Problem Set 9 Solutions

Problem 1. The primal is:

$$\begin{aligned} z &= \max \sum f_P \\ \sum_{P \ni e} f_P &\leq u_e \\ f_P &\geq 0 \end{aligned}$$

The dual is a minimization problem. Each variable in the dual corresponds to a constraint in the primal. Since there is a constraint per edge, we can name our variables y_e for each edge e. Variable y_e is positive, since the corresponding constraint in the primal is a $\leq$ constraint. The constant associated with y_e in the objective function is u_e. Each constraint in the dual corresponds to a variable in the primal. We have one variable per path and all variables are positive. Therefore the constraints in the dual are $\geq$ constraints. Thus, the dual is:

$$\begin{aligned} w &= \min \sum u_e y_e \\ \sum_{e \in P} y_e &\geq 1 \\ y_e &\geq 0 \end{aligned}$$

An English explaination of the dual: Assign each edge e to a weight y_e, such that the total weight along any path is 1. Among all such valid assignments, take the one that minimizes the weighted sum of u_e's with weights y_e.

If the values of y_e are integers, it is clear that y_e can only take values 0 and 1. In this case, the solution defines a cut over the graph, since all paths from the source to sink cross over an edge e with $y_e = 1$.

Comments from graders: Most solutions to this problem were correct.

Problem 2. Variable f_{ij} denotes the flow in edge ij.

(a) The conservation constraints on f_{ij} ensure that the flow defined by f_{ij} is a circulation. The net flow in the circulation is 1. Notice that there are no capacity constraints on the solution.

Consider the cycle decomposition of a solution f_{ij}. Each cycle C with flow f_C contributes $f_C \sum_{(ij) \in C} c_{ij}$ to the objective. Moreover the cycle contributes $|C| f_C$ to the $\sum f_{ij} = 1$ constraint. The ratio of contribution to objective to the contribution of net flow is minimum for the minimum mean-cost cycle. Thus the cycle decomposition of the solution can have only minimum mean-cost cycles.

(b) The primal is:

$$\begin{aligned} w &= \min \sum c_{ij} f_{ij} \\ \sum_j f_{ij} - f_{ji} &= 0 \quad (\forall i) \\ \sum f_{ij} &= 1 \\ f_{ij} &\geq 0 \end{aligned}$$

The dual of the above LP is:

$$\begin{aligned} z &= \max \lambda \\ y_i - y_j + \lambda &\leq c_{ij} \quad (\forall ij) \\ y_i, \lambda & \quad u/s \quad (\forall i) \end{aligned}$$

(c) The dual assigns potentials to different nodes and computes reduced costs. All the reduced costs are more than $-\lambda$. The algorithm maximizes λ, i.e., minimizes the smallest reduced cost. Recall that the sum of reduced cost over a cycle is same as the sum of original costs. So $-\lambda$ should be at least the mean cost in the minimum mean cost cycle. We will now prove that there exists a potential assignment that works for

$$-\lambda = \text{mean cost of the minimum mean cost cycle}$$

Consider a graph G^* with costs reduced by λ. This graph does not have a negative cost cycle, since such a cycle will be a smaller mean cost cycle. So there exists a potential assignment for this graph. The same potential assignment for the original graph will have atleast $-\lambda$ reduced cost.

(d) Given a λ, we can subtract it from edge costs to get graph G^*. We can check whether we can assign potential to nodes in G^* such that all reduced costs are positive, using Bellman-Ford algorithm. Such an assignment is possible for all λ smaller than the optimal λ and is not possible for other values. We can do a binary search to compute the value of λ. If we work with integer costs, we can stop the binary search once the range of investigation is reduced to $1/n^2$. This is because the difference of two unequal fractions m_1/n_1 and m_2/n_2 is at least $1/n_1n_2 \leq 1/n^2$.

Comments from graders: Some solutions to (d) did not mention how Bellman-Ford algorithm can be used to check whether a λ satisfies the dual.

Problem 3. Let the primal LP be $\min\{cx \mid Ax = b, x \geq 0\}$.

(a) The dual for this LP is $\max\{yb \mid yA \leq c\}$. Adding slack variables s, we can write the dual in the form $\max\{fz \mid Dz = e, z \geq 0\}$. Adding the constraint that primal and dual objective functions are equal, i.e., $yb = fz$, we can compute the optimal primal and dual solutions. Thus we have reduced the LP to $\min\{0 \mid Ax = b, Dz = e, x \geq 0, z \geq 0\}$ which can be written in the form $\min\{0 \mid Ax = b, x \geq 0\}$.

(b) If the transformed LP is $\min\{0 \mid Ax = b, x \geq 0\}$, its dual is $\max\{yb \mid yA \geq 0\}$.

(c) If the primal is feasible, the dual has optimum value of 0. This happens when $y = 0$.

(d) We are given a LP algorithm that solves an LP in $O((m+n)^k)$ time given the dual solution. We can solve the problem after performing the transformation in (a). The transformation takes $O(mn)$-time. Recall that the dual solution of $y = 0$ is known for the transformed problem. The primal is feasible due to strong duality. The solution to the primal will give the solution to the LP (and its dual). Thus without knowing the dual solution, we can devise an algorithm that takes $O((m+n)^k + mn) = O((m+n)^k)$ time. We assume that $k > 2$, since all elements in A need to be examined by the LP algorithm given to us.

Comments from graders: Most solutions to this problem were correct.

Problem 4. Consider the primal LP

$$\begin{aligned} w &= \max 0 \\ \pi(P - I) &= 0 \\ \sum_{i=1}^{n} \pi_i &= 1 \\ \pi &\geq 0 \end{aligned}$$

Any solution to the above LP is a stationary distribution on P. The dual of the above LP is

$$\begin{aligned} z &= \min \lambda \\ [P - I \mid 1] \begin{bmatrix} x \\ \lambda \end{bmatrix} &\geq 0 \\ x, \lambda & \quad \text{unrestricted} \end{aligned}$$

By strong duality, $\lambda < 0$ if the primal is infeasible. Assume there exists a solution to the dual with $\lambda < 0$. Consider the maximum x_i, where $x = (x_1 \ldots x_n)$. The ith constraint in the dual is therefore

$$(\text{a convex sum of } x_1, \ldots x_n) - x_i + \lambda \geq 0$$

since rows in P consist of positive entries that sum to one. By the choice of i, the LHS of the above inequality is at least λ. Therefore $\lambda \geq 0$ which is a contradiction. Thus the primal is feasible and there exists a stationary distribution π.

Comments from graders: Many solutions used Farkas' lemma instead of duality. Both the approaches are equivalent however.