The Ω-Spectrum for Brown-Peterson Cohomology

by

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ABSTRACT

TITLE OF THESIS: The Ω-spectrum for Brown-Peterson cohomology

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In this thesis we study the Brown-Peterson cohomology theory from an unstable point of view by studying its classifying spaces. This is a new approach to complex cobordism which yields significant new information. In particular, we calculate the cohomology of the classifying spaces and show they have no torsion. We then apply this to determine the homotopy type of the classifying spaces. We begin applying these results by giving a new proof of a theorem of Quillen and classifying all torsion free (localized) H-spaces.

Thesis Supervisor: F.P. Peterson

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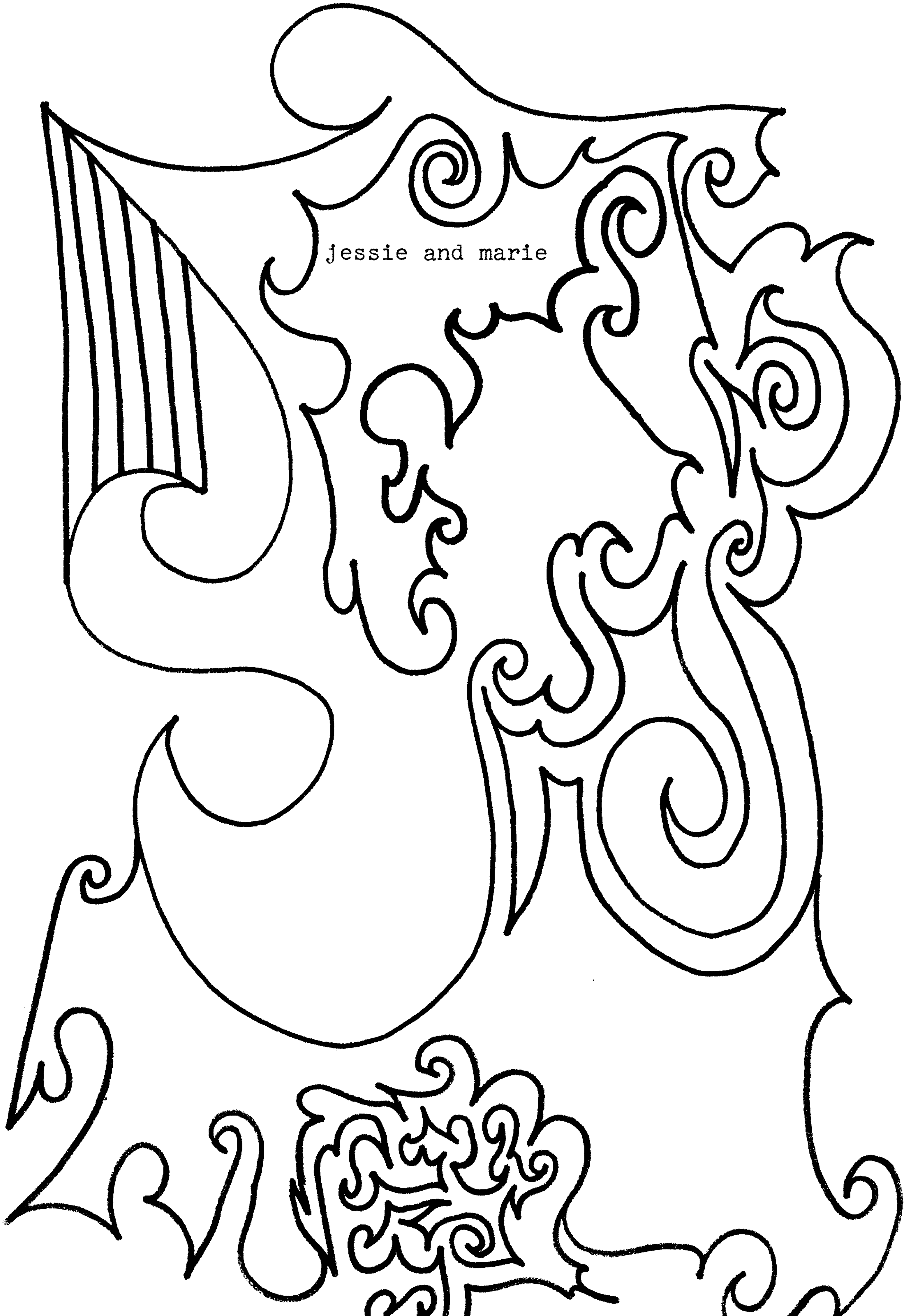

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TABLE OF CONTENTS

The Ω-Spectrum for Brown-Peterson Cohomology Part I

by W. Stephen Wilson

Introduction

BP denotes the spectrum for the Brown-Peterson cohomology, $BP^*(\cdot)$, associated with the prime p. [1, 3, 11] The spectrum can be given as an Ω-spectrum $BP=\{BP_k\}$, [2, 16], i.e. $\Omega BP_k \simeq BP_{k-1}$ and BP_k is k-1 connected for k>0. We have $BP^k(\cdot)\simeq[\cdot, BP_k]$, the unstable homotopy classes of maps. The usual way of viewing $BP^*(\cdot)$ is $BP^*(\cdot)\simeq\{\cdot, BP\}^*$, the stable homotopy classes of maps of the suspension spectrum of a space into BP. We will study the Brown-Peterson cohomology theory from an unstable point of view by studying the BP_k.

Interest in the Brown-Peterson theory stems from the fact that it is a "small" cohomology theory which determines the complex cobordism theory localized at the prime p and that all of the nice properties of complex cobordism carry over to $BP^*(\cdot)$, such as knowledge of the operation ring. Historically, everything about the Brown-Peterson theory has been as nice as could be hoped for. We will push on further in that direction. $Z_{(p)}$ is the integers localized at p, i.e., rationals with denominator prime to p.

Main Theorem (3.3) The $Z_{(p)}$ (co)homology of the zero component of BP_k has no torsion and is a polynomial algebra for k even and an exterior algebra for k odd. (k can be less than zero.)#

Using the main result of [12], the above theorem determines the Hopf algebra structure of the (co)homology.(see section 3) We begin by reviewing Larry Smith's result on the Eilenberg-

Moore spectral sequence for stable Postnikov systems.[14] We combine this with Brown and Peterson's original construction of BP ([3]) to calculate $H^*(BP_{2k+1},Z_p)$ assuming a technical lemma which we prove in section 2. In section 3 we prove the main theorem and some miscellaneous items such as lifting our result to MU.

In Part II we determine the homotopy type of the BP_k using the main theorem here.

This paper is a part of work done for my Ph.D. thesis at M.I.T. under the supervision of Professor Frank Peterson. It is my pleasure to thank Prof. Peterson for his advice, encouragement, and understanding through the last several years. I am very grateful for the quite considerable influence which he has had on my attitudes and tastes in mathematics. Thanks are also due to Larry Smith and Dave Johnson for comments on a preliminary version of this paper, in particular for pointing out a mistake in the original proof for the prime 2.

Section 1

For the remainder of the paper all coeffecient rings are assumed to be $Z_p=Z/pZ$ unless stated otherwise. In this section we show $H^*(BP_{2k+1})$ is an exterior algebra on odd dimensional generators. $H^*(BP_{2k+1})$ is a Hopf algebra, so for odd primes having odd dimensional generators is equivalent to being an exterior algebra. The general reference for Hopf algebras is [10]. We quote what we need from [14].

Let K be a product of Eilenberg-MacLane spaces. We will be concerned with the situation

A

$$\begin{tikzcd} Y \arrow[r] \arrow[d, dash, "\pi"'] & PK \arrow[d, dash] \\ X \arrow[r, "f"'] & K \end{tikzcd}$$

where all spaces are infinite loop spaces and all maps are infinite loop maps. π is the fibration induced by f from the path space PK over K. All cohomologies are thus cocommutative Hopf algebras and $H^*(K)\backslash\backslash f^*$ and $H^*(X)/\!/f^*$, the kernel and cokernel of f^* in the category of Hopf algebras are defined.

There is a natural map PH $\longrightarrow$ QH, where P and Q denote the primitives and indecomposibles respectively of a Hopf algebra H. When this is onto, H is called primitive.

<u>Lemma 1.1</u> ([14,p.69]) $H' \subset H$ a subHopf algebra over Z_p, H primitive, then H' is primitive. #

If V is a graded module, let s^qV be the graded module $(s^qV)_{n+q}=V_n$. Let V^- denote the elements of odd degree. From [14,p.95] we have a filtration of H*(Y) of diagram A such that

$$1.2 \qquad E_0H^*(Y)\simeq H^*(X)/\!/f^* \otimes E[\ldots]\otimes \frac{P[s^{-1}((Q(H^*(K)\backslash\backslash f^*))^-)]}{[s^{-1}((Q(H^*(K)\backslash\backslash f^*))^-)]^p}$$

as Hopf algebras. E and P denote exterior and polynomial algebras generated by odd and even dimensional elements respectively. E[...] is determined by $H^*(K)\backslash\backslash f^*$.

H*(K) is primitive because it is generated by cohomology operations on fundamental classes, therefore, $H^*(K)\backslash\backslash f^*$ is primitive by 1.1. So for $x \in Q(H^*(K)\backslash\backslash f^*)$ we have $x' \to x$, $x' \in P(H^*(K)\backslash\backslash f^*)$ and thus $x' \to x'' \in PH^*(K)$. For x of odd degree, x' and thus x'', are determined uniquely by x. Let $i:\Omega K \to Y$ be the inclusion of the fibre.

Lemma 1.3 ([14,p.86 and p.110]) $i^*(s^{-1}(x))=s^*(x'')$, s^* the cohomology suspension, $s^*:H^*(K) \to H^*(\Omega K)$.#

Note that if x is of odd degree then $s^*(x'')\neq 0$ by the following lemma.

Lemma 1.4 $a \in PH^*(K)$, if $s^*(a)=0$, then $a=P^t x_{2t} + \beta P^k x_{2k+1}$ ($p=2$, $a=Sq^n x_n$) where $P^i \in A$ is the i-th reduced p-th power, A is the Steenrod algebra and x_i is of degree i. #

Proof It is enough to consider $K=K(Z_p,n)$ and $a=P^I i_n$, $P^I \in A$ an Adem basis element. The proof is an argument on the excess of I and can be found in [13]. #

Brown and Peterson [3] construct BP by a series of fibrations which we now describe. Let $\mathcal{R}$ be the set of sequences of non-negative integers $(r_1,r_2,\ldots)$ which are almost all zero. Define $d(R)=\Sigma 2r_i(p^i-1)$, $\ell(R)=\Sigma r_i$ and let Δ_i be the R with $r_i=1$ and zeros everywhere else. Let V_j be the graded abelian group, free over $Z_{(p)}$, generated by $R \in \mathcal{R}$ with $\ell(R)=j$ and graded by $d(R)$. Then we have the generalized Eilenberg-MacLane spectrum $K(V_j)=\bigvee_{\ell(R)=j} S^{d(R)}K(Z_{(p)})$. BP=inverse limit X^j where we have the fibrations

$$\begin{tikzcd}
 & K(V_j) \arrow[r, "i_j"] & X^j \arrow[d] & \\
(*) & & X^{j-1} \arrow[r, "k_{j-1}"] & SK(V_j)
\end{tikzcd}$$

induced by k_{j-1}. We have an $A/A(Q_0)$ resolution for $A/A(Q_0,Q_1,\ldots) =H^*(BP)$, $d_j:M_j \to M_{j-1}$ with $H^*(K(V_j))=M_j$ and $(i_j)^*\cdot(k_j)^*=d_{j+1}$. The Q_i are the Milnor primitives. [8] (For p=2, $Q_i=P^{\Delta_{i+1}}$ in the Milnor basis.) For an $A/A(Q_0)$ generator $i_R \in H^*(K(V_j))$, $d_j(i_R)=\Sigma_i Q_i i_{R-\Delta_i}$.

The spectrum $K(V_j)$ can be given as an Ω-spectrum, $\{ K(V_j,k) = \times_{\lambda(R)=j} K(Z_{(p)},d(R)+k) \}$. The entire diagram (*) can be turned into Ω-spectra and maps of Ω-spectra. From this we get a sequence of fibrations with BP_k=inverse limit X^j.

$$\begin{tikzcd}
K(V_j,k) \arrow[r, "i_j"] & X^j \arrow[d] & \\
 & X^{j-1} \arrow[r, "k_{j-1}"] & K(V_j,k+1)
\end{tikzcd} \qquad (**)$$

We suppress the k in the notation for X^j, i_j and k_j. Note that k can be less than zero. We have $(i_j)^*\cdot(k_j)^*\cdot s^*=s^*\cdot(i_j)^*\cdot(k_j)^*$ where the i_j and k_j on the right are for BP_k and on the left for BP_{k-1}. This is because k_j for BP_{k-1} is the loop map of the k_j for BP_k. Similarly for i_j. The iterated cohomology suspension gives a map $s^*:M_j \to H^*(K(V_j,k))$ which has as its image the primitives, $PH^*(K(V_j,k))$. In general we will denote the iterated suspension by s^* and it should be clear when we mean only one. We have the following commutative diagram.

$$\begin{tikzcd}
M_j \arrow[d, "s^*"'] & M_{j+1} \arrow[l, "d_{j+1}"'] \arrow[d, "s^*"] \\
H^*(K(V_j,k)) & H^*(K(V_{j+1},k+1)) \arrow[l, "(i_j)^*\cdot(k_j)^*"]
\end{tikzcd}$$

We will often use $s^*(d_{j+1})$ for $(i_j)^*\cdot(k_j)^*$. It is given by the same formula $\Sigma Q_i \, i_{R-\Delta_i}$. In the next section we prove the following lemma.

<u>Lemma 1.5(j)</u> For k odd, if $a \ \varepsilon \ PH^{2i+1}(K(V_j,k+1))$ such that $(k_{j-1})^*(a)=0$, then there exists $b \ \varepsilon \ PH^*(K(V_{j+1},k+1))$ such that $(i_j)^*\cdot(k_j)^* (b)=s^*(a)\neq 0$. #

We use this to prove the next proposition.

Proposition 1.6(j) For k odd, $H^*(X^j)//(k_j)^*$ has no even dimensional generators. (For p=2 it is an exterior algebra.)#

Proof For j=0, $X^0=K(Z_{(p)},k)$ and all generators of $H^*(X^0)$ are in the image of $s^*:M_0=A/A(Q_0) \to H^*(K(Z_{(p)},k))$. So if x is an even dimensional generator of $H^*(K(Z_{(p)},k))$ and k is odd, then there is an odd dimensional $x' \varepsilon M_0$ with $s^*(x')=x$. We have the exact sequence

$$M_1 \xrightarrow{d_1} A/A(Q_0)=M_0 \xrightarrow{\varepsilon} A/A(Q_0,Q_1,\ldots) \to 0.$$

Thus there exists $x'' \varepsilon M_1$ with $d_1(x'')=x'$ as $\varepsilon(x')=0$ because $\varepsilon(x')$ is an odd dimensional element in $A/A(Q_0,Q_1,\ldots)$ which only has even degree elements. So $s^*(x'') \varepsilon H^*(K(V_1,k+1))$ and $(k_0)^*(s^*(x''))=s^*(d_1)\cdot s^*(x'')=s^*(d_1x'')=s^*(x')=x$ and the even dimensional generator $x \varepsilon H^*(X^0)$ goes to zero in $H^*(X^0)//(k_0)^*$. (For p=2 and x an odd dimensional generator, then $x^2=Sq^{\deg x}x$ is killed by the same argument, so we have an exterior algebra.)

By induction, assume proposition 1.6(j-1). By 1.2 we have:

$$E_0H^*(X^j)\simeq H^*(X^{j-1})//(k_{j-1})^*\otimes E[..]\otimes\frac{P[s^{-1}(Q(H^*(K(V_j,k+1))\backslash\backslash(k_{j-1})^*)^{+})]}{[s^{-1}(Q(H^*(K(V_j,k+1))\backslash\backslash(k_{j-1})^*)^{-})]^p}$$

Now by our induction assumption, all even dimensional generators look like $s^{-1}(x)$ where $x \varepsilon QH^*(K(V_j,k+1))\backslash\backslash(k_{j-1})^{*-}$. These elements map injectively to the cohomology of the fibre, see 1.3 and the remark after it. As discussed above (before 1.3), x can be represented by an $a \varepsilon PH^*(K(V_j,k+1))$ with $(k_{j-1})^*(a)=0$. Now, as a is of odd degree, from 1.5(j), there exists b such that $(i_j)^*\cdot(k_j)^*(b)=s^*(a)\neq 0$. But by 1.3, $(i_j)^*(s^{-1}(x))=s^*(a)$ and $(i_j)^*$ is injective on these even degree indecomposibles giving that $(k_j)^*(b)=s^{-1}(x)$+decomposibles. Therefore, the generator $s^{-1}(x)$ goes to a decomposible in $H^*(X^j)/\!/(k_j)^*$ and we are done. #

Corollary 1.7 For k odd, $H^*(BP_k)$ is an exterior algebra on odd dimensional generators. #

Proof. Because $K(V_j,k)$ is highly connected for high j we have $H^*(BP_k)=$ direct limit $H^*(X^j)/\!/(k_j)^*$. Because we are working with Hopf algebras, odd dimensional generators for odd primes means we have an exterior algebra. The direct limit is achieved in a finite number of stages so we have the result using 1.6. #

Section 2

We will now prove lemma 1.5(j). We have already seen that that $s^*(a)\neq 0$. (1.4)

Let A be the mod p Steenrod algebra. We define a filtration $A=F^oA\supset F^1A\supset F^2A\supset\ldots$ by giving a basis for F^sA. Given an Adem basis element, $\beta^{\varepsilon_1}P^{i_1}\beta^{\varepsilon_2}\ldots\beta^{\varepsilon_n}P^{i_n}$, it is a basis element for F^sA if $s\leq\Sigma\varepsilon_i$. For p=2 and an Adem basis element $Sq^I=Sq^{i_1}Sq^{i_2}\ldots Sq^{i_m}$, $i_j\geq 2i_{j+1}$, Sq^I is a basis element of F^sA_2 if s or more of the i_j are odd.

For our purposes it is usually more convenient to work in the Adem basis, however, the Milnor basis is a necessary excursion for p=2. For odd primes, a Milnor basis element Q^IP^R $(Q^I=Q_o^{\varepsilon_o}Q_1^{\varepsilon_1}\ldots)$ is a basis element for F^sA if $s\leq\Sigma\varepsilon_i$. For p=2, a Milnor basis element P^R is a basis element for F^sA if $R=(r_1,r_2,\ldots)$ has s or more odd r_i.

Claim 1 i) The two definitions of F^sA are the same.

ii) If $a\in F^sA$, and $b\in F^tA$, then $ab\in F^{s+t}A$. #

Sketch proof Milnor's $Q_i=P^{\Delta_i}\beta-\beta P^{\Delta_i}$. For odd primes P^{Δ_i} is in the algebra of reduced p-th powers and so can be written in the Adem basis without any β's, similarly for all P^R in the Milnor basis. The Adem relations for p odd preserve the number of β's exactly, so we see that $Q_i \varepsilon F^1A$ and not in F^2A. If we were to rewrite a Milnor basis element Q^IP^R in the Adem basis we would still have $\Sigma\varepsilon_i$ β's.

The proof of the second part just uses the fact that the Adem relations never decrease the number of Bocksteins.

The proof for p=2 is slightly more complicated and is left for the reader. #

Given a ε $PH^*(K(V_j,k))$, (any k), it can be written as $a=\sum_{\ell(R)=j} a_Ri_R$ where i_R is the fundamental class of $K(Z_{(p)},d(R)+k)$ and $a_R \varepsilon A$. If it can be written like this with each $a_R \varepsilon F^nA$, then we say a is with n Bocksteins (w.n β's). If n=1, we just say w. β's. If a is with n Bocksteins but not with n+1 β's we say a is with exactly n β's. As discussed above, Q_i is with exactly one Bockstein. Therefore by the definition of d_j and the above claim, if a ε M_j with n β's, then $S^*(d_j)(a)$ is with n+1 β's.

Claim 2 If $a=S^*(d_j)(b)$ and a is with 2 Bocksteins, then there is a b' with β's such that $S^*(d_j)(b')=a$. #

Proof First for odd primes; write $b=\sum_{\ell(R)=j} a_Ri_R$ with $a_R \varepsilon A$. $A=F^oA/F^1A \oplus F^1A$, so write $a_R=b_R+c_R$ with $b_R \varepsilon F^oA/F^1A$ and $c_R \varepsilon F^1A$. b_RQ_i then has exactly one β. c_RQ_i has more than one β. We know that $a=S^*(d_j)(b)$ has more than one β so $S^*(d_j)(\Sigma b_Ri_R)=0$. Let $b'=\Sigma c_Ri_R$.

For the prime 2 we have $Q_{i-1} = P^{\Delta_i}$ and for a Milnor basis element P^R we have $P^R P^{\Delta_i} = \sum_{r_{i+j} \text{ even}} P^{R-2^i\Delta_j+\Delta_{i+j}}$, thus increasing the filtration precisely one. Using this fact, the proof for $p=2$ is the same as for odd primes. #

Proposition 2.1(j) Given $a \varepsilon PH^*(K(V_j,k))$, a with β's such that $s^*(d_j)(a)=0$, then there exists $\bar{a} \varepsilon M_j$ such that $s^*(\bar{a})=a$ and $d_j(\bar{a})=0$. #

Proof of 1.5(j) For k and a odd, then a is with β's in $PH^*(K(V_j,k+1))$ for dimensional reasons, i.e., all of the Steenrod algebra elements used are odd dimensional, and all odd dimensional elements have β's. $(k_{j-1})^*(a)=0$ implies $s^*(d_j)(a)=0$ and we can apply proposition 2.1(j) to get $\bar{a}$ such that $s^*(\bar{a})=a$ and $d_j(\bar{a})=0$. By exactness, there exists $\bar{b} \varepsilon M_{j+1}$ such that $d_{j+1}(\bar{b})=\bar{a}$. Then $b'=s^*(\bar{b}) \varepsilon PH^*(K(V_{j+1},k+2))$ has $s^*(d_{j+1})(b')=s^*(d_{j+1})(s^*(\bar{b}))=s^*(d_{j+1}(\bar{b}))=s^*(\bar{a})=a$. So let $b=s^*(b')$, then $s^*(a)=s^*(d_{j+1})(b)$ which is what we want. #

Proposition 2.2(j) Given an a as in 2.1(j), then there exists $b \varepsilon PH^*(K(V_{j+1},k+1))$ such that $s^*(d_{j+1})(b)=a$. #

Proof See proof of 1.5(j). #

Remark Proposition 2.2(j) is really the essential feature that makes everything work. It means that exactness still holds in the unstable range for primitives with β's.

We need proposition 2.2(j-1) in the induction argument for the proof of proposition 2.1(j).

Proof of 2.1(j) This follows at once from the next proposition, just lift a up one step at a time until it is in the stable range. #

Proposition 2.3(j) Given a with β's in $PH^*(K(V_j,k))$ (any k) such that $s^*(d_j)(a)=0$, then there exists $\bar{a}$ with β's in $PH^*(K(V_j,k+1))$ such that $s^*(\bar{a})=a$ and $s^*(d_j)(\bar{a})=0$. (For $j=0$, $s^*(d_o)(a)=0$ is a vacuous condition.) #

Proof $j=0$, trivial. For $j=1$ the argument is the same as for $j>1$ except easier, so assume $j>1$. Now, trivially, there exists a' with β's such that $s^*(a')=a$. (Let $a'=\tau(a)$.) Now $s^*(d_j)(a') \in \ker s^*$ by commutativity of the following diagram. So, $s^*(d_j)(a')=P^nx_{2n}+\beta P^tx_{2t+1} \in PH^*(K(V_{j-1},k))$ by 1.4. ($p=2$, $Sq^{2n}x_{2n}$ or $Sq^{2t+1}x_{2t+1}$) $s^*(d_{j-1})(P^nx_{2n}+\beta P^tx_{2t+1})=0$ as $d_{j-1}\cdot d_j=0$.

$$\begin{tikzcd}
 & 0 & & & \bar{a}=a'-P^ny_{2n}-\beta P^ty_{2t+1} \arrow[lll] \arrow[ddddddl] \\
0 & x_{2n} \arrow[l, "s^*(d_{j-1})"'] & y_{2n} \arrow[l, "s^*(d_j)"'] & & \\
0 & x_{2t+1} \arrow[l] & y_{2t+1} \arrow[l] & & \\
0 & P^nx_{2n}+\beta P^{2t+1}x_{2t+1} \arrow[l] & & a' \arrow[ll] \arrow[ddd] & \\
 & PH^*(K(V_{j-1},k)) \arrow[d, "s^*"'] & PH^*(K(V_j,k+1)) \arrow[l] \arrow[d, "s^*"] & & \\
 & PH^*(K(V_{j-1},k-1)) \arrow[r, dash, "s^*(d_j)"'] & PH^*(K(V_j,k)) & & \\
 & 0 & & a \arrow[ll] &
\end{tikzcd}$$

Claim x_{2t+1} and x_{2n} are in the kernel of $s^*(d_{j-1})$ and x_{2t+1} is with β's and x_{2n} is with 2 β's. #

Proof $s^*(d_{j-1})(P^nx_{2n})$ is a p-th power as $P^nx_{2n}=(x_{2n})^p$. $s^*(d_{j-1})(\beta P^tx_{2t+1})$ is not, because it has a β on the left which must stay there by the Adem relations. So P^nx_{2n} and

$\beta P^t x_{2t+1}$ are each in the ker $s^*(d_{j-1})$ as they cannot give $s^*(d_{j-1})(\beta P^t x_{2t+1}) = -s^*(d_{j-1})(P^n x_{2n})$. $H^*(K(V_{j-2},k-1))$ is a free commutative algebra, so $[s^*(d_{j-1})(x_{2n})]^p = s^*(d_{j-1})(P^n x_{2n}) = 0$ implies $s^*(d_{j-1})(x_{2n})=0$. βP^t is monomorphic on $PH^{2t+1}(K(V_{j-2},k-1))$, ([13]), so $s^*(d_{j-1})(\beta P^t x_{2t+1})=0$ implies $s^*(d_{j-1})(x_{2t+1})=0$. (p=2, much easier)

a' is with β's so $s^*(d_j)(a') = P^n x_{2n} + \beta P^t x_{2t+1}$ is with 2 β's, and is equal to $\sum_R(\sum_i \lambda_i b_i)i_R + \sum_R(\sum_m \mu_m c_m)i_R$ where λ_i and μ_m are $\neq 0 \in Z_p$ and the b_i and c_m are Adem basis elements which begin with P^n and βP^t respectively. (This follows from the proof of 1.4.) From this we see that x_{2t+1} is with β's and x_{2n} is with 2 β's. (same for p=2) #

By induction on j, x_{2t+1} is with β's and in the kernel of $s^*(d_{j-1})$, so by proposition 2.2(j-1), $x_{2t+1} = s^*(d_j)(y_{2t+1})$ and likewise $x_{2n} = s^*(d_j)(y_{2n})$ where y_{2n} is with β's by an earlier claim. So $\bar{a} = a' - P^n y_{2n} - P^t y_{2t+1}$ is with β's. This is easily shown by writing out y_{2n} and y_{2t+1} using the Adem basis and then noting that $P^n y_{2n}$ and $P^t x_{2t+1}$ are still in the Adem basis form for dimensional reasons. So, we have $s^*(d_j)(\bar{a})=0$ and $s^*(\bar{a}) = s^*(a') = a$. #

Section 3

Our first objective is to compute the (co)homology of BP_{2k}. The bar construction ([4]) gives a spectral sequence of Hopf algebras: (k odd)

$$\mathrm{Tor}^{H_*(BP_k)}(Z_p,Z_p) \Rightarrow E_oH_*(\text{zero component of } BP_{k+1})$$

Now $H_*(BP_k)$ is an exterior algebra on odd dimensional generators $QH_*(BP_k)$. (Cor. 1.7) A standard computation (see [14]) gives: $\mathrm{Tor}^{H_*(BP_k)}(Z_p,Z_p) = \Gamma(s^1(QH_*(BP_k)))$ where Γ denotes the Hopf algebra dual to the polynomial algebra. Now all elements in $\Gamma(s^1(QH_*(BP_k)))$ are of even degree and the differentials change degree by one, so our spectral sequence collapses and we have: H^*(zero component of BP_{k+1}) = $[E_oH_*($zero component of $BP_{k+1})]^* = [\mathrm{Tor}^{H_*(BP_{k+1})}(Z_p,Z_p)]^* = [\Gamma(s^1(QH_*(BP_k)))]^*$ = polynomial algebra.

We will now show $H_*(BP_{k-1})$ is a polynomial algebra for k odd. Using the Eilenberg-Moore spectral sequence ([6,14]) we have $\mathrm{Tor}_{H^*(BP_k)}(Z_p,Z_p) \Rightarrow E_oH^*(BP_{k-1})$ if BP_k is simply connected. Assume it is, then the same argument just given shows $H_*(BP_{k-1})$ is a polynomial algebra. The only modification is:

$$\mathrm{Tor}_{H^*(BP_k)}(Z_p,Z_p) = \Gamma(s^{-1}(QH^*(BP_k)))$$

If BP_k, k odd, is not simply connected, then it is easy to see that one can get a splitting $BP_k \simeq (\times S^1)_{(p)} \times X$ where X is simply connected. This is because BP_k is an H-space with $Z_{(p)}$ free homotopy. Its k-invariants are therefore torsion and primitive, but $(\times S^1)_{(p)}$ has no torsion in $Z_{(p)}$ cohomology. Thus we have a spectral sequence of Hopf algebras:

$$\mathrm{Tor}_{H^*(X)}(Z_p, Z_p) = E_0H^*(\text{zero component of } BP_{k-1})$$

and our argument goes through. We have proved the following proposition.

Proposition 3.1 The mod p (co)homology of the zero component of BP_k is a polynomial algebra on even dimensional generators for k even, and an exterior algebra on odd dimensional generators for k odd. (Note that for k odd, BP_k is connected.)#

Proposition 3.2 The $Z_{(p)}$ (co)homology of BP_k has no torsion.#

Proof For k even this is trivial because $H^*(BP_k)$ has no elements in odd degrees. For k odd we view the Bockstein spectral sequence as a spectral sequence of Hopf algebras. The differentials are the higher order Bocksteins. Let β_s be the first non-trivial differential and let x be the minimum degree generator that β_s acts non-trivially on. $\beta_s(x)$ is an even dimensional primitive, contradiction, so all differentials are zero.#

We can now prove the main theorem.

Theorem 3.3 The $Z_{(p)}$ (co)homology of BP_{2k+1} is an exterior algebra and the $Z_{(p)}$ (co)homology of the zero component of BP_{2k} is a polynomial algebra.#

Proof We will do the case for polynomial algebras, the exterior case being similar. From 3.2 we know the (co)homology is free over $Z_{(p)}$ and so we can lift the mod p generators (3.1) up to it. These lifted elements generate the $Z_{(p)}$ (co)homology ring because there is no torsion and their mod p reductions generate the Z_p (co)homology. By considering the rank we can see there can be no relations and we have a polynomial algebra. #

We can now lift our result to MU. Normally the spectrum MU is given by $\{MU(n)\}$, the Thom complexes, and maps $S^2MU(n) \to MU(n+1)$. [9,15] However, if $M_n = \lim(k\to\infty)\Omega^{2k-n}MU(k)$, then $\Omega M_n \simeq M_{n-1}$ and for finite complexes $MU^n(X) = \lim(k\to\infty)[S^{2k-n}X, MU(k)] = \lim(k\to\infty)[X, \Omega^{2k-n}MU(k)] = [X, M_n]$. Thus, $\{M_n\} = MU$ as an Ω-spectrum.

Corollary 3.4 The integer (co)homology of the zero component of M_n has no torsion and is a polynomial algebra over Z for n even and an exterior algebra for n odd. #

Proof From [3] we have $MU_{(p)} \simeq \bigvee_i S^{2n_i}BP$ and so $(M_n)_{(p)} \simeq \Pi_i BP_{n+2n_i}$. By 3.3 for n even $H_*(M_n,Z)\otimes Z_{(p)} \simeq H_*(M_n,Z_{(p)}) \simeq H_*((M_n)_{(p)},Z_{(p)}) \simeq$ polynomial algebra over $Z_{(p)}$. Thus the integer homology has no torsion, and localized at every prime it is a polynomial algebra, so it is a polynomial algebra over Z. Similarly for n odd. Since there is no torsion, the same thing works for cohomology. #

Remark 1 A completely analogous theorem is true for MSO if the ring Z(1/2) is used.

Remark 2 There are several ways to determine the number of generators for 3.1, 3.3, and 3.4. The spaces BP_n and M_n are just products of rational Eilenberg-MacLane spaces when localized at $\mathbb{Q}$. (This is because their k-invariants are torsion.) Because there is no torsion, the number of generators is the same as for the rationals. As examples we have $\pi_*^S(BP) = Z_{(p)}[x_{2(p-1)},\ldots,x_{2(p^i-1)},\ldots]$ so for $2n>0$, $H^*(BP_{2n},Z_{(p)}) \simeq Z_{(p)}[s^{2n}\pi_*^S(BP)]$ and $\pi_*^S(MU) = Z[x_2,\ldots,x_{2i},\ldots]$ so for $2n>0$, $H^*(M_{2n},Z) \simeq Z[s^{2n}\pi_*^S(MU)]$.

We have shown that both the cohomology and homology of the zero component of BP_{2n} are polynomial algebras. This is a very strong statement, in fact, it determines the Hopf algebra structure of the (co)homology.

Definition A connected bicommutative Hopf algebra is called bipolynomial if both it and its dual are polynomial algebras.#

There is a bipolynomial Hopf algebra $B_{(p)}[x,2n]$ over $Z_{(p)}$ (or Z_p) which has generators $a_k(x)$ of degree $2p^k n$.[7] It is isomorphic as Hopf algebras to its own dual.

In [12] we prove the following proposition.

Proposition 3.5 If H is a bipolynomial Hopf algebra over $Z_{(p)}$ (or Z_p), then $H \simeq \otimes_j B_{(p)}[x_j,2d_j]$. (For $p=2$ and Z_2, replace $2d_j$ by d_j.) #

Using this and the counting argument of remark 2 we can just write down the Hopf algebra structure for BP_{2n}. As an example, we will do this for $n>0$. Let $\mathcal{R}_n$ be the set of sequences of non-negative integers $R=(r_1,r_2,\dots)$ with almost all $r_i=0$. Let $d(R)= 2n + \Sigma 2(p^i-1)r_i$ for our fixed prime p. We say R is prime if it cannot be written $R=pS+(n,0,0,\dots)$, $S \in \mathcal{R}_n$.

Proposition 3.6 For $n>0$, $H^*(BP_{2n},Z_{(p)}) \simeq \bigotimes_{\substack{R\in\mathcal{R}_n \\ R \text{ prime}}} B_{(p)}[x_R,d(R)]$ #

If we work over the integers and let $B[x,2d]$ be the bipolynomial Hopf algebra on generators $c_n(x)$ of degree $2dn$ with coproduct $c_n(x) \to \Sigma c_{n-j}(x)\otimes c_j(x)$ ([7]) then we have an analogous proposition. [12]

Proposition 3.7 If H is a bipolynomial Hopf algebra over Z, then $H \cong \bigotimes_j B[x_j, 2d_j]$. #

We can now apply this to $MU=\{M_n\}$. Let I_n be the set of sequences of non-negative integers $I=(i_1, i_2, \ldots)$ with $i_1 \geq n$ and almost all $i_j=0$. $(n>0)$. Let $d(I)=\Sigma_j\ 2j\ i_j$. We say I is prime if it cannot be written $I=kJ$, where $k>1$ and $J \in I_n$.

Proposition 3.8 If $\{M_k\}$ is the Ω-spectrum for MU, then for $n>0$, $H^*(M_{2n},Z) \cong \bigotimes_{I \text{ prime} \in I_m} B[x_I, d(I)]$ as Hopf algebras. #

Proof Just use 3.7 and the counting done in remark 2. #

Let S be the sphere spectrum and let $i:S \to BP$ represent $1 \in \pi_0^S(BP)$. $S=\{QS^n\}$ as an Ω-spectrum where $QX=\lim \Omega^n S^n X$. i induces maps $i_n:QS^n \to BP_n$. $H_*(QS^n)$ is given in terms of homology operations on the n dimensional generator. [5]

Proposition 3.9 Let $n>0$, the kernel of $(i_n)_*:H_*(QS^n) \to H_*(BP_n)$ is generated by homology operations on the n-dimensional class which have Bocksteins in them. #

Proposition 3.10 Let $n>0$, if $j_n:BP_n \to K(Z_{(p)},n)$ represents the generator of $H^n(BP_n, Z_{(p)})$, then the kernel of $(j_n)^*:H^*(K(Z_{(p)},n)) \to H^*(BP_n)$ is generated by cohomology operations on the n-dimensional class which have Bocksteins in them. #

Proof of 3.9 By 3.2, any homology operation which has Bocksteins in it goes to zero. Let u be a homology operation with no β's such that $ux_n \neq 0$ in $H_*(QS^n)$. As u has no β's, $u(s_*)^k x_n$ is a p-th power for some k. So $u(s_*)^k x_n = ux_{n+k} = (u'x_{n+k})^p$. Now by induction on the degree of u, $i_*(u'x_{n+k}) \neq 0$ in $H_*(BP_{n+k})$ and n+k is even since we have a p-th power. $H_*(BP_{n+k})$ is a polynomial algebra and so $[i_*(u'x_{n+k})]^p \neq 0$ and is $=i_*[u'x_{n+k}]^p = i_* u(s_*)^k x_n = i_*(s_*)^k ux_n = (s_*)^k i_*(ux_n)$ and so $i_*(ux_n) \neq 0$. #

The proof of 3.10 is similar.

The Ω-Spectrum for Brown-Peterson Cohomology Part II

by W. Stephen Wilson

Introduction

Let BP denote the spectrum for the Brown-Peterson cohomology theory, $BP^*(\cdot)$.[1,5,12] We have $BP^k(X) \cong [X, BP_k]$ where $BP = \{BP_k\}$ as an Ω-spectrum, i.e. $\Omega BP_k \simeq BP_{k-1}$.[4] In Part I [20] we determined the structure of the cohomology of BP_k. In this part we study the homotopy type of BP_k.

The structure of each BP_k is very nice and gives some insight into the cohomology theory. In particular, using it, we obtain a new proof of Quillen's theorem that $BP^*(X)$ is generated by non-negative degree elements as a module over $BP^*(S^o)$.[11] (X is a pointed finite CW complex.)

Let $Z_{(p)}$ be the integers localized at p, the prime associated with BP. We explicitly construct spaces Y_k which are the smallest possible k-1 connected H-spaces with π_* and H_* free over $Z_{(p)}$. The Y_k are the building blocks for BP_n, i.e., $BP_n \simeq \Pi_i Y_{k_i}$. In fact, one of our main theorems states that for any H-space X with π_* and H_* free over $Z_{(p)}$, then $X \simeq \Pi_i Y_{k_i}$. (This is not as H-spaces, see section 6.) To understand the spaces Y_k we need a sequence of homology theories:

$$BP_*(X) \simeq BP\langle\infty\rangle_*(X) \to \ldots \to BP\langle n+1\rangle_*(X) \to BP\langle n\rangle_*(X) \to \ldots \to BP\langle 0\rangle_*(X) = H_*(X, Z_{(p)})$$

These are constructed using Sullivan's theory of manifolds with singularities. $BP_*(S^o) \simeq Z_{(p)}[x_1, x_2 \ldots .]$ with degree of $x_i = 2(p^i - 1)$. $BP\langle n\rangle_*(S^o) = Z_{(p)}[x_1, \ldots, x_n]$ as a graded group. Let

$BP\langle n\rangle = \{BP\langle n\rangle_k\}$ be the Ω-spectrum for $BP\langle n\rangle_*(\cdot)$. For $k > 2(p^{n-1}+\ldots+p+1)$, the space $BP\langle n\rangle_k$ cannot be broken down as a product $BP\langle n\rangle_k \simeq Y\times X$ with both X and Y non-trivial. For $k \leq 2(p^n+\ldots+p+1)$, $H^*(BP\langle n\rangle_k, Z_{(p)})$ has no torsion. So, for k between these two numbers we get $Y_k \simeq BP\langle n\rangle_k$.

Main Theorem For $2(p^{n-1}+\ldots+p+1) < k \leq 2(p^n+\ldots+p+1)$

$$BP_k \simeq BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$$

and cannot be broken down further. #

The proof of this theorem exploits the fact from [20] that the $Z_{(p)}$ cohomology of BP_k has no torsion.

We begin by constructing the theories $BP\langle n\rangle_*(\cdot)$. In section 2 we review what we need about Postnikov systems. Section 3 is devoted to preliminary necessities for the proof of the main theorems in section 4. Then we state the main results and prove Quillen's theorem (section 5) and a general decomposition theorem for spaces which are p-torsion free and H-spaces when localized at p. (section 6)

In a future paper with Dave Johnson these results will be applied to study the homological dimension of $BP^*(X)$ over $BP^*(S^0)$. [21]

This paper is part of my Ph.D. thesis at M.I.T. I would like to thank my advisor, Professor Frank Peterson, for his encouragement and understanding. I would also like to acknowledge the influence of his papers on this research, in particular, [10].

Section 1 Construction of BP<n>

This section deals with Sullivan's theory of manifolds with singularities. [19] The approach we take is due to Nils Baas. This section is not intended to be an exposition on the Baas-Sullivan theory, for we only wish to use it to construct certain specific homology theories, the general case being covered in detail in [3]. Even the definitions we give will be missing major ingredients, in all cases we refer to [3].

If we dealt with the case of one singularity, P, then a manifold with singularity P would be a space $V = N\bigcup_{P\times M} cP\times M$ where N is a manifold with $\partial N=P\times M$ and cP is the cone on P. One can make a bordism group of a space using such objects in place of manifolds. An element of the bordism would be represented by a map $f:V\rightarrow X$. So, as far as bordism is concerned, one might just as well consider only the manifold N and insist that maps $f:N\rightarrow X$, when restricted to $\partial N=P\times M$, factor through the projection $P\times M \rightarrow M$. This is the approach Baas takes. When more than one singularity is considered, the definitions become quite technical. From [3]

<u>Definition</u> V is a closed decomposed manifold if there exist submanifolds $\partial_1 V,\dots,\partial_n V$ such that $\partial V=\partial_1 V\cup\dots\cup\partial_n V$ where union means identification along common part of boundary such that $\partial(\partial_i V) = (\partial_1 V\cap\partial_i V)\cup\dots\cup(\partial_{i-1}V\cap\partial_i V)\cup\emptyset\cup\dots\cup(\partial_n V\cap\partial_i V)$, which gives $\partial_i V$ the structure of a decomposed manifold. Continue, defining $\partial_k(\partial_j(\partial_i V))$, etc. #

Let $S^n=\{P_1,P_2,\dots,P_n\}$ be a fixed class of manifolds. Very loosely, A is a closed manifold of singularity type S^n if for each subset $\omega \subset \{1,2,\ldots,n\}$ there is a decomposed manifold $A(\omega)$ such that $A(\emptyset)=A$, $\partial_i A(\omega) \approx A(\omega,i)\times P_i$ if $i \notin \omega$, $\partial_i A(\omega) \approx \emptyset$ if $i \in \omega$. A singular S^n manifold in X is a map $g:A\to X$ such that $g|\partial_i A(\omega) \approx A(\omega,i)\times P_i$ factors through the projection $A(\omega,i)\times P_i \to A(\omega,i)$.

More generally, singular manifolds with boundary, singular manifolds in a pair, and a concept of bordism are all defined. (rigorously) These bordism groups are shown to give generalized homology theories, $MS^n{}_*(\cdot)$. One of the most important aspects of these theories is the relationship between $MS^n{}_*(\cdot)$ and $MS^{n+1}{}_*(\cdot)$. This will be a major tool throughout the paper There is an exact sequence

$$\begin{tikzcd}
MS^n{}_*(X) \arrow[rr, "\beta"] & & MS^n{}_*(X) \arrow[dl, "\gamma"] \\
& MS^{n+1}{}_*(X) \arrow[ul, "\delta"] &
\end{tikzcd}$$

The product of an S^n manifold with a closed manifold N gives an S^n manifold by: $(N\times A)(\omega) = N\times A(\omega)$. On a representative element $A\to X$, β is $P_{n+1}\times A \to A \to X$. Any S^n manifold A can be considered as an S^{n+1} manifold by setting $A(\omega,n+1)=\emptyset$. So $\gamma(A\to X)=(A\to X)$. For an S^{n+1} manifold A we see that $A(n+1)$ is an S^n manifold, so $\delta(f:A\to X)=f|A(n+1)\to X$. The degrees of these maps are: degree $\beta=$ dimension P_{n+1}, degree $\gamma=0$, degree $\delta=$ -dimension P_{n+1} -1. In our one singularity example, $\partial N=P\times M$, δ just restricts to M. Baas of course defines these maps rigorously, shows they are well defined and proves the exactness theorem.

Above we remarked that the product of a manifold and an

S^n manifold is again an S^n manifold. This gives us a map, $MS^o{}_*(X)\otimes MS^n{}_*(Y) \to MS^n{}_*(X\times Y)$. This is precisely the condition that tells us the spectrum associated with $MS^n{}_*(\cdot)$ is a module spectrum over the spectrum for the standard bordism theory $MS^o{}_*(\cdot)$. Further, $MS^n{}_*(X)$ is a module over $MS^o{}_*(S^o)$ and the above maps, β, γ, δ are all $MS^o{}_*(S^o)$ module maps.

We now get on to our applications. All manifolds considered above could be taken with some extra structure, and we assume them all to be U manifolds. So $MS^o{}_*(\cdot)$ is $MU_*(\cdot)$ the standard complex bordism homology theory for finite complexes. Now $MU_*(S^o) = \pi_*^S(MU) = Z[x_2,\dots,x_{2i},\dots]$ where degree $x_{2j}=2j$. We choose a representative manifold P_i for x_{2i}. Fix a prime p. $S(n,m) = \{P_i \mid i\leq m,\ i\neq p^j-1,\ j\leq n\}$. Then by all of the above, we have a homology theory $MUS(n,m)_*(\cdot)$ made from U manifolds with singularity type $S(n,m)$. For large m we have an exact sequence:

$$\begin{tikzcd}
MUS(n,m)_*(X) \arrow[rr, "\times P_{p^n-1}"] & & MUS(n,m)_*(X) \arrow[dl, "\gamma"] \\
& MUS(n-1,m)_*(X) \arrow[ul, "\delta"] &
\end{tikzcd}$$

From these exact sequences and the homotopy of MU we see that $MUS(n,m)_*(S^o) = \pi_*^S(MU)/[S(n,m)]$ where $[S(n,m)]$ is the ideal generated by $S(n,m)$. We define the homology theory $MUS(n)_*(\cdot) = \lim(m\to\infty)\ MUS(n,m)_*(\cdot)$. $MUS(n)_*(\cdot)\otimes Z_{(p)}$ is a homology theory which we will denote by $BP\langle n\rangle_*(\cdot)$ and the corresponding spectrum by $BP\langle n\rangle$. The reason for the notation is that if $BP \to MU_{(p)}$ is Quillen's map ([12]), then $BP \to MU_{(p)} \to BP\langle\infty\rangle$ clearly gives an isomorphism on homotopy and so

$BP \simeq BP\langle\infty\rangle$. Thus $BP\langle n\rangle$ is a module spectrum over BP and we have:

1.1
$$\begin{tikzcd} BP\langle n\rangle_*(X) \arrow[rr, "\beta"] & & BP\langle n\rangle_*(X) \arrow[dl, "\gamma"] \\ & BP\langle n-1\rangle_*(X) \arrow[ul, "\delta"] & \end{tikzcd}$$

with degree of $\beta = 2(p^n-1)$, degree $\gamma=0$, degree $\delta = -2p^n+1$. $BP_*(S^0) = Z_{(p)}[x_{2(p-1)},\ldots,x_{2(p^i-1)},\ldots]$. $BP\langle n\rangle_* = BP\langle n\rangle_*(S^0) = BP_*(S^0)/[x_{2(p^i-1)} \mid i>n]$ as a module over BP_*. Although BP_* acts on $BP\langle n\rangle_*(X)$, it is not known if $x_{2(p^i-1)}$ acts trivially for $i>n$.

Every spectrum can be represented as an Ω-spectrum.[4] Let $BP\langle n\rangle = \{BP\langle n\rangle_k\}$ be the Ω-spectra, i.e. $\Omega BP\langle n\rangle_k \simeq BP\langle n\rangle_{k-1}$ and $BP\langle n\rangle_k$ is k-1 connected for $k>0$. This means that $BP\langle n\rangle^k(X) = [X, BP\langle n\rangle_k]$ where $BP\langle n\rangle^*(\cdot)$ is the cohomology theory given by $BP\langle n\rangle$.

The theories $BP\langle n\rangle$ are independent of choice of manifolds P_i representing x_{2i} but seemingly dependent on the choice of generators $x_{2(p^i-1)}$ chosen for $\pi_*^S(MU)$. However, the results we obtain are independent of the choice of even these generators because the spaces $BP\langle n\rangle_k$ for different choices become homotopy equivalent when k is small enough. In addition, in [21] we show that $BP\langle 1\rangle$ is independent of choice of x_{2i}. In fact, $BP\langle 1\rangle$ is just the irreducible part of connective K-theory when localized at p.

We now permanently reindex the $x_{2(p^i-1)}$ to x_i with degree $2(p^i-1)$. From 1.1 we have a split exact sequence:

1.2 $$0 \longrightarrow BP\langle n\rangle_* \xrightarrow{x_n} BP\langle n\rangle_* \longrightarrow BP\langle n-1\rangle_* \longrightarrow 0$$

$BP\langle n\rangle_* = Z_{(p)}[x_1,\ldots,x_n]$ as a group. Again, from 1.1 for finite complexes we get a cofibration ([1]):

1.3 $$\begin{tikzcd} S^{2i}BP\langle n\rangle \arrow[r, "\beta"] & BP\langle n\rangle \arrow[d, "\gamma"] \\ i=p^n-1 & BP\langle n-1\rangle \end{tikzcd}$$

For the spaces in the Ω-spectrum this becomes a fibration:

1.4 $$\begin{tikzcd} BP\langle n\rangle_{k+j} \arrow[r, "\beta"] & BP\langle n\rangle_k \arrow[d, "\gamma"] \\ j=2(p^n-1) & BP\langle n-1\rangle_k \end{tikzcd}$$

If M is a graded module let s^kM be the graded module $(s^kM)_{k+q} = M_q$. Then,

1.5 $$\pi_*(BP\langle n\rangle_k) = s^k(BP\langle n\rangle_*) \qquad k \geq 0$$

From 1.3 we have an exact sequence:

1.6 $$\begin{tikzcd} H^*(BP\langle n\rangle) & H^*(BP\langle n\rangle) \arrow[l, "\beta^*"'] & H^*(BP\langle n-1\rangle) \arrow[l, "\gamma^*"'] \arrow[from=1-1, to=1-3, bend right, "\partial^*"'] \end{tikzcd}$$

For most of the paper, unless otherwise noted, all coeffecient groups will be Z_p where p is the fixed prime associated with the BP<n>. Let A be the mod p Steenrod algebra and Q_i the Milnor elements. [9]

Proposition 1.7 $H^*(BP\langle n\rangle) \simeq A/A(Q_0,Q_1,\ldots,Q_n) = A_n$ #

Note Baas and Madsen have a more general result which includes this, however, as this special case has a much more elementary proof we give it here.

Proof $\pi_*^S(BP\langle 0\rangle) = Z_{(p)}$, so $BP\langle 0\rangle = K(Z_{(p)})$ and $H^*(BP\langle 0\rangle)=A/A(Q_o)$. We prove the result by induction on n using 1.6. Let 1 denote the lowest dimensional class of each spectrum, then $\gamma^*(1)=1$. Assume $H^*(BP<n-1>)=A_{n-1}$. If $\gamma^*(Q_n1)=0$, then for dimensional reasons, $\partial^*(1)=\lambda Q_n1$, $0\neq\lambda \varepsilon Z_p$. If $a \varepsilon A_n$, then $0\neq aQ_n1=\partial^*(a1)$ in A_{n-1} because $A_{n-1}= A_n(1)\oplus A_n(Q_n1)$. Therefore $a1\neq0$ in $H^*(BP<n>)$. This takes care of exactness at $H^*(BP<n-1>)$. So now $H^*(BP<n>) =A_n\oplus X$ with $\beta^*:X\to X$ an isomorphism, but the degree of $\beta^* \neq0$ so $X=0$.

All we need now is $Q_n1=0 \varepsilon H^*(BP<n>)$. Our map $BP \to BP<n>$ is an isomorphism on homotopy below dimension $2(p^{n+1}-1)$ and therefore an isomorphism on cohomology in this range. $H^*(BP) \approx A/A(Q_o,Q_1,\ldots)$.[5] The dimension of Q_n is $2p^n-1$ so $Q_n1=0$.#

Section 2 Postnikov Systems

We collect here the results we need about Postnikov systems. We assume X is a simply connected CW complex. We start with the standard diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X^n \arrow[d, "g_n"'] & \\
X \arrow[ur, "\rho_n"] \arrow[r, "\rho_{n-1}"'] & X^{n-1} \arrow[r, "k_n"] \arrow[d] & K(\pi_n(X),n+1) \\
 & \vdots \arrow[d] & \\
 & X^1 = \text{point} &
\end{tikzcd}$$

2.1 Definition and existence [16]

A Postnikov system for X is a sequence of spaces $\{X^n\}$ and maps, $\{g_n:X^n \to X^{n-1}\}$, $\{\rho_n:X\to X^n\}$ such that $\rho_{n-1} \simeq g_n\cdot\rho_n$ and the fibre of g_n is $K(\pi_n(X),n)$, the Eilenberg-MacLane space. The fibration $g_n:X^n \to X^{n-1}$ is induced by a map $k_n:X^{n-1} \to K(\pi_n(X),n+1)$ from the path space of $K(\pi_n(X),n+1)$. Thus $k_n\circ g_n \simeq 0$ and $k_n \in H^{n+1}(X^{n-1},\pi_n(X))$. k_n is called the n-th k-invariant of X. Postnikov systems for simply connected CW complexes always exist and $(\rho_n)_{\#}:\pi_k(X) \to \pi_k(X^n)$ is an isomorphism for $k \leq n$ and $\pi_k(X^n)=0$ for $k>n$.

2.2 Induced maps [8]

Given $f:X \to Y$ then we have $\{f^n:X^n \to Y^n\}$ such that $f^{n-1}\cdot g_n(X) \simeq g_n(Y)\cdot f^n$, $f^n\cdot\rho_n(X) \simeq \rho_n(Y)\cdot f$ and $f_{\#}(k_n(X)) = (f^{n-1})^*(k_n(Y))$.

2.3 Loop spaces

The Postnikov system for ΩX is given by: $(\Omega X)^n = \Omega X^{n+1}$, $\rho_n(\Omega X) = \Omega\rho_{n+1}(X)$, $g_n(\Omega X) = \Omega g_{n+1}(X)$, $k_n(\Omega X) = \Omega k_{n+1}(X)$, so $k_n(\Omega X) = s^*(k_{n+1}(X)) \in H^{n+1}(\Omega X^n, \pi_n(\Omega X))$ where s^* is the cohomology suspension defined by $\delta^{-1}\cdot p^*$.

$$\begin{tikzcd} H^*(\Omega X,G) \arrow[r, "\delta", "\cong"'] & H^{*+1}(PX, \Omega X, G) & H^{*+1}(X,pt, G) \arrow[l, "p^*"'] \end{tikzcd}$$

PX is the path space fibration over X.

2.4 Product spaces

A Postnikov system for $X\times Y$ is given by $\{X^n \times Y^n\}$ with k-invariants $\{k_n(X) \times k_n(Y)\}$.

2.5 H-spaces [8]

If X is an H-space, then each X^n is an H-space, ρ_n and g_n are maps of H-spaces and $k_n \in H^{n+1}(X^{n-1}, \pi_n(X))$ is torsion and is primitive in the Hopf algebra structure induced on H^* by the multiplication in X^{n-1}. Also, if X^{n-1} is an H-space and k_n is primitive, then X^n is an H-space. If all k-invariants are primitive, then X is an H-space.

2.6 Obstruction theory [16]

If Y is CW, and we have $f_{n-1}:Y \to X^{n-1}$, then f_{n-1} lifts to $f_n:Y \to X^n$ iff $(f_{n-1})^*(k_n(X)) = 0 \in H^{n+1}(Y, \pi_n(X))$. If there exist maps $\{f_n : Y \to X^n\}$ such that $g_n(X) \cdot f_n \simeq f_{n-1}$ then there exists $f : Y \to X$ with $\rho_n(Y) \cdot f \simeq f_n$.

2.7 Construction of spaces [16]

Given a sequence of fibrations $g_n : X^n \to X^{n-1}$ with fibre $K(\pi_n, n)$ and $X^1 = pt$, then there exists a CW complex X and maps $p_n : X \to X^n$ such that $\{X^n\}$ is a Postnikov system for X.

2.8 Independent k-invariants

Assume for the rest of this section that $\pi_*(X) \otimes Z_{(p)}$ is free over $Z_{(p)}$ and the k-invariants $k_n(X)$ are torsion elements. This will always be the case in our applications. From the Serre spectral sequence of a fibration we obtain the following natural ladder of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^s(X^{s-1}, Z_{(p)}) \arrow[r, "(g_s)^*"] \arrow[d] & H^s(X^s, Z_{(p)}) \arrow[r] \arrow[d] & H^s(K(\pi_s(X), s), Z_{(p)}) \arrow[d] \\
0 \arrow[r] & H^s(X^{s-1}) \arrow[r, "(\bar{g}_s)^*"] & H^s(X^s) \arrow[r] & H^s(K(\pi_s(X), s))
\end{tikzcd}$$

2.9

$$\begin{tikzcd}
{} \arrow[r, "\tau"] & H^{s+1}(X^{s-1}, Z_{(p)}) \arrow[r, "(g_s)^*"] \arrow[d] & H^{s+1}(X^s, Z_{(p)}) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r, "\bar{\tau}"] & H^{s+1}(X^{s-1}) \arrow[r, "(\bar{g}_s)^*"] & H^{s+1}(X^s) \arrow[r] & 0
\end{tikzcd}$$

τ is the transgression. Also we obtain

2.10 $\quad H^k(X^s, G) \simeq H^k(X, G)$ for $k \leq s$.

In the dimension of our ladder, the transgression is related to the k-invariant map k_s by $\tau \cdot s^* = k_s{}^*$. This motivates the following definitions.

For $x \in H^s(K(\pi_s(X), s), Z_{(p)})$, a free generator, $\tau(x)$ will be called a k-invariant of X. If $\tau(x) = 0$, it is called dependent.

The k-invariant $\tau(x)$ is independent and hits a p-torsion generator if and only if $\rho\cdot\tau(x) = \bar{\tau}\cdot\rho(x) \neq 0$ where ρ is the mod p reduction. If the k-invariants, $\tau(x)$, of ΩX, hit p-torsion generators, then there is a y with $s^*(y)=x$, and so $s^*(\tau(y)) = \tau(x)$ showing that the k-invariants $\tau(y)$ of X also hit p-torsion generators. (Remember that we have restricted ourselves to spaces with torsion k-invariants.) If $H^*(X,Z_{(p)})$ has no p torsion, then all p torsion generators of $H^{s+1}(X^{s-1},Z_{(p)})$ are hit by k-invariants. This is true because the coker $\tau \simeq H^{s+1}(X^s,Z_{(p)}) \subset H^{s+1}(X^{s+1},Z_{(p)}) \simeq H^{s+1}(X,Z_{(p)})$ which is free, all by 2.9 and 2.10.

2.11 Localization [18]

Usually we will work with localized spaces, i.e. spaces with $\pi_*(X)$ a $Z_{(p)}$ module. For simply connected spaces or H-spaces, the localization $X_{(p)}$ and a mod p equivalence $X \to X_{(p)}$ can be built by 2.7 using $\pi_*(X)\otimes Z_{(p)}$ for homotopy groups and $k_n(X)\otimes Z_{(p)}$ as the k-invariants. We get that

$$H_*(X,Z)\otimes Z_{(p)} \simeq H_*(X,Z_{(p)}) \simeq H_*(X_{(p)},Z_{(p)}) \simeq H_*(X_{(p)},Z).$$

2.12 Irreducible spaces

If a space cannot be written as a non-trivial product of spaces it will be called irreducible (indecomposible). If X is connected and ΩX is irreducible, then X must also be irreducible. If X is a localized space with $\pi_*(X)$ free (over $Z_{(p)}$) then if X^{s-1} in the Postnikov system for X is irreducible and all of the s k-invariants are independent, then X^s is also irreducible.

Section 3 The Map

Before we can prove the main theorem,

$$BP_k \simeq BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$$

for $k \leq 2(p^n+\ldots+p+1)$, we need the maps $BP_k \to BP\langle j\rangle_{k+2(p^j-1)}$. The natural transformation $BP_*(\cdot) \to BP\langle n\rangle_*(\cdot)$ gives us the map $BP_k \to BP\langle n\rangle_k$ which is onto in homotopy. If we obtain the map $BP_k \to BP\langle j\rangle_{k+2(p^j-1)}$ for $k=2(p^{j-1}+\ldots p+1)$ then we have it for all $k \leq 2(p^{j-1}+\ldots+p+1)$ by taking the loop map. We can then combine these maps to give a map

$$BP_k \to BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)} \quad \text{for } k \leq 2(p^n+\ldots+p+1).$$

We fix $k=2(p^{j-1}+\ldots+p+1)$ and construct a map $BP_k \to BP\langle j\rangle_{k+2(p^j-1)}$ by the following series of lemmas.

Lemma 3.1 There is an element $x_j \in H^{k+2(p^j-1)}(BP\langle j\rangle_k, Z_{(p)})$ such that $x_j : BP\langle j\rangle_k \to K(Z_{(p)}, k+2(p^j-1))$ is onto in homotopy, $k \leq 2(p^{j-1}+\ldots+p+1)$. #

Before proceeding, we need to state a lemma which we will prove later.

Let i_k be the generator of $H^k(BP\langle j-1\rangle_k)$.

Lemma 3.2 For $k>2(p^{j-1}+\ldots+p+1)$, $Q_j i_k \neq 0$ in $H^*(BP\langle j-1\rangle_k)$. For $k=2(p^{j-1}+\ldots+p+1)$, $H^i(BP\langle j-1\rangle_k) = 0$ for $i=k+2p^j-1 =$ dimension $Q_j i_k = pk+1$. #

Proof of 3.1 We go to the fibration 1.4.

3.3

$$\begin{tikzcd}
BP\langle j\rangle_s \arrow[r, "\beta"] & BP\langle j\rangle_k \arrow[d, "\gamma"'] \\
 & BP\langle j-1\rangle_k
\end{tikzcd}$$

$s=k+2(p^j-1)$

$k=2(p^{j-1}+\ldots+p+1)$

$BP\langle j\rangle_s$ is s-1 connected and $\pi_s(BP\langle j\rangle_s) \simeq H^s(BP\langle j\rangle_s, Z_{(p)}) \simeq Z_{(p)}$.

To show β^* is onto in dimension s we look at the Serre spectral sequence for the fibration 3.3. In this range we have the Serre exact sequence:

$$3.4 \qquad H^s(BP\langle j\rangle_k, Z_{(p)}) \xrightarrow{\beta^*} H^s(BP\langle j\rangle_s, Z_{(p)}) \to H^{s+1}(BP\langle j-1\rangle_k, Z_{(p)})$$

We have $k=2(p^{j-1}+\ldots+p+1)$ and so $s+1=k+2(p^j-1)+1$. By 3.2 and the numbers we are using, the last term is zero and so β^* is onto. If $x_j \in H^s(BP\langle j\rangle_k, Z_{(p)})$ is such that $\beta^*(x_j)$ is the generator, and $S^s \to BP\langle j\rangle_s$ represents $1 \in \pi_s(BP\langle j\rangle_s) = Z_{(p)}$, then the composition $S^s \to BP\langle j\rangle_s \to BP\langle j\rangle_k \xrightarrow{x_j} K(Z_{(p)}, s)$ induces an isomorphism on H^s, so therefore x_j is onto in homotopy. #

Lemma 3.5 There is a map $f_j : BP_k \to BP\langle j\rangle_{k+2(p^j-1)}$ for $k \le 2(p^{j-1}+\ldots+p+1)$ such that $(f_j)_\#$ is onto, $(k \ge -2(p^j-1))$

$$(f_j)_\# : \pi_{k+2(p^j-1)}(BP_k) \to \pi_{k+2(p^j-1)}(BP\langle j\rangle_{k+2(p^j-1)}) \simeq Z_{(p)} \qquad \#$$

Proof It is enough to prove this for $k=2(p^{j-1}+\ldots+p+1)$. We have a map $BP_k \to BP\langle j\rangle_k \to K(Z_{(p)}, k+2(p^j-1))$ from lemma 3.1. Each of these maps is onto in homotopy so the composite is too. $K(Z_{(p)}, k+2(p^j-1))$ is the first non-trivial term of the Postnikov system for $BP\langle j\rangle_{k+2(p^j-1)}$. We know that the k-invariants of this space are torsion by 2.5 and that its homotopy is free over $Z_{(p)}$ by construction. (1.5) The main theorem of [20] gives us that $H^*(BP_k, Z_{(p)})$ has no torsion. Obstructions to lifting the map to a map of the type we want are therefore torsion elements in $H^{q+1}(BP_k, \pi_q(BP\langle j\rangle_k))$, (2.6) which has no torsion. Therefore we see that we can lift the map. #

<u>Corollary 3.6</u> For $k \leq 2(p^n+\ldots+p+1)$ there is a map

$BP_k \to BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$ which composed with projections is onto in homotopy for $\pi_*(BP\langle n\rangle_k)$ and $\pi_{k+2(p^j-1)}(BP\langle j\rangle_{k+2(p^j-1)})$. #

Before proving 3.2 we will make an observation which we need in the next section.

Consider the map $\beta:BP\langle j\rangle_s \to BP\langle j\rangle_k$, $s=k+2(p^j-1)$. We have $\beta_\#:K(Z_{(p)},s) \to K(\pi_s(BP\langle j\rangle_k),s)$. $(\beta_\#)^*$ is onto in $Z_{(p)}$ cohomology. Pick a generator $x \in H^s(K(\pi_s(BP\langle j\rangle_k),s),Z_{(p)})$ such that $(\beta_\#)^*(x)$ is a generator. We wish to study the k-invariant $\tau(x)$. Above we showed that for $k \leq 2(p^{j-1}+\ldots+p+1)$ there was such a k-invariant which was dependent. Here we wish to show the following lemma.

<u>Lemma 3.7</u> For $k>2(p^{j-1}+\ldots+p+1)$, the above k-invariant $\tau(x)$ is independent and hits a p-torsion generator. #

<u>Proof</u> Using the naturality of the mod p version of 2.9 we have:

3.8

$$\begin{tikzcd}
 & Z_p \simeq H^s(BP\langle j\rangle_s) & & \\
0 \arrow[r] & H^s(K(Z_{(p)},s)) \arrow[u, "\simeq"] \arrow[r, "\simeq"] & H^s(K(Z_{(p)},s)) \arrow[r, "\bar{\tau}"] & 0 \\
{} \arrow[r] & H^s((BP\langle j\rangle_k)^s) \arrow[u, "\beta^*"'] \arrow[r] \arrow[d, "\simeq"] & H^s(K(\pi_s(BP\langle j\rangle_k),s)) \arrow[u, "(\beta_\#)^*"'] \arrow[r, "\bar{\tau}"] & {} \\
 & H^s(BP\langle j\rangle_k) & &
\end{tikzcd}$$

As in 2.8, $\tau(x)$ is independent and hits a p-torsion generator iff $\bar{\tau}(\rho(x))\neq 0$, ρ the mod p reduction. Because $(\beta_{\#})^*(x)$ is a generator, this is equivalent to β^* not being onto in 3.8. Again we go to the Serre exact sequence for 1.4.

$$3.9 \qquad H^s(BP\langle j\rangle_k) \xrightarrow{\beta^*} H^s(BP\langle j\rangle_s) \xrightarrow{\bar{\tau}} H^{s+1}(BP\langle j-1\rangle_k)$$

We know from the proof of 1.7 that for k very large $\bar{\tau}(i_s) = \lambda Q_j\, i_k$, $\lambda\neq 0$. By 3.2, for $k>2(p^{j-1}+\ldots+p+1)$ we know $Q_j\, i_k \neq 0$ so $\bar{\tau}(i_s) = \lambda Q_j\, i_k \neq 0$ in this range. So, in 3.9 we see that β^* is not onto and $\tau(x)$ for such an x is an independent p-torsion generating k-invariant. #

We will need the following in our proof of 3.2.

Lemma 3.10 $Q_{n+1} = \lambda\beta P^{p^n+\ldots+p+1} (\text{mod } A(Q_o,\ldots,Q_n))$, $\lambda\neq 0 \in Z_p$. #

Note For p=2, just consider $P^i = Sq^{2i}$.

Proof The lowest non-zero odd dimensional element of $A/A(Q_o,\ldots Q_n)$ is Q_{n+1}. From [9], $Q_i = [P^{p^{i-1}}, Q_{i-1}]$, so $Q_{n+1} = P^{p^n}Q_n - Q_nP^{p^n} = -Q_nP^{p^n} = -(P^{p^{n-1}}Q_{n-1} - Q_{n-1}P^{p^{n-1}})P^{p^n}$ $= Q_{n-1}P^{p^{n-1}}P^{p^n}$ (as the dimension of $Q_{n-1}P^{p^n}$ is less than the dimension of Q_{n+1} and also odd, so it is zero) = ... = $(-1)^{n+1}Q_oP^1\ldots P^{p^{n-1}}P^{p^n}$ (mod $A(Q_o,\ldots,Q_n)$). Let $k_n=1+p+\ldots+p^{n-1}$, all that is left to show is: (note that $Q_o=\beta$)

Claim $P^{k_n}P^{p^n} = \lambda P^{k_{n+1}}$, $\lambda\neq 0 \in Z_p$. #

Proof By the Adem relations,

$$P^{k_n} P^{p^n} = \sum_{t=0}^{k_{n-1}} (-1)^{k_n+t} P^{k_{n+1}-t} P^t \begin{pmatrix} (p-1)(p^n-t)-1 \\ k_n - pt \end{pmatrix} .$$

So all we need is for the binomial coeffecient to be zero mod p for $0 < t \leq k_{n-1}$ and $\neq 0$ for $t=0$. First we reindex, let $s+t = k_{n-1}$. Then $(p-1)(p^n-t)-1 = (p-1)(p^n-k_{n-1}+s)-1 = (p-1)p^n - (p-1)k_{n-1} + (p-1)s - 1$. Now $(p-1)k_{n-1} = p^{n-1} - 1$, so this is $(p-1)p^n - p^{n-1} + 1 + (p-1)s - 1 = (p-2)p^n + (p-1)p^{n-1} + (p-1)s$. $k_n - pt = k_n - pk_{n-1} + ps = 1 + ps$. So our coeffecient is:

$$\begin{pmatrix} (p-2)p^n + (p-1)p^{n-1} + (p-1)s \\ 1 + ps \end{pmatrix}$$

We want to show this is 0 for $0 \leq s < k_{n-1}$ and $\neq 0$ for $s = k_{n-1}$.

From [17], if $a = \Sigma a_i p^i$, $b = \Sigma b_i p^i$, a_i and $b_i < p$, then mod p $\begin{pmatrix} a \\ b \end{pmatrix} = \Pi_i \begin{pmatrix} a_i \\ b_i \end{pmatrix}$. So for $s < p^{n-2}$ our binomial coeffecient is $\begin{pmatrix} p-2 \\ 0 \end{pmatrix} \begin{pmatrix} p-1 \\ 0 \end{pmatrix} \begin{pmatrix} (p-1)s \\ 1+ps \end{pmatrix}$ but $(p-1)s < 1+ps$, so it is zero for $s < p^{n-2}$. Set $s = p^{n-2} + s_1 \geq p^{n-2}$. We get:

$$\begin{pmatrix} (p-2)p^n + (p-1)p^{n-1} + (p-1)p^{n-2} + (p-1)s_1 \\ p^{n-1} + 1 + ps_1 \end{pmatrix}$$

$$= \begin{pmatrix} p-2 \\ 0 \end{pmatrix} \begin{pmatrix} p-1 \\ 1 \end{pmatrix} \begin{pmatrix} p-1 \\ 0 \end{pmatrix} \begin{pmatrix} (p-1)s_1 \\ 1 + ps_1 \end{pmatrix}$$

for $s_1 < p^{n-3}$ this is zero again. Let $s_1 = p^{n-3} + s_2$. Continue like this until we get:

$$\begin{pmatrix} (p-2)p^n + (p-1)p^{n-1} + \ldots + (p-1)p + (p-1)s_{n-2} \\ p^{n-1} + p^{n-2} + \ldots + p^2 + 1 + ps_{n-2} \end{pmatrix}$$

where $0 \leq s_{n-2} \leq 1$. For $s_{n-2} = 0$, this is zero again as it is

$$= \qquad \binom{p-2}{0}\binom{p-1}{1}\cdots\binom{p-1}{1}\binom{p-1}{0}\binom{0}{1}$$

For $s_{n-2} = 1$, we get

$$\binom{p-2}{0}\binom{p-1}{1}\cdots\cdots\binom{p-1}{1}\binom{p-1}{1} = (-1)^n.$$

This finishes the proof of 3.10. #

<u>Proof of 3.2</u> For large k, $Q_j\ i_k \neq 0$ in $H^*(BP\langle j-1\rangle_k)$ because $H^*(BP\langle j-1\rangle) = A/A(Q_0, \ldots, Q_{j-1})$. (1.7) The Eilenberg-Moore spectral sequence [15]

3.11 $\quad \mathrm{Tor}_{H^*(BP\langle j-1\rangle_{k+1})}(Z_p, Z_p) \Rightarrow H^*(BP\langle j-1\rangle_k)$

collapses in dimensions < pk and on indecomposibles, $s^*: QH^*(BP\langle j-1\rangle_{k+1}) \to QH^*(BP\langle j-1\rangle_k)$ is an isomorphism in this range. For $k > 2(p^{j-1} + \ldots + p + 1)$, dimension $Q_j\ i_k < pk$ so $Q_j i_k \neq 0$.

For $k = 2(p^{j-1} + \ldots + p + 1)$, the E_2 term of 3.10 has one element in dimension pk+1 (for p=2, none), $s^{-1}(Q_j\ i_{k+1})$. All $Q_i\ i_{k+1} = 0$ for $i < j$ so by 3.10 this is $s^{-1}(\lambda\beta P^{p^{j-1}+\ldots+p+1}\ i_{k+1})$, s^{-1} corresponds to the cohomology suspension ([14]).

$s^*(\beta P^{p^{j-1}+\ldots+p+1}\ i_{k+1}) = \beta P^{k/2}\ i_k = \beta(i_k)^p = 0$, so $s^{-1}(Q_j i_{k+1})$ is hit by a differential and the result follows. #

Section 4 Proofs

In the last section we constructed a map (3.6)

$$4.1 \quad BP_k \to BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)} \quad \text{for } k \leq 2(p^n+\ldots+p+1).$$

If this map is a homotopy equivalence for some $k>0$ then it is a homotopy equivalence for all $k \leq 2(p^n+\ldots+p+1)$. To see this, look at the diagram for $f: X \to Y$

$$\begin{tikzcd}
\pi_*(\Omega X) \arrow[r, phantom, "\simeq"] \arrow[d, "\Omega f_\#"] & \pi_{*+1}(X) \arrow[d, "f_\#"] \\
\pi_*(\Omega Y) \arrow[r, phantom, "\simeq"] & \pi_{*+1}(Y)
\end{tikzcd}$$

If either $\Omega f_\#$ or $f_\#$ is an isomorphism then so is the other and then they are both homotopy equivalences because our spaces are the homotopy type of CW complexes.

We will prove the homotopy equivalences for the $k = k_n = 2(p^{n-1}+\ldots+p+1) + 1$, ($k_o = 1$) by induction on the Postnikov system. As a plausibility argument, as well as the fact that we need it, we prove the following lemma.

Lemma 4.2 The homotopy is the same on both sides of 4.1. #

Proof $\pi_*(BP_k) \simeq s^k(Z_{(p)}[x_1, x_2, \ldots])$.

$$\pi_*(BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}) = \pi_*(BP\langle n\rangle_k) \bigoplus_{j>n} \pi_*(BP\langle j\rangle_{k+2(p^j-1)})$$

$$= s^k(Z_{(p)}[x_1, \ldots, x_n]) \bigoplus_{j>n} s^{k+2(p^j-1)}(Z_{(p)}[x_1, \ldots, x_j]). \quad (1.5)$$

Our isomorphism takes a $Z_{(p)}$ generator on the right hand side,

$s^{k+2(p^j-1)}[(x_1)^{i_1} \ldots (x_j)^{i_j}]$ to $s^k[(x_1)^{i_1} \ldots (x_{j-1})^{i_{j-1}}(x_j)^{i_j+1}]$.#

Recall that $k_n = 2(p^{n-1} + \ldots + p + 1) + 1$ $(k_o = 1)$.

Statement P(n,s) $i = k_n + 2(p^j-1)$

$$f^{k_n+s} : (BP_{k_n})^{k_n+s} \rightarrow (BP\langle n\rangle_{k_n})^{k_n+s} \times \prod_{j>n} (BP\langle j\rangle_i)^{k_n+s}$$

is a homotopy equivalence. #

4.3 Statement P(n,s) implies a similar statement for any $k < k_{n+1}$ replacing k_n.

Statement K(n,s) All k-invariants $\tau(x)$ in

$$H^{k_n+s+2}((BP\langle n\rangle_{k_n})^{k_n+s}, Z_{(p)})$$

are independent and hit p-torsion generators. (see 2.8) #

4.4 K(n,s) implies that all k-invariants $\tau(x)$ in $H^{k+s+2}((BP\langle n\rangle_k)^{k+s}, Z_{(p)})$ are independent and hit p-torsion generators for $k \geq k_n$.

Statement A

$$\left.\begin{array}{ll} P(n,s) & s \leq m \\ K(n,s) & s \leq m \end{array}\right\} \Rightarrow K(n+1,m)$$

#

Statement B

1) $K(n+j,s)$ $s \leq m$ $j \geq 0$

2) $P(n,m)$

$$\Rightarrow P(n,m+1)$$

#

4.5 Now, to get things started, observe that statement P(n,0) is true for all n as it just reduces to $K(Z_{(p)}, k_n) \xrightarrow{\simeq} K(Z_{(p)}, k_n)$. Also, statement K(0,s) is trivially true for all s because $BP\langle 0\rangle_{k_0=1}$ is just the circle localized at p and has no k-invariants.

Lemma 4.6 Statements A and B imply statements P(n,s) and K(n,s) for all n and s. #

Proof Claim (t) a) P(n,m) is true for $m \leq t$, all n.

b) K(n,m) is true for m < t, all n. #

Claim (t) is true for t = 0 by 4.5. We will show claim (t) => claim (t+1). By 4.5 we know K(0,t) is true, applying statement A n times we have K(n,t), therefore we have K(n,t) for all n giving us b) of claim (t+1). Now, applying statement B we obtain P(n,t+1) for all n. This proves claim (t+1), so, by induction, claim (t) is true for all t and we are done.#

Now we will prove statements A and B. In the next section we will explore some of the consequences of P(n,s) and K(n,s).

Proof of statement A Consider the fibration 1.4

$$\begin{tikzcd} BP\langle n+1\rangle_i \arrow[r, "\beta"] & BP\langle n+1\rangle_k \arrow[d, "\gamma"] \\ & BP\langle n\rangle_k \end{tikzcd}$$

$i = k_{n+1} + 2(p^{n+1}-1)$ $k = k_{n+1}$

and the induced maps on the Postnikov systems: $q = k + s + 1$

4.7
$$\begin{tikzcd} (BP\langle n+1\rangle_i)^{k+s} \arrow[r, "\beta^{q-1}"] \arrow[d, "k'"] & (BP\langle n+1\rangle_k)^{k+s} \arrow[r, "\gamma^{q-1}"] \arrow[d, "k"] & (BP\langle n\rangle_k)^{k+s} \arrow[d, "k''"] \\ K(\pi_q(BP\langle n+1\rangle_i),q+1) \arrow[r, "\beta_\#"] & K(\pi_q(BP\langle n+1\rangle_k),q+1) \arrow[r, "\gamma\#"] & K(\pi_q(BP\langle n\rangle_k),q+1) \end{tikzcd}$$

$\beta_{\#}$ and $\gamma_{\#}$ give the split short exact sequence 1.2, 1.5. We know that the k-invariants in $H^{q+1}((BP\langle n\rangle_k)^{k+s}, Z_{(p)})$ are independent and hit p torsion generators for $s \leq m$ by statement $K(n,s)$, $s \leq m$ of A and comment 4.4; equivalently, $(\bar{\tau}'')_q$ is injective:

4.8

$$\begin{tikzcd}
H^q(K(\pi_q(BP\langle n\rangle_k),q)) \arrow[r, "(\bar{\tau}'')_q"] \arrow[d, "(\gamma_{\#})^*"'] & H^{q+1}((BP\langle n\rangle_k)^{k+s}) \arrow[d, "(\gamma^{q-1})^*"'] \\
H^q(K(\pi_q(BP\langle n+1\rangle_k),q)) \arrow[r, "(\bar{\tau})_q"] \arrow[d, "(\beta_{\#})^*"'] & H^{q+1}((BP\langle n+1\rangle_k)^{k+s}) \arrow[d, "(\beta^{q-1})^*"'] \\
H^q(K(\pi_q(BP\langle n+1\rangle_i),q)) \arrow[r, "(\bar{\tau}')_q"] & H^{q+1}((BP\langle n+1\rangle_i)^{k+s})
\end{tikzcd}$$

Assume for a moment that γ^{q-1} pulls these k-invariants in $H^{q+1}((BP\langle n\rangle_k)^{k+s}, Z_{(p)})$ back to independent p-torsion generating k-invariants in $H^{q+1}((BP\langle n+1\rangle_k)^{k+s}, Z_{(p)})$, i.e. $(\bar{\tau})_q \cdot (\gamma_{\#})^* = (\gamma^{q-1})^* \cdot (\bar{\tau}'')_q$ is injective in 4.8. Then the first possible dependent k-invariant is of the type discussed in 3.7. There, it was shown to be an independent p-torsion generating k-invariant. Assume for some minimum $s \leq m$ that we have a dependent k-invariant, or one which is not a p-torsion generator, equivalently, assume there is an $x \in \ker (\bar{\tau})_q$, therefore $q > i$. By what we have assumed about the k-invariants pulling back, x is not in the image of $(\gamma_{\#})^*$. Thus, by the split exactness of homotopy, and therefore the $(\gamma_{\#})^*$, $(\beta_{\#})^*$ sequence of 4.8, $(\beta_{\#})^*(x) = y \neq 0$. Now using the result 2.3 about the k-invariants of loop spaces, $(s^*)^r \cdot (\bar{\tau}')_q = (\bar{\tau})_{q-r} \cdot (s^*)^r$, $r=i-k=2(p^{n+1}-1)$. By our minimality assumption on s, $(\bar{\tau})_{q-r}$ is injective so $0 \neq (\bar{\tau})_{q-r}\cdot(s^*)^r(y) = (s^*)^r\cdot(\bar{\tau}')_q(y) =$ $(s^*)^r\cdot(\bar{\tau}')_q\cdot(\beta_\#)^*(x) = (s^*)^r\cdot(\beta^{q-1})^*\cdot(\bar{\tau})_q(x)$, contradicting $(\bar{\tau})_q(x) = 0$. #

All we need now is to show that $(\bar{\tau})_q\cdot(\gamma_\#)^* = (\gamma^{q-1})^*\cdot(\bar{\tau}'')_q$ is injective. We have the maps $(k+s+1=q)$

$$(BP_k)^{k+s} \xrightarrow{F^{q-1}} (BP\langle n+1\rangle_k)^{k+s} \xrightarrow{\gamma^{q-1}} (BP\langle n\rangle_k)^{k+s}$$

If we show that $(F^{q-1})^*\cdot(\gamma^{q-1})^*\cdot(\bar{\tau}'')_q$ is injective, we will be through. Using statement $P(n,s)$, $s \leq m$, from our given in A, we see that this is true if $G^*\cdot(\bar{\tau}'')_{k_n+s+1}$ is injective (as $k=k_{n+1} > k_n$), G the projection:

$$(BP\langle n\rangle_{k_n})^{k_n+s} \times \prod_{j>n} (BP\langle j\rangle_k)^{k_n+s} \longrightarrow (BP\langle n\rangle_{k_n})^{k_n+s}$$

This follows trivially from statement $K(n,s)$, $s \leq m$. #

Proof of Statement B By 1) of statement B and 2.4 on k-invariants of product spaces, all of the k-invariants on the right hand side of $P(n,m)$ are independent and hit p-torsion generators except possibly a zero k-invariant if $m=2p^j-3$ which corresponds by construction (3.6) to a dependent k-invariant on the left hand side of $P(n,m)$. Now by $P(n,m)$, $(f^{k_n+m})^*$ is an isomorphism and so pulls back all of the independent p-torsion generating k-invariants to independent p-torsion generating k-invariants in $(BP_{k_n})^{k_n+m}$. This determines all of the k-invariants on the left hand side because we know the homotopy is the same on both sides.(4.2) So by this, (and 3.6 if $m=2p^j-3$) $f_\#$ on π_{k_n+m+1} must be an isomorphism. Thus $(f^{k_n+m+1})_\#$ is an isomorphism on π_* giving us $P(n,m+1)$. #

Section 5 Statement of Results

In section 4 we proved the main theorem: $k \leq 2(p^n+\dots+p+1)$

$$BP_k \simeq BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^i-1)}$$

The main theorem of [20] says: The $Z_{(p)}$ (co)homology of the connected part of BP_k has no torsion and is a polynomial algebra for k even and an exterior algebra for k odd. The map above is a map of H-spaces for $k < 2(p^n+\dots+p+1)$ so we have the following corollary.

Corollary 5.1 For $k < 2(p^n+\dots+p+1)$, the $Z_{(p)}$ (co)homology of the connected part of $BP\langle n\rangle_k$ has no torsion and is a polynomial algebra for k even and an exterior algebra for k odd. For $k=2(p^n+\dots+p+1)$, $H^*(BP\langle n\rangle_k, Z_{(p)})$ has no torsion and is a polynomial algebra. (Note that for $k > 0$ or k odd < 0, $BP\langle n\rangle_k$ is connected.) #

Note For $k=2(p^n+\dots+p+1)$, $H_*(BP\langle n\rangle_k, Z_{(p)})$ is not a polynomial algebra.

At the rationals, the space $BP\langle n\rangle_k$ is just a product of Eilenberg-MacLane spaces. So, since there is no torsion, the number of generators over $Z_{(p)}$ is the same as over $\mathbb{Q}$. As an example, for k even, $0 < k \leq 2(p^n+\dots+p+1)$, we have

$$H^*(BP\langle n\rangle_k, Z_{(p)}) = Z_{(p)}[s^k\pi_*^S(BP\langle n\rangle)] = Z_{(p)}[s^k(Z_{(p)}[x_1,\dots,x_n])].$$

For k even and less than $2(p^n+\ldots+p+1)$, the (co)homology Hopf algebras of 5.1 are bipolynomial, that is, both it and its dual are polynomial algebras. Such Hopf algebras are studied in [13]. There, such a Hopf algebra is shown to be isomorphic to a tensor product of the Hopf algebras $B_{(p)}[x,2d]$ studied in [7]. $B_{(p)}[x,2d]$, as an algebra, is a polynomial algebra over $Z_{(p)}$ on generators $a_k(x)$ of degree $2p^k d$. As a Hopf algebra it is isomorphic to its own dual.

Letting $R(n,k)$ be the set of all n-tuples of non-negative integers, $R=(r_1,\ldots,r_n)$ with $d(R)=2k+\Sigma 2(p^i-1)r_i$. R is called prime if it cannot be written $R=pR'+(k,0,\ldots,0)$ with $R' \varepsilon R(n,k)$. Then, as a further example, we have the following corollary from [13] and the counting done above.

Corollary 5.2 For $0 < k < p^n+\ldots+p+1$ as Hopf algebras:

$$H^*(BP\langle n\rangle_{2k}, Z_{(p)}) \approx \bigotimes_{\substack{R\ \varepsilon\ R(n,k)\\ R\ \text{prime}}} B_{(p)}[x_R, d(R)] \qquad \#$$

We now utilize statement $K(n,s)$; all k-invariants $\tau(x)$ in $H^{k_n+s+2}((BP\langle n\rangle_{k_n})^{k_n+s}, Z_{(p)})$ are independent and hit p torsion generators, $k_n=2(p^{n-1}+\ldots+p+1)+1$. This implies that $BP\langle n\rangle_{k_n}$ cannot be written as a non-trivial product. (2.12)

Corollary 5.3 For $k > 2(p^{n-1}+\ldots+p+1)$, $BP\langle n\rangle_k$ is irreducible.#

Using the fact that $k_n+2(p^j-1) \geq k_j$ for $j > n$ we have now completed the proof of the main theorem.

Theorem 5.4 For $k \leq 2(p^n+\ldots+p+1)$

$$BP_k \simeq BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$$

and for $k > 2(p^{n-1}+\ldots+p+1)$, this decomposition is as irreducibles.#

Note For $k < 2(p^n+\ldots+p+1)$ this is as H-spaces.

Now letting $k \leq 2(p^{n-1}+\ldots+p+1)$ and using two versions of 5.4 we have

$$BP_k \simeq BP\langle n\rangle_k \times \text{OTHER} \qquad \text{and}$$

$$BP_k \simeq BP\langle n-1\rangle_k \times BP\langle n\rangle_{k+2(p^n-1)} \times \text{OTHER}.$$

From this we get the following corollary.

Corollary 5.5 For $k \leq 2(p^{n-1}+\ldots+p+1)$

$$BP\langle n\rangle_k \simeq BP\langle n-1\rangle_k \times BP\langle n\rangle_{k+2(p^n-1)} \qquad \#$$

Note For $k < 2(p^{n-1}+\ldots+p+1)$ this is as H-spaces.

This gives us the point where the fibration 1.4 becomes trivial. Again, using $BP_k \simeq BP\langle n\rangle_k \times \text{OTHER}$ for $k \leq 2(p^n+\ldots+p+1)$ and the fact that for finite complexes $BP\langle n\rangle^k(X)=0$ for high k we get 5.6.

Corollary 5.6 i) $BP^k(X) \to BP\langle n\rangle^k(X)$ is onto for $k \leq 2(p^n+\ldots+p+1)$.
ii) $BP^*(X) \to BP\langle n\rangle^*(X)$ is onto in all but a finite number of dimensions. #

We now apply 5.6 to prove Quillen's theorem. The problem was first studied in [6].

Theorem 5.7 (Quillen) Let X be a finite CW complex, then $BP^*(X)$ is generated as a $BP^*(S^0)$ module by elements of non-negative degree. #

Proof If $u \in BP^k(X)$ and $k < 0$, we will show u is a finite sum

$\sum_{i>0} x_i u_i = u$, $u_i \in BP^{k+2(p^i-1)}(X)$ and $x_i \in BP^*(S^o) = Z_{(p)}[x_1,\ldots,x_i,\ldots]$ of degree $-2(p^i-1)$. By downward induction on the degree of u we will be done.

Consider the maps

$$BP^*(X) \xrightarrow{g_n} BP\langle n\rangle^*(X) \xrightarrow{f_n} BP\langle n-1\rangle^*(X)$$

Find n such that $g_n(u) \neq 0$ but $f_n \circ g_n(u) = g_{n-1}(u) = 0$. Such an n exists because n=0 gives $g_o(u) \in H^k(X,Z_{(p)})=0$ as $k < 0$, and for n high enough $BP^k(X) \simeq BP\langle n\rangle^k(X)$, by the finiteness of X.

Dual to 1.1 we have an exact sequence and commuting diagram:

$$\begin{tikzcd}
BP^{k+2(p^n-1)}(X) \arrow[r, "x_n"] \arrow[d, "g_n"'] & BP^k(X) \arrow[d, "g_n"'] & \\
BP\langle n\rangle^{k+2(p^n-1)}(X) \arrow[r, "x_n"] & BP\langle n\rangle^k(X) \arrow[r, "f_n"] & BP\langle n-1\rangle^k(X) \\
u' \arrow[r] & g_n(u) \arrow[r] & 0
\end{tikzcd}$$

As $f_n(g_n(u))=0$ there exists u' with $x_n u' = g_n(u)$ by exactness. But now, by 5.6 and $2(p^n+\ldots+p+1) \geq k+2(p^n-1)$ for $k < 0$ we have that g_n is onto in dimension $k+2(p^n-1)$ and so pick u_n $BP^{k+2(p^n-1)}(X)$ with $g_n(u_n)=u'$. Then by commutativity, $g_n(x_n u_n) = g_n(u)$. Now continue this process using $u-x_n u_n$. By the finiteness of X, $BP^{k+2(p^j-1)}(X)$ will be zero for large j and we will get our finte sum $u = \sum_{i>0} x_i u_i$ and be done. #

The spaces $BP\langle n\rangle_k$ are most useful in the range $2(p^{n-1}+\ldots+p+1) < k \leq 2(p^n+\ldots+p+1)$ where they are both irreducible and torsion free. In the next section, we will identify these with spaces that have perhaps a more tangible description.

Section 6 Torsion free H-spaces

All modules will be over $Z_{(p)}$, and, until further notice, all coeffecients will be $Z_{(p)}$. In this section we will study torsion free H-spaces. Our immediate goal is to construct and study the following spaces.

Proposition 6.1 There exists an irreducible k-1 connected H-space Y_k which has $H^*(Y_k)$ and $\pi_*(Y_k)$ both free over $Z_{(p)}$ and such that each stage of the Postnikov system is irreducible.#

Proof We will build up a Postnikov system for Y_k and use 2.7. We drop the subscript k. Clearly we must start the Postnikov system with $Y^k = K(Z_{(p)},k)$. We will now just build up a Postnikov system by killing off the torsion in cohomology as effeciently as possible. $\pi_*(Y^k)$ is free over $Z_{(p)}$ and $H^j(Y^k)$ has no torsion for $j \leq k+1$. Y^k is an H-space. Assume we have constructed the s-1 stage, Y^{s-1} for $s > k$ such that $\pi_*(Y^{s-1})$ is free and $H^j(Y^{s-1})$ has no torsion for $j \leq s$. Assume also that Y^{s-1} has an H-space structure. $H^{s+1}(Y^{s-1}) \simeq F \oplus T$ where F is the free part and T is the torsion part. It is finitely generated so it is isomorphic to $(Z_{(p)})^{n_o} \underset{i>0}{\oplus} (Z_{p^i})^{n_i}$, where $(G)^n=G\oplus\ldots\oplus G$ n times. Using the torsion generators, this isomorphism determines a map:

$$Y^{s-1} \to \prod_{n=\sum_{i>0} n_i \text{ times}} K(Z_{(p)},s+1) = K(F_n, s+1), \quad F_n=(Z_{(p)})^n.$$

Let this map be the s k-invariant, k_s. This constructs the space Y^s as the induced fibration. k_s is torsion and so it is primitive because there is no torsion in lower dimensions, therefore by 2.5, Y^s is an H-space. Recall the $Z_{(p)}$ sequence 2.9

$$0 \to H^s(Y^{s-1}) \xrightarrow{(g_s)^*} H^s(Y^s) \to H^s(K(F_n,s)) \overset{\tau}{\to} H^{s+1}(Y^{s-1}) \to H^{s+1}(Y^s) \to 0$$

Using $k_s^* = \tau \cdot s^*$ we see that all of our "k-invariants" $\tau(x)$ are independent and hit p-torsion generators by construction. Coker $(g_s)^*$ is a subgroup of a free group and so is free giving us:

$$0 \to H^s(Y^{s-1}) \xrightarrow{(g_s)^*} H^s(Y^s) \to \text{coker } (g_s)^* \to 0$$

with both ends free by our induction hypothesis. Therefore $H^s(Y^s)$ is free. $H^{s+1}(Y^s)$ is coker τ = coker $(k_s)^*$ which by construction is F, so free. By the isomorphism 2.10, $H^j(Y^s) \approx H^j(Y^{s-1})$, $j < s$, we have $H^j(Y^s)$ is free for $j \leq s+1$. Also $\pi_*(Y^s)$ is free by construction. Because we have used the minimum number of $Z_{(p)}$'s for $\pi_s(Y^s)$, if Y^{s-1} is irreducible, then so is Y^s. (2.12) #

This would give us a space Y_k with the properties we specified, but we want more than this from Y_k.

Lemma 6.2 Y_k as in 6.1 is unique up to homotopy type. #

Proof In the proof of 6.1 if we choose a different isomorphism $H^{s+1}(Y^{s-1}) \approx (Z_{(p)})^{n_o} \underset{j>0}{\oplus} (Z_{p^i})^{n_i} = F \oplus T$. This would give us a different k-invariant and then possibly a different space Y_*^s. We would like to know that really $Y_*^s \approx Y^s$. Our result will follow.

Lemma 6.3 Let G be an isomorphism $G:T \approx T$. If we have surjections $g,f:F_n \to T$, then we can find an isomorphism $h:F_n \approx F_n$ such that the diagram commutes:

$$\begin{tikzcd} F_n \arrow[r, "g"] \arrow[d, "h"'] & T \arrow[d, "G"] \\ F_n \arrow[r, "f"] & T \end{tikzcd}$$ #

Proof Given generators x_i, $0 \leq i \leq n$ of F_n we can pick y_i $f^{-1}(G(g(x_i)))$ because f is onto. Map the generator x_i to y_i and extend to get our map h which commutes by our choice of y_i. If we tensor the diagram with Z_p, then g, G, f are all isomorphisms of vector spaces. This forces h to be an isomorphism mod p, but since F_n is free over $\mathbb{Z}_{(p)}$, h is also an isomorphism. #

Note If we tried to work over Z, the last step of the above proof would fail and we would not be able to prove the uniqueness of the sort we want.

We can now return to our proof that $Y_*{}^s \simeq Y^s$. Our choice of k-invariants k_s, $k'_s : Y^{s-1} \to K(F_n, s+1)$ really corresponds to a self isomorphism $G:H^{s+1}(Y^{s-1})$. As the k-invariants are torsion and G is an isomorphism when restricted to the torsion subgroup T, we can apply the lemma. Let $g=(k_s)^*$, $f=(k'_s)^*$. This gives us a self isomorphism $h^*;H^{s+1}(K(F_n,s+1))$ which can always be realized by a homotopy equivalence $h:K(F_n,s+1) \to K(F_n,s+1)$. This has the property that $h \cdot k'_s \simeq k_s$ so we have a map $Y_*{}^s \to Y^s$ over the identity $Y^{s-1} \to Y^{s-1}$, which is just $h_\#$ on π_s and therefore a homotopy equivalence. #

This completes our construction of the spaces Y_k. Unfortunately, this construction tells us nothing about $\pi_*(Y_k)$ and $H^*(Y_k)$ except that they are free. However, to remedy that, we have the following.

Proposition 6.4 For $2(p^{n-1}+\ldots+p+1)<k\leq 2(p^n+\ldots+p+1)$, $BP\langle n\rangle_k \simeq Y_k$.#

Proof $H^*(BP\langle n\rangle_k)$ has no torsion by 5.1. Statement $K(n,s)$ (section 4) just tells us that $BP\langle n\rangle_k$ is built up exactly as we constructed Y_k and so by 6.2 we are done. #

We know $\pi_*(BP\langle n\rangle_k)$ and $H^*(BP\langle n\rangle_k)$ so we now know the same for Y_k. We can now prove the main theorem of this section.

Theorem 6.5 If X is a simply connected CW H-space with $\pi_*(X)$ and $H^*(X, Z_{(p)})$ free and locally finitely generated over $Z_{(p)}$, then $X \simeq \prod_i Y_{k_i}$. #

Remark 1 The simply connected assumption is not necessary because one can just split off a bunch of circles localized at p. $Y_1 = (S^1)_{(p)}$. Then, what is left is still an H-space, see the next remark.

Remark 2 The reason for the H-space hypothesis is that we want torsion k-invariants (2.5). Since spaces with π_* and H^* free are H-spaces if their k-invariants are torsion we could have used the hypothesis that X must have torsion k-invariants instead. Note that our homotopy equivalence is not as H-spaces.

Remark 3 There are really two ways we could have approached this theorem. We did not need to construct the spaces Y_k as we did, but actually use the spaces $BP\langle n\rangle_k$ and what we know about them. The proof is the same and nowhere would we need our knowledge of $\pi_*(BP\langle n\rangle_k)$. Then, using the theorem and our knowledge of $BP\langle n\rangle_k$ we could imply the existence of a unique Y_k of the type we constructed. However, we could just do what we have done, construct the Y_k and prove the theorem in ignorance of $\pi_*(Y_k)$, that is, forgetting 6.4. Then, afterward, we can note that $BP\langle n\rangle_k$ (for the appropriate n) satisfies the criterion of the theorem and is irreducible, thus getting $BP\langle n\rangle_k \simeq Y_k$. This is the prettiest way and gives a better understanding of the space $BP\langle n\rangle_k$, or Y_k, depending on how one looks at it.

Remark 4 The ideal space to apply the theorem to would be BP_k but alas, it was necessary to prove this special case before we could make the identification $Y_k \simeq BP\langle n\rangle_k$. $BP\langle 1\rangle$ corresponds to the spectrum constructed in [10] and so we do get the result of Sullivan and Peterson:

$$BU_{(p)} \simeq \prod_{i=0}^{p-2} \Omega^{2i} Y_{2(p-1)} = \prod_{i=1}^{p} Y_{2i}$$

Proof of 6.5 As always, we do everything by induction on the Postnikov system, but first we need the map $X \to \Pi_i Y_{k_i}$. The construction is similar to that for the main theorem except easier because X is only a theoretical space. We revert back to mod p cohomology for the proof. We start with the mod p version of the sequence 2.9.

6.6
$$0 \to H^s(X^{s-1}) \xrightarrow{g^*} H^s(X^s) \xrightarrow{i^*} H^s(K(\pi_s(X),s)) \xrightarrow{\bar{\tau}} H^{s+1}(X^{s-1}) \to H^{s+1}(X^s) \to 0$$

Choose $V^s \subset H^s(X^s) = H^s(X)$ such that $i^*: V^s \to \ker \bar{\tau}$. Let r_s be the rank of V^s. This determines a map $f'_s: X \to K((Z_p)^{r_s},s)$. $H^*(X,Z_{(p)})$ has no p torsion so f'_s lifts first into the product of r_s copies of $K(Z_{(p)},s)$ and then into the product of r_s copies of Y_s, denoted rY_s. (It lifts by 2.6 because Y_s has p-torsion k-invariants and free homotopy.) So we have $f_s: X \to rY_s$ such that the image of $(f_s)^*$ in dimension s is V^s. Let $f = \prod_s f_s : X \to \prod_s rY_s = Y$.

Claim: f is a homotopy equivalence. #

Proof By induction on the Postnikov system assume $f^s : X^s \to Y^s = \prod_{k \leq s} (rY_k)^s$ is a homotopy equivalence. $(X^1 = Y^1 = pt)$

Let $f_\#$ be the induced map $K(\pi_{s+1}(X), s+1) \to K(\pi_{s+1}(Y), s+1)$. f^s is a homotopy equivalence, if $f_\#$ is too, then $f^{s+1}{}_\# : \pi_*(X^{s+1}) \to \pi_*(Y^{s+1})$ is an isomorphism and so f^{s+1} is a homotopy equivalence. $f_\#$ is a homotopy equivalence iff

$(f_\#)^* : H^{s+1}(K(\pi_{s+1}(Y), s+1)) \to H^{s+1}(K(\pi_{s+1}(X), s+1))$ is an

isomorphism. Now

$$K(\pi_{s+1}(Y), s+1) = K(\pi_{s+1}(\prod_{k \leq s} rY_k), s+1) \times K((Z_{(p)})^{r_{s+1}}, s+1) = K \times K'$$

and $H^{s+1}(K \times K') = H^{s+1}(K) \oplus H^{s+1}(K')$. Ker $\bar{\tau}_Y = H^{s+1}(K')$ by the construction of the Y_k, i.e., all k-invariants are independent and hit p-torsion generators. Using the naturality of 2.9 we have:

6.7

$$\begin{tikzcd}
{} \arrow[r] & H^{s+1}(X^{s+1}) \arrow[r, "i_X^*"] & H^{s+1}(K(\pi_{s+1}(X), s+1)) \arrow[r, "\bar{\tau}"] & H^{s+2}(X^s) \arrow[r] & {} \\
{} \arrow[r] & H^{s+1}(Y^{s+1}) \arrow[u, "f^* = (f^{s+1})^*"'] \arrow[r, "i_Y^*"] & H^{s+1}(K(\pi_{s+1}(Y), s+1)) \arrow[u, "(f_\#)^*"] \arrow[r, "\bar{\tau}"] & H^{s+2}(Y^s) \arrow[u, "(f^s)^*"'] \arrow[r] & {} \\
& H^{s+1}(\prod_{k \leq s} rY_s) \oplus H^{s+1}(rY_{s+1}) \arrow[u, "\approx"] \arrow[r] & H^{s+1}(K) \oplus H^{s+1}(K') \arrow[u, "\approx"] \arrow[r] & {}
\end{tikzcd}$$

Now $(i_Y)^*:H^{s+1}(rY_{s+1}) \longrightarrow \ker \bar{\tau}_Y = H^{s+1}(K')$ and by construction of f_{s+1}, $f^*=(f^{s+1})^*:H^{s+1}(rY_{s+1}) \longrightarrow V^{s+1}$. By commutativity, $(f_\#)^*:\ker \bar{\tau}_Y \to \ker \bar{\tau}_X$. $\bar{\tau}_Y | H^{s+1}(K)$ is injective and by our construction of the Y_k it hits every possible element in cohomology, i.e., all that reduce from torsion elements in $Z_{(p)}$ cohomology. f^s is a homotopy equivalence by induction so by commutativity of 6.7 $\bar{\tau}_X$ also hits all possible elements and we have isomorphism on the ends of diagram 6.8 giving us the desired isomorphism by the five lemma.

6.8
$$\begin{tikzcd}
0 \arrow[r] & V^{s+1} \arrow[r] & H^{s+1}(K(\pi_{s+1}(X),s+1)) \arrow[r] & \text{image } \bar{\tau}_X \arrow[r] & 0 \\
0 \arrow[r] & H^{s+1}(rY_{s+1}) \arrow[r] \arrow[u, "\cong"'] & H^{s+1}(K') \oplus H^{s+1}(K) \arrow[r] \arrow[u, "(f_\#)^*"] & \text{image } \bar{\tau}_Y \arrow[r] \arrow[u, "(f^s)^*"] & 0
\end{tikzcd}$$

#

Using similar techniques to those above we obtain the next lemmas.

Lemma 6.9 Any map $Y_k \to Y_k$ which induces an isomorphism on $\pi_k(Y_k) = Z_{(p)}$ is a homotopy equivalence. #

Lemma 6.10 For $k > 2(p^{n-1}+\ldots+p+1)$, any map $BP\langle n\rangle_k \to BP\langle n\rangle_k$ which induces an isomorphism on π_k is a homotopy equivalence.#

A map $f:X \to Y$ of simply connected CW complexes is said to be a mod p homotopy equivalence if $f^* : H^*(Y) \longrightarrow H^*(X)$ is an isomorphism, the coeffecients being Z_p.

Mod p homotopy type is the equivalence relation given by this, written $X \underset{p}{\simeq} Y$. If $\pi_*(X) \otimes Z_{(p)}$ is locally finitely generated over $Z_{(p)}$ then $X \underset{p}{\simeq} Y$ is equivalent to $X_{(p)} \simeq Y_{(p)}$.

$X \underset{p}{\sim} X_{(p)}$. This gives us the following corollary.

Corollary 6.11 Let X be a CW complex such that $X_{(p)}$ satisfies the condition of 6.2, then

$$X \underset{p}{\sim} \Pi_i \, Y_{k_i} \qquad \#$$

Note that $Y_\infty \simeq BP\langle\infty\rangle \simeq BP$ and we get an unpublished result of Peterson.

Corollary 6.12 (Peterson) Given a spectrum X with no p-torsion in either $H^*(X, Z_{(p)})$ or $\pi_*^S(X)$, with $\pi_*^S(X) \otimes Z_{(p)}$ locally finitely generated and zero below some dimension, then

$$X \underset{p}{\sim} \bigvee_i S^{k_i} BP \qquad \#$$

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BIOGRAPHICAL NOTE

W. Stephen Wilson were born on November 11, 1946, which used to be Armistice Day so they always let school out then. Later it became Veteran's Day, which doesn't have quite the same ring to it. More recently Veteran's Day has been relocated to convenient mondays, so they don't always let school out on the right day anymore.

Completely unaware of the future of our birthday anniversary, (see if people remember V-E day after WW III) we were raised in total ignorance of everything in far western Kansas. (St. Francis)(the most north-westernest town in the state)

We graduated from high school in 1964 and came to M.I.T. where we have been since then. In 1968 we finished the requirements for an S.B. in mathematics and spent the next 15 months on an N.S.F. Graduate Fellowship, receiving our S.M. in mathematics in September, 1969. Because our country needed us, we served as a full-time teaching assistant during the next year and a half-time teaching assistant the year after that. This last year we have been an N.S.F. Graduate Fellow again.

Although we have built up considerable seniority in the last eight years, our petition to become a tenured student was rejected. Students beware, this could happen to you.

No biographical note for us would be complete without mentioning Jessie and Marie, but there is so much to say.