Introduction to Algorithms
6.046J/18.401J/SMA5503

Lecture 10

Prof. Erik Demaine

Balanced search trees

Balanced search tree: A search-tree data structure for which a height of $O(\lg n)$ is guaranteed when implementing a dynamic set of n items.

Examples:

- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees

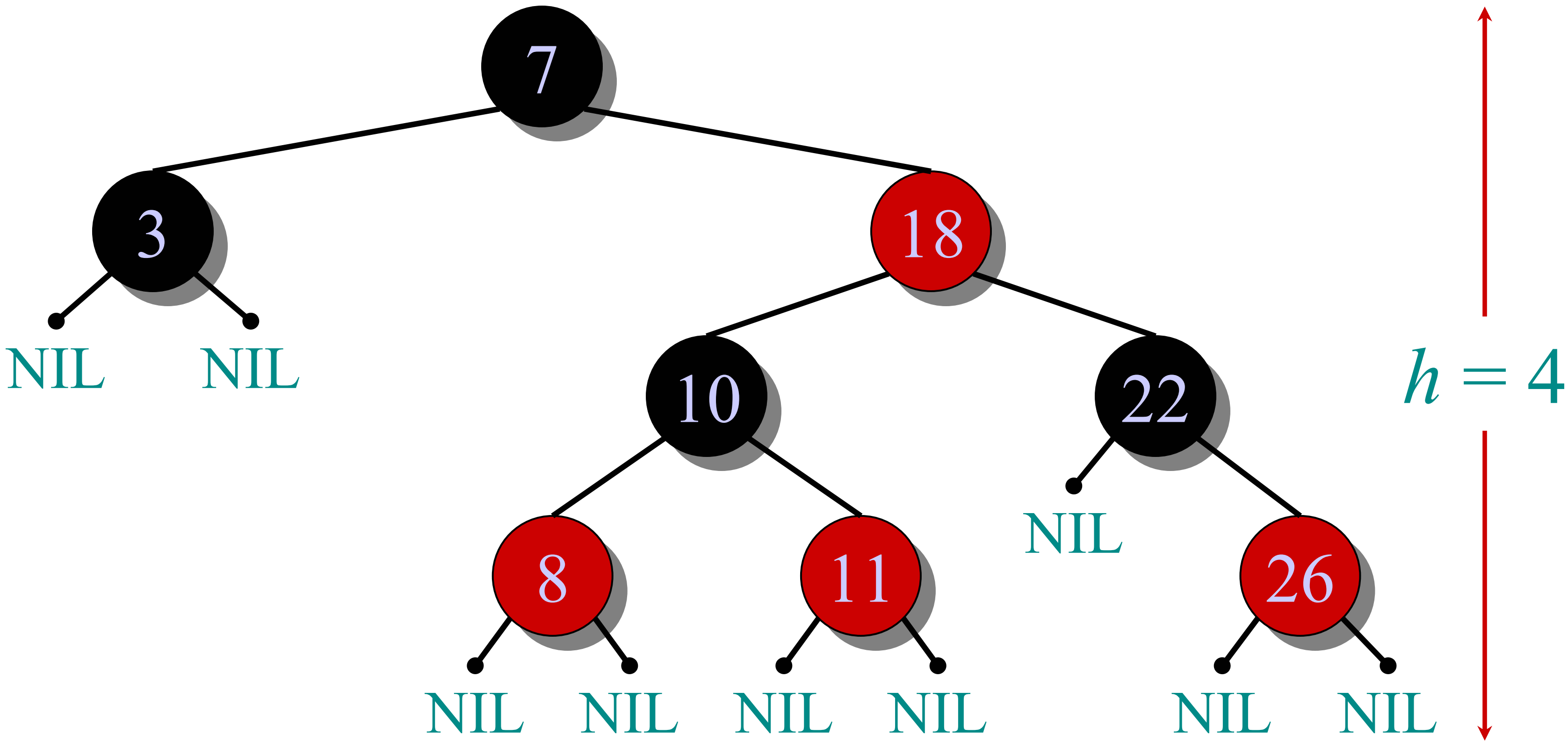
Red-black trees

This data structure requires an extra one-bit color field in each node.

Red-black properties:

1. Every node is either red or black.
2. The root and leaves (NIL's) are black.
3. If a node is red, then its parent is black.
4. All simple paths from any node x to a descendant leaf have the same number of black nodes = black-height(x).

Example of a red-black tree



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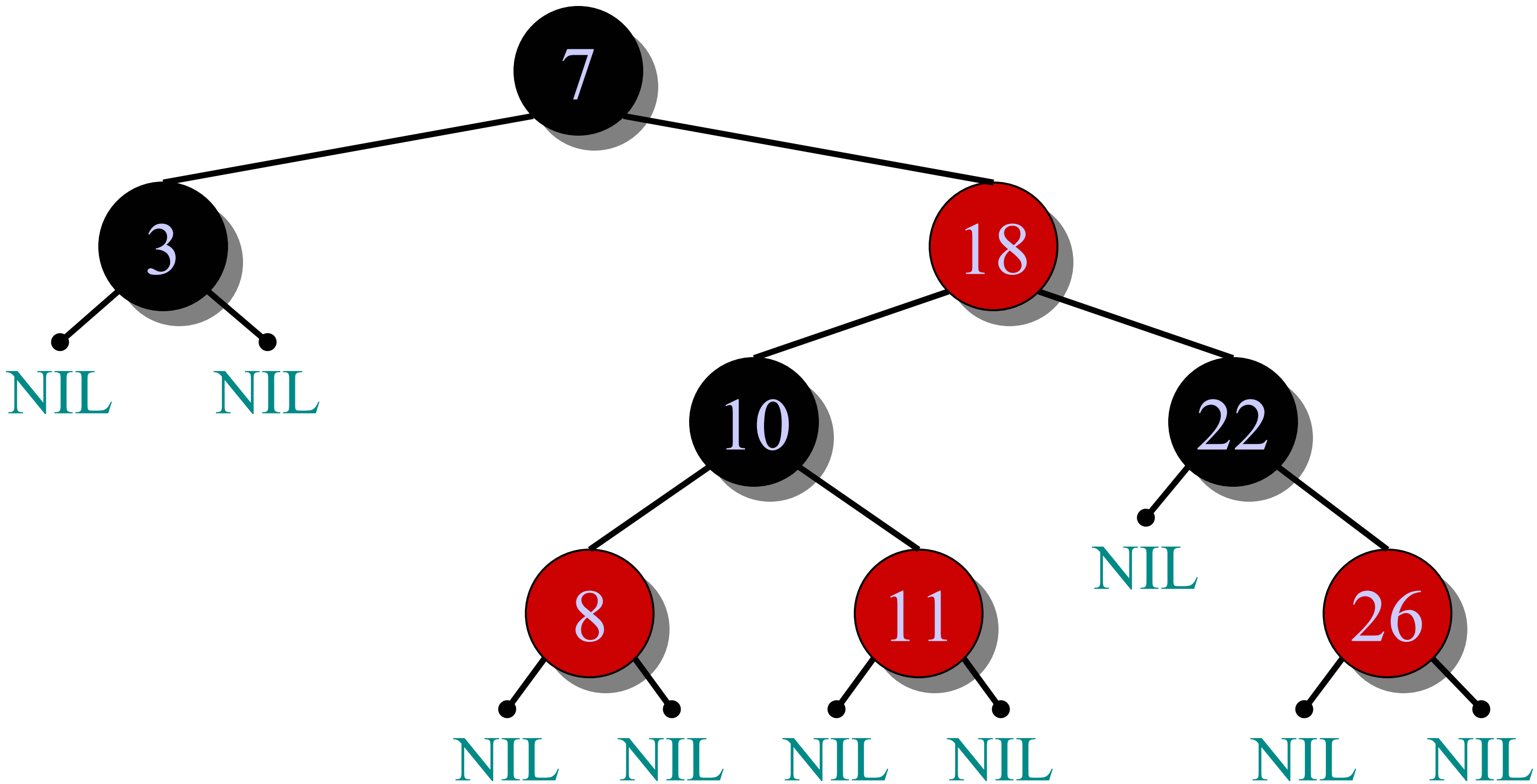

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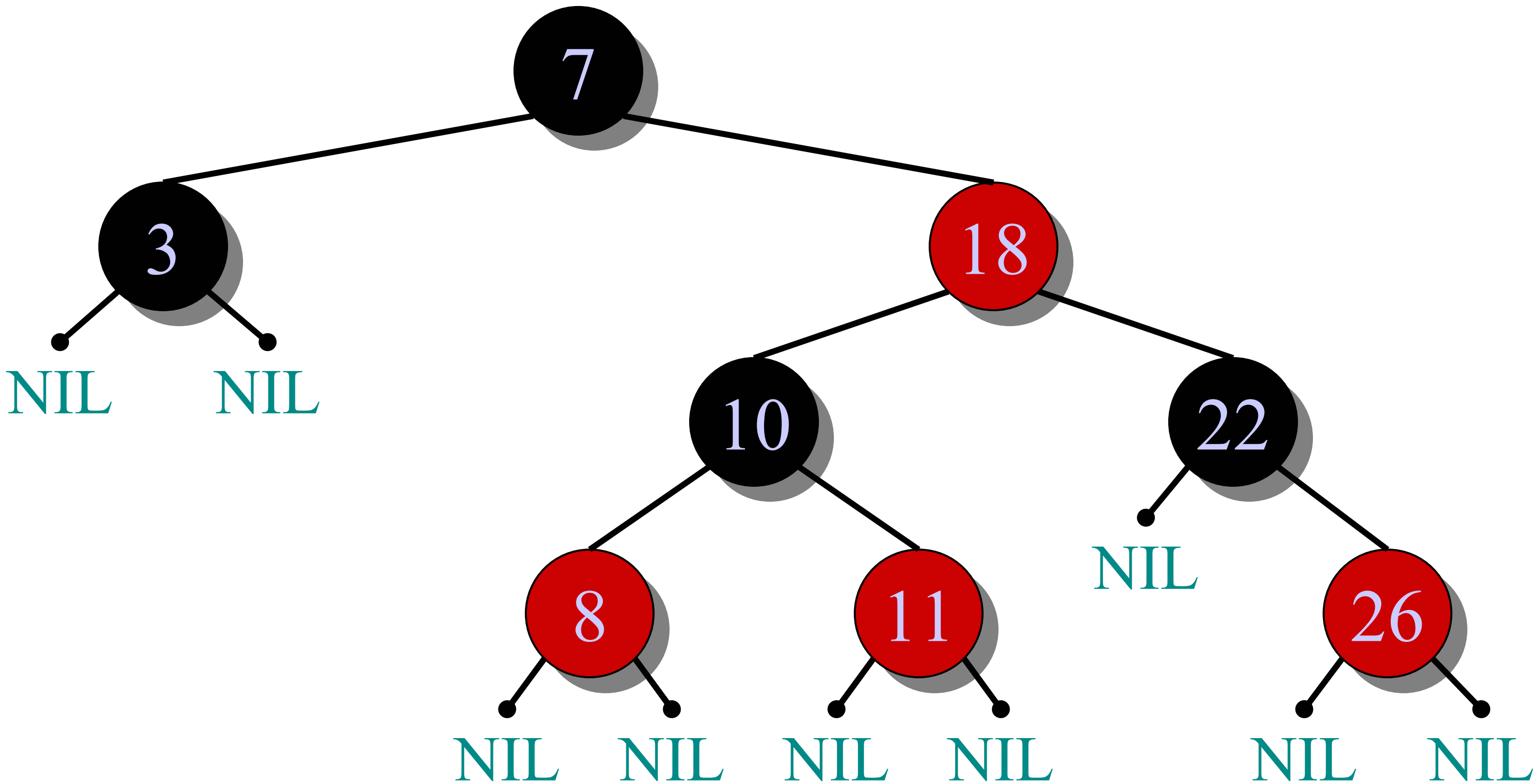

2. The root and leaves (NIL's) are black.

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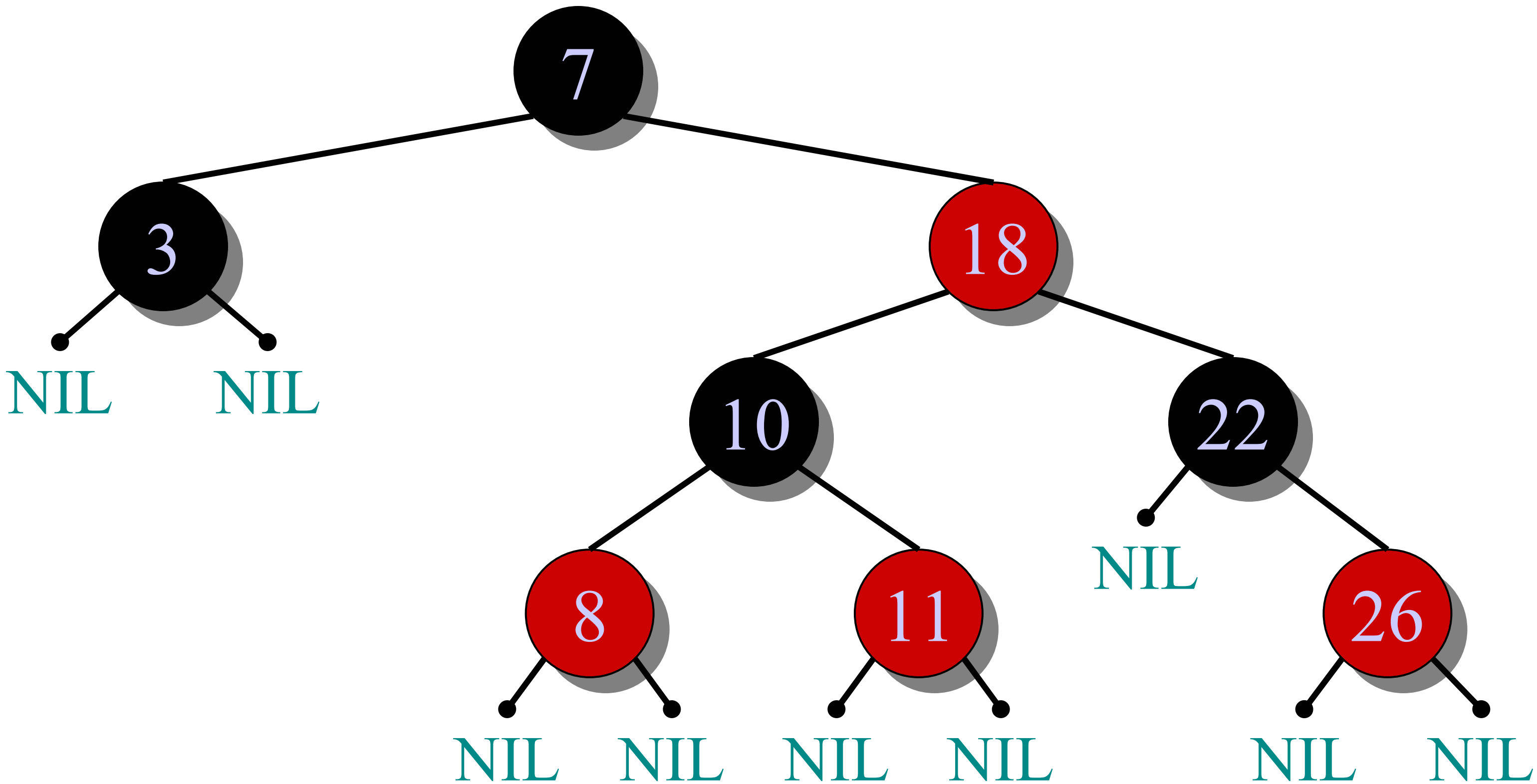

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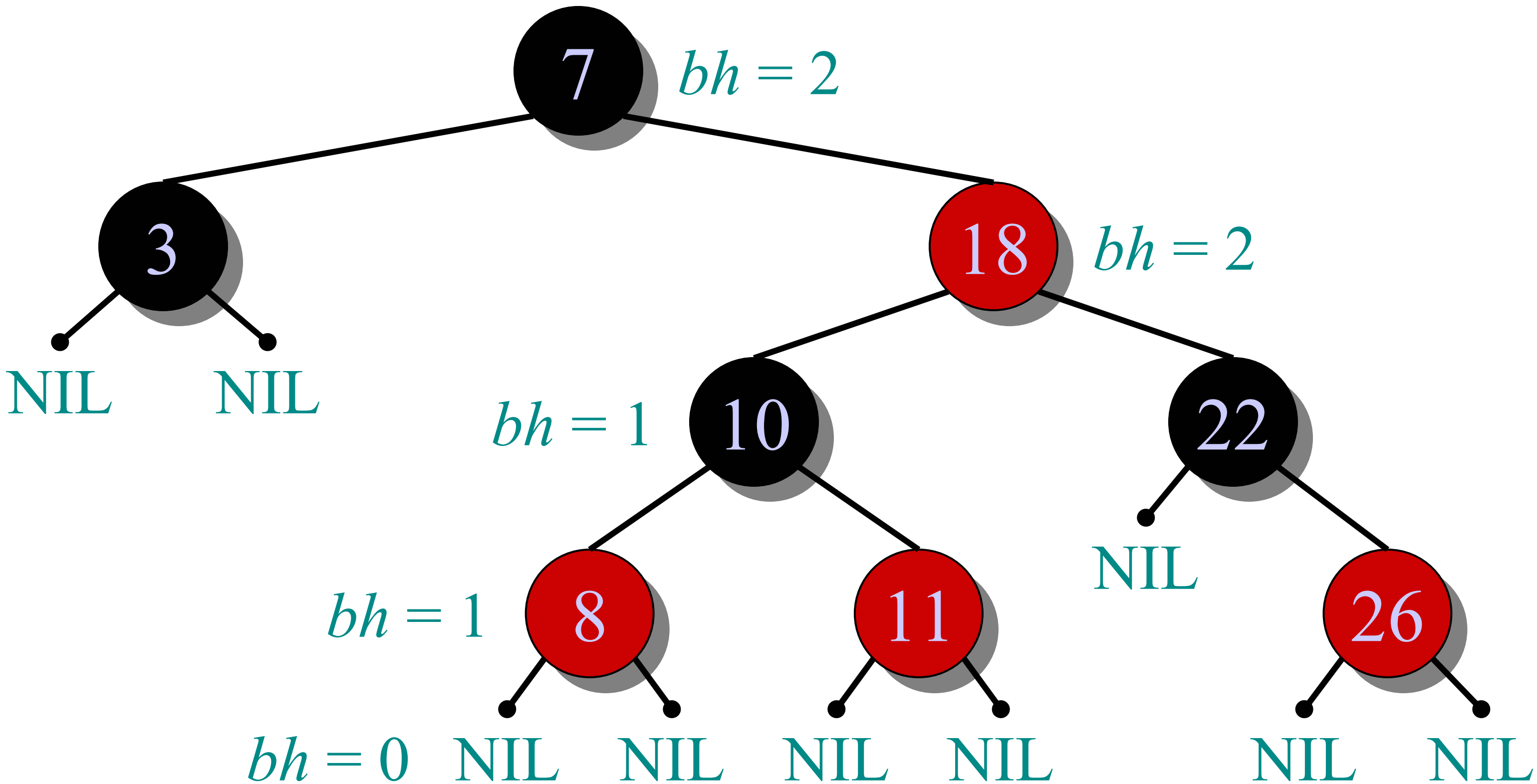


4. All simple paths from any node x to a descendant leaf have the same number of black nodes = *black-height*(x).

Height of a red-black tree

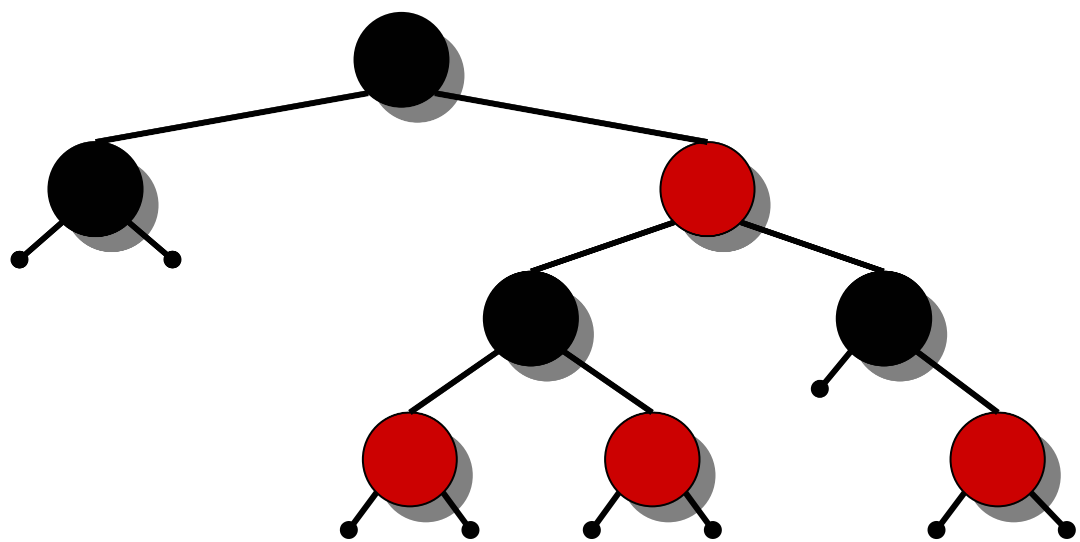
Theorem. A red-black tree with n keys has height

$$h \leq 2 \lg(n + 1).$$

Proof. (The book uses induction. Read carefully.)

INTUITION:

- Merge red nodes into their black parents.



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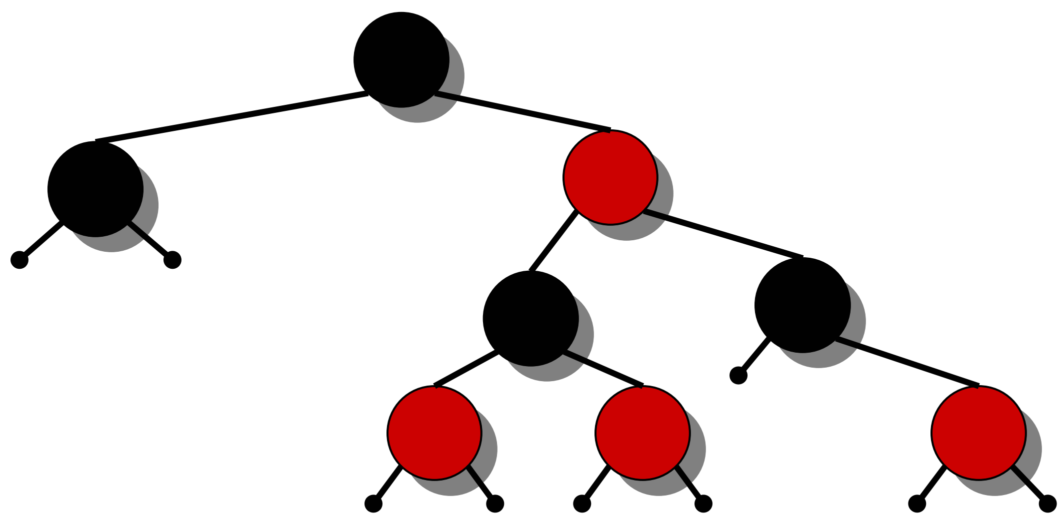

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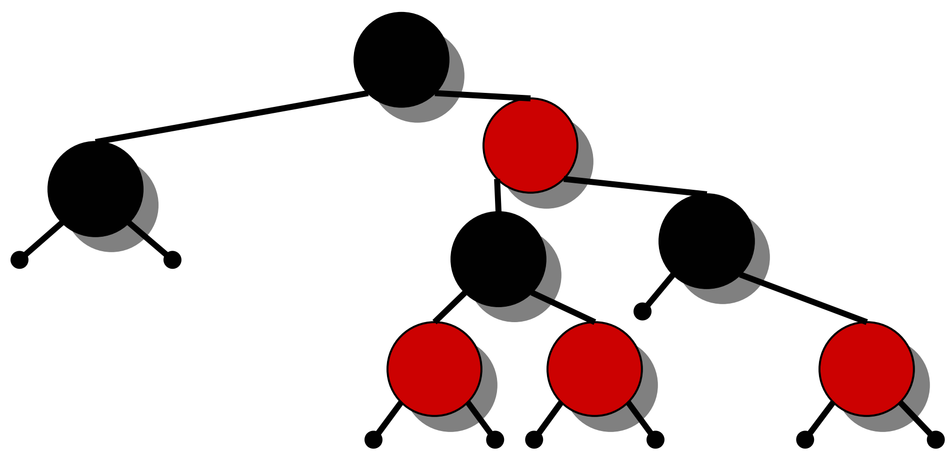

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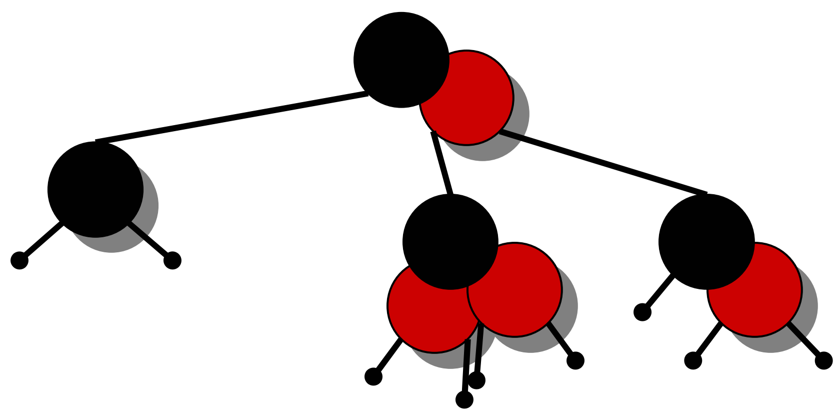

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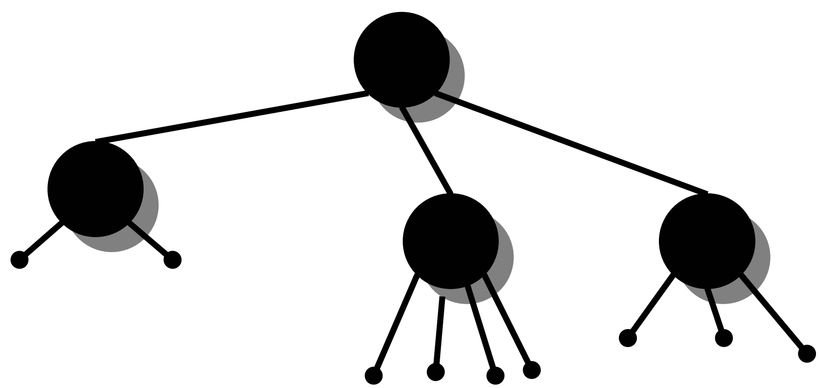

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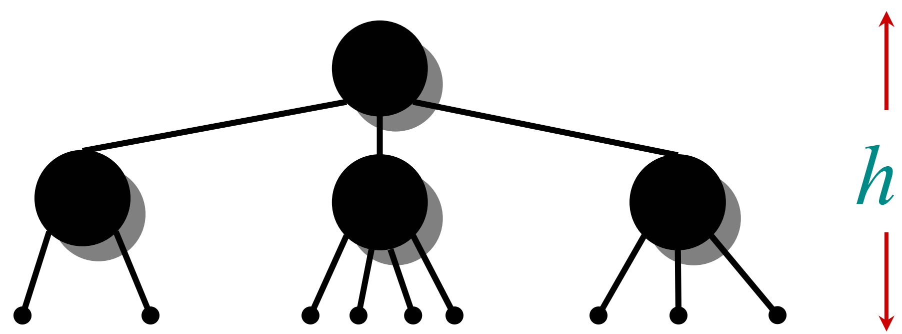

- Merge red nodes into their black parents.
- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth h' of leaves.

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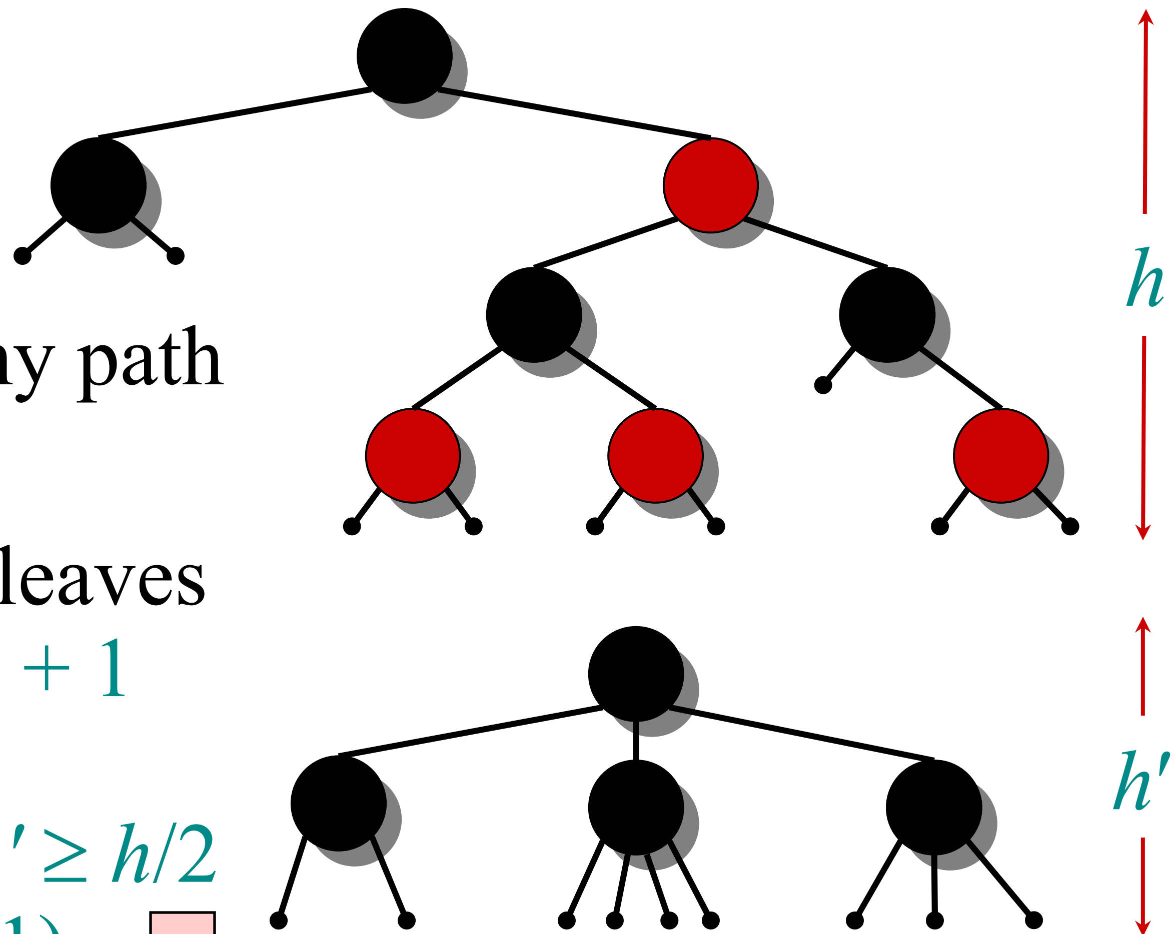
Proof (continued)

- We have $h' \geq h/2$, since at most half the leaves on any path are red.
- The number of leaves in each tree is $n + 1$
  $\Rightarrow n + 1 \geq 2^{h'}$
  $\Rightarrow \lg(n + 1) \geq h' \geq h/2$
  $\Rightarrow h \leq 2\lg(n + 1)$. □



Query operations

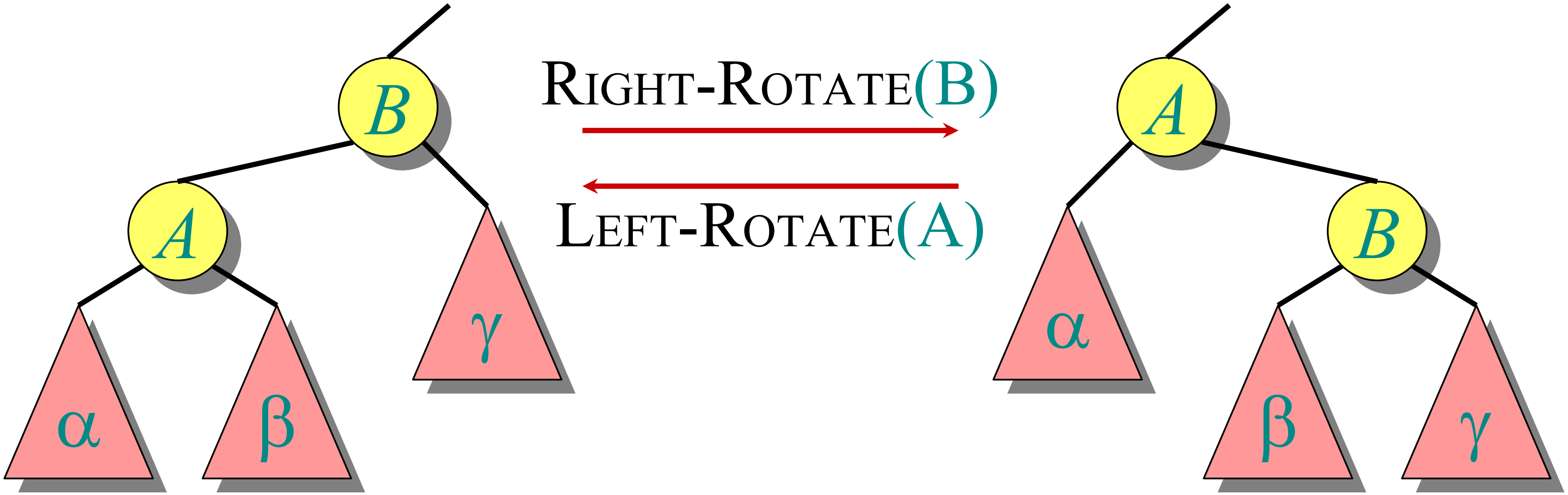
Corollary. The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR all run in $O(\lg n)$ time on a red-black tree with n nodes.

Modifying operations

The operations INSERT and DELETE cause modifications to the red-black tree:

- the operation itself,
- color changes,
- restructuring the links of the tree via ***"rotations"***.

Rotations



Rotations maintain the inorder ordering of keys:

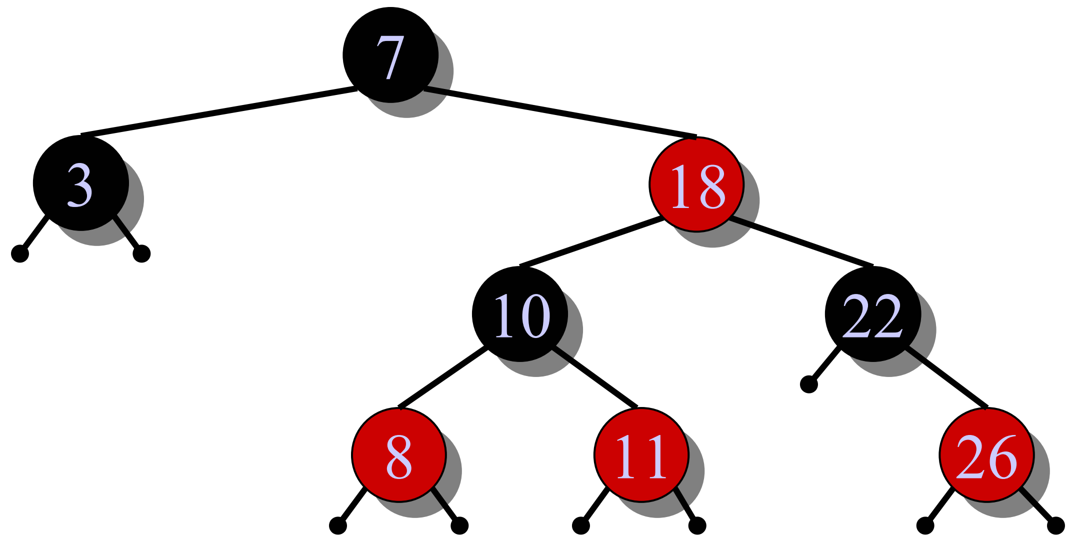
- $a \in \alpha, b \in \beta, c \in \gamma \Rightarrow a \le A \le b \le B \le c$.

A rotation can be performed in $O(1)$ time.

Insertion into a red-black tree

IDEA: Insert *x* in tree. Color *x* red. Only red-black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

Example:



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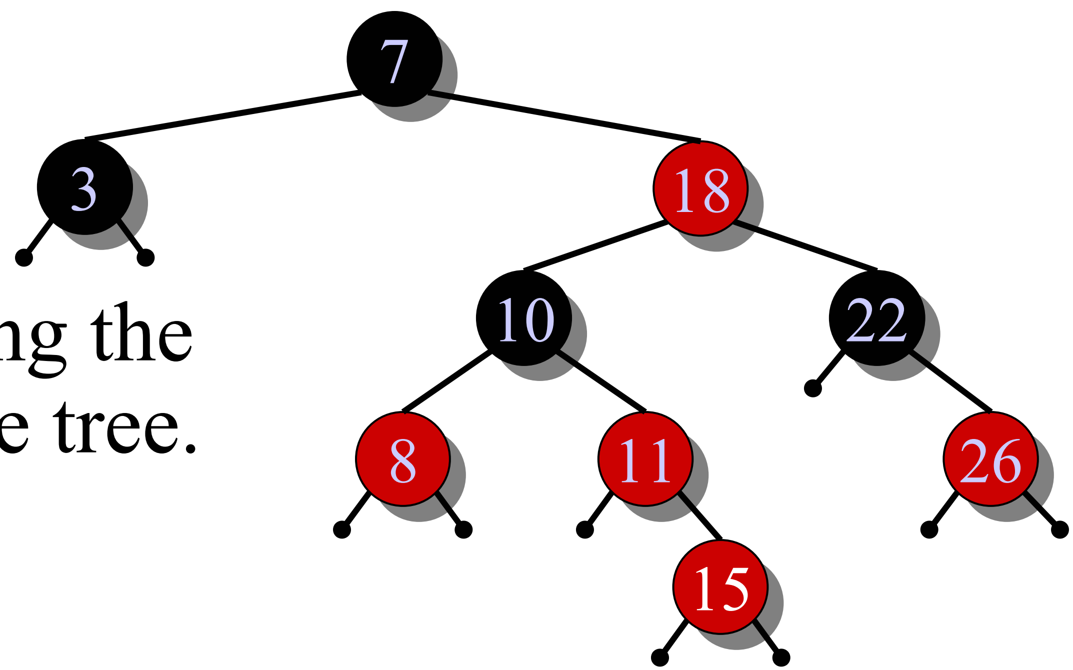
Example:

- Insert $x = 15$.
- Recolor, moving the violation up the tree.



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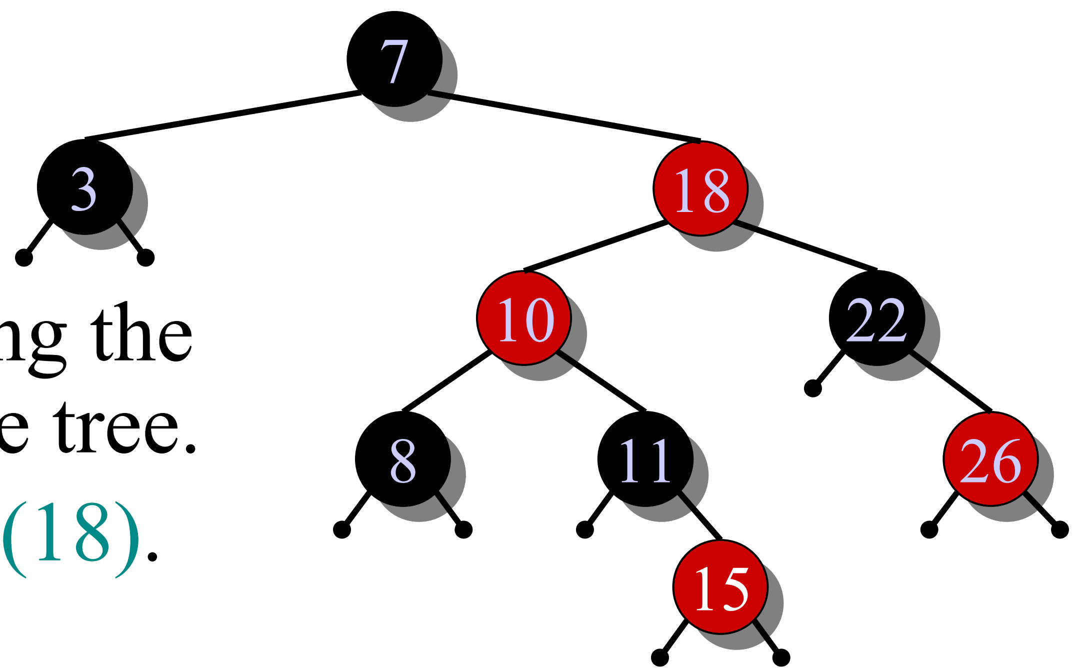
- RIGHT-ROTATE(18).



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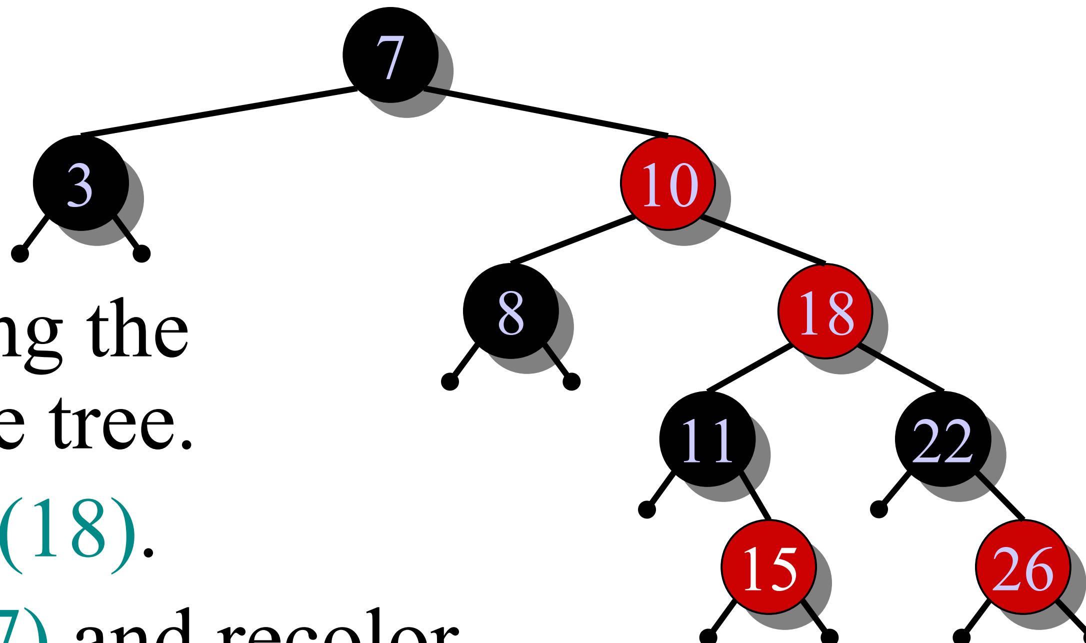
- LEFT-ROTATE(7) and recolor.



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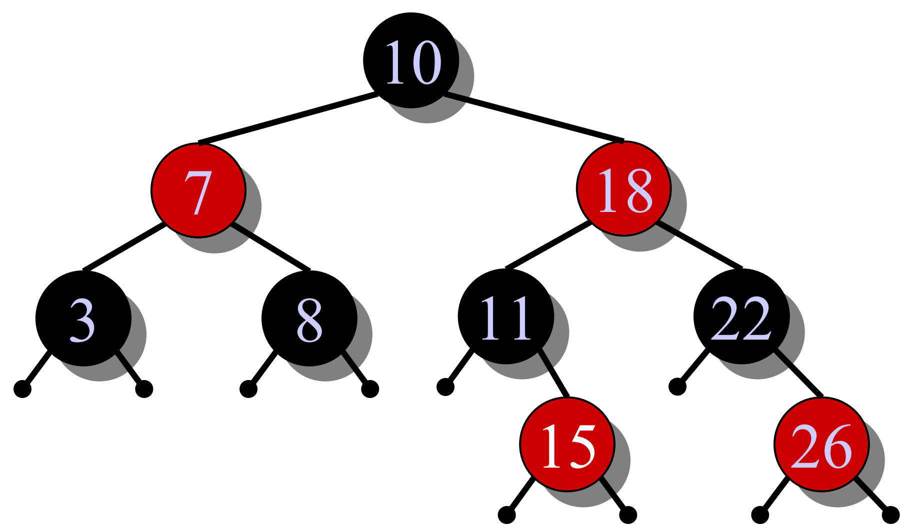
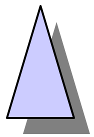
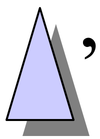


Pseudocode

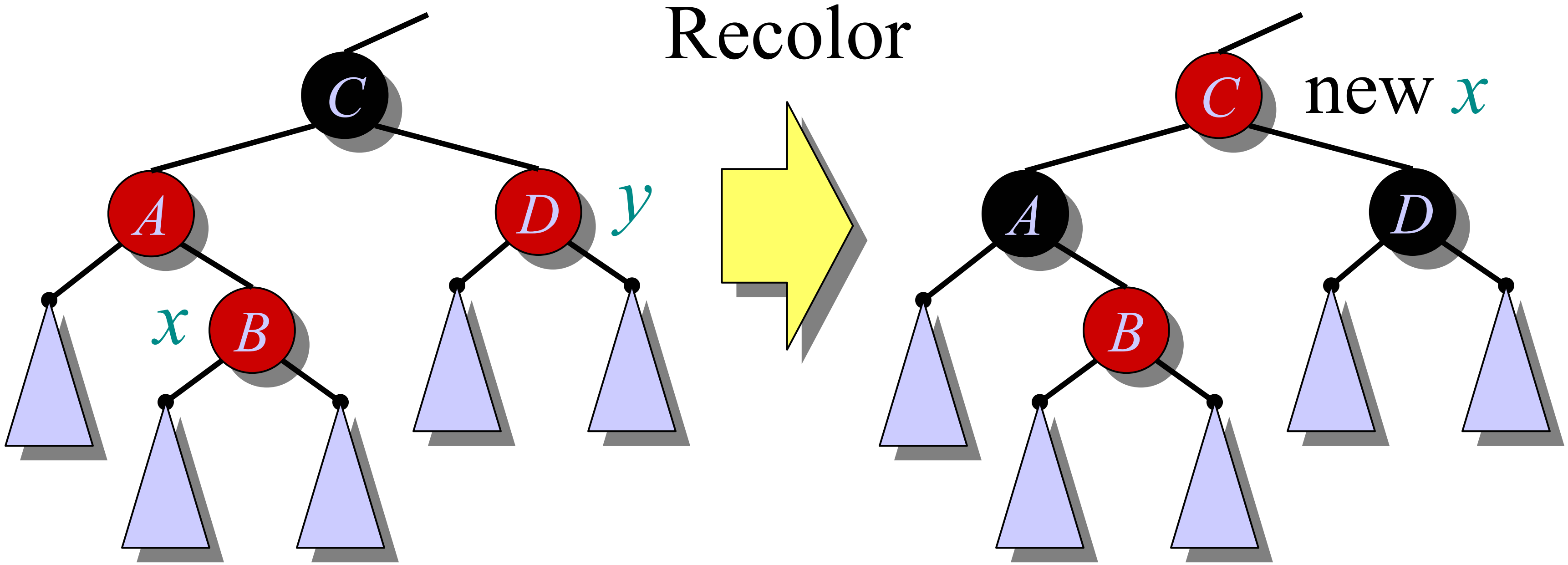
```
RB-INSERT(T, x)
    TREE-INSERT(T, x)
    color[x] ← RED        ▷ only RB property 3 can be violated
    while x ≠ root[T] and color[p[x]] = RED
        do if p[x] = left[p[p[x]]]
            then y ← right[p[p[x]]]        ▷ y = aunt/uncle of x
                 if color[y] = RED
                  then ⟨Case 1⟩
                  else  if x = right[p[x]]
                          then ⟨Case 2⟩    ▷ Case 2 falls into Case 3
                        ⟨Case 3⟩
            else ⟨"then" clause with "left" and "right" swapped⟩
    color[root[T]] ← BLACK
```

Graphical notation

Let  denote a subtree with a black root.

All 's have the same black-height.

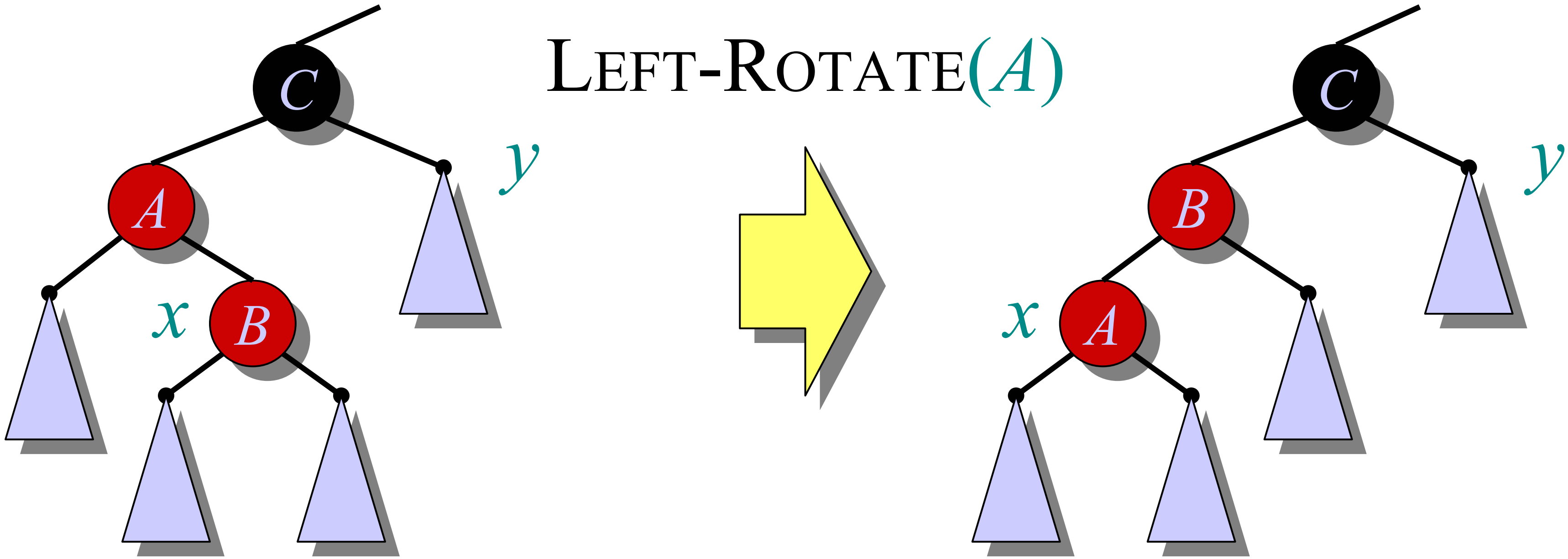
Case 1



(Or, children of *A* are swapped.)

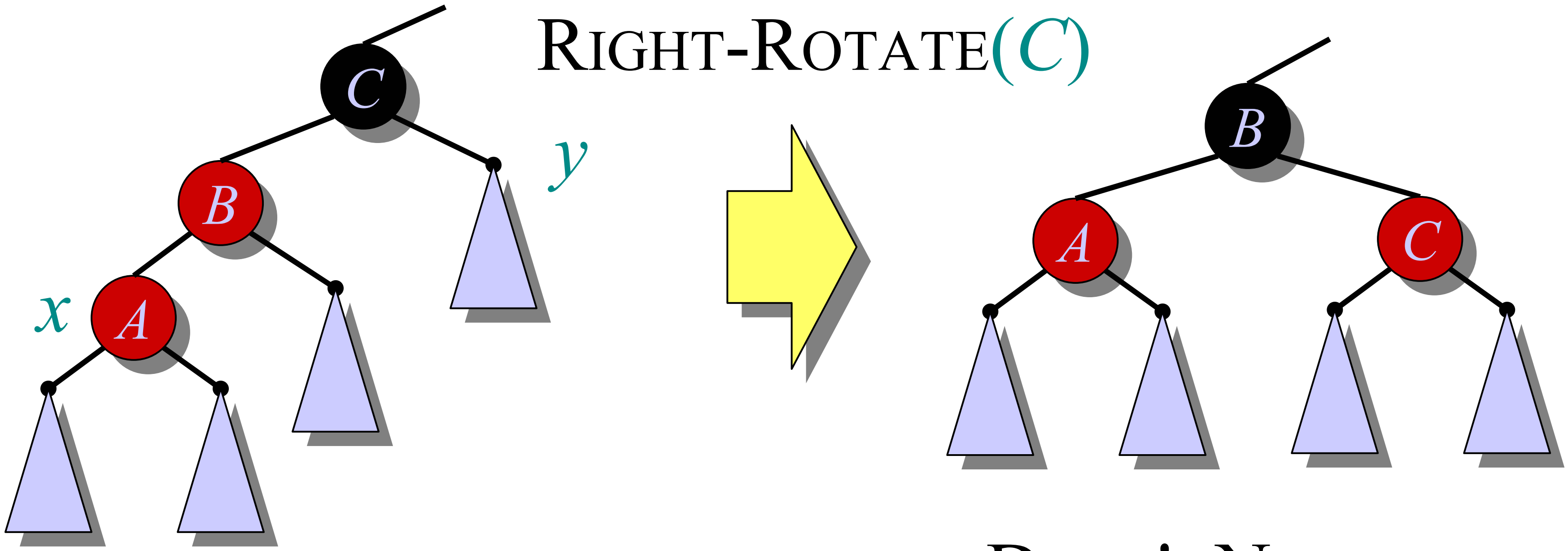
Push *C*'s black onto *A* and *D*, and recurse, since *C*'s parent may be red.

Case 2



Transform to Case 3.

Case 3



Done! No more violations of RB property 3 are possible.

Analysis

- Go up the tree performing Case 1, which only recolors nodes.
- If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

Running time: $O(\lg n)$ with $O(1)$ rotations.

RB-DELETE — same asymptotic running time and number of rotations as RB-INSERT (see textbook).