Introduction to Algorithms

6.046J/18.401J/SMA5503

Lecture 22

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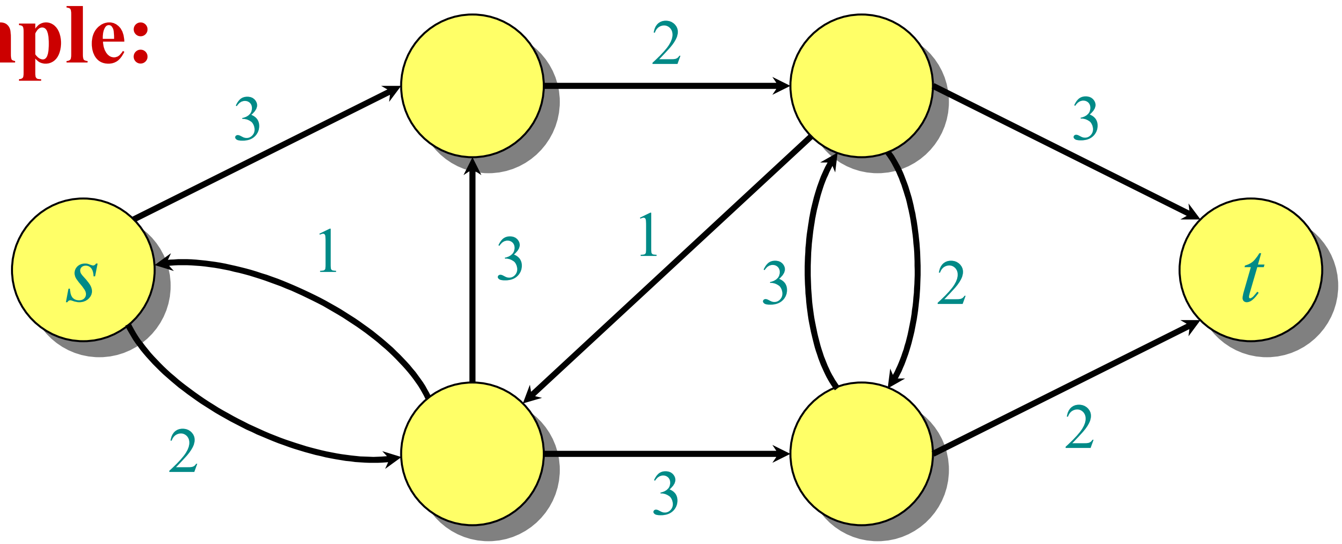
Flow networks

Definition. A ***flow network*** is a directed graph $G = (V, E)$ with two distinguished vertices: a ***source*** s and a ***sink*** t. Each edge $(u, v) \in E$ has a nonnegative ***capacity*** $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.

Example:



Flow networks

Definition. A ***positive flow*** on G is a function $p : V \times V \to \mathbb{R}$ satisfying the following:

- ***Capacity constraint:*** For all $u, v \in V$,

$$0 \le p(u, v) \le c(u, v).$$

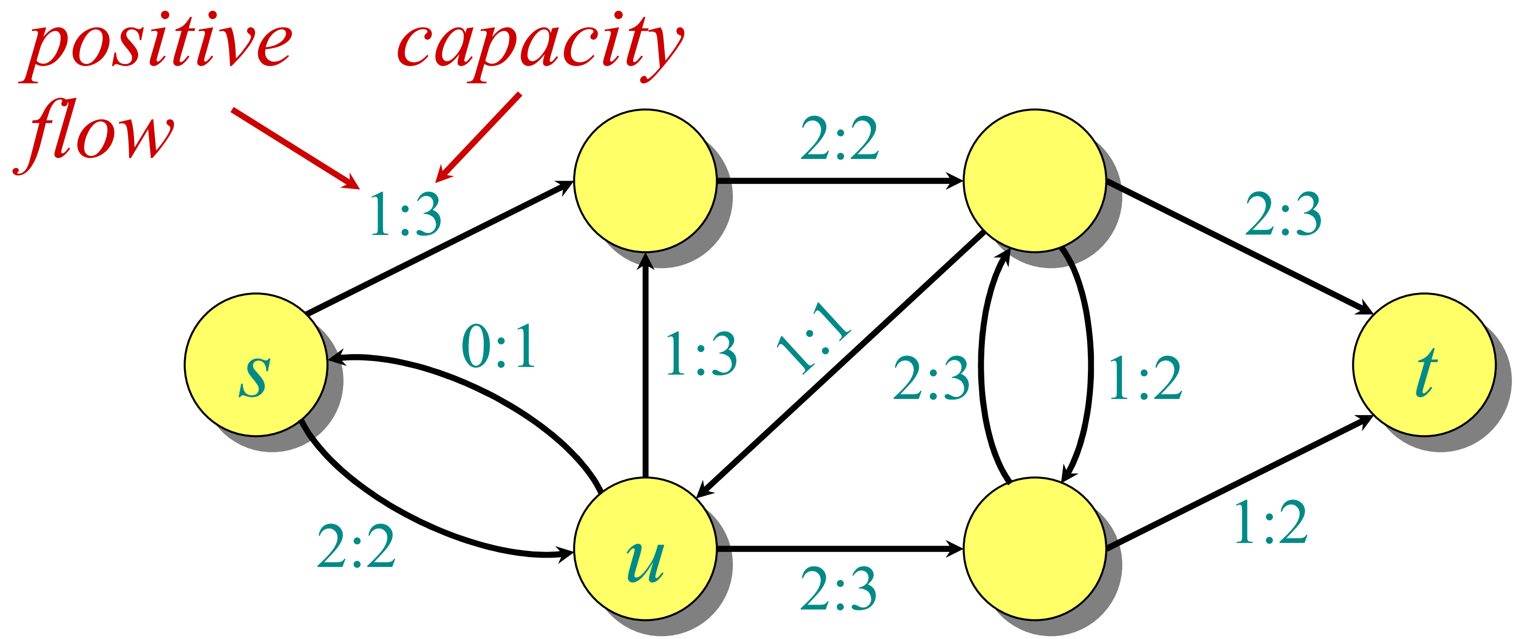
- ***Flow conservation:*** For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) = 0.$$

The ***value*** of a flow is the net flow out of the source:

$$\sum_{v \in V} p(s,v) - \sum_{v \in V} p(v,s).$$

A flow on a network



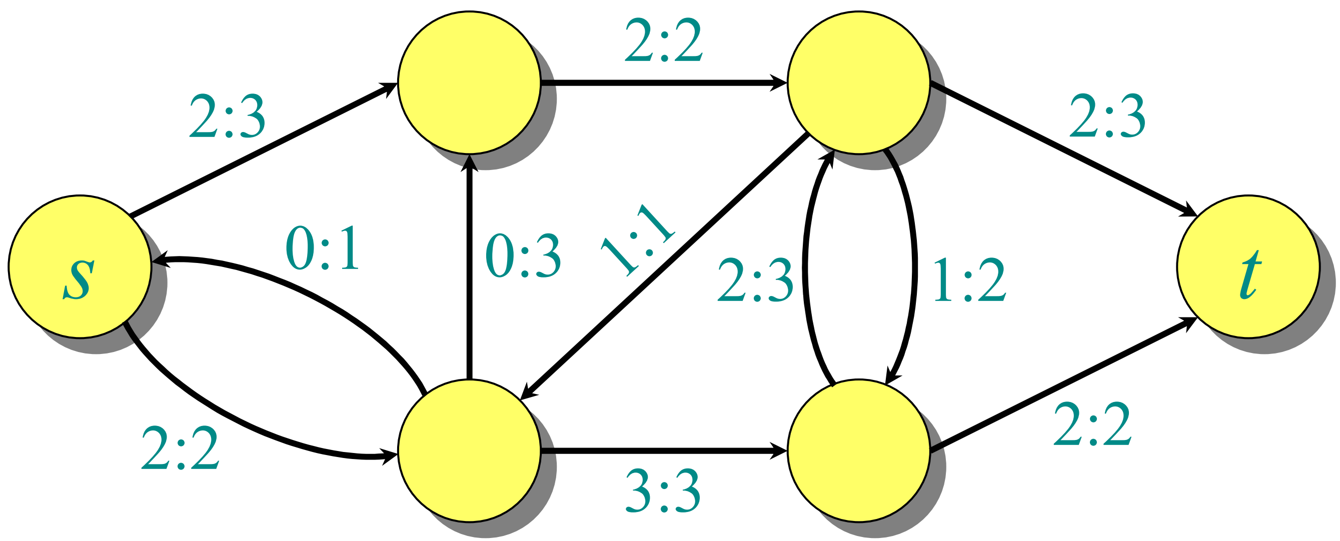
Flow conservation (like Kirchoff's current law):

- Flow into u is $2 + 1 = 3$.
- Flow out of u is $0 + 1 + 2 = 3$.

The value of this flow is $1 - 0 + 2 = 3$.

The maximum-flow problem

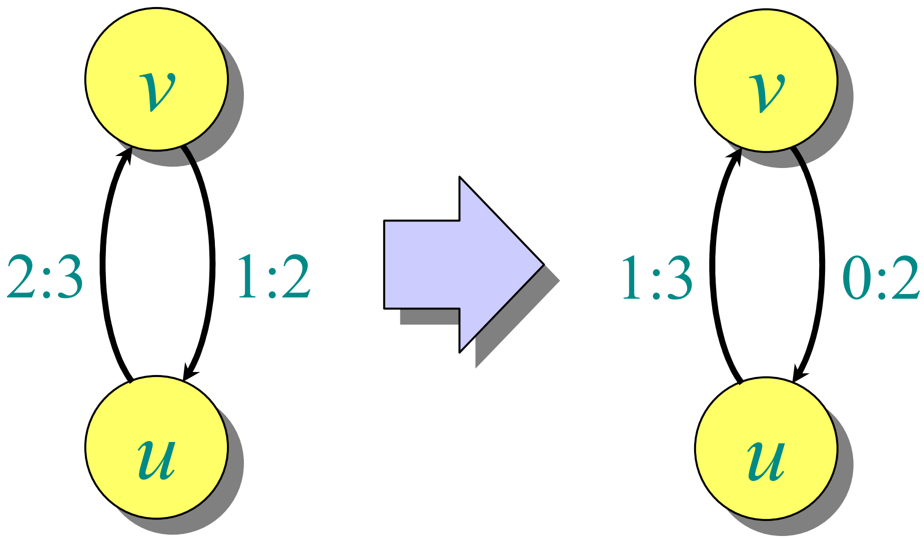
Maximum-flow problem: Given a flow network *G*, find a flow of maximum value on *G*.



The value of the maximum flow is 4.

Flow cancellation

Without loss of generality, positive flow goes either from *u* to *v*, or from *v* to *u*, but not both.



Net flow from *u* to *v* in both cases is 1.

The capacity constraint and flow conservation are preserved by this transformation.

INTUITION: View flow as a *rate*, not a *quantity*.

A notational simplification

IDEA: Work with the net flow between two vertices, rather than with the positive flow.

Definition. A ***(net) flow*** on G is a function $f : V \times V \rightarrow \mathbb{R}$ satisfying the following:

- ***Capacity constraint:*** For all $u, v \in V$,
$$f(u, v) \leq c(u, v).$$
- ***Flow conservation:*** For all $u \in V - \{s, t\}$,
$$\sum_{v \in V} f(u, v) = 0.$$
← *One summation instead of two.*
- ***Skew symmetry:*** For all $u, v \in V$,
$$f(u, v) = -f(v, u).$$

Equivalence of definitions

Theorem. The two definitions are equivalent.

Proof. ($\Rightarrow$) Let $f(u, v) = p(u, v) - p(v, u)$.

- ***Capacity constraint:*** Since $p(u, v) \le c(u, v)$ and $p(v, u) \ge 0$, we have $f(u, v) \le c(u, v)$.
- ***Flow conservation:***

$$\sum_{v \in V} f(u,v) = \sum_{v \in V} \big(p(u,v) - p(v,u)\big)$$
$$= \sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u)$$

- ***Skew symmetry:***

$$\begin{aligned} f(u, v) &= p(u, v) - p(v, u) \\ &= -\big(p(v, u) - p(u, v)\big) \\ &= -f(v, u). \end{aligned}$$

Proof (continued)

($\Leftarrow$) Let

$$p(u, v) = \begin{cases} f(u, v) & \text{if } f(u, v) > 0, \\ 0 & \text{if } f(u, v) \le 0. \end{cases}$$

- ***Capacity constraint:*** By definition, $p(u, v) \ge 0$. Since $f(u, v) \le c(u, v)$, it follows that $p(u, v) \le c(u, v)$.
- ***Flow conservation:*** If $f(u, v) > 0$, then $p(u, v) - p(v, u) = f(u, v)$. If $f(u, v) \le 0$, then $p(u, v) - p(v, u) = -f(v, u) = f(u, v)$ by skew symmetry. Therefore,

$$\sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) = \sum_{v \in V} f(u,v). \quad \square$$

Notation

Definition. The ***value*** of a flow f, denoted by $|f|$, is given by

$$|f| = \sum_{v \in V} f(s,v) = f(s,V).$$

Implicit summation notation: A set used in an arithmetic formula represents a sum over the elements of the set.

- **Example** — flow conservation: $f(u, V) = 0$ for all $u \in V - \{s, t\}$.

Simple properties of flow

Lemma.

- $f(X, X) = 0$,
- $f(X, Y) = -f(Y, X)$,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \emptyset$. ■

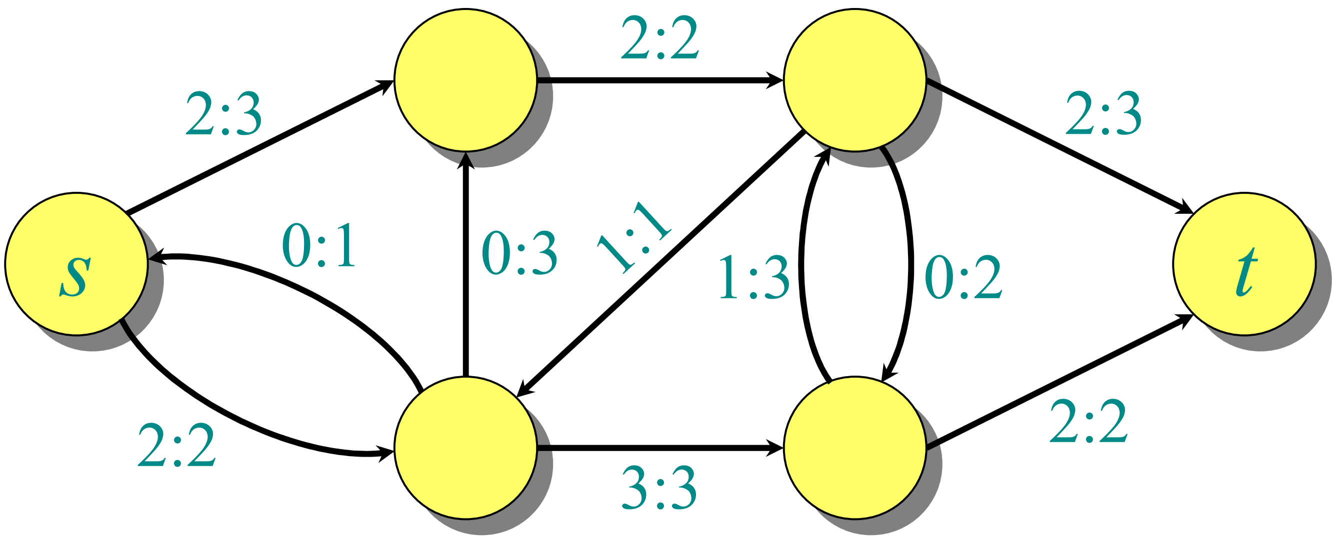
Theorem. $|f| = f(V, t)$.

Proof.

$$\begin{aligned} |f| &= f(s, V) \\ &= f(V, V) - f(V-s, V) \\ &= f(V, V-s) \\ &= f(V, t) + f(V, V-s-t) \\ &= f(V, t). \quad \blacksquare \end{aligned}$$

Omit braces.

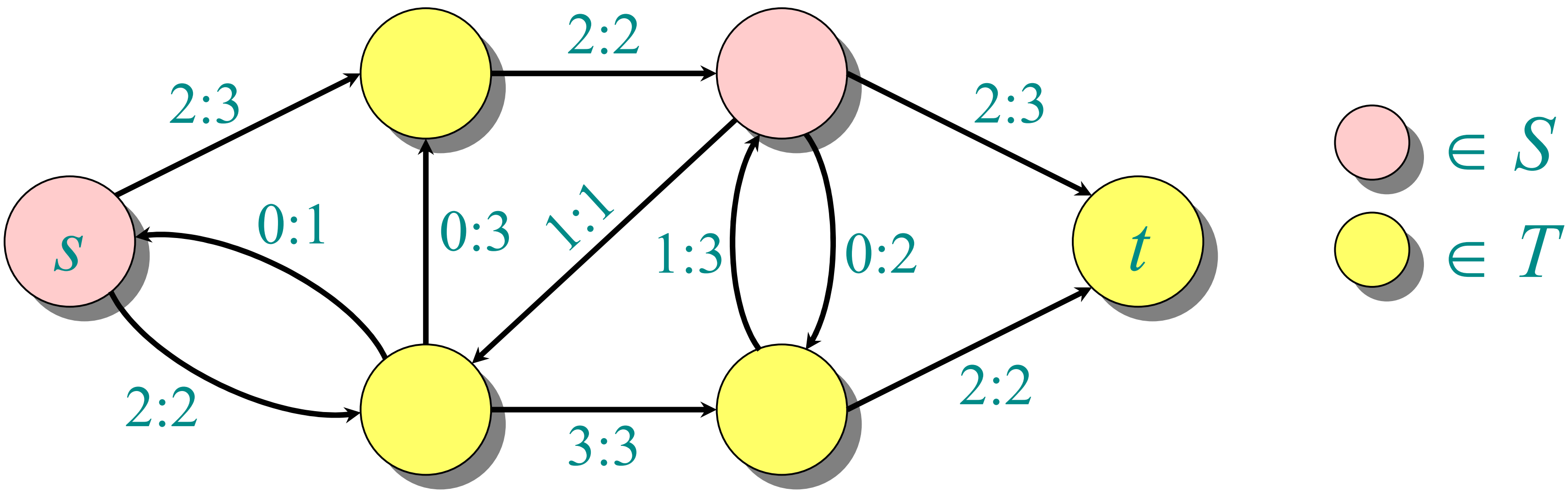
Flow into the sink



$|f| = f(s, V) = 4$ $f(V, t) = 4$

Cuts

Definition. A ***cut*** (S, T) of a flow network $G = (V, E)$ is a partition of V such that $s \in S$ and $t \in T$. If f is a flow on G, then the ***flow across the cut*** is $f(S, T)$.



$$f(S, T) = (2 + 2) + (-2 + 1 - 1 + 2)$$
$$= 4$$

Another characterization of flow value

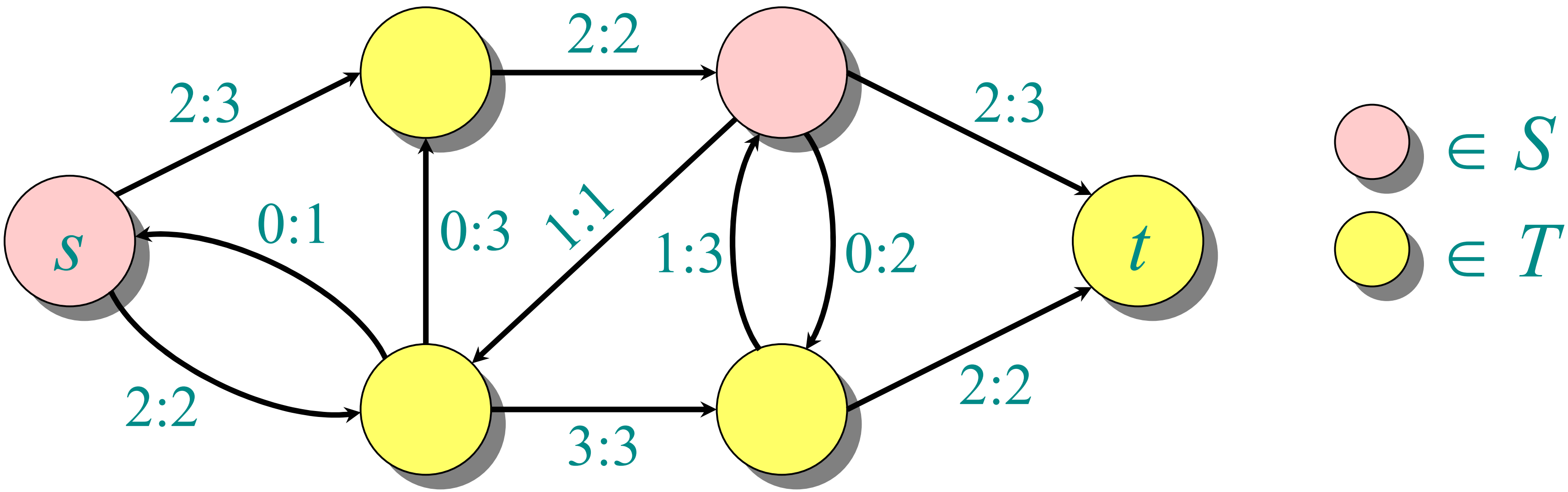
Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.

Proof.

$$\begin{aligned} f(S, T) &= f(S, V) - f(S, S) \\ &= f(S, V) \\ &= f(s, V) + f(S - s, V) \\ &= f(s, V) \\ &= |f|. \quad \blacksquare \end{aligned}$$

Capacity of a cut

Definition. The ***capacity of a cut*** (S, T) is $c(S, T)$.



$$c(S, T) = (3 + 2) + (1 + 2 + 3)$$
$$= 11$$

Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

Proof.

$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u\in S}\sum_{v\in T} f(u,v) \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v) \\
&= c(S,T)\,. \quad \blacksquare
\end{aligned}
$$

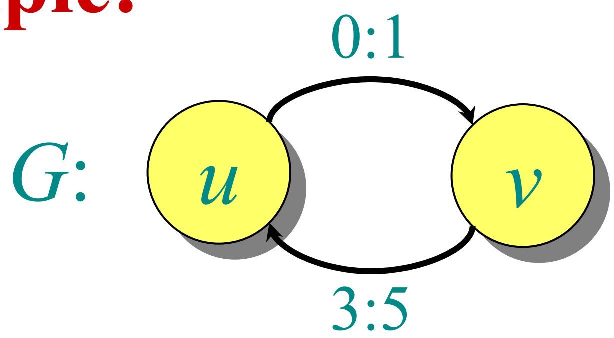
Residual network

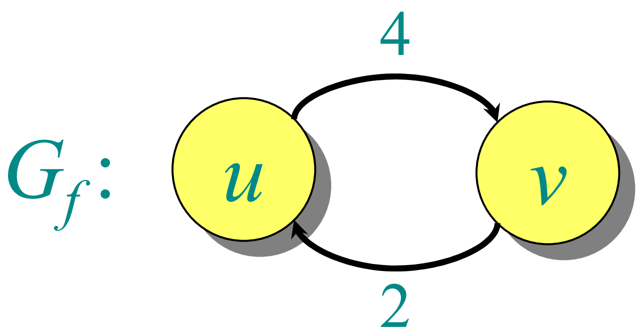
Definition. Let f be a flow on $G = (V, E)$. The ***residual network*** $G_f(V, E_f)$ is the graph with strictly positive ***residual capacities***

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

Edges in E_f admit more flow.

Example:





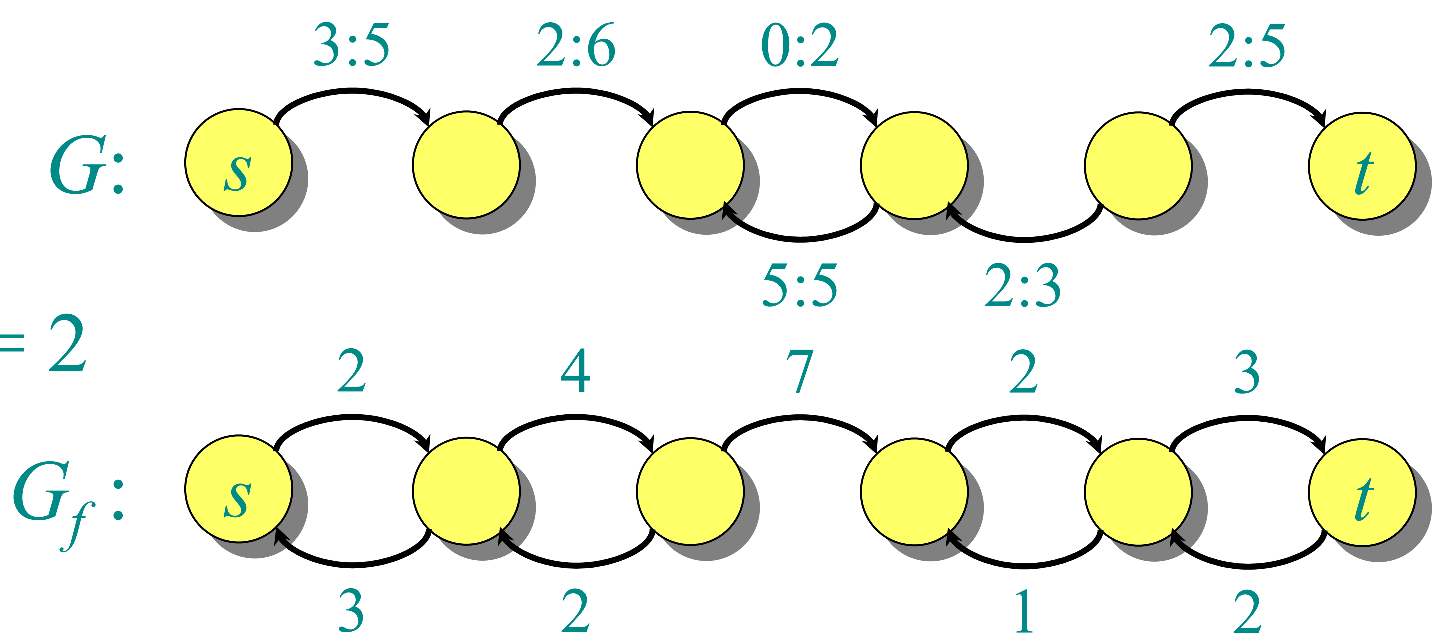
Lemma. $|E_f| \leq 2|E|$. ∎

Augmenting paths

Definition. Any path from s to t in G_f is an ***augmenting path*** in G with respect to f. The flow value can be increased along an augmenting path p by $c_f(p) = \min_{(u,v)\in p} \{c_f(u,v)\}$.

Ex.:

$c_f(p) = 2$



Max-flow, min-cut theorem

Theorem. The following are equivalent:

1. f is a maximum flow.

2. f admits no augmenting paths.

3. $|f| = c(S, T)$ for some cut (S, T).

Proof (and algorithms). Next time. □