Introduction to Algorithms

6.046J/18.401J/SMA5503

Lecture 2

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Solving recurrences

- The analysis of merge sort from ***Lecture 1*** required us to solve a recurrence.
- Recurrences are like solving integrals, differential equations, etc.
  - Learn a few tricks.
- ***Lecture 3***: Applications of recurrences.

Substitution method

The most general method:

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.

Example: $T(n) = 4T(n/2) + n$

- [Assume that $T(1) = \Theta(1)$.]
- Guess $O(n^3)$. (Prove O and Ω separately.)
- Assume that $T(k) \leq ck^3$ for $k < n$.
- Prove $T(n) \leq cn^3$ by induction.

Example of substitution

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\le 4c(n/2)^3 + n \\ &= (c/2)n^3 + n \\ &= cn^3 - ((c/2)n^3 - n) \leftarrow \textit{desired} - \textit{residual} \\ &\le cn^3 \leftarrow \textit{desired} \end{aligned}$$

whenever $(c/2)n^3 - n \ge 0$, for example, if $c \ge 2$ and $n \ge 1$.

← *residual*

Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick c big enough.

This bound is not tight!

A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

Assume that $T(k) \le ck^2$ for $k < n$:

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4cn^2 + n \\
&= O(n) \quad \textit{\textbf{Wrong!}} \text{ We must prove the I.H.} \\
&= cn^2 - (-n) \quad [\text{ desired} - \text{residual }] \\
&\le cn^2
\end{aligned}
$$

for ***no*** choice of $c > 0$. Lose!

A tighter upper bound!

IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$.

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\le 4(c_1 (n/2)^2 - c_2 (n/2)) + n \\ &= c_1 n^2 - 2c_2 n + n \\ &= c_1 n^2 - c_2 n - (c_2 n - n) \\ &\le c_1 n^2 - c_2 n \quad \text{if } c_2 > 1. \end{aligned}$$

Pick c_1 big enough to handle the initial conditions.

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion tree method is good for generating guesses for the substitution method.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$T(n)$$

Example of recursion tree

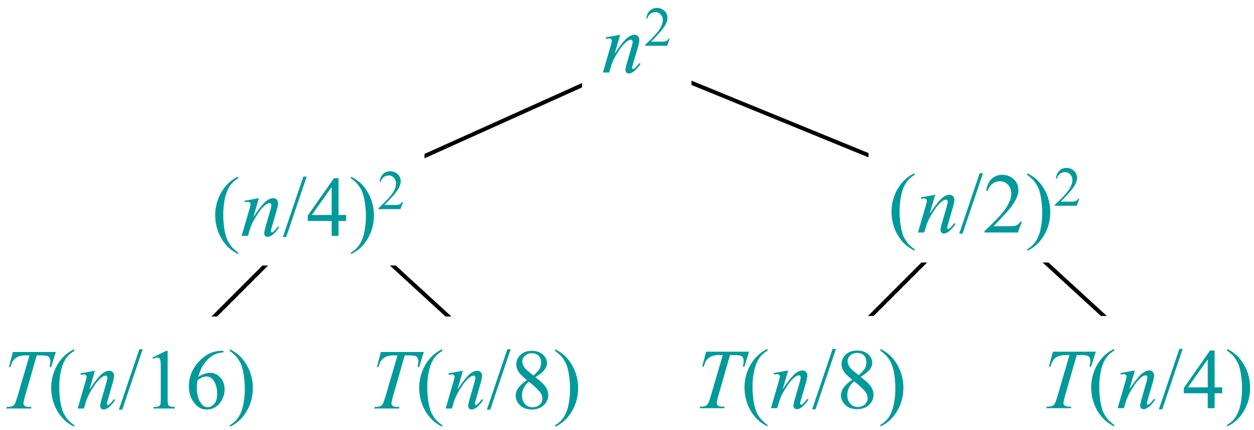
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$$T(n/4) \qquad\qquad T(n/2)$$

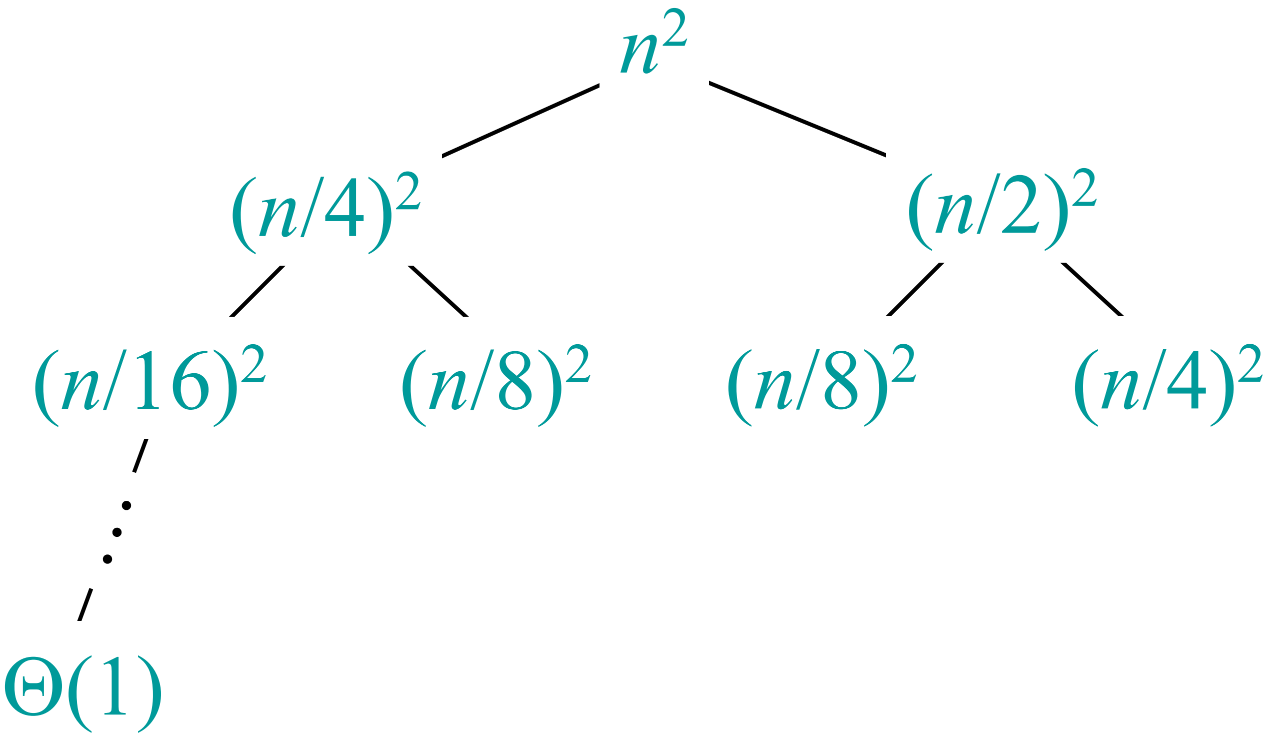
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



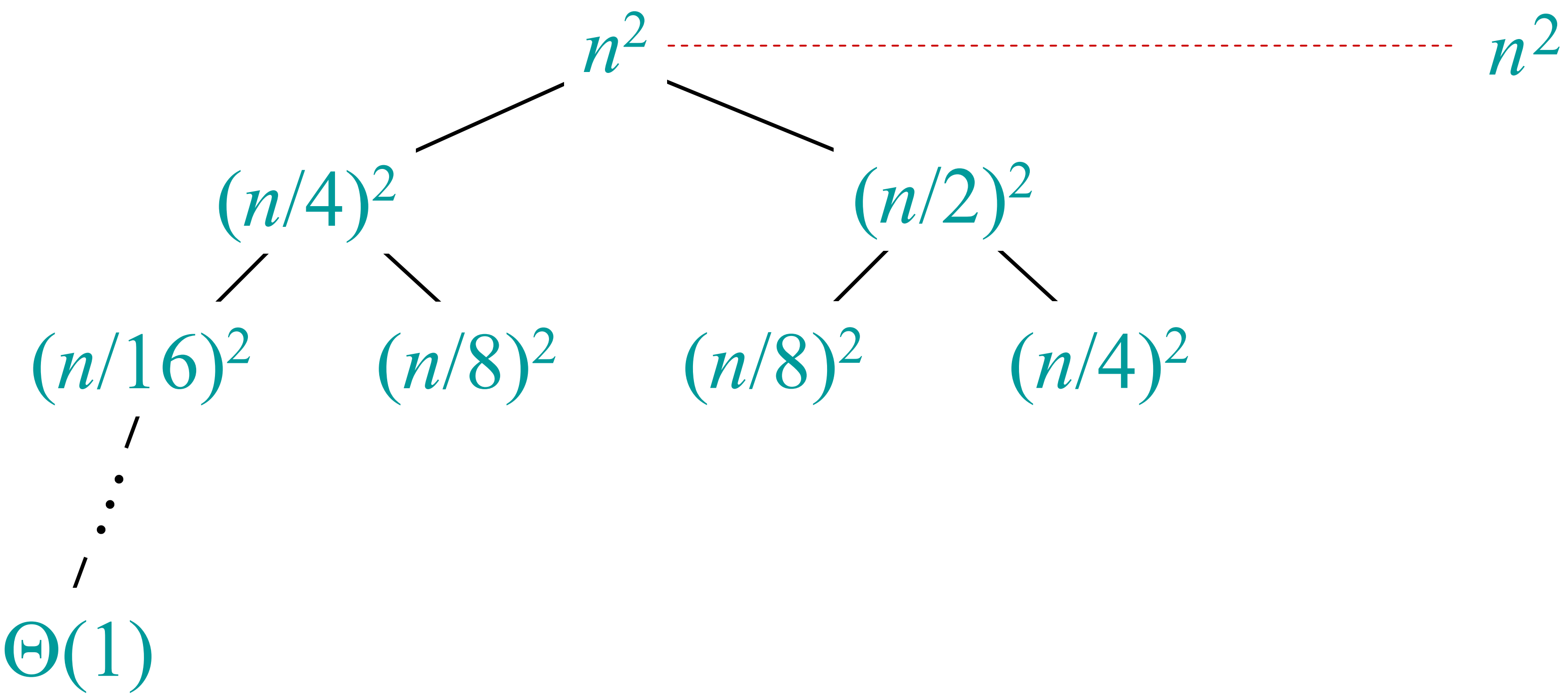
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



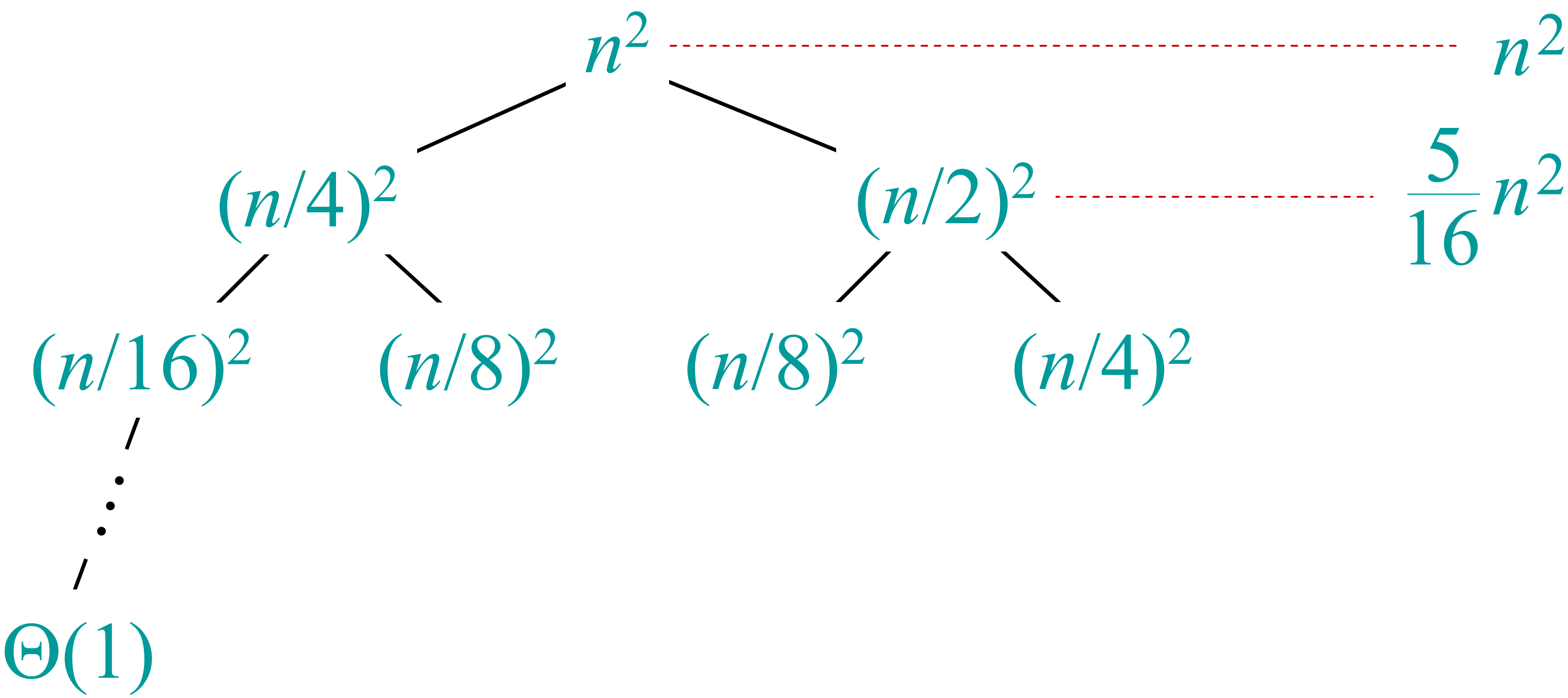
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



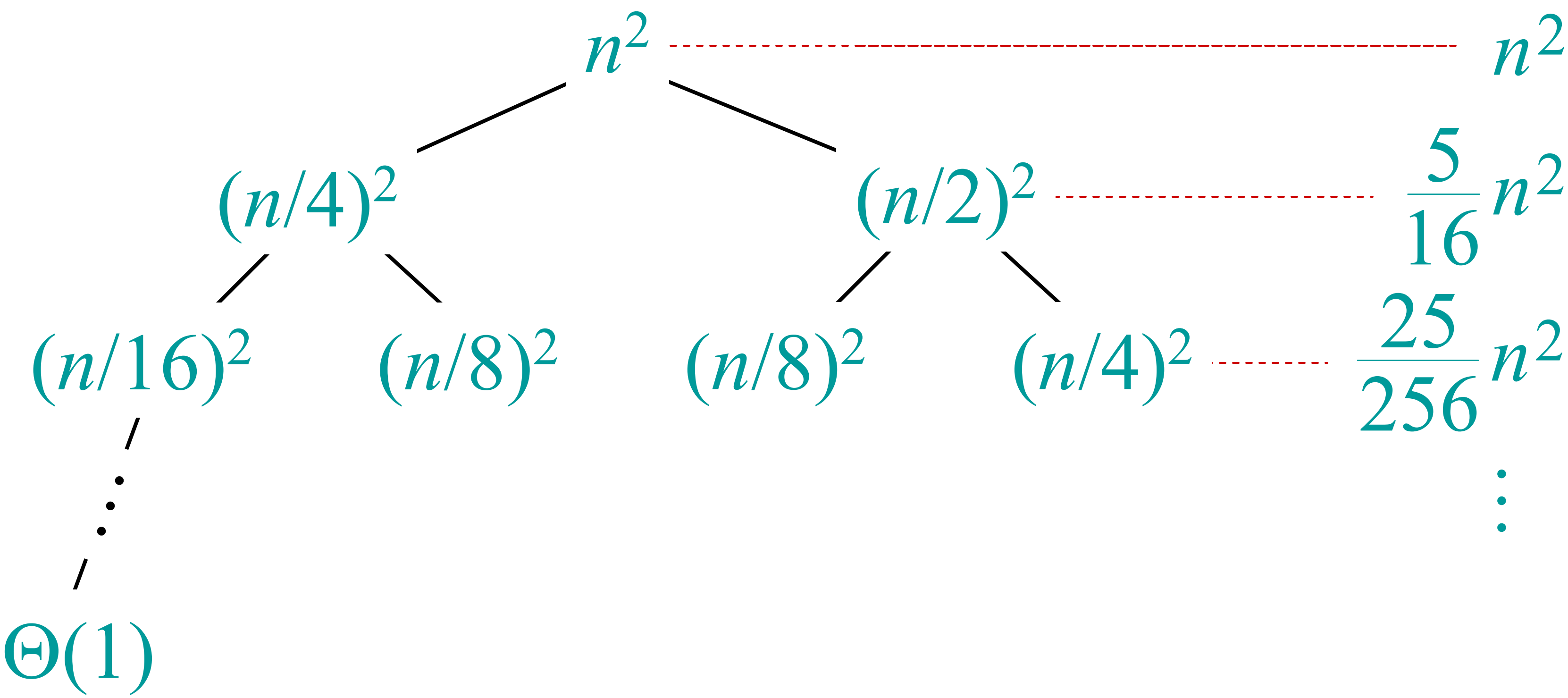
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



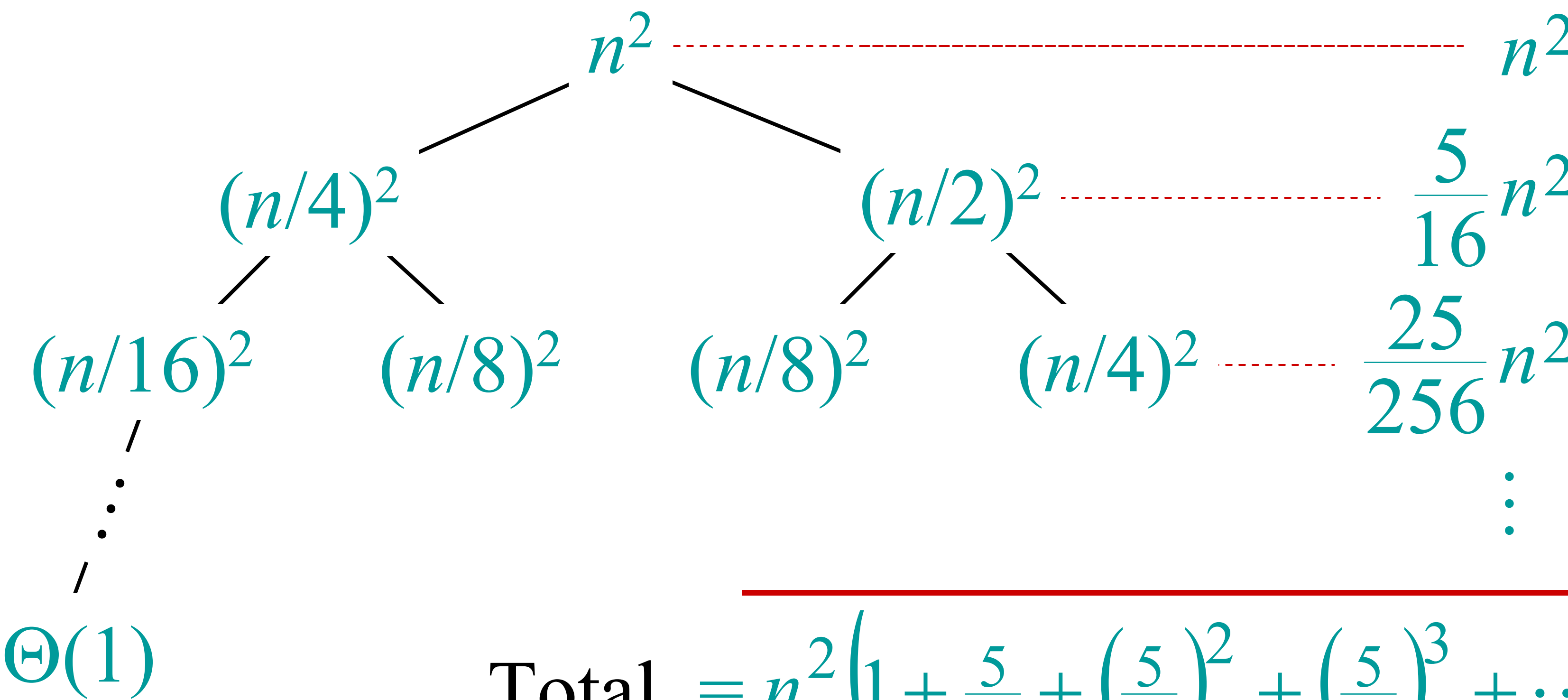
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Total $= n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$

$= \Theta(n^2)$ *geometric series*

The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n) \,,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \leq c f(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.

Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n$.

CASE 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2)$.

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^2$.

CASE 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n)$.

Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(cn/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.

Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.

General method (Akra-Bazzi)

$$T(n) = \sum_{i=1}^{k} a_i T(n/b_i) + f(n)$$

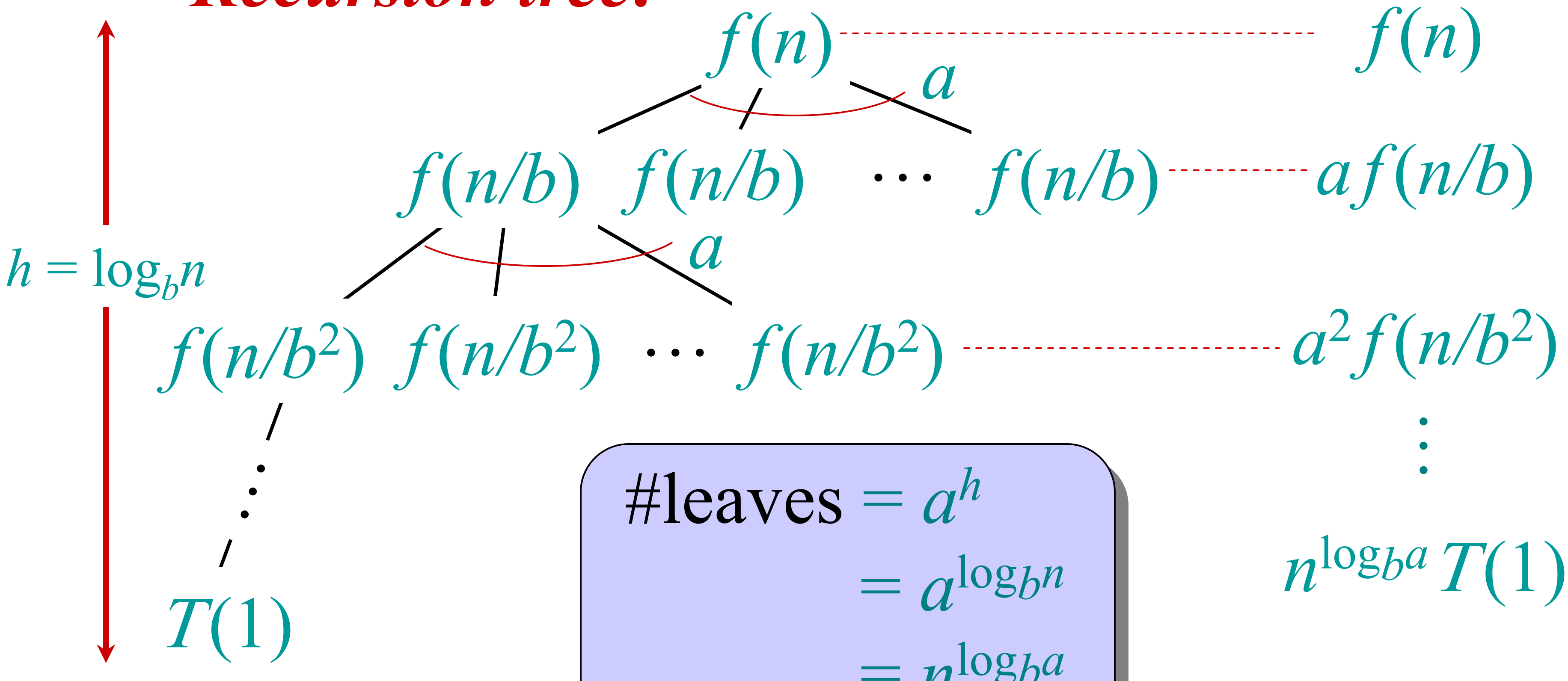
Let p be the unique solution to

$$\sum_{i=1}^{k} \left(a_i / b_i^p\right) = 1.$$

Then, the answers are the same as for the master method, but with n^p instead of $n^{\log_b a}$. (*Akra and Bazzi also prove an even more general result*.)

Idea of master theorem

Recursion tree:



$$\#\text{leaves} = a^h = a^{\log_b n} = n^{\log_b a}$$

Idea of master theorem

Recursion tree:

$f(n)$ ... $f(n)$

a

$f(n/b)$ $f(n/b)$ $\cdots$ $f(n/b)$ ... $af(n/b)$

a

$h = \log_b n$

$f(n/b^2)$ $f(n/b^2)$ $\cdots$ $f(n/b^2)$ ... $a^2 f(n/b^2)$

$\vdots$

$T(1)$

CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

$n^{\log_b a} T(1)$

$\Theta(n^{\log_b a})$

Idea of master theorem

Recursion tree:

$$f(n) \quad \cdots\cdots \quad f(n)$$

a

$$f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b) \quad \cdots\cdots \quad a\,f(n/b)$$

a

$$f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2) \quad \cdots\cdots \quad a^2 f(n/b^2)$$

$h = \log_b n$

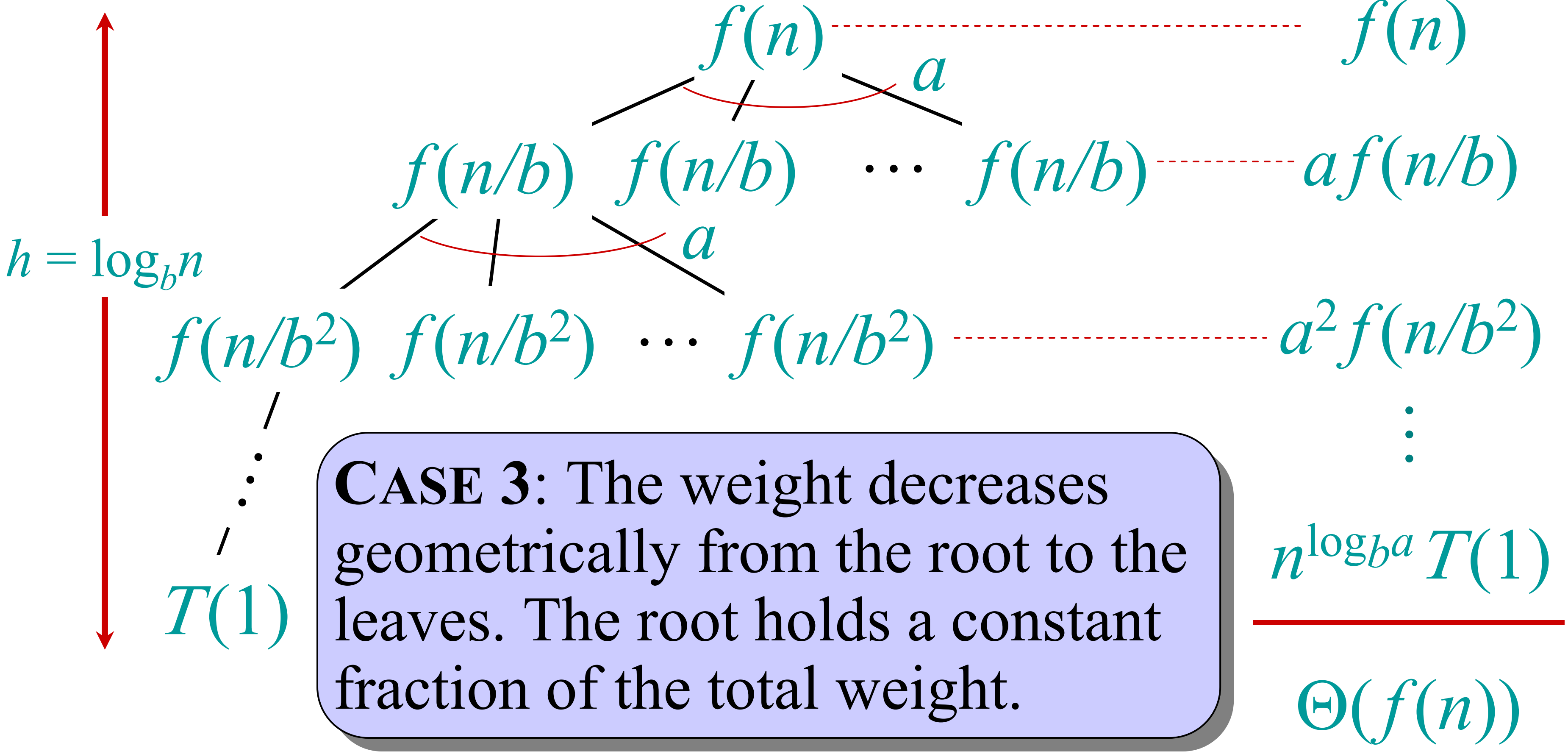
$$T(1) \qquad\qquad n^{\log_b a}\,T(1)$$

CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

$$\Theta(n^{\log_b a} \lg n)$$

Idea of master theorem

Recursion tree:



CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

Conclusion

- Next time: applying the master method.
- For proof of master theorem, see CLRS.

Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

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