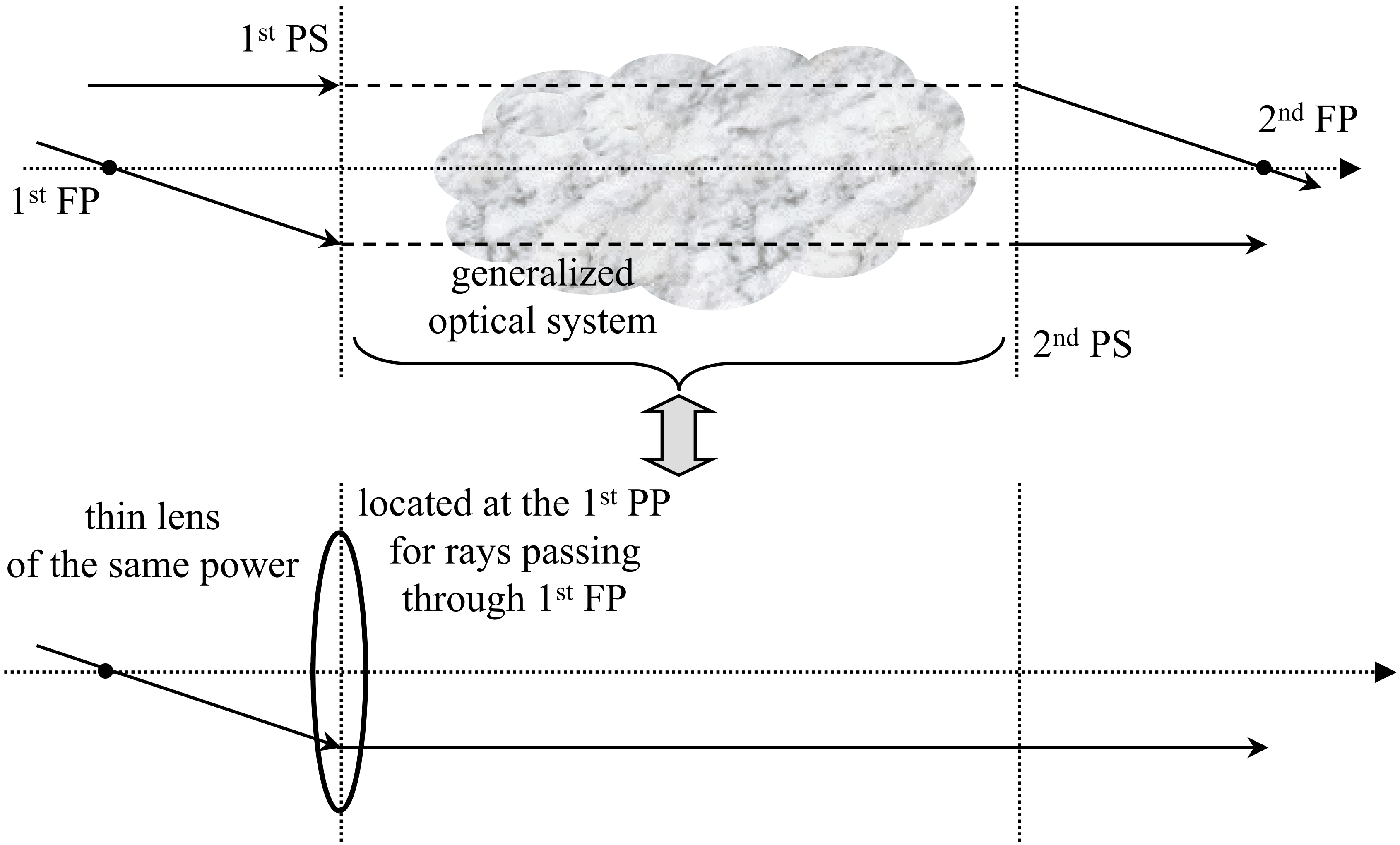
$$\frac{1}{(\mathrm{EFL})} \equiv P = P_1 + P_2 \qquad (\mathrm{BFL}) = (\mathrm{EFL}) = (\mathrm{FFL})$$

- The principal planes coincide with the (collocated) glass surfaces
- The rays bend precisely at the thin lens plane (=collocated glass surfaces & PP)

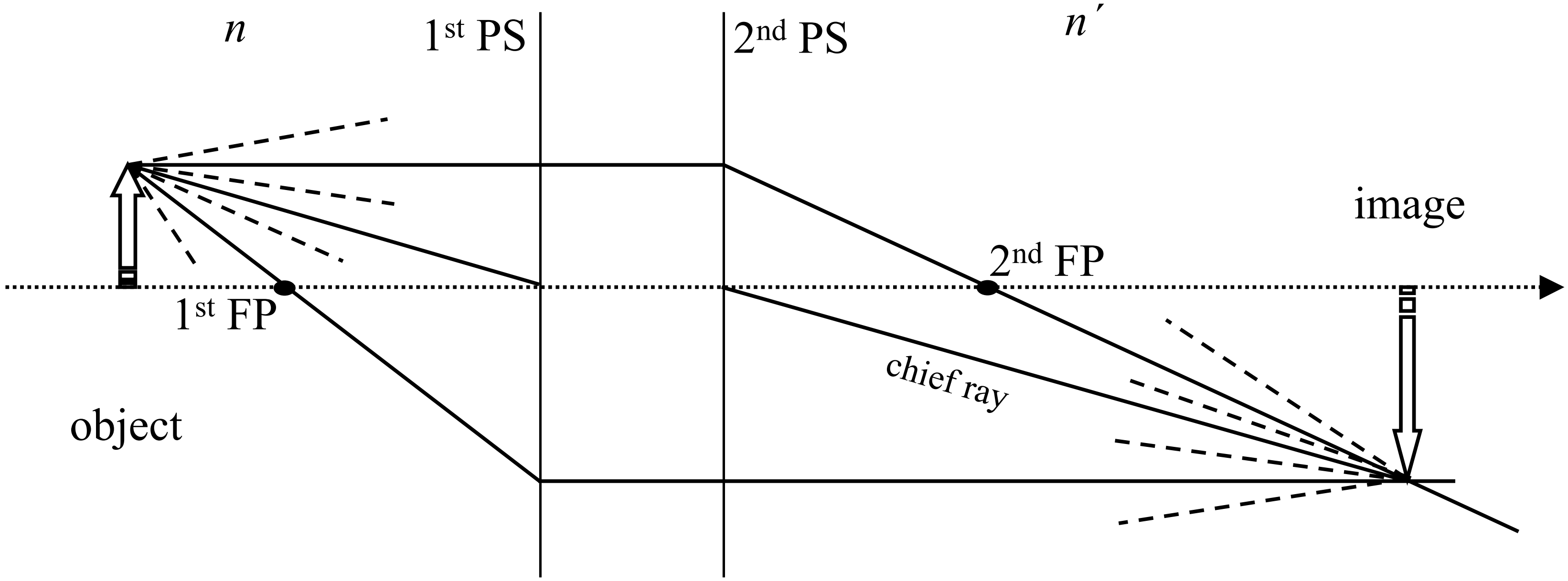
The significance of principal planes /1



The significance of principal planes /2

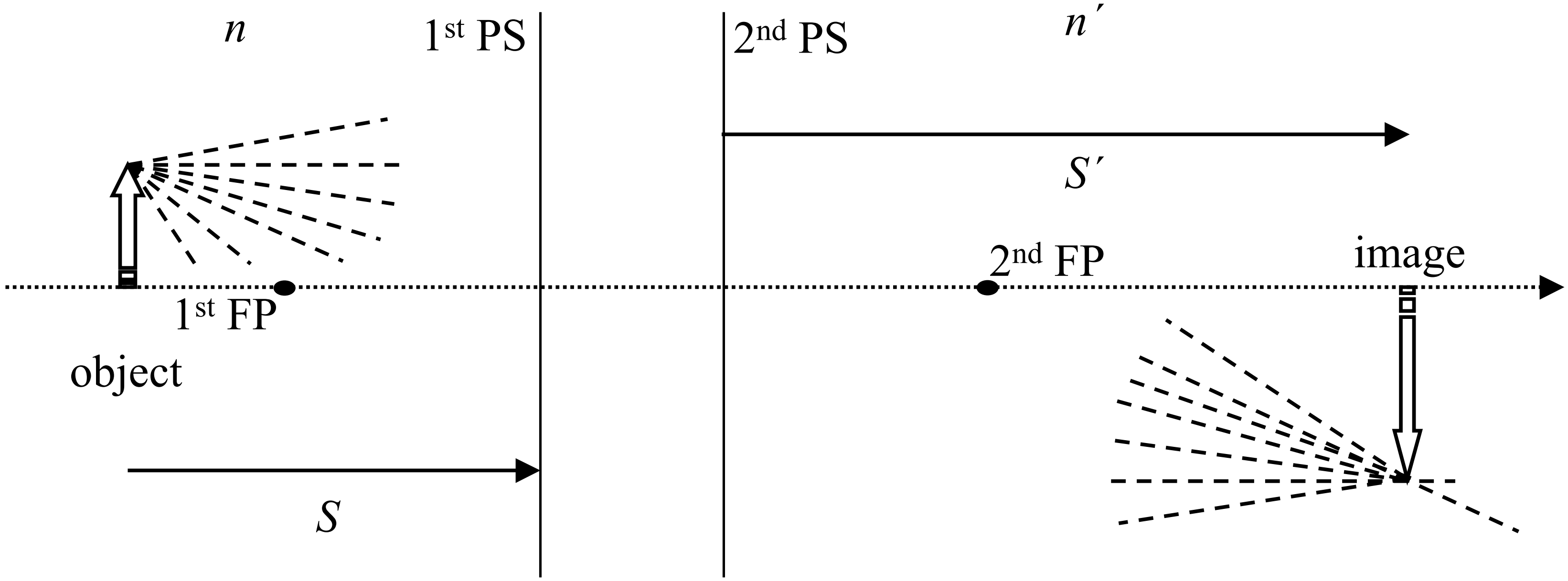


Imaging condition: ray-tracing



- Image point is located at the common intersection of **all** rays which emanate from the corresponding object point
- The two rays passing through the two focal points and the chief ray can be ray-traced directly

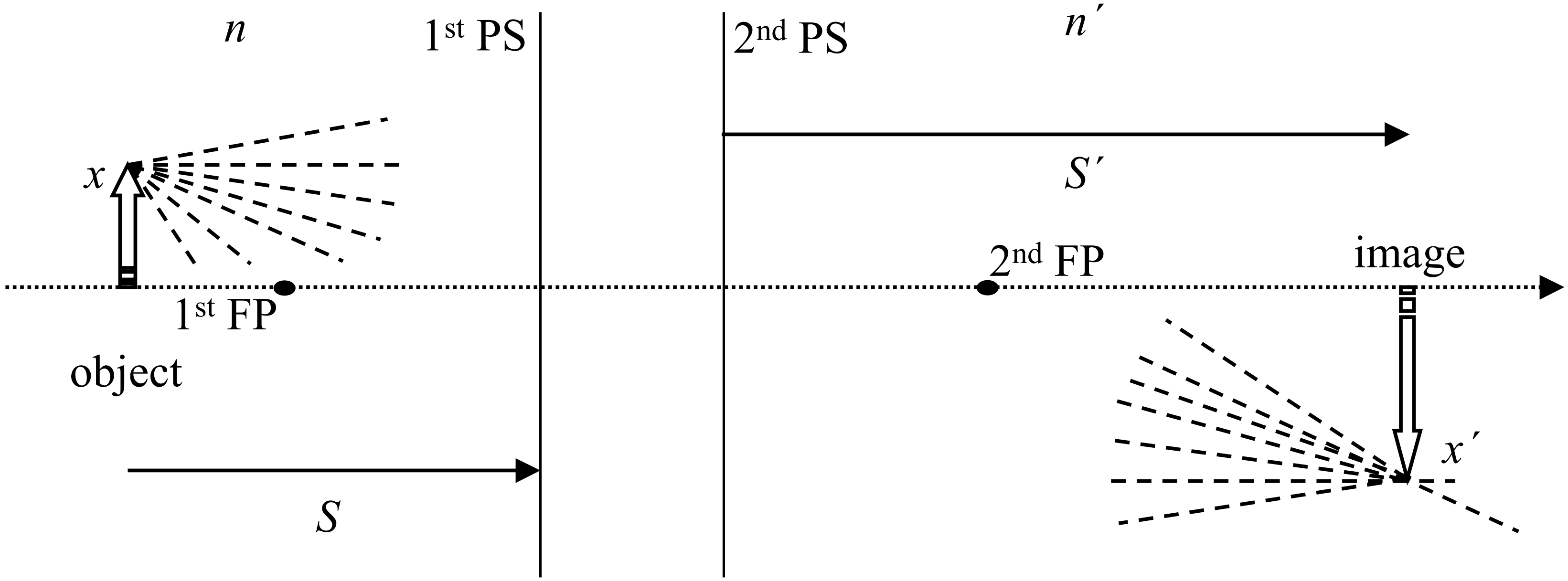
Imaging condition: matrix form /1



system matrix

$$\begin{pmatrix} 1 & 0 \\ S'/n' & 1 \end{pmatrix}\begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ S/n & 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{PS}{n} & -P \\ \dfrac{S'}{n'}+\dfrac{S}{n}-\dfrac{PSS'}{nn'} & 1-\dfrac{PS'}{n'} \end{pmatrix}$$

Imaging condition: matrix form /2

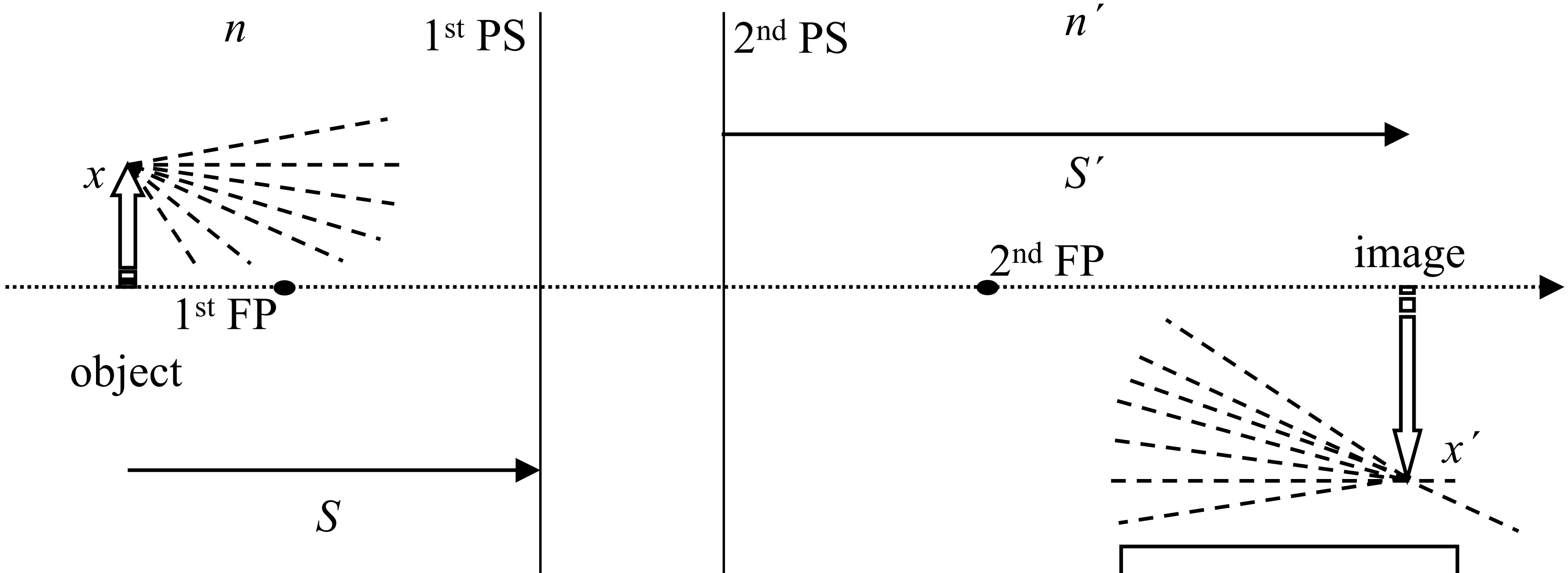


Imaging condition:
Output coordinate x' must not depend on entrance angle γ

$$\begin{pmatrix} n'\gamma' \\ x' \end{pmatrix} = \begin{pmatrix} 1-\frac{PS}{n} & -P \\ \frac{S'}{n'}+\frac{S}{n}-\frac{PSS'}{nn'} & 1-\frac{PS'}{n'} \end{pmatrix} \begin{pmatrix} n\gamma \\ x \end{pmatrix}$$

$$\frac{S'}{n'}+\frac{S}{n}-\frac{PSS'}{nn'} \longrightarrow \mathbf{= 0}$$

Imaging condition: matrix form /3



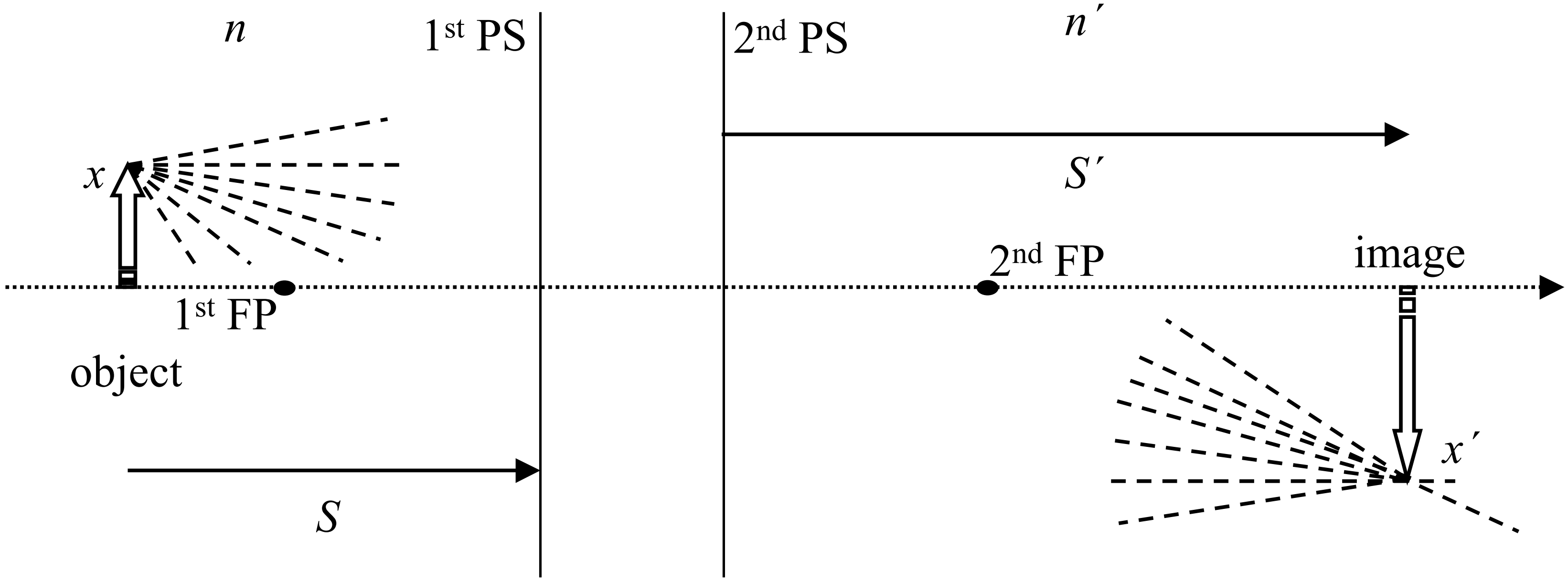
Imaging condition:

$$\frac{S'}{n'}+\frac{S}{n}-\frac{PSS'}{nn'}=0 \Leftrightarrow \boxed{\frac{n}{S}+\frac{n'}{S'}=P}$$

system immersed in air,
$n=n'=1$;
power $P=1/f$

$$\Leftrightarrow \boxed{\frac{1}{S}+\frac{1}{S'}=\frac{1}{f}}$$

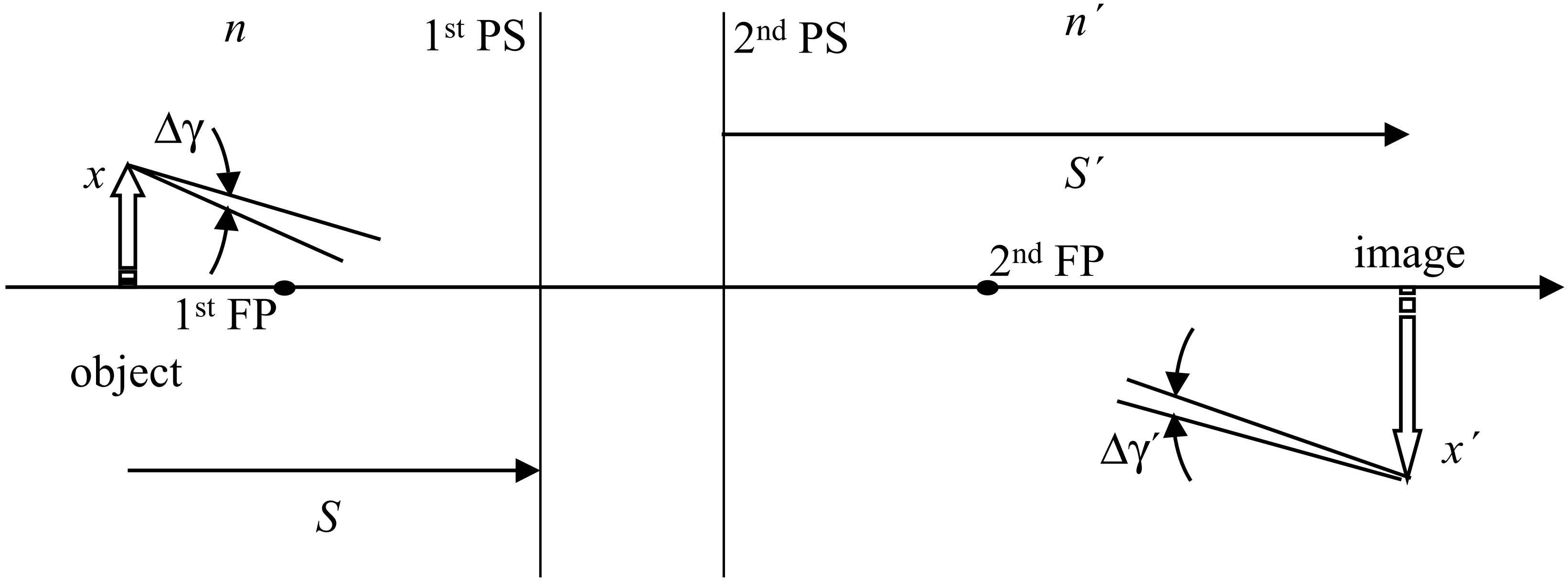
Lateral magnification



(assume imaging condition is satisfied)

$$\begin{pmatrix} n'\gamma' \\ x' \end{pmatrix} = \begin{pmatrix} 1-\dfrac{PS}{n} & -P \\ 0 & 1-\dfrac{PS'}{n'} \end{pmatrix} \begin{pmatrix} n\gamma \\ x \end{pmatrix} \Rightarrow \boxed{m_x \equiv \frac{x'}{x} = 1-\frac{PS'}{n'}}$$

Angular magnification



(assume imaging condition is satisfied)

$$\begin{pmatrix} n'\gamma' \\ x' \end{pmatrix} = \begin{pmatrix} 1-\dfrac{PS}{n} & -P \\ 0 & 1-\dfrac{PS'}{n'} \end{pmatrix} \begin{pmatrix} n\gamma \\ x \end{pmatrix} \Rightarrow \boxed{m_{\mathrm{a}} \equiv \frac{\Delta\gamma'}{\Delta\gamma} = \frac{n}{n'}\left(1-\frac{PS}{n}\right)}$$

Generalized imaging conditions

$$\underbrace{\begin{pmatrix} n'\alpha' \\ x' \end{pmatrix}}_{\text{image}} = \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{\substack{\text{system} \\ \text{matrix}}} \underbrace{\begin{pmatrix} n\alpha \\ x \end{pmatrix}}_{\text{object}}$$

Power: $P = -M_{12} \neq 0$

Imaging condition: $M_{21} = 0$

Lateral magnification: $m_x = M_{22}$

Angular magnification: $m_{\mathrm{a}} = \dfrac{n}{n'} M_{11}$