Problem Set 10 Solutions

Problem 1. By applying the polar operator $*$ to P, we get

$$P^* = \{zx \mid zx \leq 1 \; \forall x \in P\} \tag{1}$$

The set P^{**} is defined as

$$\begin{aligned} P^{**} &= \{w \mid wz \leq 1 \; \forall z \in P^*\} \\ &= \{w \mid zw \leq 1 \; \forall z \in P^*\} \end{aligned} \tag{2}$$

(a) Consider any $x \in P$. We know that $zx \leq 1$ for all $z \in P^*$, from (1). By plugging in $w = x$ in (2), we get $x \in P^{**}$. Thus $P \subseteq P^{**}$.

Consider any $y \notin P$. There exists a separating hyperplane $z \in P^*$ such that $zy > 1$ and $zx \leq 1$, $\forall x \in P^*$. From (1), we know that $z \in P^*$. Therefore $y \notin P^{**}$, by the definition of P^{**} given in (2). Thus $P^{**} \subseteq P$.

(b) Let x^* be the value of x that optimizes $\{\max \; wx | x \in P\}$. If w is not in P^*, then $wx^* > 1$. But $zx^* \leq 1$ for all z in P^*, so x^* gives a hyperplane that separates w and P^*. Thus an optimization algorithm over P gives a separation algorithm over P^*.

(c) Given an optimization algorithm over P, we can separate over P^*. Given the separation algorithm for P^*, we can optimize over P^*. Again, the optimization algorithm for P^* is a separation algorithm for P^{**}. From part(a), we know that $P^{**} = P$. Thus we can separate over P if we can optimize over P.

Comments from graders: Some solutions did not prove that $P^{**} \subseteq P$ in part (a). Other parts were easy.

Problem 2. Notice that while this problem asked for a continuous algorithm, which is not useful for today's computers, it is possible to implement it by taking discrete steps.

The rest of this solution was shamelessly stolen verbatim from Michel Goemans.

(a) Since $x_j = s_j/\alpha$ we have

$$\sum_j a_{ij} x_j = \sum_j a_{ij} s_j / \alpha = \frac{\sum_j a_{ij} s_j}{\alpha}.$$

Also since $\alpha = \min(\sum a_{ij} s_j)$ we have that for any i, $\sum a_{ij} s_j \geq \alpha$ which implies that

$$\frac{\sum_j a_{ij} s_j}{\alpha} \geq 1.$$

We know that initially $s_j = 0$ and at any time we only increase s_j so $s_j \geq 0$. That along with $a_{ij} \geq 0$ gives us $\alpha \geq 0$. Since $x_j = s_j/\alpha$, $x_j \geq 0$ so x_j is primal feasible. A similar argument shows that y_i is dual feasible.
Since y_i is dual feasible, $\sum y_i = \sum t_i/D$ is a lower bound on the objective function value. By induction, $LB = \max(LB, \frac{\sum t_i}{D})$ is also a lower bound on the objective function value.

(b) Following the hint, first we will show that initially the inequality holds. We start with $s_j = 0$ so $\sum t_i = \sum_{i=1}^{m} e^0 = m$. The rest of the argument is inductive. Whenever the inequality is strict, by continuity, the inequality will be preserved. So we assume the inequality holds at equality at time T, that is $\sum t_i = me^{-\sum s_j/LB}$. To show that it holds just after T, we need to show that the increase of the LHS is upper bounded by the increase of the RHS. At time T, suppose k is the index attaining the maximum in definition of D (that is, the algorithm increases s_k). (The argument is similar if the maximum is attained for several indices, but we omit this case for simplicity.)
The rate of increase of the LHS is given by

$$\frac{d\sum_i t_i}{ds_k} = -\left(\sum_i t_i a_{ik}\right) = -d_k = -D,$$

by definition of k. On the other hand,

$$\frac{d(me^{-\sum_j s_j/LB})}{ds_k} = -\left(me^{-\sum_j s_j/LB}\right)\frac{d(\sum_j s_j/LB)}{ds_k}.$$

Using the fact that the inequality holds at equality at time T and the fact that $\frac{dLB}{ds_k} \geq 0$, we can rewrite the above as:

$$\frac{d(me^{-\sum_j s_j/LB})}{ds_k} = -\left(\sum_i t_i\right)\frac{d(\sum_j s_j/LB)}{ds_k} \geq -\left(\sum_i t_i\right)/LB.$$

To show that the rate of increase of the LHS is at most the rate of increase of the RHS, we need to show that $D \geq \left(\sum_i t_i\right)/LB$, which follows from the definition of LB.

(c) As time goes to infinity we continue to increase the s_j's. The LB also increases, but because it is a bound for the primal, we know that if a solution x to the primal is feasible, it will always be a bound for LB. So $\sum s_j/LB$ approaches infinity and by (b), $\sum t_i$ must tend to zero.

(d) From part (b) we can see that $\sum s_j \leq -LB \ln \frac{\sum t_i}{m}$. Thus

$$\sum_j x_j = \frac{\sum_j s_j}{\alpha} \leq LB\frac{\ln(\sum_i t_i/m)}{-\alpha} = LB\frac{\ln(\sum_i t_i/m)}{\max \ln t_i}.$$

Since t_i is nonnegative, trivially $\ln t_i \leq \ln(\sum_i t_i)$. Let $f = \sum_i t_i$. Thus,

$$\sum_j x_j \leq LB\frac{\ln f - \ln m}{\ln f} = LB(1 - \frac{\ln m}{\ln f}).$$

Since f tends to 0, $\ln f$ tends to $-\infty$ and thus $\sum_j x_j$ tends to the lower bound LB.

Comments from graders: Solutions that did not solve part (b) using derivatives or limits of differences were usually wrong. Some solutions assumed that $d(LB)/d(s_k) = 0$. Some solutions did not solve the limit obtained in part (d) correctly.