Massachusetts Institute of Technology
Department of Physics

Physics 8.962 Spring 2002

Number-Flux Vector and Stress-Energy Tensor

1 Introduction

These notes supplement Section 3 of the 8.962 notes "Introduction to Tensor Calculus for General Relativity." Having worked through the formal treatment of vectors, one-forms and tensors, we are ready to evaluate two particularly useful and important examples, the number-flux four-vector and the stress-energy (or energy-momentum) tensor for a gas of particles. A good elementary discussion of these objects is given in chapter 4 of Schutz, *A First Course in General Relativity*; more advanced treatments are in chapters 5 and 22 of MTW. Some of the mathematical material presented here is formalized in Section 4 of the 8.962 notes; to avoid repetition we will present the computations here in a locally flat frame (orthonormal basis with locally vanishing connection) frame rather than in a general basis. However, the final results are tensor equations valid in any basis.

2 Number-Flux Four-Vector for a Gas of Particles

We wish to describe the fluid properties of a gas of noninteracting particles of rest mass m starting from a microscopic description. In classical mechanics, we would describe the system by giving the spatial trajectories $\underline{x}_a(t)$ where a labels the particle and t is absolute time. (An underscore is used for 3-vectors; arrows are reserved for 4-vectors. While the position $\underline{x}_a$ isn't a true tangent vector, we retain the common notation here.) The number density and number flux density are

$$n = \sum_a \delta^3(\underline{x} - \underline{x}_a(t)) \ , \quad \underline{J} = \sum_a \delta^3(\underline{x} - \underline{x}_a(t)) \frac{d\underline{x}_a}{dt} \tag{1}$$

where the Dirac delta function has its usual meaning as a distribution:

$$\int d^3x\, f(\underline{x})\, \delta^3(\underline{x} - \underline{y}) = f(\underline{y}) \ . \tag{2}$$

In order to get well-defined quantities when relativistic motions are allowed, we attempt to combine the number and flux densities into a four-vector $\vec{N}$. The obvious generalization of equation (1) is

$$\vec{N} = \sum_a \delta^3(\underline{x} - \underline{x}_a(t)) \frac{d\vec{x}_a}{dt} \ . \tag{3}$$

However, this is not suitable because time and space are explicitly distinguished: $(t, \underline{x})$. A first step is to insert one more delta function, with an integral (over time) added to cancel it:

$$\vec{N} = \sum_a \int dt'\, \delta^4(x - x_a(t')) \frac{d\vec{x}_a}{dt'} \ . \tag{4}$$

The four-dimensional Dirac delta function is to be understood as the product of the three-dimensional delta function with $\delta(t - t_a(t')) = \delta(x^0 - t')$:

$$\delta^4(x - y) \equiv \delta(x^0 - y^0)\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3) \ . \tag{5}$$

Equation (4) looks promising except for the fact that our time coordinate t' is frame-dependent. The solution is to use a Lorentz-invariant time for each particle — the proper time along the particle's worldline. We already know that particle trajectories in spacetime can be written $x^a(\tau)$. We can change the parametrization in equation (4) so as to obtain a Lorentz-invariant object, a four-vector:

$$\vec{N} = \sum_a \int d\tau\, \delta^4(x - x_a(\tau)) \frac{d\vec{x}_a}{d\tau} \ . \tag{6}$$

2.1 Lorentz Invariance of the Dirac Delta Function

Before accepting equation (6) as a four-vector, we should be careful to check that the delta function is really Lorentz-invariant. We can do this without requiring the existence of a globally inertial frame (something that doesn't exist in the presence of gravity!) because the delta function picks out a single spacetime point and so we may regard spacetime integrals as being confined to a small neighborhood over which locally flat coordinates may be chosen with metric $\eta_{\mu\nu}$ (the Minkowski metric).

To prove that $\delta^4(x - y)$ is Lorentz invariant, we note first that it is nonzero only if $x^\mu = y^\mu$. Now suppose we that perform a local Lorentz transformation, which maps dx^μ to $dx^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_\nu dx^\nu$ and d^4x to $d^4\bar{x} = |\det \Lambda|\, d^4x$. Clearly, $\delta^4(\bar{x} - \bar{y})$ is nonzero only if

$x^{\bar{\mu}} = y^{\bar{\mu}}$ and hence only if $x^\mu = y^\mu$. From this it follows that $\delta^4(\bar{x} - \bar{y}) = S\delta^4(x - y)$ for some constant S. We will show that $S = 1$.

To do this, we write the Lorentz transformation in matrix notation as $\bar{x} = \Lambda x$ and we make use the definition of the Dirac delta function:

$$f(\bar{y}) = \int d^4\bar{x}\, \delta^4(\bar{x} - \bar{y}) f(\bar{x}) = \int d^4x\, |\det\Lambda|\, S\delta^4(x - y) f(\Lambda x) = S\, |\det\Lambda|\, f(\bar{y}) \ . \quad (7)$$

Lorentz transformations are the group of coordinate transformations which leave the Minkowski metric invariant, $\eta = \Lambda^T\eta\Lambda$. Now, $\det\eta = -1$, from which it follows that $|\det\Lambda| = 1$. From equation (7), $S = 1$ and the four-dimensional Dirac delta function is Lorentz-invariant (a Lorentz scalar).

As an aside, $\delta^4(x)$ is *not* invariant under arbitrary coordinate transformations, because d^4x isn't invariant in general. (It is invariant only for those transformations with $|\det\Lambda| = 1$). In part 2 of the notes on tensor calculus we show that $|\det g|^{1/2}d^4x$ is fully invariant, so we should multiply the Dirac delta function by $|\det g|^{-1/2}$ to make it invariant under general coordinate transformations. In the special case of an orthonormal basis, $g = \eta$ so that $|\det g| = 1$.

3 Stress-Energy Tensor for a Gas of Particles

The energy and momentum of one particle is characterized by a four-vector. For a gas of particles, or for fields (e.g. electromagnetism), we need a rank $(2, 0)$ tensor which combines the energy density, momentum density (or energy flux — they're the same) and momentum flux or stress. The stress-energy tensor is symmetric and is defined so that

$$\mathsf{T}(\tilde{e}^\mu, \tilde{e}^\nu) = T^{\mu\nu} \text{ is the flux of momentum } p^\mu \text{ across a surface of constant } x^\nu \ . \quad (8)$$

It follows (Schutz chapter 4) that in an orthonormal basis T^{00} is the energy density, T^{0i} is the energy flux (energy crossing a unit area per unit time), and T^{ij} is the stress (i-component momentum flux per unit area per unit time crossing the surface $x^j =$ constant. The stress-energy tensor is especially important in general relativity because it is the source of gravity. It is important to become familiar with it.

The components of the number-flux four-vector $N^\nu = \vec{N}(\tilde{e}^\nu)$ give the flux of particle number crossing a surface of constant x^ν (with normal one-form $\tilde{e}^\nu$). From this, we can obtain the stress-energy tensor following equation (6). Going from number (a scalar) to momentum (a four-vector) flux is simple: multiply by $\vec{p} = m\vec{V} = md\vec{x}/d\tau$. Thus,

$$\mathsf{T} = \sum_a \int d\tau\, \delta^4(x - x_a(\tau)) m\vec{V}_a \otimes \vec{V}_a \ . \quad (9)$$

4 Uniform Gas of Non-Interacting Particles

The results of equations (6) and (9) include a discrete sum over particles. To go to the continuum, or fluid, limit, we suppose that the particles are so numerous that the sum of delta functions may be replaced by its average over a small spatial volume. To get the number density measured in a locally flat (orthonormal) frame we must undo some of the steps leading to equation (6). Using the fact that $dt/d\tau = \gamma$, comparing equations (3) and (6) shows that we need to evaluate

$$\sum_a \int d\tau \, \delta^4(x - x_a(\tau)) = \sum_a {\gamma_a}^{-1} \, \delta^3(\underline{x} - \underline{x}_a(t)) \ . \tag{10}$$

Now, aside from the factor γ_a^{-1}, integrating equation (10) over a small volume ΔV and dividing by ΔV would yield the local number density. However, we must also keep track of the velocity distribution of the particles. Let us suppose that the velocities are randomly sampled from a (possibly spatially or temporally varying) three-dimensional velocity distribution $f(\underline{x}, \underline{v}, t)$ normalized so that, in an orthonormal frame,

$$\int d^3v \, f(\underline{x}, \underline{v}, t) = 1 \ . \tag{11}$$

To make the velocity distribution Lorentz-invariant takes a little more work which we will not present here; the interested reader may see problem 5.34 of the *Problem Book in Relativity and Gravitation* by Lightman, Press, Price, and Teukolsky.

In an orthonormal frame with flat spacetime coordinates, the result becomes

$$\sum_a \int d\tau \, \delta^4(x - x_a(\tau)) = n(x) \int d^3v \, \gamma^{-1} \, f(x, \underline{v}) \ . \tag{12}$$

Using $\vec{V} = \gamma(1, \underline{v})$ and substituting into equation (3), we obtain the number-flux four-vector

$$\vec{N} = (n, \underline{J}) \ , \quad \underline{J} = n(x) \int d^3v \, f(x, \underline{v}) \underline{v} \ . \tag{13}$$

Although this result has been evaluated in a particular Lorentz frame, once we have it we could transform to any other frame or indeed to any basis, including non-orthonormal bases.

The stress-energy tensor follows in a similar way from equations (9) and (12). In a local Lorentz frame,

$$T^{\mu\nu} = mn(x) \int d^3v \, f(x, \underline{v}) \, \frac{V^\mu V^\nu}{V^0} \ . \tag{14}$$

If there exists a frame in which the velocity distribution is isotropic (independent of the direction of the three-velocity), the components of the stress-energy tensor are particularly simple in that frame:

$$T^{00} \equiv \rho = \int d^3v \, f(x, \underline{v}) \, \gamma m n(x) \ , \quad T^{0i} = T^{i0} = 0 \ ,$$
$$T^{ij} = p\delta^{ij} \text{ where } p \equiv \frac{1}{3} \int d^3v \, f(x, \underline{v}) \gamma m n(x) v^2 \ . \tag{15}$$

Here ρ is the energy density (γm is the energy of a particle) and p is the pressure.

Equation (15) isn't Lorentz-invariant. However, we can get it into the form of a spacetime tensor (an invariant) by using the tensor basis plus the spatial part of the metric:

$$\mathsf{T} = \rho \vec{e}_0 \otimes \vec{e}_0 + p\eta^{ij} \vec{e}_i \otimes \vec{e}_j \ . \tag{16}$$

We can make further progress by noting that the pressure term may be rewritten after defining the projection tensor

$$\mathsf{h} = \mathsf{g}^{-1} + \vec{e}_0 \otimes \vec{e}_0 \tag{17}$$

since $g^{\mu\nu} = \eta^{\mu\nu}$ in an orthonormal basis and therefore $h^{00} = \eta^{00} + 1 = 0$, $h^{0i} = h^{i0} = 0$ and $h^{ij} = \delta^{ij}$. The tensor h projects any one-form into a vector orthogonal to $\vec{e}_0$. Combining results, we get

$$\mathsf{T} = (\rho + p)\vec{e}_0 \otimes \vec{e}_0 + p \, \mathsf{g}^{-1} \ . \tag{18}$$

Equation (18) is in the form of a tensor, but it picks out a preferred coordinate system through the basis vector $\vec{e}_0$. To eliminate this remnant of our nonrelativistic starting point, we note that, for any four-velocity $\vec{U}$, there exists an orthonormal frame (the instantaneous local inertial rest frame) in which $\vec{U} = \vec{e}_0$. Thus, if we identify $\vec{U}$ as the fluid velocity, we obtain our final result, the stress-energy tensor of a perfect gas:

$$\mathsf{T} = (\rho + p)\vec{U} \otimes \vec{U} + p \, \mathsf{g}^{-1} \quad \text{or} \quad T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p \, g^{\mu\nu} \tag{19}$$

If the sleight-of-hand of converting $\vec{e}_0$ to $\vec{U}$ seems unconvincing (and it is worth checking!), the reader may apply an explicit Lorentz boost to the tensor of equation (18) with three-velocity U^i/U^0 to obtain equation (19). We must be careful to remember that ρ and p are scalars (the proper energy density and pressure in the fluid rest frame) and $\vec{U}$ is the fluid velocity four-vector.

From this result, one may be tempted to rewrite the number-flux four-vector as $\vec{N} = n\vec{U}$ where $\vec{U}$ is the same fluid 4-velocity that appears in the stress-energy tensor. This is valid for a perfect gas, whose velocity distribution is isotropic in a particular frame, where n would be the proper number density. However, in general T^{0i} is nonzero in the frame in which $N^i = 0$, because the energy of particles is proportional to γ but the number is not. Noting that the kinetic energy of a particle is $(\gamma - 1)m$, we could have a net flux of kinetic energy (heat) even if there is no net flux of momentum. In other words, energy may be conducted by heat as well as by advection of rest mass. This leads to a fluid velocity in the stress-energy tensor which differs from the velocity in the number-flux 4-vector.

Besides heat conduction, a general fluid has a spatial stress tensor differing from $p\delta^{ij}$ due to shear stress provided by, for example, shear viscosity.

An example where these concepts and techniques find use is in the analysis of fluctuations in the cosmic microwave background radiation. When the radiation (photon) field begins to decouple from the baryonic matter (hydrogen-helium plasma) about 300,000 years after the big bang, anisotropies in the photon momentum distribution develop which lead to heat conduction and shear stress. The stress-energy tensor of the radiation field must be computed by integrating over the full non-spherical momentum distribution of the photons. Relativistic kinetic theory is one of the ingredients needed in a theoretical calculation of cosmic microwave background anisotropies (Bertschinger & Ma 1995, *Astrophys. J.* **455**, 7).