Problem Set 8

Due: November 16, 1999.

Problem 1. Suppose you are given two polyhedra $P = \{x \mid Ax \leq b\}$ and $Q = \{x \mid Dx \leq d\}$.

(a) Using our polyhedral techniques, prove that if the polyhedra have empty intersection (i.e. no point is in both) then there is a separating hyperplane for P and Q (that is, a c such that $cx < cy$ for $x \in P$ and $y \in Q$).

(b) Suppose you have a linear programming algorithm. Argue that with this algorithm you can either find a point in $P \cap Q$ or find their separating hyperplane c.

Problem 2. Consider the problem of minimizing cx over a polyhedron P. Prove the following:

(a) x is optimal if and only if $cy \geq 0$ for every feasible direction y (that is, direction y such that $x + \epsilon y \in P$ for sufficiently small positive ϵ).

(b) A feasible solution is the unique optimum if and only if $cy > 0$ for every feasible direction y.

(c) Under the simplex algorithm, if the reduced cost of every nonbasic variable is **positive**, then x is the unique optimum.

Problem 3. In class we defined a polyhedron as an intersection of finitely many halfspaces. A related concept is a **polytope:** the set of convex combinations of finitely many points.

(a) Prove that any polytope is a polyhedron. **Hint:** the polytope is an intersection of many halfspaces (why?). What if such a halfspace intersects less than n of the defining points?

(b) Prove that in any **bounded** polyhedron (that is, one where every ray from inside eventually hits a boundary), every point is a convex combination of vertices. **Hint:** use induction on the number of satisfied constraints.

(c) Prove that any element of a bounded n-dimensional polyhedron can be expressed as a convex combination of at most $n+1$ vertices of the polyhedron. **Hint:** consider the set of all possible representation of x as convex combinations.

Problem 4. One thing that makes LP hard in practice is **degeneracy** in the input problem. Given an LP, a vertex is **degenerate** if more than n constraints are tight at that vertex. Consider the simplex algorithm for a minimization LP in standard form and suppose you are at a vertex x.

(a) Prove that if x is the unique optimum and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

(b) Prove that if x is any nondegenerate, nonoptimum vertex, then there is a pivot step to a strictly better solution (so no cycling pivots occur).

Problem 5. One (theoretical) way to eliminate degeneracy is *perturbation*. Suppose we take a standard form ($Ax = b,\ x \geq 0$) problem, and replace b_i by $b_i + \epsilon^i$.

(a) Assuming that the rows of A are linearly independent, prove that for all sufficiently small ϵ, there are no degenerate vertices.

(b) Show that for sufficiently small ϵ, an optimum basic feasible solution for the perturbed problem is also an optimum (under the same basis) for the original problem.

** **(c)** Prove that if the polytope you are optimizing over is full-dimensional, the perturbed problem is still feasible.