Introduction to Algorithms

6.046J/18.401J/SMA5503

Lecture 11

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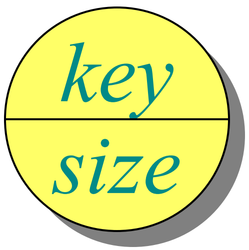
Dynamic order statistics

OS-SELECT(i, S): returns the ith smallest element in the dynamic set S.

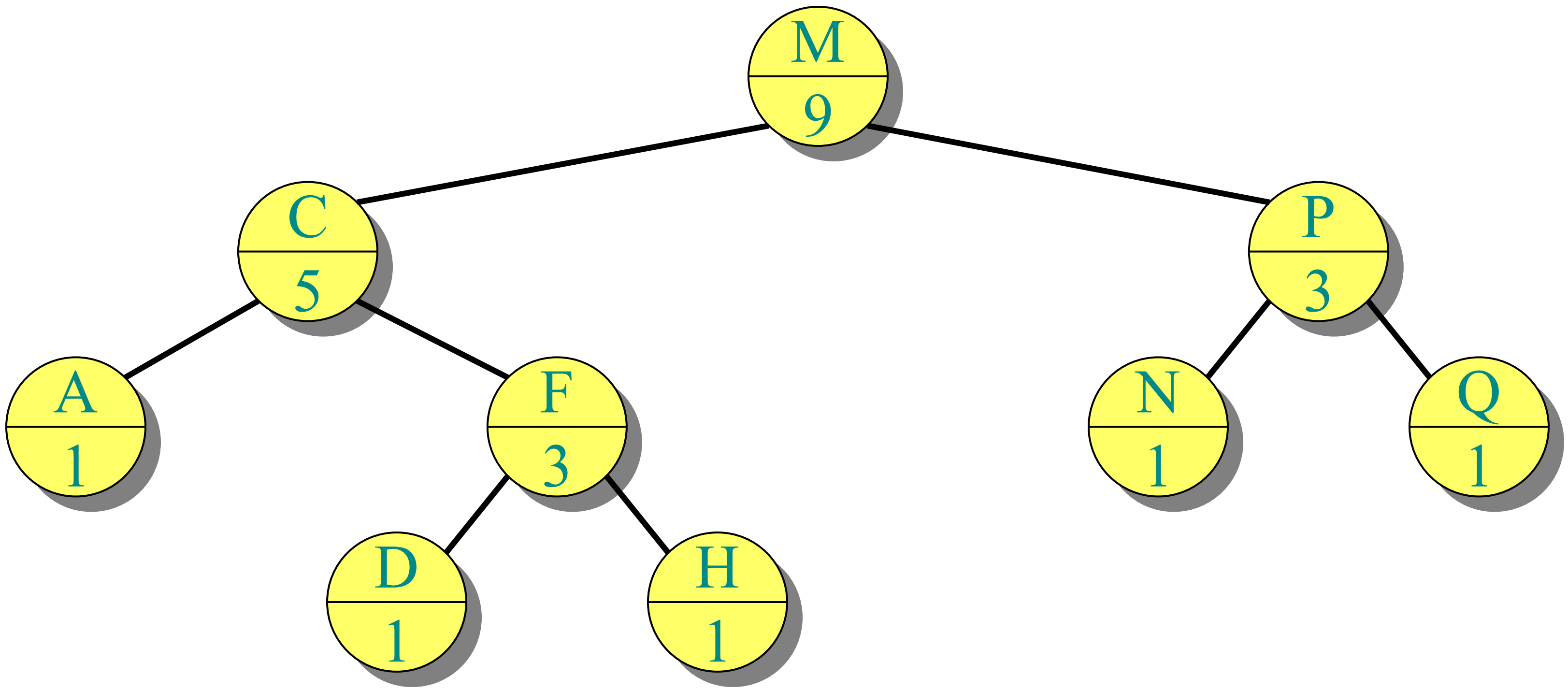
OS-RANK(x, S): returns the rank of $x \in S$ in the sorted order of S's elements.

IDEA: Use a red-black tree for the set S, but keep subtree sizes in the nodes.

Notation for nodes:



Example of an OS-tree



$$size[x] = size[left[x]] + size[right[x]] + 1$$

Selection

Implementation trick: Use a ***sentinel*** (dummy record) for NIL such that $size[\text{NIL}] = 0$.

OS-SELECT(x, i) ▷ ith smallest element in the subtree rooted at x

$k \leftarrow size[left[x]] + 1$ ▷ $k = \text{rank}(x)$

if $i = k$ **then return** x

if $i < k$

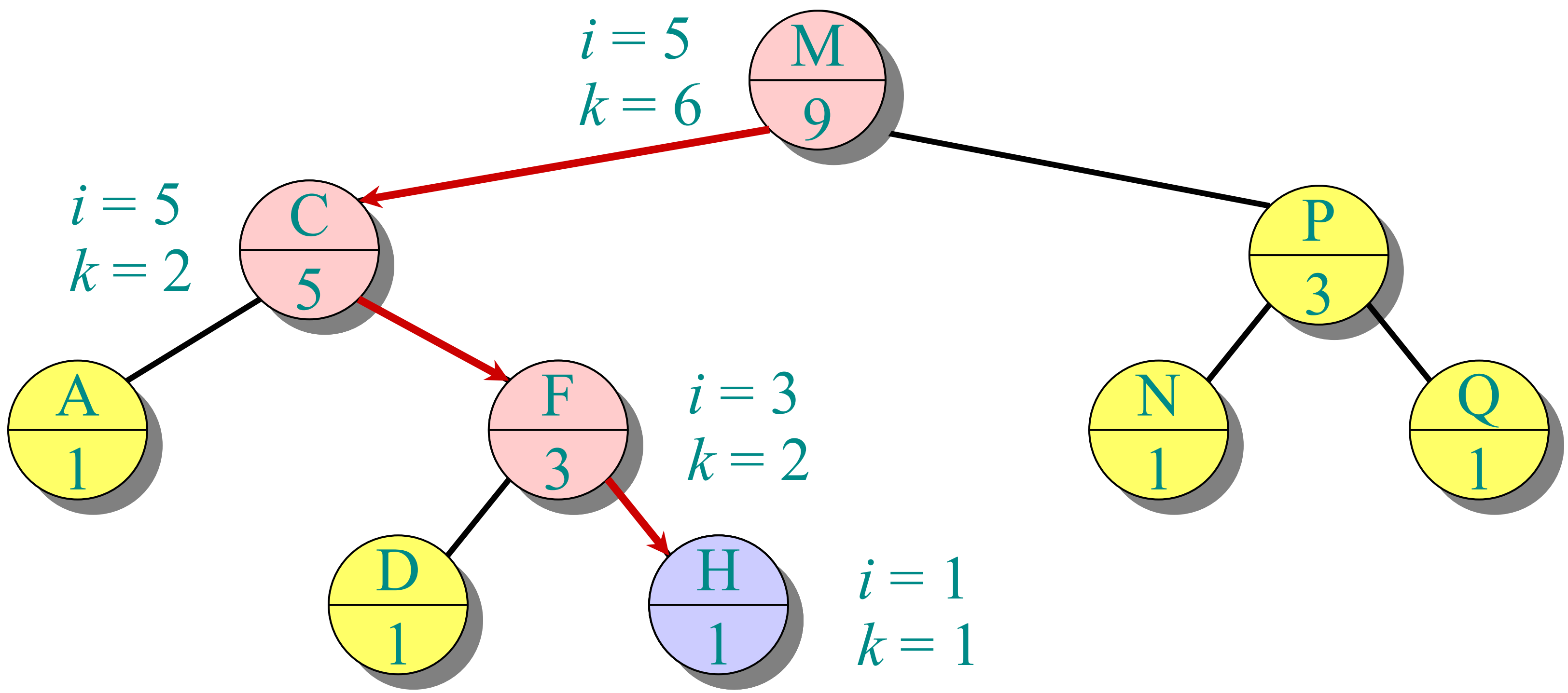
  then return OS-SELECT$(left[x], i)$

  else return OS-SELECT$(right[x], i - k)$

(OS-RANK is in the textbook.)

Example

OS-SELECT(*root*, 5)



Running time $= O(h) = O(\lg n)$ for red-black trees.

Data structure maintenance

Q. Why not keep the ranks themselves in the nodes instead of subtree sizes?

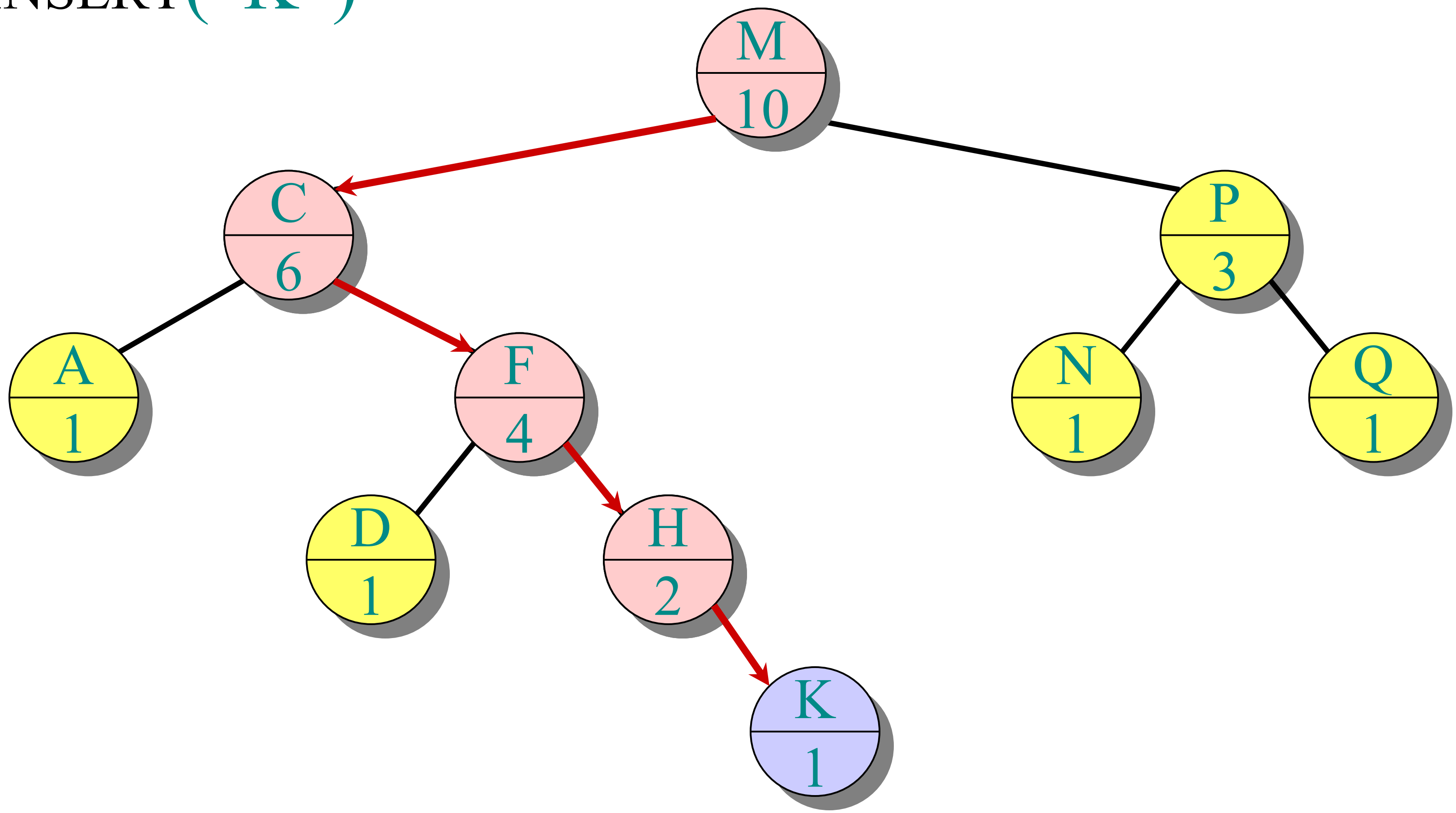
A. They are hard to maintain when the red-black tree is modified.

Modifying operations: INSERT and DELETE.

Strategy: Update subtree sizes when inserting or deleting.

Example of insertion

INSERT("K")

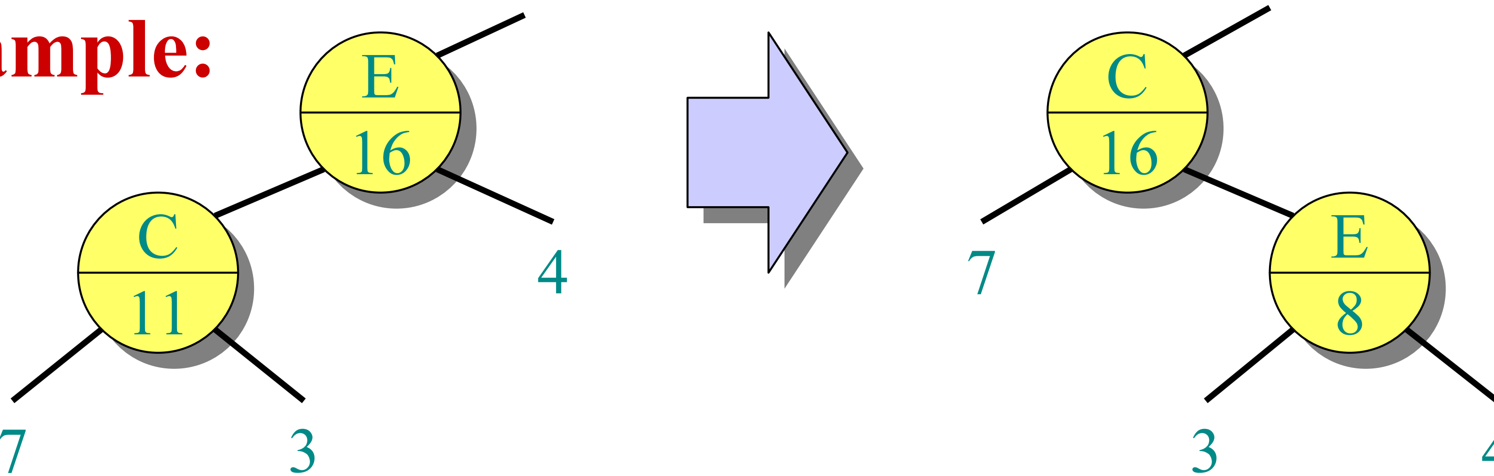


Handling rebalancing

Don't forget that RB-INSERT and RB-DELETE may also need to modify the red-black tree in order to maintain balance.

- *Recolorings*: no effect on subtree sizes.
- *Rotations*: fix up subtree sizes in $O(1)$ time.

Example:



∴ RB-INSERT and RB-DELETE still run in $O(\lg n)$ time.

Data-structure augmentation

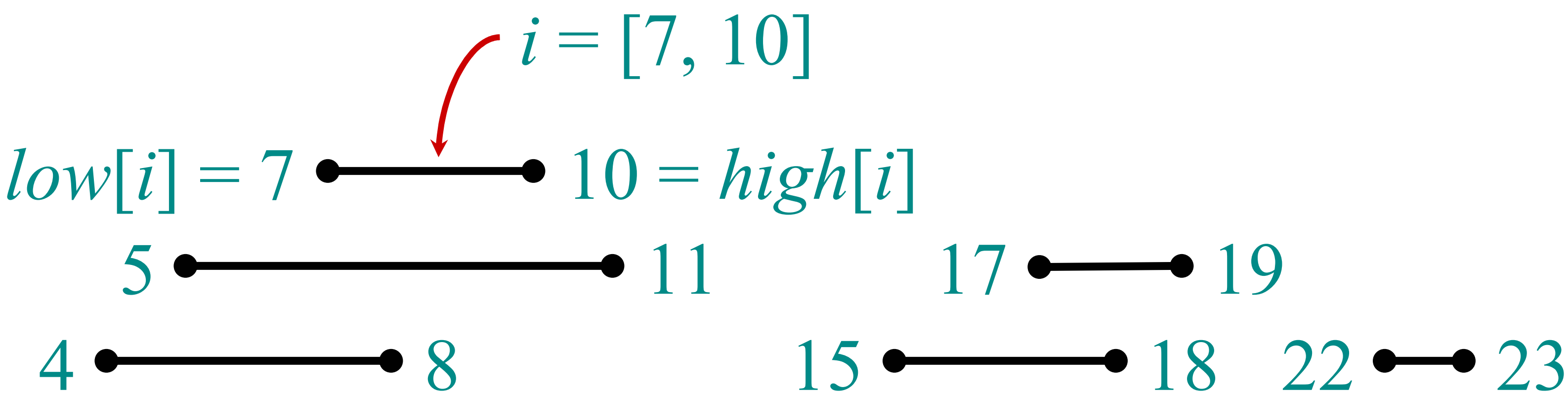
Methodology: (*e.g., order-statistics trees*)

1. Choose an underlying data structure (*red-black trees*).
2. Determine additional information to be stored in the data structure (*subtree sizes*).
3. Verify that this information can be maintained for modifying operations (*RB-INSERT, RB-DELETE — don't forget rotations*).
4. Develop new dynamic-set operations that use the information (*OS-SELECT and OS-RANK*).

These steps are guidelines, not rigid rules.

Interval trees

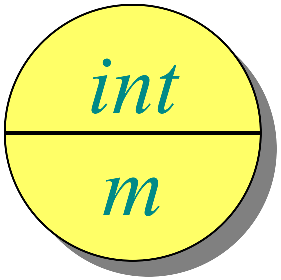
Goal: To maintain a dynamic set of intervals, such as time intervals.



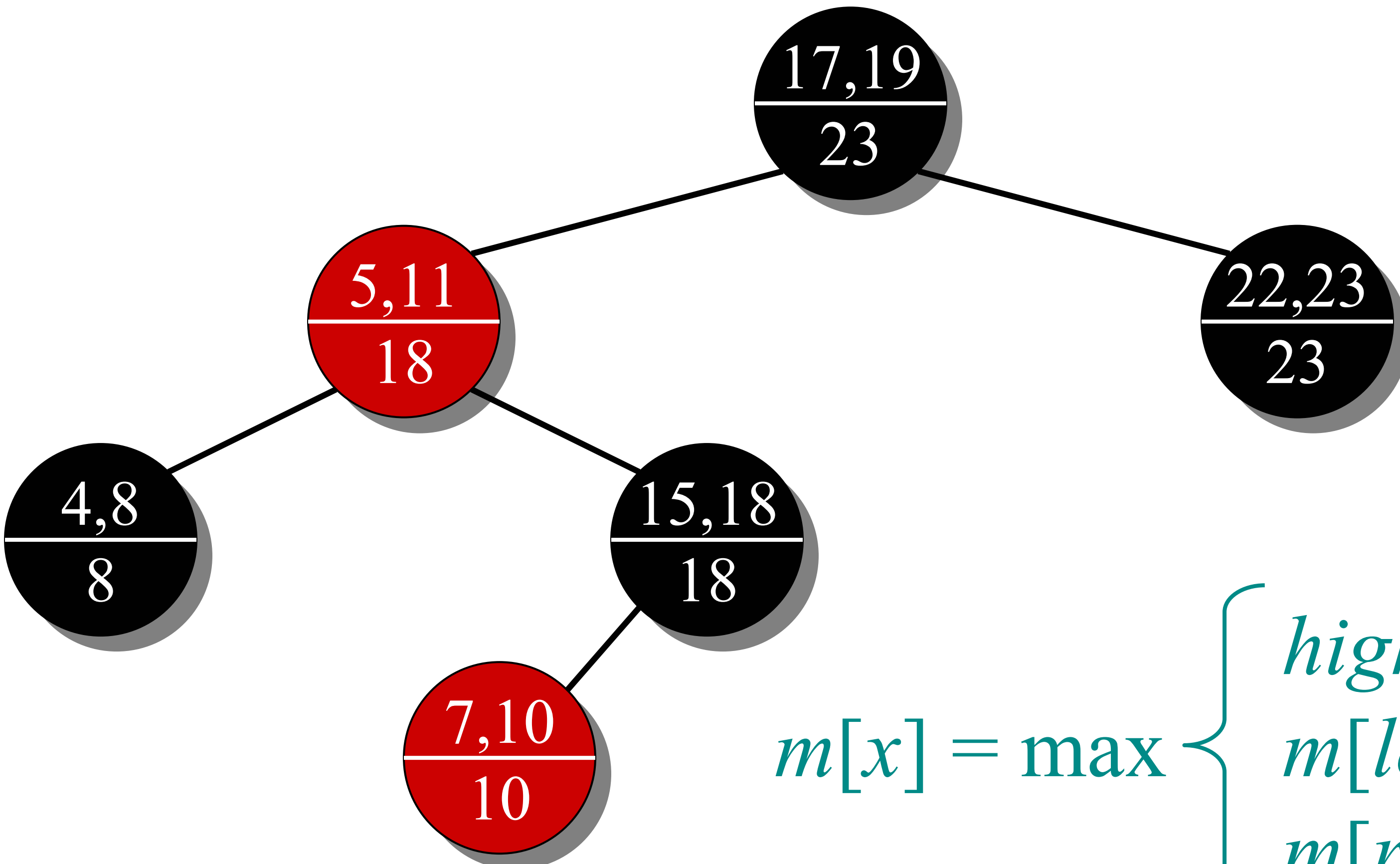
Query: For a given query interval i, find an interval in the set that overlaps i.

Following the methodology

1. *Choose an underlying data structure.*
   - Red-black tree keyed on low (left) endpoint.
2. *Determine additional information to be stored in the data structure.*
   - Store in each node x the largest value $m[x]$ in the subtree rooted at x, as well as the interval $int[x]$ corresponding to the key.



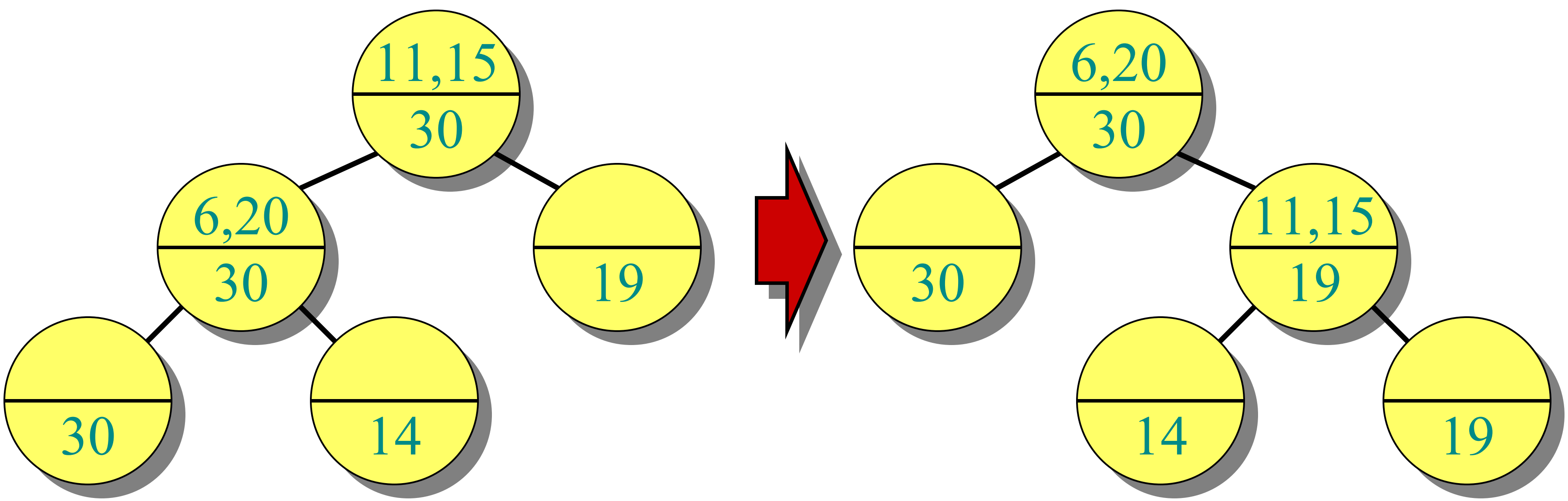
Example interval tree



$$m[x] = \max \begin{cases} high[int[x]] \\ m[left[x]] \\ m[right[x]] \end{cases}$$

Modifying operations

3. *Verify that this information can be maintained for modifying operations.*
   - INSERT: Fix m's on the way down.
   - Rotations — Fixup $= O(1)$ time per rotation:



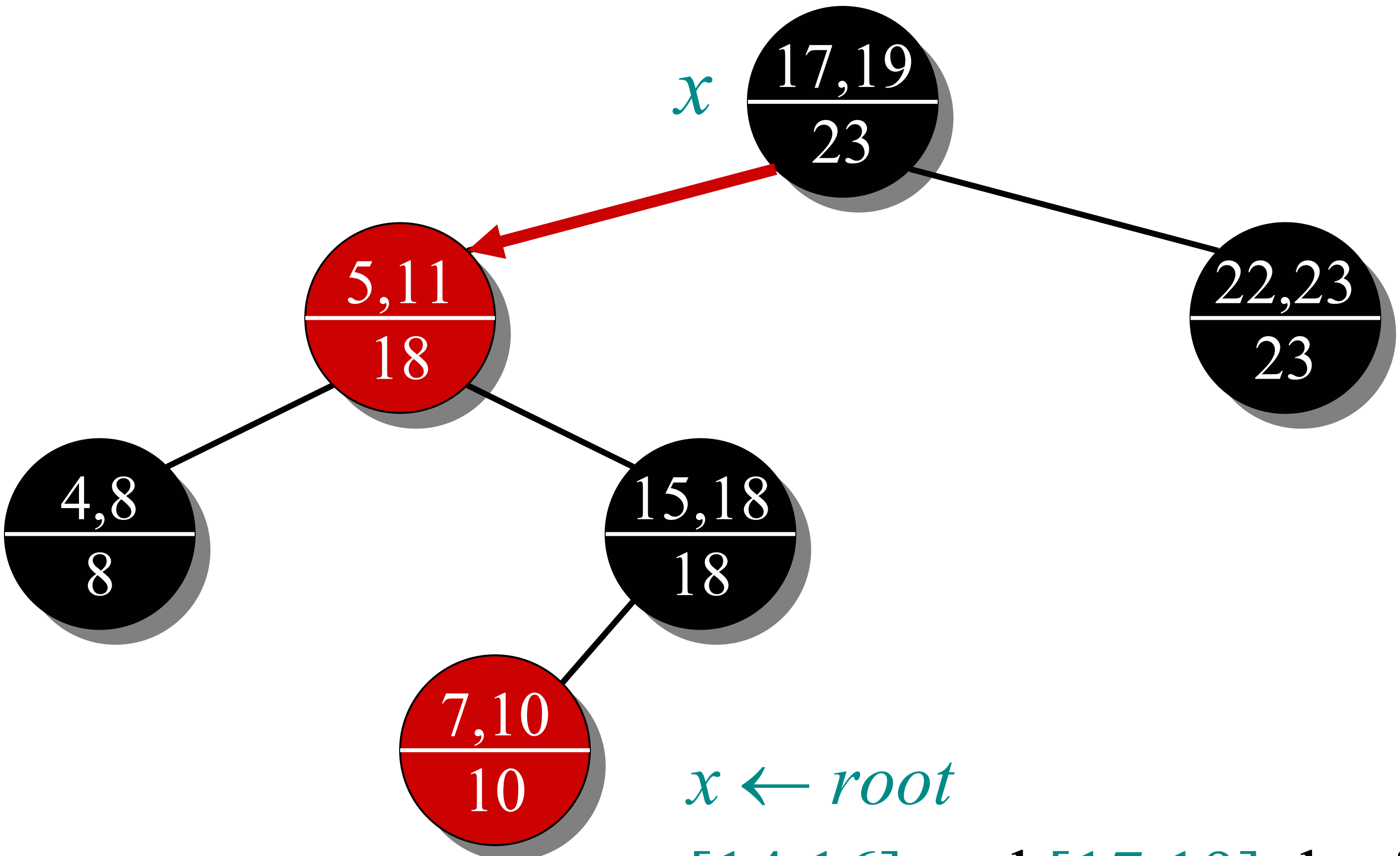
Total INSERT time $= O(\lg n)$; DELETE similar.

New operations

4. Develop new dynamic-set operations that use the information.

```
INTERVAL-SEARCH(i)
    x ← root
    while x ≠ NIL and (low[i] > high[int[x]]
                       or low[int[x]] > high[i])
        do ▷ i and int[x] don't overlap
            if left[x] ≠ NIL and low[i] ≤ m[left[x]]
                then x ← left[x]
                else x ← right[x]
    return x
```

Example 1: Interval-Search([14,16])



$x \leftarrow root$

[14,16] and [17,19] don't overlap

$14 \leq 18 \Rightarrow x \leftarrow left[x]$

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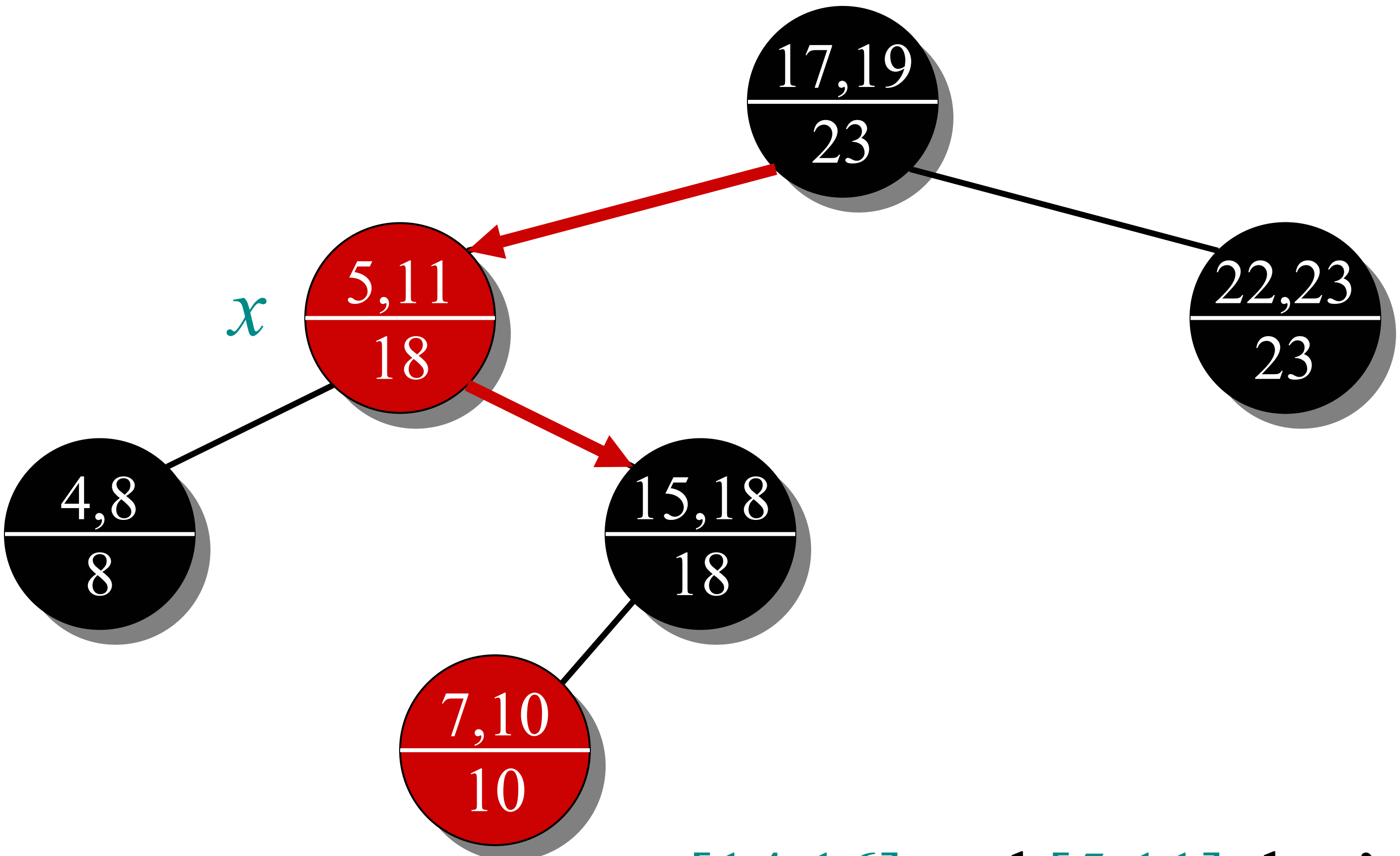
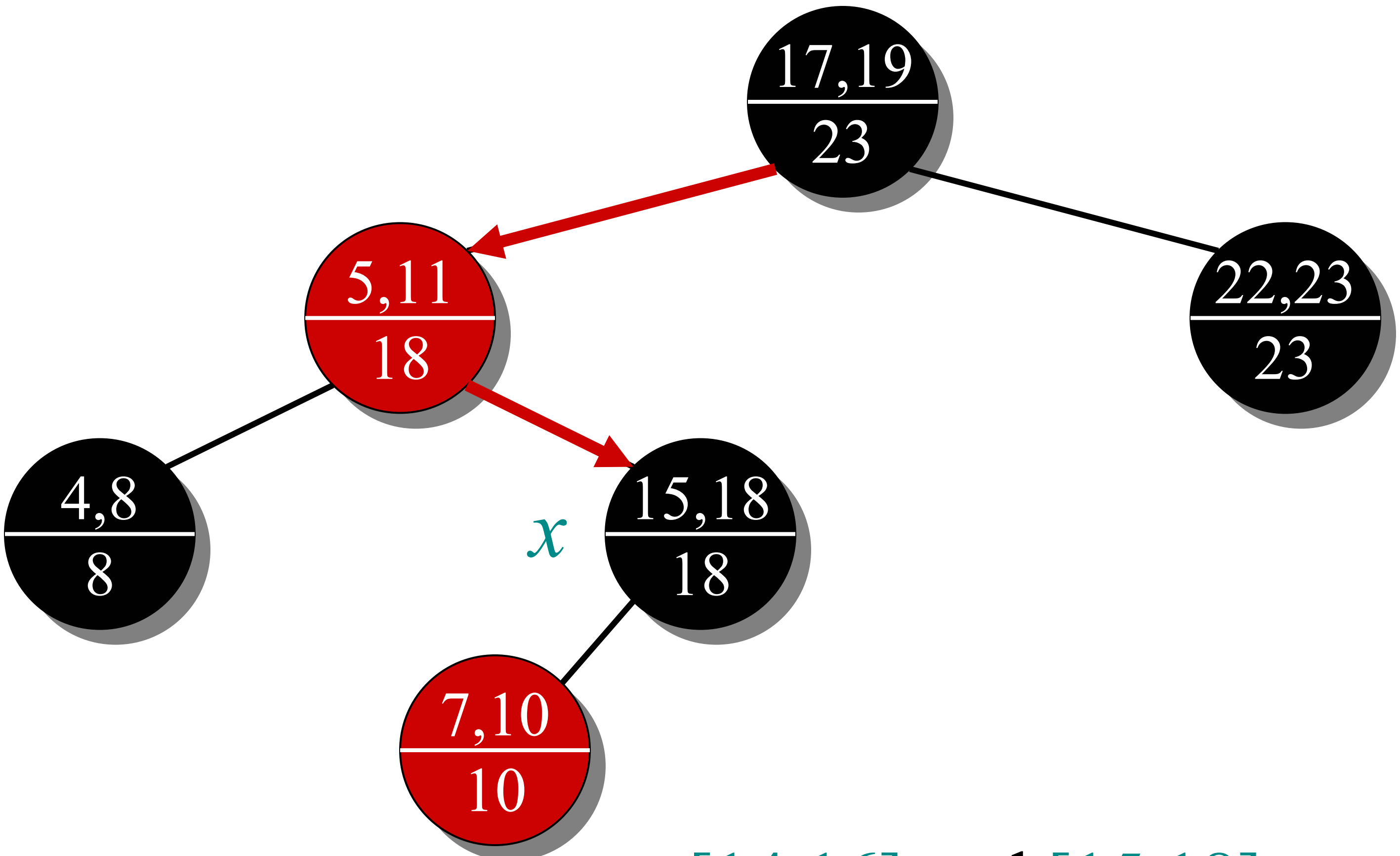


[14,16] and [5,11] don't overlap

$14 > 8 \Rightarrow x \leftarrow right[x]$

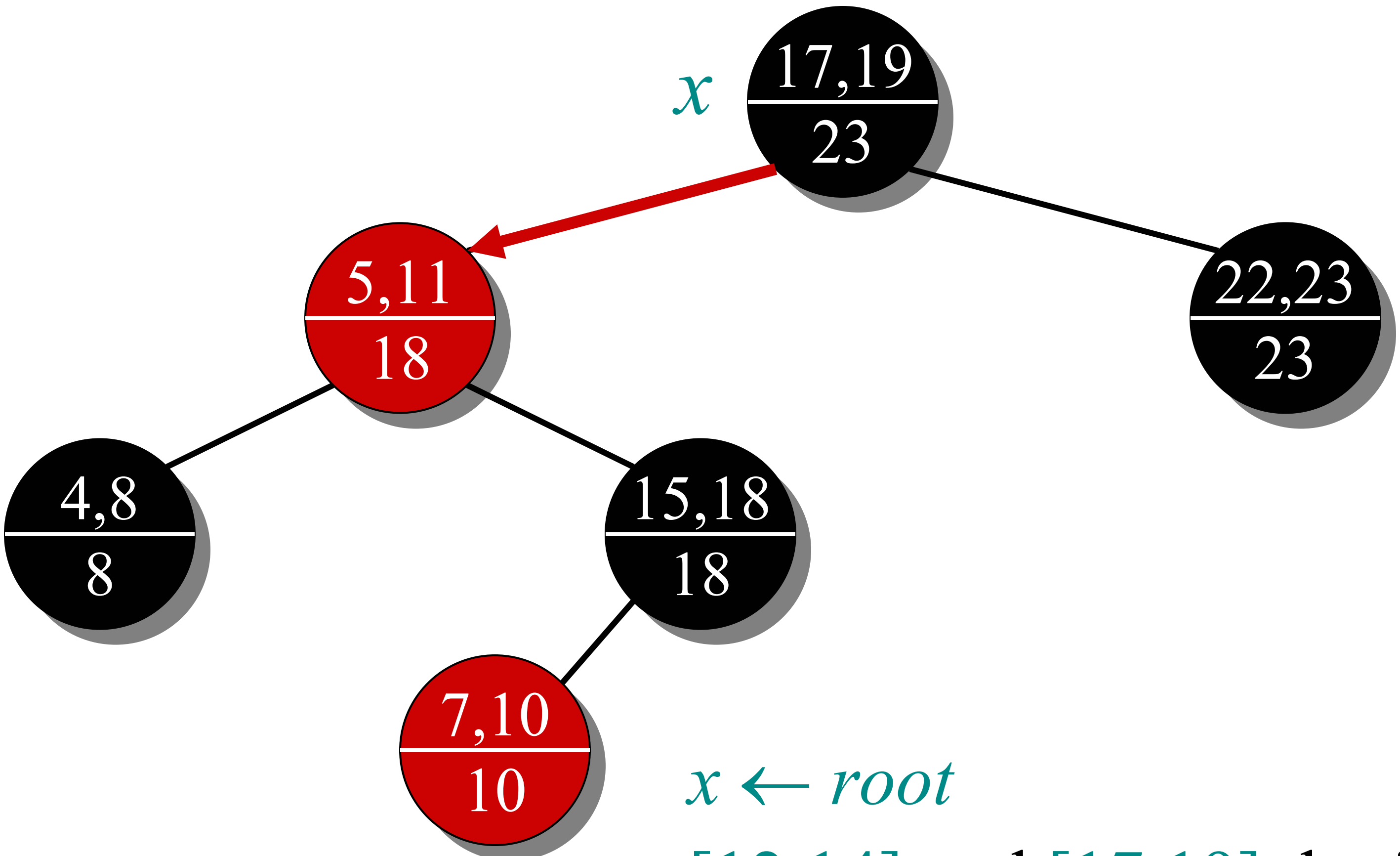
Example 1: INTERVAL-SEARCH([14,16])



[14,16] and [15,18] overlap

return [15,18]

Example 2: INTERVAL-SEARCH([12,14])

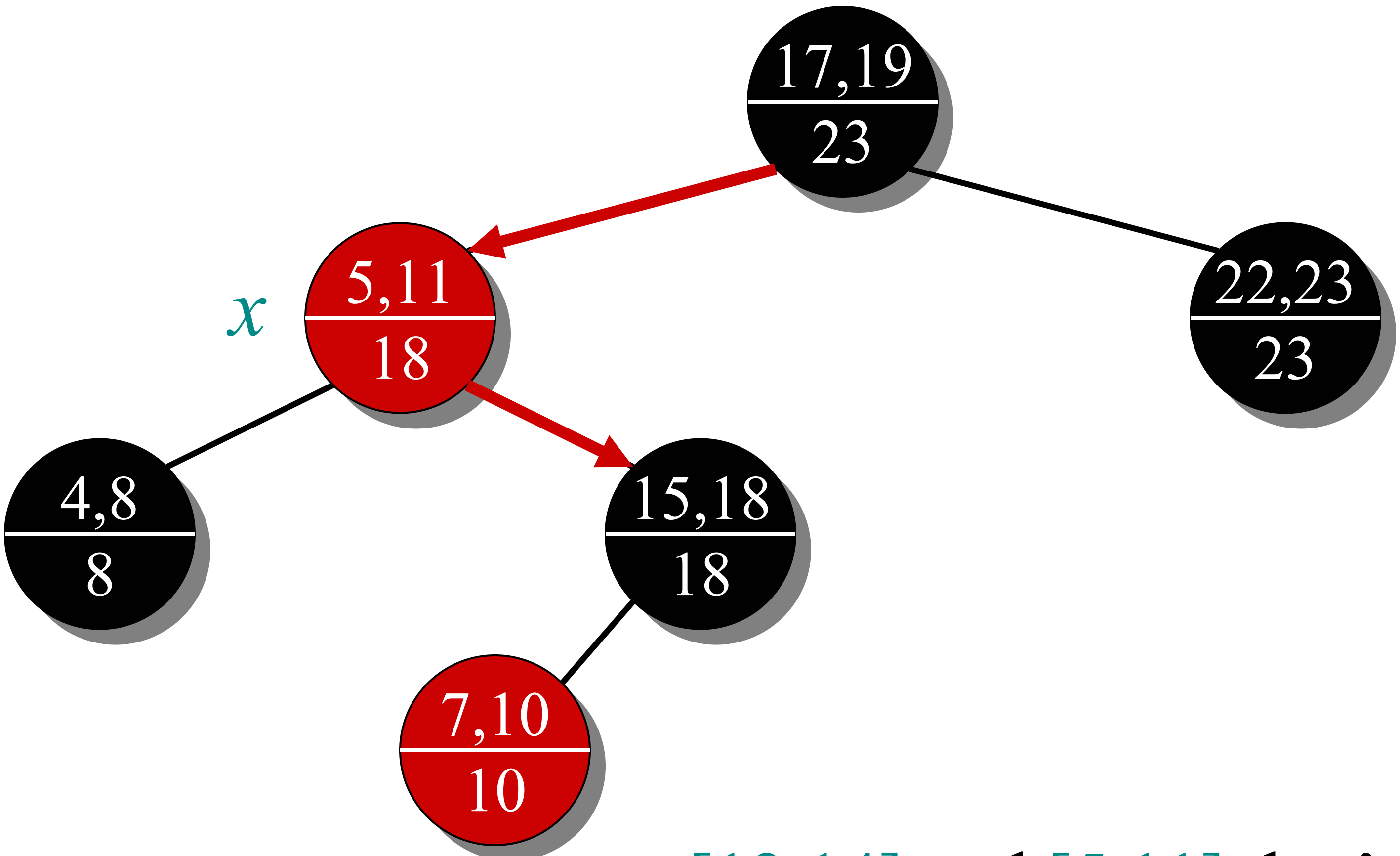


$x \leftarrow root$

[12,14] and [17,19] don't overlap

$12 \le 18 \Rightarrow x \leftarrow left[x]$

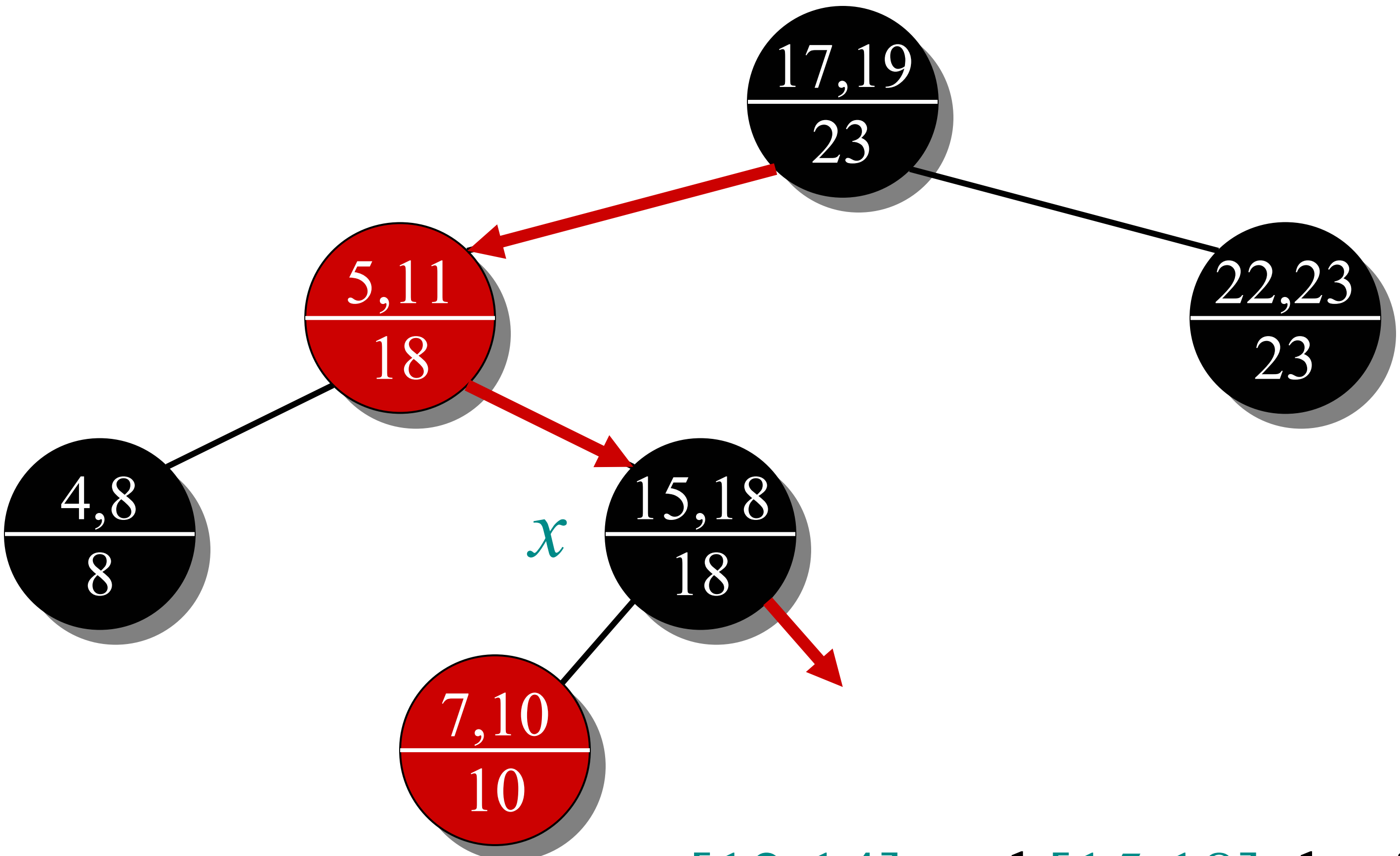
Example 2: INTERVAL-SEARCH([12,14])



[12,14] and [5,11] don't overlap

$12 > 8 \Rightarrow x \leftarrow right[x]$

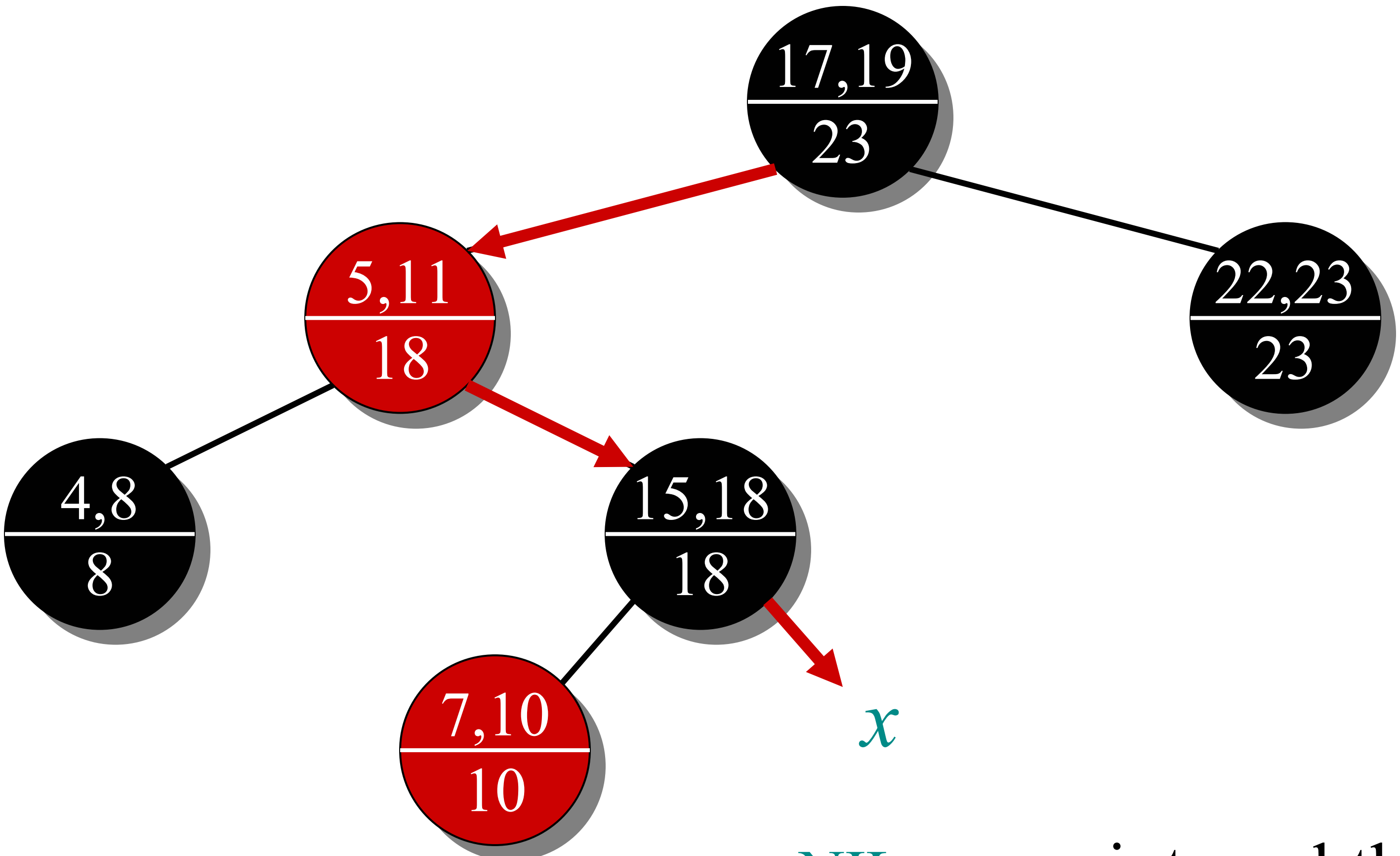
Example 2: INTERVAL-SEARCH([12,14])



[12,14] and [15,18] don’t overlap

$12 > 10 \Rightarrow x \leftarrow right[x]$

Example 2: INTERVAL-SEARCH([12,14])



x = NIL ⇒ no interval that overlaps [12,14] exists

Analysis

Time $= O(h) = O(\lg n)$, since INTERVAL-SEARCH does constant work at each level as it follows a simple path down the tree.

List *all* overlapping intervals:

- Search, list, delete, repeat.
- Insert them all again at the end.

Time $= O(k \lg n)$, where k is the total number of overlapping intervals.

This is an ***output-sensitive*** bound.

Best algorithm to date: $O(k + \lg n)$.

Correctness

Theorem. Let L be the set of intervals in the left subtree of node x, and let R be the set of intervals in x's right subtree.

- If the search goes right, then

$$\{ i' \in L : i' \text{ overlaps } i \} = \emptyset.$$

- If the search goes left, then

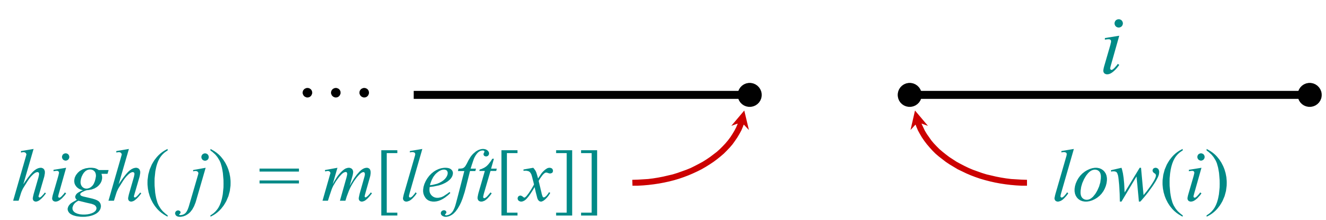
$$\{i' \in L : i' \text{ overlaps } i \} = \emptyset$$
$$\Rightarrow \{i' \in R : i' \text{ overlaps } i \} = \emptyset.$$

In other words, it's always safe to take only 1 of the 2 children: we'll either find something, or nothing was to be found.

Correctness proof

Proof. Suppose first that the search goes right.

- If $left[x] = \text{NIL}$, then we're done, since $L = \emptyset$.
- Otherwise, the code dictates that we must have $low[i] > m[left[x]]$. The value $m[left[x]]$ corresponds to the right endpoint of some interval $j \in L$, and no other interval in L can have a larger right endpoint than $high(j)$.



- Therefore, $\{i' \in L : i' \text{ overlaps } i\} = \emptyset$.

Proof (continued)

Suppose that the search goes left, and assume that

$$\{i' \in L : i' \text{ overlaps } i\} = \emptyset.$$

- Then, the code dictates that $low[i] \le m[left[x]] = high[j]$ for some $j \in L$.
- Since $j \in L$, it does not overlap i, and hence $high[i] < low[j]$.
- But, the binary-search-tree property implies that for all $i' \in R$, we have $low[j] \le low[i']$.
- But then $\{i' \in R : i' \text{ overlaps } i\} = \emptyset$. $\square$