4 Curvature

We introduce curvature by considering parallel transport around a general (non-geodesic) closed curve. In flat space, in a globally flat coordinate system (for which the connection vanishes everywhere), parallel transport leaves the components of a vector unchanged. Thus, in flat space, transporting a vector around a closed curve returns the vector to its starting point unchanged. Not so in a nonflat space. This change under a closed cycle is called an “anholonomy.”

Consider, for example, a sphere. Suppose that we have a vector pointing east on the equator at longitude 0°. We parallel transport the vector eastward on the equator by 180°. At each point on the equator the vector points east. Now the vector is parallel transported along a line of constant longitude over the pole and back to the starting point on the equator. At each point on this second part of the curve, the vector points at right angles to the curve, and its direction never changes. Yet, at the end of the curve, at the same point where the curve started, the vector points west!

The reader may imagine that the example of the sphere is special because of the sharp changes in direction made in the path. However, parallel transport around any
smooth closed curve results in an anholonomy on a sphere. For example, consider a latitude circle away from the equator. Imagine you are an airline pilot flying East from Boston. If you were flying on a great circle route, you would soon be flying in a south-east direction. If you parallel transport a vector along a geodesic, its direction relative to the tangent vector (direction of motion) does not change, i.e. $\nabla_V (A \cdot V) = 0$ for parallel transport of $A$ along tangent $V$. Parallel transport implies $\nabla_V A = 0$; moreover, $\nabla_V V = 0$ for a geodesic. However, a constant-latitude circle is not a geodesic, hence $\nabla_V V \neq 0$. In order to maintain a constant latitude, you will have to constantly steer the airplane north compared with a great circle route. Consequently, the angle between $\vec{A}$ (which is parallel-transported) and the tangent changes: $\nabla_V (\vec{A} \cdot V) = A \cdot (\nabla_V V)$. A nonzero rotation accumulates during the trip, leading to a net rotation of $\vec{A}$ around a closed curve.

We can refine this into a definition of curvature as follows. Suppose that our closed curve consists of four infinitesimal segments: $d\vec{x}_1$, $d\vec{x}_2$, $-d\vec{x}_1$, and $-d\vec{x}_2$. In a flat space this would be called a parallelogram and the difference $d\vec{A}$ between the final and initial vectors would vanish. In a curved space we can create a parallelogram by taking two pairs of coordinate lines and choose $d\vec{x}_1$ and $d\vec{x}_2$ to point along the coordinate lines (e.g. in directions $\vec{e}_1$ and $\vec{e}_2$). Parallel transport around a closed curve gives a change in the vector $d\vec{A}$ that must be proportional to $\vec{A}$, to $d\vec{x}_1$, and to $d\vec{x}_2$. Remarkably, it is proportional to nothing else. Therefore, $d\vec{A}$ is given by a rank $(1,3)$ tensor called the Riemann curvature tensor:

$$d\vec{A}(\cdot) \equiv -\mathbf{R}(\cdot, \vec{A}, d\vec{x}_1, d\vec{x}_2) = -\vec{e}_\mu R^\mu{}_{\nu\alpha\beta} A^\nu d\vec{x}_1^\alpha d\vec{x}_2^\beta.$$  (29)
The dots indicate that a one-form is to be inserted; recall that a vector is a function of a one-form. The minus sign is purely conventional and is chosen for agreement with MTW. Note that the Riemann tensor must be antisymmetric on the last two slots because reversing them amounts to changing the direction around the parallelogram, i.e., swapping the final and initial vectors $\vec{A}$, hence changing the sign of $d\vec{A}$.

All standard GR textbooks show that equation (29) is equivalent to the following important result known as the Ricci identity

\[ (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) A^\mu = R^\mu_{\nu \alpha \beta} A^\nu \text{ in a coordinate basis}. \quad (30) \]

In a non-coordinate basis, there is an additional term on the left-hand side, $-\nabla_C A$ where $\mathcal{C} \equiv [\tilde{e}_\alpha, \tilde{e}_\beta]$. This commutator vanishes for a coordinate basis (eq. 12).

Equation (30) is a remarkable result. In general, there is no reason whatsoever that the derivatives of a vector field should be related to the vector field itself. Yet the difference of second derivatives is not only related to, but is linearly proportional to the vector field! This remarkable result is a mathematical property of metric spaces with connections. It is equivalent to the statement that parallel transport around a small closed parallelogram is proportional to the vector and the oriented area element (eq. 29).

Equation (30) is similar to equation (11). The torsion tensor and Riemann tensor are geometric objects from which one may build a theory of gravity in curved spacetime. In general relativity, the torsion is zero and the Riemann tensor holds all of the local information about gravity.

It is straightforward to determine the components of the Riemann tensor using equation (30) with $\vec{A} = \tilde{e}_\nu$. The result is

\[ R^\mu_{\nu \alpha \beta} = \partial_\alpha \Gamma^\mu_{\nu \beta} - \partial_\beta \Gamma^\mu_{\nu \alpha} + \Gamma^\mu_{\kappa \alpha} \Gamma^\kappa_{\nu \beta} - \Gamma^\mu_{\kappa \beta} \Gamma^\kappa_{\nu \alpha} \text{ in a coordinate basis}. \quad (31) \]

Note that some authors (e.g., Weinberg 1972) define the components of Riemann with opposite sign. Our sign convention follows Misner et al (1973), Wald (1984) and Schutz (1985).

Note that the Riemann tensor involves the first and second partial derivatives of the metric (through the Christoffel connection in a coordinate basis). Weinberg (1972) shows that the Riemann tensor is the only tensor that can be constructed from the metric
tensor and its first and second partial derivatives and is linear in the second derivatives. Recall that one can always define locally flat coordinates such that $\Gamma^\mu_{\nu\lambda} = 0$ at a point. However, one cannot choose coordinates such that $\Gamma^\mu_{\nu\lambda} = 0$ everywhere unless the space is globally flat. The Riemann tensor vanishes everywhere if and only if the manifold is globally flat. This is a very important result.

If we lower an index on the Riemann tensor components we get the components of a $(0,4)$ tensor:

$$R_{\mu\nu\kappa\lambda} = g_{\mu\kappa} R^\alpha_{\nu\lambda\alpha} = \frac{1}{2} (g_{\mu\lambda,\nu\kappa} - g_{\mu\kappa,\nu\lambda} + g_{\nu\kappa,\mu\lambda} - g_{\nu\lambda,\mu\kappa}) + g_{\alpha\beta} \left( \Gamma^\alpha_{\mu\lambda} \Gamma^\beta_{\nu\kappa} - \Gamma^\alpha_{\mu\kappa} \Gamma^\beta_{\nu\lambda} \right) ,$$

(32)

where we have used commas to denote partial derivatives for brevity of notation: $g_{\mu\lambda,\nu\kappa} \equiv \partial_\kappa \partial_\nu g_{\mu\lambda}$. In this form it is easy to determine the following symmetry properties of the Riemann tensor:

$$R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu} = -R_{\nu\mu\kappa\lambda} , \quad R_{\mu\nu\kappa\lambda} + R_{\mu\kappa\lambda\nu} + R_{\mu\lambda\nu\kappa} = 0 .$$

(33)

It can be shown that these symmetries reduce the number of independent components of the Riemann tensor in four dimensions from $4^4$ to 20.

### 4.1 Bianchi identities, Ricci tensor and Einstein tensor

We note here several more mathematical properties of the Riemann tensor that are needed in general relativity. First, by differentiating the components of the Riemann tensor one can prove the **Bianchi identities**:

$$\nabla_\sigma R^\mu_{\nu\kappa\lambda} + \nabla_\nu R^\mu_{\lambda\sigma\kappa} + \nabla_\kappa R^\mu_{\nu\sigma\lambda} = 0 .$$

(34)

Note that the gradient symbols denote the covariant derivatives and not the partial derivatives (otherwise we would not have a tensor equation). The Bianchi identities imply the vanishing of the divergence of a certain $(2,0)$ tensor called the Einstein tensor. To derive it, we first define a symmetric contraction of the Riemann tensor, known as the Ricci tensor:

$$R_{\mu\nu} \equiv R^\kappa_{\mu\kappa\nu} = R_{\kappa\nu\mu} = \partial_\kappa \Gamma^\kappa_{\mu\nu} - \partial_\mu \Gamma^\kappa_{\kappa\nu} + \Gamma^\kappa_{\kappa\lambda} \Gamma^\lambda_{\mu\nu} - \Gamma^\kappa_{\mu\lambda} \Gamma^\lambda_{\kappa\nu} .$$

(35)

One can show from equations (33) that any other contraction of the Riemann tensor either vanishes or is proportional to the Ricci tensor. The contraction of the Ricci tensor is called the Ricci scalar:

$$R \equiv g^{\mu\nu} R_{\mu\nu} .$$

(36)

Contracting the Bianchi identities twice and using the antisymmetry of the Riemann tensor one obtains the following relation:

$$\nabla_\nu G^\mu_{\nu} = 0 , \quad G^\mu_{\nu} \equiv R^\mu_{\nu} - \frac{1}{2} g^{\mu\nu} R = G^\mu_{\nu} .$$

(37)
The symmetric tensor $G^\mu\nu$ that we have introduced is called the *Einstein tensor*. Equation (37) is a mathematical identity, not a law of physics. Through the Einstein equations it provides a deep illustration of the connection between mathematical symmetries and physical conservation laws.

**References**


