16.100 Homework Assignment # 2
Fall 2003
Due: Monday, September 15, 9am

Reading Assignment

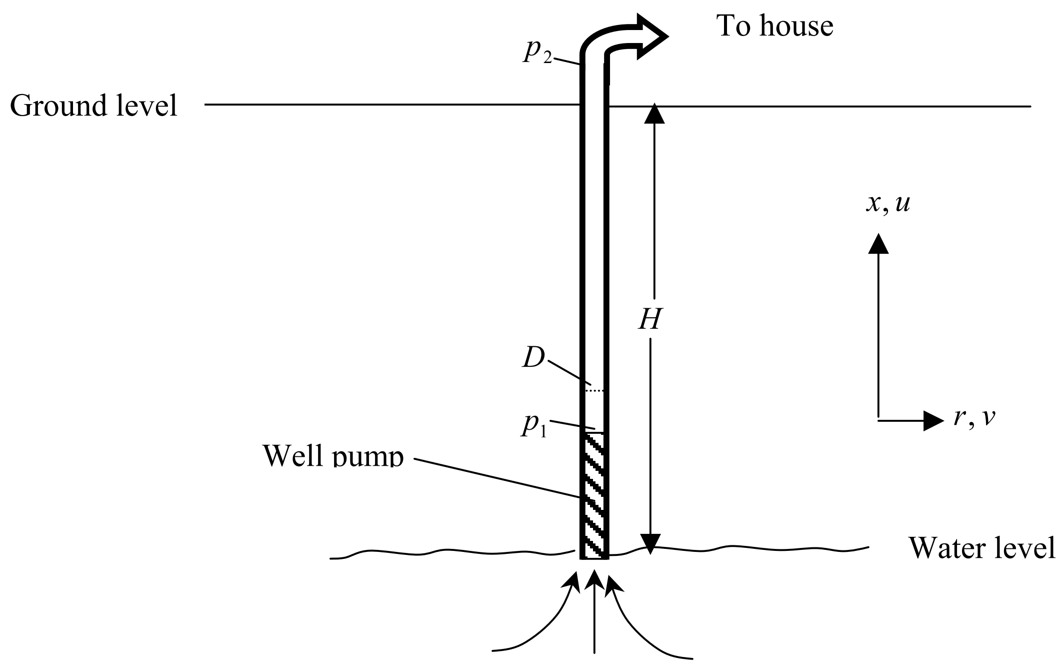
Anderson, 3rd edition: Chapter 15, pages 713 – 730, 738 – 740
Chapter 16, pages 745 – 751, 781 – 786

Anderson, 2nd edition: Chapter 15, pages 637 – 655
Chapter 16, pages 669 – 676, 705 - 709

Problem 1 (60%)

While many people around the country have a public water supply, the house I live in has a well. Last year, I had the unfortunate experience of having my well pump break down and spent a late night with a local well repairer. It was actually pretty interesting to see how the well was designed and was the inspiration for this problem.

The well pump I have is called a submersible pump. It sits below ground and pumps the water up from a reservoir to the house like the figure below:



a) Assuming that the flow in the pipe is laminar and steady, derive the velocity distribution, u=u(r) that would exist in the pipe if the changes in the velocity are negligible in the x-direction. Start from the incompressible, steady, axisymmetric Navier-Stokes equations which are:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) \right] - \rho g$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial r} = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial^2 v}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r} \right) - \frac{v}{r^2} \right]$$

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0$$

b) Depending on the water flow needs of the house, the pump must be able to supply a certain mass flow, $\dot{m}_{req}$. Derive the pressure drop $p_1 - p_2$ which exists at this flow rate. Your final result should only be in terms of H, D, g, ρ, μ, and $\dot{m}_{req}$. How does the pressure drop vary with increasing pipe diameter, D?

c) The pump at my house is 300 feet below ground and is connected to a 1 inch diameter pipe. The flow rate needed is 10 gallons per minute. What is the pressure drop in this case? How much of the pressure drop is due to skin friction and how much is due to gravitational effects?

d) Estimate the minimum amount of power that the pump would require for the conditions in part c).

Problem 2 (20%)

The compressible, viscous momentum equations (i.e. the Navier-Stokes equations) given in vector notation are:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

where the viscous stresses are given by Stokes law (Equations 15.5 – 15.10 in 3rd Edition of Anderson). Assuming the flow is incompressible and that the dynamic viscosity is constant, show that the incompressible form of the Navier-Stokes equations can be written in vector notation as:

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \mu \nabla^2 \vec{V}$$

Problem 3 (20%)

a) Starting from the incompressible Navier-Stokes equations, derive the following 'Bernoulli-like' equation:

$$\rho \frac{\partial \vec{V}}{\partial t} + \nabla\left(p + \tfrac{1}{2}\rho\left|\vec{V}\right|^2 \right) = \rho\vec{V} \times \vec{\omega} - \mu\nabla \times \vec{\omega}$$

The following vector calculus identities might be helpful:

$$\left(\vec{V} \cdot \nabla\right)\vec{V} = \nabla\left(\tfrac{1}{2}\left|\vec{V}\right|^2 \right) - \vec{V} \times \vec{\omega}$$
$$\nabla^2\vec{V} = \nabla\left(\nabla \cdot \vec{V}\right) - \nabla \times \vec{\omega}$$

b) Show that the total pressure, $p + \tfrac{1}{2}\rho\left|\vec{V}\right|^2$, is constant along a streamline in a steady, inviscid flow.

c) Show that the total pressure is constant everywhere in a steady, inviscid, and irrotational flow.