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AUXILIARY FIELD APPROACH TO EFFECTIVE POTENTIALS

By

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Abstract

A non-perturbative effective potential of $\lambda\phi^4$ theory at finite temperature was obtained and renormalized by using the Hubbard-Stratonovich transformation (or auxiliary field approach). This effective potential exhibits renormalization group invariance and agrees with the perturbative one-loop effective potential at the tree level. It can also recover zero-temperature non-perturbative effective potentials obtained from the Functional Schroedinger Picture approach and the Gaussian Effective Potential approach. A formal expression of the contribution to the partition function of $\lambda\phi^4$ theory beyond leading order saddle point contribution was also obtained. This effective potential was shown to be able to drive inflation in the new inflationary scenario of the early universe. We also calculated the effective potential of the two component $\lambda\phi^4$ theory and studied its relevance to the relativistic Bose-Einstein condensation. The same method was also applied to the O(N) invariant Gross-Neveu model and the renormalization of the non-perturbative effective potential was accomplished. Attempts were also made to apply the same method to Scalar QED.

Thesis Supervisor: Arthur Kerman Professor of Physics.

To My Family:

My Father, My Mother

and My Two Sisters

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Table of Contents

Chapter I

Introduction

The effective potential for a field theory is a very useful tool in the studies of spontaneous symmetry breaking since it incorporates quantum effects into the classical potential. These quantum effects may change the vacuum structure of the classical theory, as was first pointed out by Jona-Lasinio ([1]) and later by other authors ([2]). The effective potential is traditionally calculated by summing infinite series of Feynman graphs at zero momentum ([2]), or by the perturbative, loop-expansion technique. This approach is mainly confined to the one-loop calculation since higher-loop calculations are extremely difficult to accomplish, although several authors have successfully calculated two-loop or higher order effective potentials. However, when the relevant renormalized coupling constant in a theory becomes very large, this perturbative approach becomes questionable although it is essentially a perturbative expansion in terms of the product of Planck's constant and the renormalized coupling constant. Because of the widespread application of spontaneous symmetry breaking, especially for its application to the finite temperature phase transition, much effort has been put into nonperturbative ways of calculating effective potential. Among them are the Gaussian Effective Potential approach ([3], [4] and [5]) and Functional Schroedinger Picture approach ([6]).

The Gaussian Effective Potential was first introduced by Stevenson ([3]). It is based on the intuitive ideas familiar in quantum mechanics. It is conceptually more straightforward than the standard generating functional approach. It is a variational approach which uses a normalized Gaussian wave functional as the trial ground state. This trial ground state is centered around an arbitrary constant classical background field $\bar{\phi}$. The trial wave functional also contains a variable mass parameter. Actually this trial wave functional is the free field vacuum with a variable mass parameter. The effective potential, for a given background field $\bar{\phi}$, is obtained by minimizing the energy of the system with respect to the variable mass parameter, where the energy is the expectation value of the Hamiltonian in the trial ground state. In evaluating this expectation value, the Hamiltonian is rewritten in terms of the usual annihilation and creation operators which obey the usual commutation relations (for bosonic case). Quantum fluctuations are incorporated into the effective potential due to the fact that these minima normally differ from their classical counterparts. With a crude approximation, finite temperature effective potential can also be obtained in this formalism by minimizing the free energy of the system rather than energy ([5]).

The idea of the Functional Schroedinger Picture approach is essentially the same as that of the Gaussian Effective Potential. The main difference is that the Functional Schroedinger Picture approach evaluates the expectation value of the Hamiltonian with the functional integral method by replacing the operators of the Hamiltonian by functional differentiations. The main advantage of this approach over the Gaussian Effective Potential approach is that it is possible to extend it to the nonconstant background field case. However, it is unclear how to extend it to the finite temperature case. As a comparison, although it is possible to obtain Gaussian Effective Potential at finite temperature, the effective potential is evaluated with the real-time operators which obey the usual commutation relations. Today, this cumbersome method is replaced to a large extent by the finite temperature field theory formulated covariantly in terms of the Feynman functional path integrals. It is in light of this that we want to obtain a nonperturbative approach to the effective potential which can be extended to finite temperature within the functional path integral formalism.

In next chapter we will illustrate the main idea of our auxiliary field approach by calculating the free energy of $\lambda\phi^4$ theory at finite temperature. Keeping only the leading order saddle point contribution to the partition function, we carried out the renormalization of the free energy in a way essentially the same as the method used in the variational approach in Functional Schroedinger Picture. But we extended previous results in that temperature is easily incorporated into the formalism.

We will focus on the second order contribution to the free energy in Chapter III. A formal expression for the contribution to the partition function of $\lambda\phi^4$ theory beyond leading order saddle point contribution was obtained and its renormalizability was discussed. We will derive the finite temperature effective potential of $\lambda\phi^4$ theory in Chapter IV. This is just an extension of Chapter II to the case of nonzero background field. The potential implication of this effective potential to inflation in the new inflationary scenario will be studied in Chapter V. In Chapter VI we will calculate the effective potential of the two component $\lambda\phi^4$ theory and study relativistic Bose-Einstein condensation. The same method will also be applied to Gross-Neveu model and renormalization of the effective potential will be accomplished in Chapter VII. Finally, we made attempts to extend our auxiliary field approach to scalar QED in Chapter VIII.

Chapter II

$\lambda\phi^4$ Theory at Finite Temperature

II.1 Free Energy

We begin our investigation with the $\lambda\phi^4$ theory described by the following Lagrangian in Euclidean space:

$$\begin{aligned}\mathcal{L}_E(\phi) &= \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) \\ &= -\frac{1}{2}(\frac{\partial\phi}{\partial\tau})^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\end{aligned} \tag{2.1}$$

where

$$V(\phi) = \frac{m^2}{2}\phi^2 + \lambda\phi^4 \tag{2.2}$$

The partition function of a system of bosons interacting through this Lagrangian at temperature $T = 1/\beta$ is

$$\begin{aligned}Z &= N' \int_{peri} D[\phi] exp\left(\int_0^\beta d\tau \int d^3x \mathcal{L}_E\right) \\ &= N' \int_{peri} D[\phi] exp\left\{-\int d^4x \left[\frac{1}{2}(\frac{\partial\phi}{\partial\tau})^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \lambda\phi^4\right]\right\}\end{aligned} \tag{2.3}$$

where $\int d^4x \equiv \int_0^\beta d\tau \int d^3x$ and

$$N' = \exp\left(-V\int \frac{d^3p}{(2\pi)^3}\sum_n \ln\beta\right) \tag{2.4}$$

which is a dimensional, temperature dependent constant ([7]). Here V is the volume to which the system is confined.

The only troublesome term in the above Lagrangian is the $\lambda\phi^4$ term, which prevents us from doing a gaussian functional integral. In order to overcome this difficulty, we consider the two cases: **(a)** $\lambda < 0$ and **(b)** $\lambda > 0$ separately.

(a) $\lambda < 0$

In this case, the functional integral (2.3) is ill defined since it is manifestly divergent. In addition, since classical potential is unbounded from below, intuition suggests that the theory is unstable. Since we will encounter this problem again when we derive the effective potential, we will defer our discussions on these questions to Chapter IV.

We can remove the $\lambda\phi^4$ term by making use of the following identity:

$$(\det A)^{-\frac{1}{2}} \int \frac{dx_1 \cdots dx_n}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}x_i A_{ij}^{-1} x_j + x_i J_i} = e^{\frac{1}{2} J_i A_{ij} J_j} \tag{2.5}$$

If we regard $-2\lambda\delta^4(x-x')$ as A_{ij} and $\phi^2(x)$ as J_i, then the $\lambda\phi^4$ term can be eliminated at the sacrifice of introducing an extra functional integral of the auxiliary field $\sigma(x)$.

$$Z = N \int_{peri} D[\phi] D[\sigma] exp\left\{ \int d^4x \left[-\phi(x) \left(-\frac{1}{2}\frac{\partial^2}{\partial\tau^2} - \frac{1}{2}\nabla^2 + \frac{1}{2}m^2 - \sigma(x) \right) \phi(x) + \frac{1}{4\lambda}\sigma^2(x) \right] \right\} \tag{2.6}$$

with $N \equiv N'(\det A)^{-\frac{1}{2}}$. Note $(\det A)^{-\frac{1}{2}}$ is an infinite, temperature independent constant. It merely represents an infinite multiplicity in each energy state in the partition function. This multiplicity represents a certain underlying degree of freedom of the system which has no effect on the physics of the system. Indeed, a constant multiplied with the partition function has no effect on the average energy of the system. Therefore, we can drop the $(\det A)^{-\frac{1}{2}}$ factor in front of the partition function.

Upon doing the gaussian functional integral of $\phi(x)$, we obtain

$$\begin{aligned} Z &= N' \int_{peri} D[\sigma] e^{-\frac{1}{2} tr \ln \left[(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2 - 2\sigma(x))\delta(x-x') \right] + \int d^4x \frac{1}{4\lambda}\sigma^2(x)} \\ &= N' \int_{peri} D[\sigma] e^{-\frac{1}{2} tr \ln [\hat{p}_0^2 + \hat{p}^2 + m^2 - 2\hat{\sigma}] + \int d^4x \frac{1}{4\lambda}\sigma^2(x)} \end{aligned} \tag{2.7}$$

In the last line above we switch to the operator notation which will facilitate our derivations later. Apparently, the above functional integral in $\sigma(x)$ is convergent since $\lambda < 0$. So far we have transformed the original partition function from a functional integral of the $\phi(x)$ field into that of a new auxiliary field $\sigma(x)$. This new functional integral is to be evaluated with the method of saddle point integration by expanding $\sigma(x)$ around $\hat{\sigma}_0$, where the exponent attains its global maximum.

$$
\begin{aligned}
Z = & N' e^{-\frac{1}{2} tr\ln[\hat{p}_0^2+\hat{p}^2+m^2-2\hat{\sigma}_0]+\int d^4x \frac{1}{4\lambda}\sigma_0^2(x)} \\
& \times \int_{peri} D[\delta\sigma] e^{tr\left[\frac{\delta\hat{\sigma}}{\hat{p}_0^2+\hat{p}^2+m^2-2\hat{\sigma}_0}\right]+\frac{1}{2\lambda}\int d^4x \sigma_0(x)\delta\sigma(x)} \\
& \times e^{tr\left[\frac{1}{\hat{p}_0^2+\hat{p}^2+m^2-2\hat{\sigma}_0}\delta\hat{\sigma}\frac{1}{\hat{p}_0^2+\hat{p}^2+m^2-2\hat{\sigma}_0}\delta\hat{\sigma}\right]+\frac{1}{4\lambda}\int d^4x \delta\sigma^2(x)}
\end{aligned}
\tag{2.8}
$$

By definition, the terms linear in $\delta\sigma(x)$ have to vanish. We want to drop terms quadratic in $\delta\sigma(x)$ for now, which we will deal with later. Thus, we are only left with the zeroth order term in $\delta\sigma$. Suppose the saddle point for $\sigma(x)$ field is translational invariant, then we can replace $\sigma_0(x)$ by a constant σ_0. Under this assumption, the operator $\hat{p}_0^2+\hat{p}^2+m^2-2\hat{\sigma}_0$ is diagonal in the four momentum space.

$$
\begin{aligned}
\ln Z = \ln Z_0 &= -\frac{1}{2} tr\ln[\hat{p}_0^2+\hat{p}^2+m^2-2\hat{\sigma}_0] + \int d^4x \frac{1}{4\lambda}\sigma_0^2 + \ln N' \\
&= -\frac{1}{2}\int d^4x \int d^4p < x|p > \ln[p_0^2+p^2+m^2-2\sigma_0] < p|x > + \int d^4x \frac{1}{4\lambda}\sigma_0^2 + \ln N'
\end{aligned}
$$

Inserting $< x|p >= \frac{1}{\sqrt{\beta(2\pi)^3}} e^{i\mathbf{p}\cdot\mathbf{x}+i\omega_n\tau}$ and $\int d^4x = \beta V$ into the above equation, where matsubara frequency $\omega_n = \frac{2\pi n}{\beta}$ and V is the volume to which the system of bosons is confined, we get

$$
-\frac{1}{2}V\int \frac{d^3p}{(2\pi)^3}\sum_n \ln[(2\pi n)^2+\beta^2\omega^2] + V\int \frac{d^3p}{(2\pi)^3}\sum_n \ln\beta + \frac{1}{4\lambda}\beta V\sigma_0^2 \tag{2.9}
$$

where the second term is a infinite temperature dependent constant, which cancels the $\ln N'$ term (see [7] and equation (2.4)). Here we have defined an effective mass $\bar{m}$ and its on-shell energy ω by,

$$
\omega \equiv \sqrt{\mathbf{p}^2+\bar{m}^2} \equiv \sqrt{\mathbf{p}^2+m^2-2\sigma_0} \tag{2.10}
$$

Notice that

$$\ln[(2\pi n)^2 + \beta^2\omega^2] = \int_1^{\beta^2\omega^2} \frac{d\theta^2}{\theta^2 + (2\pi n)^2} + \ln[1 + (2\pi n)^2] \tag{2.11}$$

where last term can be ignored since it is temperature independent. Furthermore,

$$\sum_{-\infty}^{\infty} \frac{1}{n^2 + (\theta/2\pi)^2} = \frac{2\pi^2}{\theta}\left(1 + \frac{2}{e^\theta - 1}\right) \tag{2.12}$$

With the help of these two identities we get the following expression for the partition function,

$$Z = e^{V\int \frac{d^3p}{(2\pi)^3}[-\frac{1}{2}\beta\omega - \ln(1-e^{-\beta\omega})] + \frac{1}{4\lambda}\beta V\sigma_0^2} \tag{2.13}$$

The free energy density of the system easily follows from the partition function,

$$\begin{aligned} f &\equiv -\frac{1}{\beta V}\ln Z \\ &= \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2}\omega + \frac{1}{\beta}\ln(1 - e^{-\beta\omega})\right] - \frac{1}{4\lambda}\sigma_0^2 \end{aligned} \tag{2.14}$$

The two terms in square bracket above have exactly the same expression as the free energy density of the ideal boson gas. The first term, $\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\omega$, is the zero-point energy. In the case of ideal boson gas, this term can be dropped since it is just a pure infinite constant. However, in our case, the zero-point energy depends on an effective mass $\bar{m}$, which is to be determined from the first order terms in $\delta\sigma$ in equation (2.8). Later we will see that $\bar{m}^2$ depends on temperature, which means we are not allowed to drop this zero-point energy term. Consequently, we need to deal with the infinities associated with it. This problem could be solved since the coupling constant and the mass which appear in the Lagrangian are bare. As in perturbation approach, these bare parameters are momentum- cutoff dependent. Here we make an assumption that the σ_0 also depends on the cutoff and it approaches infinity in such a way that it cancels the infinities in bare mass m^2, leaving $\bar{m}^2$ finite. It can be shown later that $\bar{m}^2$ is indeed finite. The last term in the above expression consists of bare coupling constant and σ_0, which are both infinite, we hope this term will cancel the infinities from the zero-point energy. Note the second term in the square bracket is finite, which we call function $\mathcal{F}(\beta, \bar{m}^2)$ for convenience.

$$\mathcal{F}(\beta, \bar{m}^2) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\beta} \ln\left(1 - e^{-\beta\sqrt{\mathbf{p}^2+\bar{m}^2}}\right) \tag{2.15}$$

Before we embark on the renormalization of the free energy density, we come back to the first order terms in $\delta\sigma$. The collection of the first order terms are,

$$\begin{aligned}
& tr\left[\frac{\delta\hat{\sigma}}{\hat{p}_0^2 + \hat{p}^2 + m^2 - 2\hat{\sigma}_0}\right] + \frac{1}{2\lambda}\int d^4x \sigma_0(x)\delta\sigma(x) \\
&= \int d^4x \left[< x|\frac{1}{\hat{p}_0^2 + \hat{p}^2 + m^2 - 2\hat{\sigma}_0}|x> + \frac{1}{2\lambda}\sigma_0(x)\right]\delta\sigma(x) \\
&= \int d^4x \left[\frac{1}{\beta}\int \frac{d^3p}{(2\pi)^3}\sum_n \frac{1}{n^2(2\pi/\beta)^2 + p^2 + m^2 - 2\sigma_0} + \frac{1}{2\lambda}\sigma_0(x)\right]\delta\sigma(x)
\end{aligned}$$

By definition, the sum in square bracket should be zero. After doing the sum in n, which is the same as in equation (2.12), we get,

$$\sigma_0 = \int \frac{d^3p}{(2\pi)^3} \frac{-2\lambda}{\omega}\left[\frac{1}{2} + \frac{1}{e^{\beta\omega} - 1}\right] \tag{2.16}$$

The second term above is a finite integral. Here we define a finite function $f(\beta, \bar{m}^2)$ for later convenience,

$$f(\beta, \bar{m}^2) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{-2}{\omega} \frac{1}{e^{\beta\omega} - 1} \tag{2.17}$$

Thus we have obtained the free energy density equation (2.14), with σ_0 being determined by equation (2.16).

(**b**) $\lambda > 0$

In this case in order to obtain a functional integral of $\sigma(x)$ which is manifestly convergent, the equation (2.6) has to be replaced by

$$Z = N \int_{peri} D[\phi]D[\sigma]exp\left\{ \int d^4x \left[-\phi(x) \left(-\frac{1}{2}\frac{\partial^2}{\partial\tau^2} - \frac{1}{2}\nabla^2 + \frac{1}{2}m^2 - i\sigma(x) \right) \phi(x) - \frac{1}{4\lambda}\sigma^2(x) \right] \right\} \tag{2.18}$$

with $N \equiv N'(detA)^{-\frac{1}{2}}$ and $2\lambda\delta^4(x-x')$ as A_{ij}.

Upon doing the gaussian functional integral of $\phi(x)$ and expanding $\sigma(x)$ around the saddle point $\sigma_0(x)$, we obtain,

$$\begin{aligned} Z = & N' e^{-\frac{1}{2}tr\ln[\hat{p}_0^2+\hat{p}^2+m^2-2i\hat{\sigma}_0] - \int d^4x \frac{1}{4\lambda}\sigma_0^2(x)} \\ & \times \int_{peri} D[\delta\sigma] e^{tr\left[\frac{i\delta\hat{\sigma}}{\hat{p}_0^2+\hat{p}^2+m^2-2i\hat{\sigma}_0}\right] - \frac{1}{2\lambda}\int d^4x \sigma_0(x)\delta\sigma(x)} \\ & \times e^{-tr\left[\frac{1}{\hat{p}_0^2+\hat{p}^2+m^2-2i\hat{\sigma}_0}\delta\hat{\sigma}\frac{1}{\hat{p}_0^2+\hat{p}^2+m^2-2i\hat{\sigma}_0}\delta\hat{\sigma}\right] - \frac{1}{4\lambda}\int d^4x \delta\sigma^2(x)} \end{aligned} \tag{2.19}$$

Looking at the linear term, we can realize that the saddle point σ_0 should be purely imaginary. If we set $\sigma_0 = -i\sigma_0'$, then in terms of σ_0', we can recover all our previous expressions in the case of $\lambda < 0$ for the zeroth order and first order terms in $\delta\sigma$. Since later it will be shown that renormalization is entirely determined by the zeroth order and first order terms, we conclude that renormalization is unaffected by the sign change of λ.

The reason why $\sigma(x)$ can become purely imaginary is that the exponent of equation (2.18) is diagonal in σ. Therefore, the functional integral can be considered as a infinite product of ordinary integrals in each $\sigma(x)$. Each of these integrals is an analytic function of $\sigma(x)$, therefore, we can deform the contour of the integration. To be specific, we can arbitrarily deform the integration contour when $Re\sigma$ is finite. When $Re\sigma \to \pm\infty$, $Im\sigma$ has to be zero in order to ensure the convergence of the functional integral (2.18).

II.2 Renormalization

Now we embark on the task of renormalization. Since the sign change of λ has no effects on renormalization, we consider the case $\lambda < 0$ only. To show clearly how renormalization is accomplished, I purposely choose to be pedestrian in some parts of the derivations. we begin with the first order equation (2.16). The first term in that equation is divergent, we want to single out the divergent terms explicitly in order to perform renormalization,

$$
\begin{aligned}
&\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega} \\
&= \frac{4\pi}{(2\pi)^3}\int_0^{\Lambda} p^2 dp \frac{1}{\sqrt{p^2+\bar{m}^2}} \\
&= \frac{4\pi}{(2\pi)^3}\left\{\frac{1}{2}p\sqrt{p^2+\bar{m}^2} - \frac{1}{2}\bar{m}^2 \ln(p+\sqrt{p^2+\bar{m}^2})\right\}\Big|_0^{\Lambda}
\end{aligned}
$$

We insert the two integration limits into the above expression and expand it into powers of $\bar{m}/\Lambda$, we keep explicitly only the finite and divergent terms, then we obtain,

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega} = \frac{1}{4\pi^2}\left\{\Lambda^2 + \frac{1}{2}\bar{m}^2 \ln\frac{\bar{m}^2}{\alpha\Lambda^2}\right\} + O\left(\frac{1}{\Lambda^2}\right) \tag{2.20}$$

with $\ln\alpha \equiv 2\ln 2 - 1$. Consequently, we can rewrite the first order equation (2.16) as,

$$m^2 - 2\sigma_0 = \frac{\lambda}{2\pi^2}\left\{\Lambda^2 + \frac{1}{2}\bar{m}^2 \ln\frac{\bar{m}^2}{\alpha\Lambda^2}\right\} + m^2 - 2\lambda f(\beta, \bar{m}^2)$$

or

$$\bar{m}^2 = m^2 + \frac{\lambda}{2\pi^2}\Lambda^2 + \frac{\lambda}{4\pi^2}\bar{m}^2 \ln\frac{\bar{m}^2}{\Sigma^2} + \frac{\lambda}{4\pi^2}\bar{m}^2 \ln\frac{\Sigma^2}{\alpha\Lambda^2} - 2\lambda f(\beta, \bar{m}^2) \tag{2.21}$$

Where we have split the logarithm ic term by introducing an arbitrary mass scale Σ^2, this splitting is the key in our scheme of renormalization. If we collect linear terms of $\bar{m}^2$ into the left hand side of the equation, then we have

$$\bar{m}^2\left[1 - \frac{\lambda}{4\pi^2}\ln\frac{\Sigma^2}{\alpha\Lambda^2}\right] = m^2 + \frac{\lambda}{2\pi^2}\Lambda^2 + \frac{\lambda}{4\pi^2}\bar{m}^2 \ln\frac{\bar{m}^2}{\Sigma^2} - 2\lambda f(\beta, \bar{m}^2)$$

From this equation we can see if we define the following renormalized mass and coupling constant,

$$\begin{aligned} \lambda_R(\Sigma^2) &\equiv \frac{\lambda/(2\pi^2)}{1-\frac{\lambda}{4\pi^2}\ln\frac{\Sigma^2}{\alpha\Lambda^2}} \\ \mu^2(\Sigma^2) &\equiv \frac{m^2+\lambda\Lambda^2/(2\pi^2)}{1-\frac{\lambda}{4\pi^2}\ln\frac{\Sigma^2}{\alpha\Lambda^2}} \end{aligned} \tag{2.22}$$

then we can obtain the following renormalized version of the first order equation since every variable in this equation: $\mu^2(\Sigma^2)$, $\lambda_R(\Sigma^2)$ and Σ^2, is finite.

$$\bar{m}^2 = \mu^2(\Sigma^2) + \frac{1}{2}\lambda_R(\Sigma^2)\bar{m}^2\ln\frac{\bar{m}^2}{\Sigma^2} - 4\pi^2\lambda_R(\Sigma^2)f(\beta,\bar{m}^2) \tag{2.23}$$

Finite solution(s) to $\bar{m}^2$ can be obtained by solving this equation if they exist. We will call this equation the constraint equation later.

In the above definition of renormalized mass and coupling constant, the bare mass and coupling constant depend on Λ in such a way that they make $\mu^2(\Sigma^2)$ and $\lambda_R(\Sigma^2)$ finite. The relationship between the renormalized mass and coupling constant with the bare ones are temperature independent, which is expected for a renormalizable field theory. Note the renormalized mass and coupling constant run with Σ^2. When splitting the logarithm term in equation (2.21), we could as well have chosen $\Sigma^2 = \mu^2$, then everywhere Σ^2 will be replaced by μ^2 and the equations defining the renormalized mass and coupling constant will be,

$$\begin{aligned} \lambda_R &\equiv \frac{\lambda/(2\pi^2)}{1-\frac{\lambda}{4\pi^2}\ln\frac{\mu^2}{\alpha\Lambda^2}} \\ \mu^2 &\equiv \frac{m^2+\lambda\Lambda^2/(2\pi^2)}{1-\frac{\lambda}{4\pi^2}\ln\frac{\mu^2}{\alpha\Lambda^2}} \end{aligned} \tag{2.24}$$

The explicit appearance of μ^2 in the logarithm requires it to be positive. However, the renormalized coupling constant can be any real value. This set of renormalized mass and coupling constant are not running anymore and are determined "uniquely". (In practice, we choose a set of numbers for them). With the help of equations (2.22) and (2.24), we can easily establish the following relationship by calculating $\mu^2(\Sigma^2) - \mu^2$ and $\lambda_R(\Sigma^2) - \lambda_R$,

$$\mu^2(\Sigma^2) = \frac{\mu^2}{1 - \frac{\lambda_R}{2}\ln\frac{\Sigma^2}{\mu^2}} \qquad \lambda_R(\Sigma^2) = \frac{\lambda_R}{1 - \frac{\lambda_R}{2}\ln\frac{\Sigma^2}{\mu^2}} \tag{2.25}$$

These set of equations tell us how $\mu^2(\Sigma^2)$ and $\lambda_R(\Sigma^2)$ run with μ^2 and λ as Σ^2 changes its value. We will later show that this extra freedom, or this group of renormalization adds no new physics to our system. Thus we will drop the explicit argument Σ^2 in λ_R and μ^2 later. (However, we still use equation (2.22) rather than (2.24) for the renormalization.)

Now we come back to perform renormalization of the free energy density equation (2.14). As before, we first single out the divergent terms in the zero-point energy. The result is

$$\int \frac{d^3p}{(2\pi)^3}\omega = \frac{1}{8\pi^2}\left\{\Lambda^4 + \Lambda^2\bar{m}^2 + \frac{1}{4}\bar{m}^4\ln\frac{\bar{m}^2}{\gamma\Lambda^2}\right\} + O\left(\frac{1}{\Lambda^2}\right) \tag{2.26}$$

with $\ln\gamma \equiv 2\ln 2 - \frac{1}{2}$. The free energy in equation (2.14) becomes

$$\begin{aligned} f = &\frac{1}{16\pi^2}\left[\Lambda^4 + \Lambda^2\bar{m}^2 + \frac{1}{4}\bar{m}^4\ln\frac{\bar{m}^2}{\gamma\Lambda^2}\right] \\ &- \frac{1}{16\lambda}(m^2 - \bar{m}^2)^2 + \mathcal{F}(\beta, \bar{m}^2) \end{aligned} \tag{2.27}$$

where use have been made of the definition of $\bar{m}^2$ to replace the σ_0 in the free energy. As mentioned before, the last term above is finite, thus we hope the infinities associated with the bare mass and bare coupling constant will kill the explicit infinities above.

To this end we need to invert the relationship between bare parameters and renormalized parameters,

$$\lambda(\Lambda) = \lambda_R/A \qquad m^2(\Lambda) = \frac{1}{2\pi^2}(\mu^2 - \lambda_R\Lambda^2)/A \tag{2.28}$$

where A is an infinite constant defined by

$$A \equiv \frac{1}{2\pi^2} - \frac{\lambda_R}{4\pi^2}\ln\frac{\alpha\Lambda^2}{\mu^2} \tag{2.29}$$

It is interesting to note that for any fixed renormalized mass and coupling constant, the bare coupling constant approaches 0^- as Λ goes to infinity unless the renormalized mass is 0. Now we substitute the above expressions of λ and m^2 into equation (2.27),

$$\begin{aligned} f = & \frac{1}{16\pi^2}\left[\Lambda^4 + \Lambda^2\bar{m}^2 + \frac{1}{4}\bar{m}^4 \ln\frac{\bar{m}^2}{\gamma\Lambda^2}\right] + \mathcal{F}(\beta, \bar{m}^2) \\ & - \frac{A}{16}\left[\frac{\bar{m}^4 - 2\bar{m}^2(\mu^2 - \lambda_R\Lambda^2)/(2\pi^2 A) + (\mu^2 - \lambda_R\Lambda^2)^2/(4\pi^4 A^2)}{\lambda_R}\right] \end{aligned} \tag{2.30}$$

The Λ^4 term is a pure constant, which we can drop. The last term in the second square bracket also gives us an infinite constant, which is independent of $\bar{m}^2$, thus we can also drop this term. Then

$$\begin{aligned} f = & \frac{1}{16\pi^2}\Lambda^2\bar{m}^2 + \frac{1}{64\pi^2}\bar{m}^4 \ln\frac{\bar{m}^2}{\gamma\Lambda^2} + \mathcal{F}(\beta, \bar{m}^2) \\ & - \frac{A}{16}\frac{\bar{m}^4}{\lambda_R} + \frac{\bar{m}^2(\mu^2 - \lambda_R\Lambda^2)/(2\pi^2)}{8\lambda_R} \end{aligned} \tag{2.31}$$

Inserting equation (2.29) into the above expression, we can see that the $\bar{m}^2\Lambda^2$ and $\ln\Lambda^2$ divergences cancel, thus we obtain the following renormalized version of the free energy.

$$f = \frac{1}{64\pi^2}\bar{m}^4\left(\ln\frac{\bar{m}^2}{\mu^2} - \frac{1}{2}\right) - \frac{1}{32\pi^2\lambda_R}(\bar{m}^4 - 2\bar{m}^2\mu^2) + \mathcal{F}(\beta, \bar{m}^2) \tag{2.32}$$

The explicit appearance of $1/\lambda_R$ in the free energy density shows that this approach is non-perturbative.

In summary, with the definition of the renormalized mass and coupling constant, equation (2.22), we accomplished renormalization of both the constraint equation (2.16) and the free energy density (2.14). The free energy density depends on temperature both explicitly through β and implicitly through $\bar{m}^2$. For a fixed pare of renormalized mass and coupling constant, the constraint equation (2.23) relates effective mass $\bar{m}^2$ to temperature, with this relation we can obtain the relationship between the free energy density and temperature.

II.3 Discussion

In the following we want to further elucidate the relationship between the constraint equation and the equation for free energy. The constraint equation is obtained by setting $\frac{\delta G}{\delta \sigma_0} = 0$. Here for simplicity we denote the exponent in the partition function (2.7) as G. If we replace the variable σ_0 by $\bar{m}^2$, which is just shifted from σ_0 by a bare mass, which can be regarded as a constant under this circumstance, we get $\frac{\delta G}{\delta \bar{m}^2} = 0$. However, in our calculation for the free energy density , we essentially kept only the leading term of G in the expansion of

$$G = G_0 + \delta\sigma \frac{\delta G}{\delta \sigma}|_{\sigma=\sigma_0} \tag{2.33}$$

This G_0 has the same functional form of $\bar{m}$ as G does, thus we expect to regain the constraint equation if we take derivative of equation (2.32) with respect to $\bar{m}^2$. Making use of the relationship,

$$\frac{\partial \mathcal{F}(\beta, \bar{m}^2)}{\partial \bar{m}^2} = -\frac{1}{4} f(\beta, \bar{m}^2) \tag{2.34}$$

we can easily check that

$$\frac{\partial f}{\partial \bar{m}^2} = 0 \qquad \Rightarrow$$

$$-\frac{1}{4} f(\beta, \bar{m}^2) + \frac{1}{32\pi^2} \bar{m}^2 \left(\ln \frac{\bar{m}^2}{\Sigma^2} - \frac{1}{2} \right) + \frac{1}{64\pi^2} \bar{m}^2 - \frac{1}{16\pi^2} \frac{1}{\lambda_R} (\bar{m}^2 - \mu^2) = 0$$

which is nothing but the constraint equation (2.23). Note the f above denotes free energy density, not the finite, temperature dependent function $f(\beta, \bar{m}^2)$. Since the free energy density (2.32) is an even function of $\bar{m}$, it always has an extremum at $\bar{m} = 0$. If we just keep track of the constraint equation (2.23), then we will lose the other solution $\bar{m} = 0$.

Now we need to discuss whether $\bar{m}$ is allowed to be zero. In our renormalization process, it is implicitly assumed that the $\bar{m}$ can not be zero due to the explicit appearance of terms like $\ln \bar{m}^2$. However, if it is zero, we could have adopted another renormalization procedure. Before renormalization, the free energy density (2.14) and the constraint equation (2.16) become the following equations when $\bar{m} = 0$:

$$f = \mathcal{F}(\beta, 0) - \frac{1}{16\lambda} m^4 \qquad \frac{m^2}{2\lambda} = f(\beta, 0) \tag{2.35}$$

In order to renormalize the above two equations, we just need to define the bare mass and the bare coupling constant to be the renormalized ones. On the other hand, if we take the limit of $\bar{m} \to 0$, then, up to irrelevant constants, we can recover the above two equations from (2.32) and (2.23) by appropriately defining the renormalized mass and coupling constant in (2.35). Therefore, we can safely consider the case of $\bar{m} = 0$ by taking the limit of $\bar{m} \to 0$ in equations (2.32) and (2.23).

Now we take up the point of the running renormalized mass and coupling constant, we want to show that this degree of freedom is illusionary by showing that Σ has now effect on both the free energy and the constraint equation. (or on $\bar{m}^2$ since the purpose of the constraint equation is to solve for $\bar{m}^2$.) Quite generally, we have,

$$\begin{aligned} \frac{df}{d\Sigma^2} &= \frac{\partial f}{\partial \Sigma^2} + \frac{\partial f}{\partial \bar{m}^2}\frac{d\bar{m}^2}{d\Sigma^2} + \frac{\partial f}{\partial \lambda_R}\frac{d\lambda_R}{d\Sigma^2} + \frac{\partial f}{\partial \mu^2}\frac{d\mu^2}{d\Sigma^2} \\ &= -\frac{1}{64\pi^2}\frac{\bar{m}^4}{\Sigma^2} + \frac{\partial f}{\partial \bar{m}^2}\frac{d\bar{m}^2}{d\Sigma^2} + \frac{1}{32\pi^2\lambda_R^2}[\bar{m}^4 - 2\bar{m}^2\mu^2]\frac{d\lambda_R}{d\Sigma^2} + \frac{\bar{m}^2}{16\pi^2\lambda_R}\frac{d\mu^2}{d\Sigma^2} \end{aligned} \tag{2.36}$$

If we insert into the above equation the following identities, which can be easily established by taking derivative with respect to Σ^2 to equation (2.25),

$$\begin{aligned} \frac{d\lambda_R}{d\Sigma^2} &= \frac{1}{2}\lambda_R^2\frac{1}{\Sigma^2} \\ \frac{d\mu^2}{d\Sigma^2} &= \frac{1}{2}\mu^2\lambda_R\frac{1}{\Sigma^2} \end{aligned} \tag{2.37}$$

and $\frac{\partial f}{\partial \bar{m}^2} = 0$, all the terms in equation (2.36) cancel out and we reach the conclusion that

$$\frac{df}{d\Sigma^2} = 0$$

In the language of renormalization group, we have a β function of

$$\beta(\lambda_R) \equiv \Sigma \frac{d\lambda_R}{d\Sigma} = \lambda_R^2 \tag{2.38}$$

The positivity of the β function suggests that this theory is not an asymptotically free theory.

If we take derivative with respect to Σ^2 on both sides of the constraint equation, we obtain,

$$\left[1 - \frac{\lambda_R}{2}\left(1 + \ln\frac{\bar{m}^2}{\Sigma^2}\right) + 4\pi^2\lambda_R \frac{df(\beta, \bar{m}^2)}{d\bar{m}^2}\right] \frac{d\bar{m}^2}{d\Sigma^2}$$
$$= \frac{d\mu^2}{d\Sigma^2} + \left(\frac{\bar{m}^2}{2}\ln\frac{\bar{m}^2}{\Sigma^2} - 4\pi^2 f(\beta, \bar{m}^2)\right)\frac{d\lambda_R}{d\Sigma^2} - \frac{1}{2}\lambda_R \frac{\bar{m}^2}{\Sigma^2}$$

Inserting equations (2.37) into the above equation and making use of the constraint equation, we find the right hand side of the equation is zero, thus we proved that $\bar{m}^2$ is invariant with respect to Σ^2 also:

$$\frac{d\bar{m}^2}{d\Sigma^2} = 0$$

We can similarly prove the term

$$\frac{1}{64\pi^4}\frac{1}{\lambda_R A}(\mu^2 - \lambda_R \Lambda^2)^2$$

that we dropped in equation (2.30) when we renormalize the free energy is not only an infinite constant, but also a constant invariant under the running of Σ^2. By inspecting the free energy density (2.32), we are tempted to add a "constant" term $-\mu^4/(32\pi^2\lambda_R)$ into the free energy density to complete the square. However, this additional term is not invariant under the running of Σ^2. In later discussions, unless explicitly stated, we will stick to the renormalization (2.24), where the Σ^2 degree of freedom is removed by setting $\Sigma^2 = \mu^2$, then we can add the $-\mu^4/(32\pi^2\lambda_R)$ term and the free energy density becomes

$$f = \frac{1}{64\pi^2}\bar{m}^4\left(\ln\frac{\bar{m}^2}{\mu^2} - \frac{1}{2}\right) - \frac{1}{32\pi^2\lambda_R}(\bar{m}^2 - \mu^2)^2 + \mathcal{F}(\beta, \bar{m}^2) \tag{2.39}$$

If we want to plot it as a function of temperature T, we need to solve the constraint equation (2.23) for $\bar{m}^2$ at various temperatures. However, the constraint equation has no real solution above a critical temperature for both $\lambda_R > 0$ and $\lambda_R < 0$. Below that temperature, it usually has two solutions to $\bar{m}$ (This is always true in the case of $\lambda_R > 0$ and true for a certain interval of temperatures in the case of $\lambda_R < 0$. See Fig. 2.1(a) as an example, where we have two intersections from both sides of the constraint equation). For reasons explained later, we keep the bigger one of the two solutions. As an example, the solution to $\bar{m}$ as a function of temperature T (with $\lambda_R = 1$ and $\mu^2 = 10$) is plotted in Fig. 2.1(b). The critical temperature in this case is $T = 2.17$.

We can give an explanation to this phenomenon by plotting the free energy density as a function of both T and $\bar{m}$, namely, we ignore the constraint equation for the moment and treat T and $\bar{m}$ as independent variables. First, we list the following four properties of the function $F(\beta, \bar{m}^2)$:

$$F(\beta, \bar{m}^2) \longrightarrow 0^- \quad as \quad \beta \to \infty$$

$$F(\beta, \bar{m}^2) \longrightarrow -\infty \quad as \quad \beta \to 0$$

$$F(\beta, \bar{m}^2) \longrightarrow \quad finite \quad F(\beta, 0) \quad as \quad \bar{m}^2 \to 0$$

$$F(\beta, \bar{m}^2) \quad increases \quad as \quad \bar{m}^2 \quad increases$$

Starting with the zero temperature, if we look at the free energy density, equation (2.39), we learn from the first property above that only the first two terms in the free energy density contribute. In the $\bar{m} \to 0$ region, the first term approaches 0. Thus, in that region, depending on the sign of λ_R, the free energy is either an valley or hill. In the region of large $\bar{m}$, the first term dominate, the free energy density goes up as $\bar{m}$ increases.

As temperature increases, in the $\bar{m} \to 0$ region, the $F(\beta, \bar{m}^2)$ term will dominate over the other two terms, and the free energy density will form a valley eventually in this region regardless of the sign of λ_R according to the second property above. In the region of large $\bar{m}$, even at high temperature, the first term still dominates and the free energy density goes up as $\bar{m}$ increases, as before. The above analysis explains the "restoration of symmetry" as the temperature increases, as exhibited in Fig. 2.2(a)-2.2(d) for the case of $\lambda_R > 0$ (with $\lambda_R = 1$ and $\mu^2 = 10$) and in Fig. 2.3(a)-2.3(d) for the case of $\lambda_R < 0$ (with $\lambda_R = -1$ and $\mu^2 = 10$).

The constraint equation results from the partial derivative of the free energy density with respect to $\bar{m}$. Thus as soon as the free energy density has no non-zero extrema with respect to $\bar{m}$, the constraint equation has no solution. Fig. 2.2(c) shows that the free energy doesn't have any non-zero extremum above $T = 2.17$, the critical temperature in Fig. 2.1(b). Below the critical temperature, if the constraint equation has two solutions to $\bar{m}$, then the smaller one corresponds to the local maximum between the minimum at $\bar{m} = 0$ and the other minimum which is the bigger solution. This explains why we kept only the bigger solution when relating $\bar{m}$ to temperature.

The above analysis suggests us that we should add up the two contributions around the two minima as we calculate free energy density, thus, we have

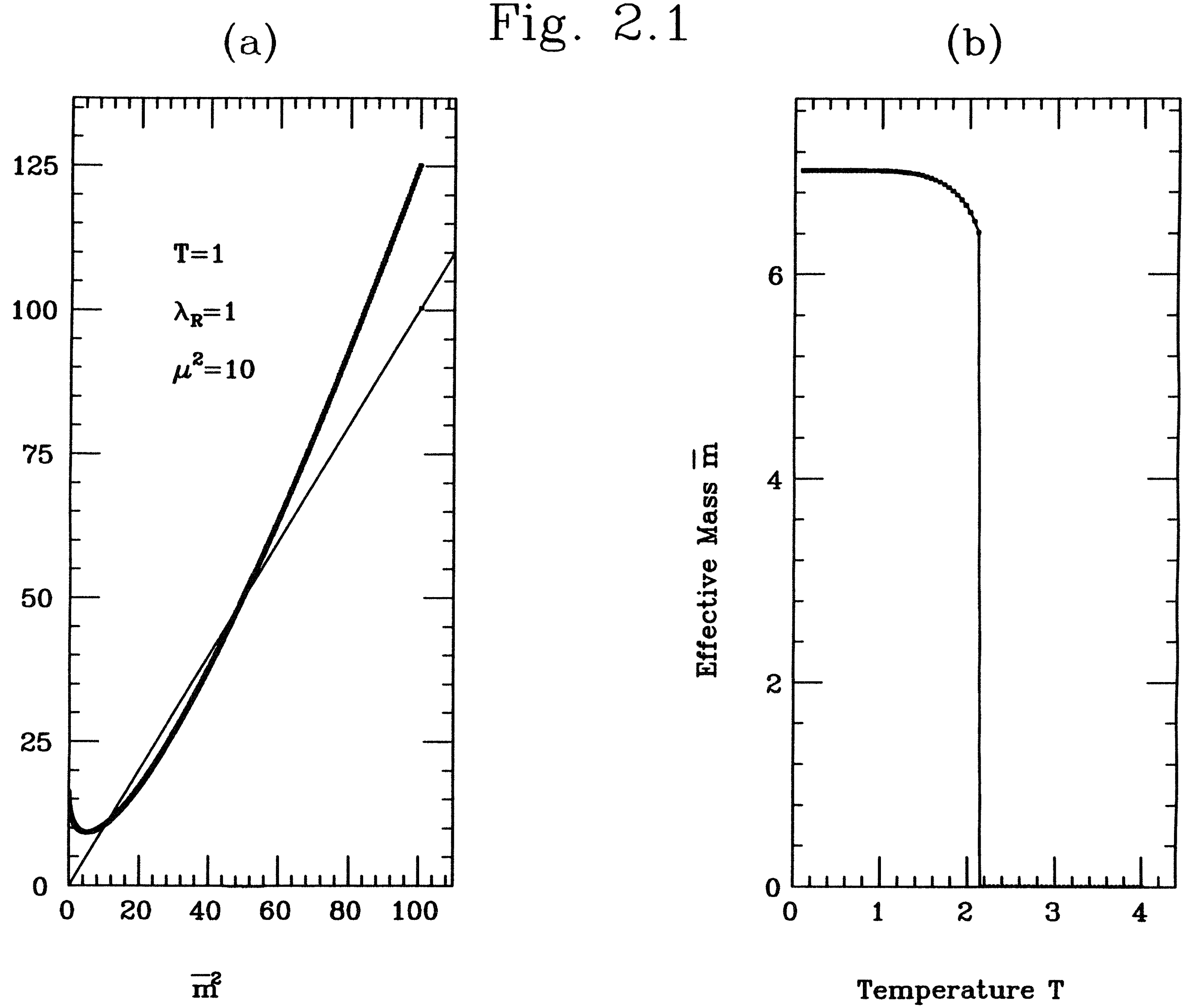
$$f = \frac{1}{64\pi^2}\bar{m}^4\left(\ln\frac{\bar{m}^2}{\mu^2} - \frac{1}{2}\right) - \frac{1}{32\pi^2\lambda_R}(\bar{m}^2 - \mu^2)^2 + \mathcal{F}(\beta, \bar{m}^2) + F(\beta, 0) - \frac{\mu^4}{32\pi^2\lambda_R} \tag{2.40}$$

Note that when $\lambda_R < 0$, $\bar{m} = 0$ is a maximum of the free energy density at low temperatures. Therefore the above improvement doesn't apply to this situation. In addition, the above expression is only an improvement in the temperature (or $\bar{m}$) region where the constraint equation has a solution. As soon as we fail to obtain a solution to $\bar{m}$ from the constraint equation, we can approximate the free energy density by

$$f = F(\beta, 0) - \frac{\mu^4}{32\pi^2 \lambda_R} \tag{2.41}$$

Actually we will show in Chapter IV that even before the constraint equation stops having solution(s) to $\bar{m}^2$, the free energy density given by $\bar{m} = 0$, equation (2.41), is already lower than the free energy density given by the solutions to the constraint equation. In other words, the free energy density, equation (2.41), should be deployed earlier than stated above. However, equation (2.40) is always valid unless the constraint equation has no solutions any more, under which circumstance we are left with equation (2.41) only.

Fig. 2.1


(a)
T=1
λ_R=1
μ^2=10
$\overline{m}^2$
(b)
Effective Mass $\overline{m}$
Temperature T

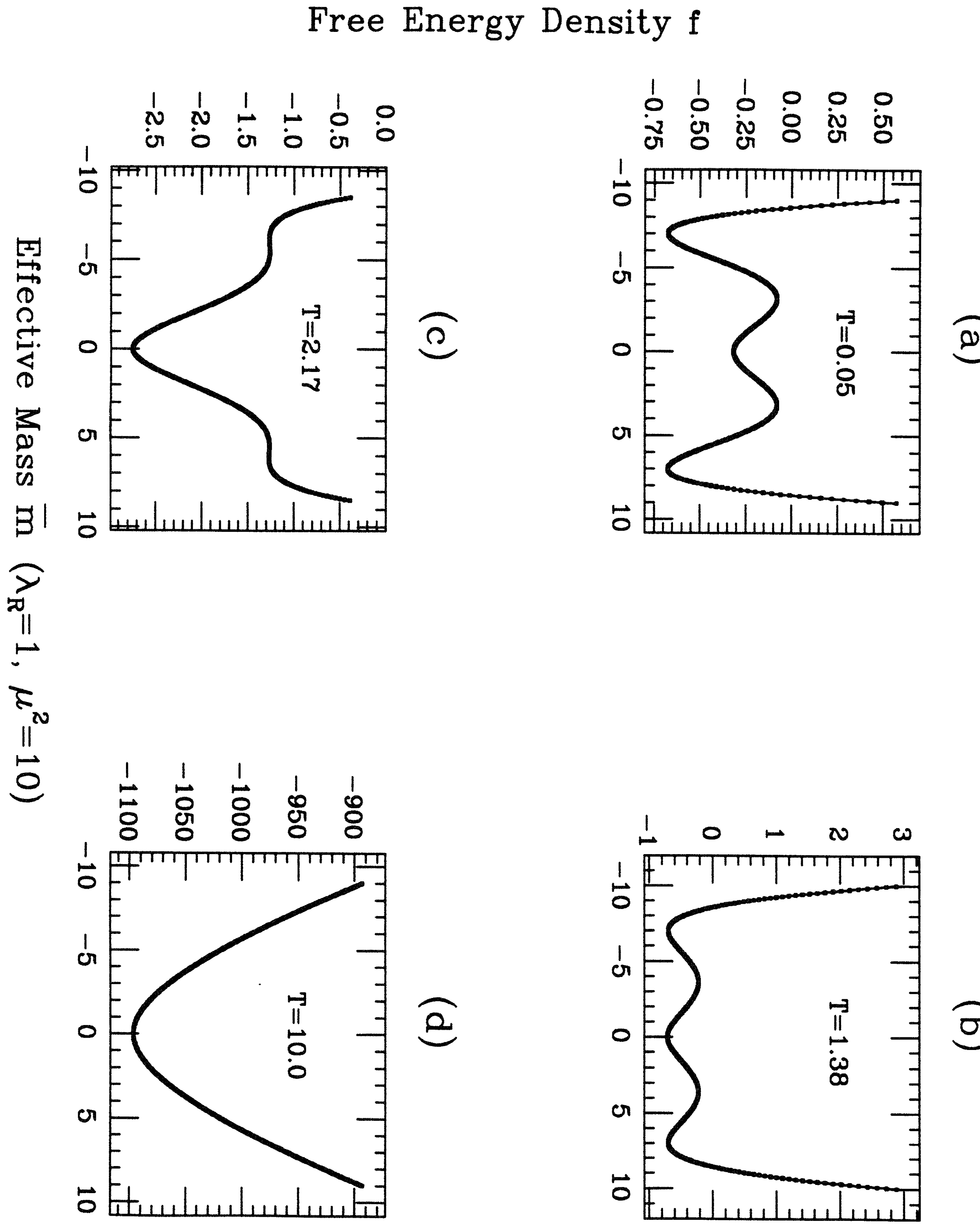
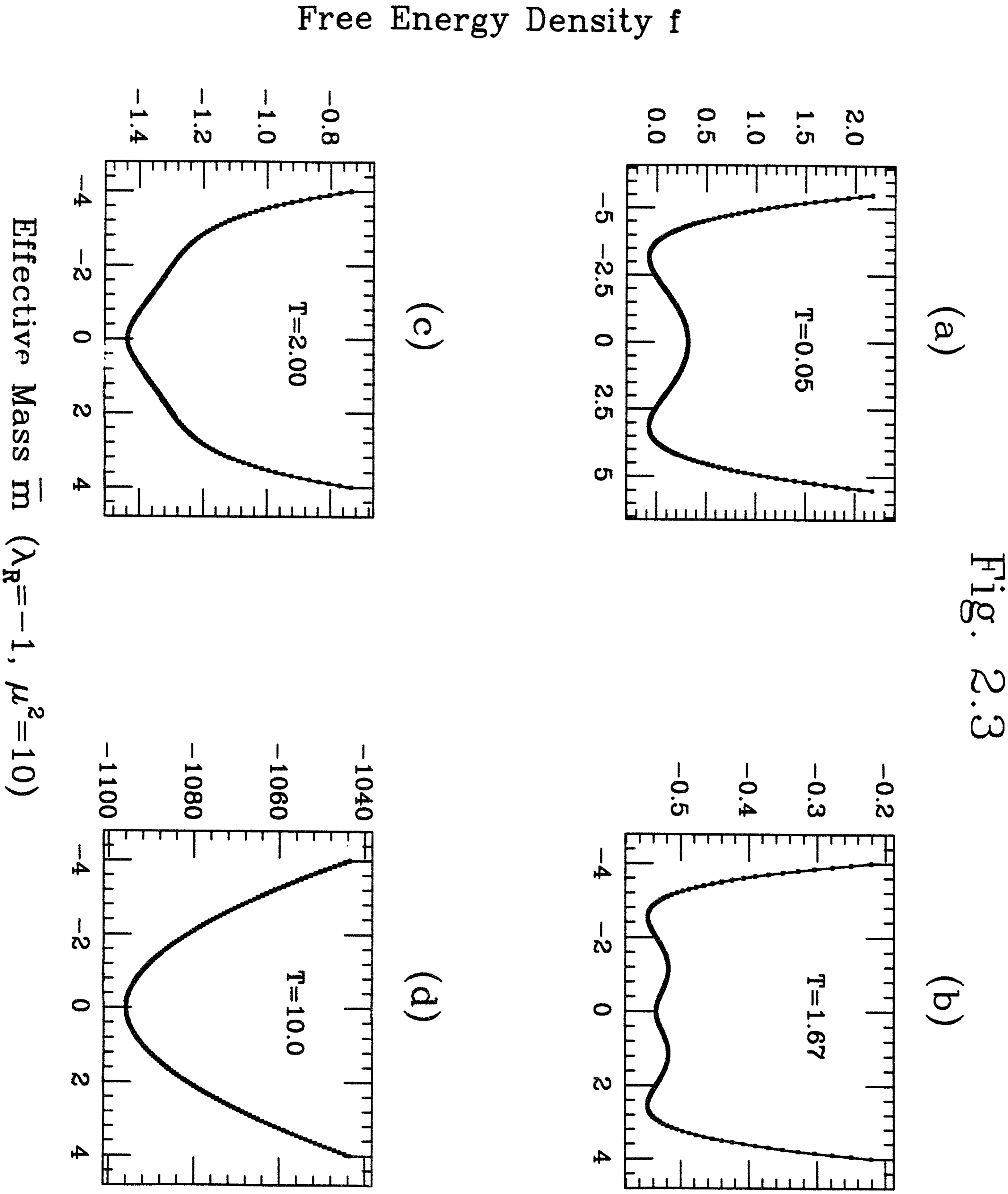

Free Energy Density f
(a)
T=0.05
(b)
T=1.38
(c)
T=2.17
(d)
T=10.0
Effective Mass $\bar{m}$ (λ_R=1, μ^2=10)

Fig. 2.2



Free Energy Density f
(a)
T=0.05
(b)
T=1.67
(c)
T=2.00
(d)
T=10.0
Effective Mass $\bar{m}$ (λ_R=−1, μ^2=10)

Fig. 2.3

II.4 Nonuniform σ_0

Now we break translational invariance by assuming σ_0 is nonuniform in space, namely, $\sigma_0 = \sigma_0(\mathbf{x})$. We want to accomplish renormalization in this case.

Our derivations up to equation (2.8) are general. Thus, our first order constraint equation is

$$\sigma_0(\mathbf{x}) = (-2\lambda) < x|\frac{1}{\hat{p}_0^2 + \hat{p}^2 + \hat{\bar{m}}^2}|x > \tag{2.42}$$

where we defined $\bar{m}^2(\mathbf{x}) \equiv m^2 - 2\sigma_0(\mathbf{x})$ and it takes over the degree of freedom represented by $\sigma_0(\mathbf{x})$. We can do the trace in τ (or p_0) as before since we break only the three dimensional, translational invariance, which has no effect on the fourth component: either τ or p_0. We obtain,

$$\sigma_0(\mathbf{x}) = (-\lambda) < \mathbf{x}|\frac{1}{\hat{\omega}}|\mathbf{x} > +\lambda\tilde{f}(\beta, \mathbf{x}) \tag{2.43}$$

with $\hat{\omega}(\mathbf{x}) = \sqrt{p^2 + \bar{m}^2(\mathbf{x})}$ and $\tilde{f}(\beta, \mathbf{x}) \equiv < \mathbf{x}| \left[\frac{-2}{\hat{\omega}(e^{\beta\hat{\omega}}-1)}\right] |\mathbf{x} >$. Whether or not $\bar{m}^2$ is uniform, this term is always finite since it decreases exponentially as ω approaches infinity in any given representation. This term corresponds to the finite function $f(\beta, \bar{m}^2)$ in the uniform case. The first term in equation (2.43) is divergent. We want to separate it into finite and divergent parts. This can be achieved through an operator expansion around $\hat{\bar{m}}^2 = \Sigma^2 \hat{I}$, where Σ^2 is an arbitrary positive constant. If we secretly take the trace in three momentum space, by counting the powers of momentum, we can determined which terms are finite or divergent. In the following we write out the two divergent terms explicitly and denote the finite part of $< \mathbf{x}|1/\hat{\omega}|\mathbf{x} >$ by $R_f(\beta, \mathbf{x})$.

$$\sigma_0(\mathbf{x}) = -\lambda < \mathbf{x}|\frac{1}{\sqrt{\hat{p}^2 + \Sigma^2}}|\mathbf{x} > +\frac{1}{2\lambda} < \mathbf{x}|\frac{1}{(\hat{p}^2 + \Sigma^2)^{3/2}}|\mathbf{x} > -\lambda R_f(\beta, \mathbf{x}) + \lambda\tilde{f}(\beta, \mathbf{x}) \tag{2.44}$$

Making use of equation (2.20) and

$$\begin{aligned}<\mathbf{x}|[\hat{p}^2+\Sigma^2]^{-3/2}|\mathbf{x}> &= -2\frac{\delta}{\delta\Sigma^2}<\mathbf{x}|[\hat{p}^2+\Sigma^2]^{-1/2}|\mathbf{x}> \\ &= -\frac{1}{2\pi^2}\left[\frac{1}{2}\ln\frac{\Sigma^2}{\alpha\Lambda^2}+\frac{1}{2}\right] \\ &= -\frac{1}{4\pi^2}\left[1+\ln\frac{\Sigma^2}{\alpha\Lambda^2}\right]\end{aligned} \tag{2.45}$$

we can obtain the following renormalized version for the constraint equation with the renormalized mass and coupling constant defined exactly the same as those by equation (2.22).

$$\bar{m}^2(\mathbf{x}) = \mu^2 + \frac{1}{2}\lambda_R(\bar{m}^2(\mathbf{x})-\Sigma^2) + 4\pi^2\lambda_R R_f(\beta,\mathbf{x}) - \tilde{f}(\beta,\mathbf{x}) \tag{2.46}$$

The leading term from the saddle point integral gives us the following expression for the free energy,

$$F = \frac{1}{2\beta}tr\ln[\hat{p}_0^2+\hat{p}^2+\bar{m}^2(\mathbf{x})] - \frac{1}{4\lambda\beta}\int d^4x\sigma_0^2(\mathbf{x}) \tag{2.47}$$

We can evaluate the trace in the fourth component as before,

$$F = \frac{1}{2}tr_3\hat{\omega} + V\tilde{F}(\beta,\mathbf{x}) - \frac{1}{16\lambda}\int d^3x(m^2-\bar{m}^2(\mathbf{x}))^2 \tag{2.48}$$

where we defined a finite function $\tilde{F}(\beta,\mathbf{x}) \equiv \frac{1}{V\beta}tr_3\ln\left(1-e^{-\beta\hat{\omega}}\right)$, which is equivalent to the finite function $F(\beta,\bar{m}^2)$ in the uniform case. The first term in equation (2.48) is divergent, separating divergent parts from the finite part by the expansion of $\bar{m}^2(\mathbf{x})$ around Σ^2, we obtain the following expression for the free energy density,

$$\begin{aligned}f = \tilde{F}(\beta,\mathbf{x}) &+ \frac{1}{2}R_F(\beta,\mathbf{x}) + \frac{1}{4}(\hat{\bar{m}}^2-\Sigma^2)<\mathbf{x}|[\hat{p}^2+\Sigma^2]^{-1/2}|\mathbf{x}> \\ &- \frac{1}{16}(\hat{\bar{m}}^2-\Sigma^2)^2<\mathbf{x}|[\hat{p}^2+\Sigma^2]^{-3/2}|\mathbf{x}> - \frac{1}{16\lambda}(m^2-\bar{m}^2(\mathbf{x}))^2\end{aligned} \tag{2.49}$$

where we defined the finite part of $tr_3\hat{\omega}$ by $V\cdot R_F(\beta,\mathbf{x})$. We have also dropped the term $<\mathbf{x}|\sqrt{\hat{p}^2+\mu^2}|\mathbf{x}>$ since it just contributes to the free energy by an infinite constant.

Inserting the divergences from equations (2.20) and (2.45) and dropping irrelevant infinite constants, we arrive at

$$
\begin{aligned}
f = \tilde{F}(\beta, \mathbf{x}) + \frac{1}{2} R_F(\beta, \mathbf{x}) + \frac{1}{64\pi^2}(\bar{m}^4(\mathbf{x}) - 2\bar{m}^2(\mathbf{x})\Sigma^2) + \frac{1}{64\pi^2}\bar{m}^4(\mathbf{x})\ln\frac{\Sigma^2}{\alpha\Lambda^2} \\
+ \frac{1}{16\pi^2}\bar{m}^2(\mathbf{x})\Lambda^2 - \frac{1}{16\lambda}(m^2 - \bar{m}^2(\mathbf{x}))^2
\end{aligned}
\tag{2.50}
$$

Note the last three terms are very similar to the divergent terms encountered in equation (2.30) of the uniform case. If we adopt the same renormalization defined by equations (2.22) in the uniform case, we obtain the following renormalized free energy density by replacing the bare parameters by the renormalized ones.

$$
f = \tilde{F}(\beta, \mathbf{x}) + \frac{1}{2} R_F(\beta, \mathbf{x}) + \frac{1}{64\pi^2}(\bar{m}^4(\mathbf{x}) - 2\bar{m}^2(\mathbf{x})\Sigma^2) - \frac{\bar{m}^4(\mathbf{x})}{32\pi^2\lambda_R} + \frac{\bar{m}^2(\mathbf{x})\mu^2}{16\pi^2\lambda_R} \tag{2.51}
$$

The renormalization group analysis applies here also. As before, if we choose $\Sigma = \mu$, then we can add an additional term to the free energy density to complete a square with the last two terms. The free energy density is still connected to the constraint equation by the partial derivative of $\bar{m}^2(\mathbf{x})$.

Chapter III

Second Order Contribution to the Free Energy

III.1 Formal Result

Now we embark on the calculation of the second order contribution to the free energy. According to equation (2.8), we need to calculate,

$$Z_2 \equiv \int_{peri} D[\delta\sigma] e^{tr\left[\frac{1}{\hat{p}_0^2+\hat{p}^2+m^2-2\hat{\sigma}_0}\delta\hat{\sigma}\frac{1}{\hat{p}_0^2+\hat{p}^2+m^2-2\hat{\sigma}_0}\delta\hat{\sigma}\right]+\frac{1}{4\lambda}\int d^4x\delta\sigma^2(x)} \tag{3.1}$$

The first term in the exponent can be easily seen to be,

$$\int d^4xd^4x'\delta\sigma(x) < x|\frac{1}{L(\hat{p})}|x' >< x'|\frac{1}{L(\hat{p})}|x > \delta\sigma(x') \tag{3.2}$$

with $L(\hat{p}) \equiv \frac{1}{\hat{p}_0^2+\hat{p}^2+\bar{m}^2}$. Here multiplied with $\delta\sigma(x)$ and $\delta\sigma(x')$ are two matrix elements, $< x|\frac{1}{L(\hat{p})}|x' >$ and $< x'|\frac{1}{L(\hat{p})}|x >$, of the same operator $\frac{1}{L(\hat{p})}$. We want to convert it to one matrix element of another operator. As can be seen later, this can be accomplished by the introduction of relative and total momentum. By inserting complete set of four momentum states, we have

$$< x|\frac{1}{L(\hat{p})}|x' >< x'|\frac{1}{L(\hat{p})}|x >= \int d^4pd^4p'\left[\frac{1}{(2\pi)^4}\right]^2\frac{1}{L(p)}\frac{1}{L(p')}e^{i(p-p')(x'-x)}$$

Here $\int dp_0$ means $\frac{2\pi}{\beta}\sum_n$ since $p_0 = \frac{2\pi n}{\beta}$

Now let's define total momentum $P = p - p'$ and relative momentum $q = p + p'$. Here we could also define total momentum and relative momentum the other way around by calling $p + p'$ the total momentum. However, just for the convenience to the derivations in section

III.2, we choose this definition here. If we have chosen the other definition, all the arguments and derivations in III.2 are valid as well. But they are valid in a less intuitive way. With these total and relative momenta, the above expression can be rewritten as,

$$\begin{aligned}&\int d^4Pd^4q\frac{1}{16}\left[\frac{1}{(2\pi)^4}\right]^2\frac{1}{L((P+q)/2)}\frac{1}{L((P-q)/2)}e^{iP(x'-x)}\\&=\frac{1}{(2\pi)^4}<x|f(\hat{P})|x'>\end{aligned} \tag{3.3}$$

with

$$f(\hat{P})\equiv\frac{1}{16}\int d^4q\frac{1}{L((\hat{P}+q)/2)}\frac{1}{L((\hat{P}-q)/2)} \tag{3.4}$$

So far we have converted the product of two matrix elements of the same operator $\frac{1}{L(\hat{p})}$ into one matrix element of the operator $f(\hat{P})$. Thus, after the functional integral in equation (3.1) is done, Z_2 becomes,

$$Z_2=e^{-\frac{1}{2}tr\ln\left[\frac{-2}{(2\pi)^4}f(\hat{P})-\frac{1}{2\lambda}\right]}$$

Finally, taking the trace in configuration space, we get,

$$Z_2=e^{-\frac{1}{2}\beta V\int\frac{d^4P}{(2\pi)^4}\ln\left[\frac{-2}{(2\pi)^4}f(P)-\frac{1}{2\lambda}\right]} \tag{3.5}$$

Again we want to emphasize that a four momentum integral $\int d^4p$ means $\frac{2\pi}{\beta}\sum_n\int d^3p$. Note in this final result we have $-\frac{1}{2\lambda}$ in it. From chapter II we know that $\lambda\to 0^-$ unless $\lambda_R=0$. Thus, we hope that the first term in the exponent will cancel the infinity of $-\frac{1}{2\lambda}$. To this end we need to calculate the integral in the definition of $f(P)$ in next section.

III.2 Renormalization

In the definition of $f(P)$, equation (3.4), we have discrete sum due to q_0. This is sum is done is Appendix B. Thus,

$$f(P) = \frac{1}{16} \int d^3q \left[\frac{\omega_+ + \omega_-}{\omega_+\omega_-[(\omega_+ + \omega_-)^2 + \omega_n^2]} + F_1 + F_2 \right]$$

with ω_+, ω_-, F_1 and F_2 defined in Appendix B and $\omega_n = P_0$. Then the second order contribution to the partition function becomes,

$$Z_2 = exp\left\{ -\frac{1}{2} V \sum_n \int \frac{d^3P}{(2\pi)^3} \ln \left[\int \frac{d^3q}{(2\pi)^3} \left[\frac{\omega_+ + \omega_-}{[-\omega_n^2 - (\omega_+ + \omega_-)^2]\omega_+\omega_-} + F_1 + F_2 \right] - \frac{1}{2\lambda} \right] \right\} \tag{3.6}$$

where we have rescaled q to $2q$ in order to absorb a 1/8 factor. In contrast to Appendix B, this rescaling changes the definitions of of ω_+ and ω_- to,

$$\omega_\pm \equiv \sqrt{\bar{m}^2 + (\frac{\mathbf{P}}{2} \pm \mathbf{q})^2} \tag{3.7}$$

We are interested in renormalizing the argument of the logarithm . Because of the exponential terms in F_1 and F_2, the integrations of F_1 and F_2 are finite. Therefore, we are only interested in the calculation of integration of d^3q over the first term. To obtain an analytical expression for this integration, we first need to calculate the following integral,

$$Int \equiv \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\Omega^2 - (\omega'_+ + \omega'_-)^2 + i\epsilon} \frac{\omega'_+ + \omega'_-}{\omega'_+\omega'_-} \tag{3.8}$$

here we assume Ω is positive. Note this integral is essentially the same as the first term in the argument of logarithm in equation (3.6) except for a sign change in front of Ω^2. This sign change is introduced on purpose. With this sign change, we can separate integral (3.8) into principal part and imaginary part. It turns out that we can obtain an analytical expression for the imaginary part. Then, with the help of dispersion relations, we can obtain both the final result for integral (3.8) and the integral in the argument of logarithm in equation (3.6), namely, the integral with correct sign. In other words, we change the sign in front of Ω^2 just as a means to obtain the result for the integral with the correct sign. In equation (3.8) p' plays the role of q in equation (3.6). This change is introduced for notational clarity later. Consequently, $\omega'_{\pm}$ are defined as (3.7) with the replacement of q by p'.

As is outlined above, we devote the rest of this section to the calculation of the aforementioned integrals. Making using of the following identities,

$$\begin{aligned} \frac{1}{\Omega^2 - (\omega'_+ + \omega'_-)^2 + i\epsilon} &= P\frac{1}{\Omega^2 - (\omega'_+ + \omega'_-)^2} - i\pi\delta(\Omega^2 - (\omega'_+ + \omega'_-)^2) \\ \delta(x^2 - a^2) &= \frac{1}{2|a|}[\delta(x-a) + \delta(x+a)] \end{aligned} \tag{3.9}$$

we get the following expression for the imaginary part of the integral,

$$\begin{aligned} Im(Int) &= -2\pi \int \frac{d^3p'}{(2\pi)^3} \frac{1}{4\omega'_+\omega'_-} \delta(\Omega - (\omega'_+ + \omega'_-)) \\ &= -(2\pi)^4 \int \frac{d^3p'}{(2\pi)^3} \frac{d^3P'}{(2\pi)^3} \frac{1}{4\omega'_+\omega'_-} \delta(\Omega - (\omega'_+ + \omega'_-))\delta(\mathbf{P}' - \mathbf{P}) \\ &= -(2\pi)^4 \int \frac{d^3p'_1}{(2\pi)^3} \frac{d^3p'_2}{(2\pi)^3} \frac{1}{4\omega'_+\omega'_-} \delta(\Omega - (\omega'_+ + \omega'_-))\delta(\mathbf{P}' - \mathbf{P}) \end{aligned} \tag{3.10}$$

In the second step above we introduced an three dimensional integral d^3P' on purpose in order to change $\omega'_{\pm}$ to

$$\omega'_{\pm} \equiv \sqrt{\bar{m}^2 + (\frac{\mathbf{P}'}{2} \pm \mathbf{p}')^2}$$

In the last step we introduced momentum $\mathbf{p}'_1 \equiv \mathbf{P}'/2 + \mathbf{p}'$ and $\mathbf{p}'_2 \equiv \mathbf{P}'/2 - \mathbf{p}'$. Namely, we "disintegrate" the total momentum $\mathbf{P}'$ and relative momentum $\mathbf{p}'$ into their "original" component and consequently $\omega'_{\pm}$ become ω'_1 and ω'_2 respectively.

The reason why we want to introduce $\mathbf{p}'_1$ and $\mathbf{p}'_2$ is that they enable us to deploy the following identity

$$\int d^4p'\delta(p'^2 - \bar{m}^2)\theta(p'_0) = \int d^3p' \frac{1}{2\sqrt{\mathbf{p}'^2 + \bar{m}^2}} \tag{3.11}$$

to transform the three dimensional integrals in equation (3.10) to four dimensional integrals in Minkowski space. The above identity puts particle 1 and particle 2 on mass shell, namely

$$p'_{10} = \omega'_+ \qquad p'_{20} = \omega'_- \tag{3.12}$$

Now we change variables p'_1 and p'_2 back to the relative momentum p' and the total momentum P'. However, this time it is a four dimensional transformation and we introduced p'_0 and P'_0 for the first time. Due to equation (3.12), we can replace $\delta(\Omega - (\omega'_+ + \omega'_-))$ by $\delta(\Omega - P'_0)$, which kills the d^4P' integral. Consequently, we are left with

$$\begin{aligned}
Im(Int) &= -2\pi \int \frac{d^4p'}{(2\pi)^3} \delta(p'^2_+ - \bar{m}^2)\delta(p'^2_- - \bar{m}^2)\theta(p'_{+0})\theta(p'_{-0}) \\
&= -2\pi \int \frac{d^4p'}{(2\pi)^3} \delta(\frac{P^2}{4} + Pp' + p'^2 - \bar{m}^2) \\
&\quad \times \delta(\frac{P^2}{4} - Pp' + p'^2 - \bar{m}^2)\theta(\frac{\Omega}{2} + p'_0)\theta(\frac{\Omega}{2} - p'_0) \\
&= -\pi \int \frac{d^4q}{(2\pi)^3} \delta(Pq)\delta(\frac{P^2}{4} + q^2 - \bar{m}^2)\theta(\frac{\Omega}{2} + q_0)\theta(\frac{\Omega}{2} - q_0)
\end{aligned}$$

where we change p' back to q as in equation (3.6) for notational simplicity. Here P is the four momentum $(\Omega, \mathbf{P})$ in Minkowski space and $P^2 = \Omega^2 - \mathbf{P}^2$. As far as no confusions arise, we define

$$q = |\mathbf{q}| \qquad P = |\mathbf{P}| \qquad u = \frac{\mathbf{P} \cdot \mathbf{q}}{Pq}$$

Then in polar coordinates we have,

$$\begin{aligned}
Im(Int) &= -\frac{2\pi^2}{(2\pi)^3} \int_{-1}^{+1} du \int_0^\infty q^2 dq \int_{-\infty}^{+\infty} dq_0 \; \delta(\Omega q_0 - qPu) \\
&\quad \times \delta(q_0^2 - q^2 + \frac{\Omega^2}{4} - \frac{P^2}{4} - \bar{m}^2) \; \theta(\frac{\Omega}{2} + q_0)\theta(\frac{\Omega}{2} - q_0) \\
&= -\frac{2\pi^2}{(2\pi)^3\Omega} \int_{-1}^{+1} du \int_0^\infty q^2 dq \; \theta(\frac{\Omega}{2} + \frac{qPu}{\Omega}) \\
&\quad \times \theta(\frac{\Omega}{2} - \frac{qPu}{\Omega}) \; \delta(\frac{qPu}{\Omega}^2 - q^2 + \frac{\Omega^2}{4} - \frac{P^2}{4} - \bar{m}^2) \\
&= -\frac{2\pi^2}{(2\pi)^3\Omega} \int_{-1}^{+1} \frac{du}{1 - \frac{P^2u^2}{\Omega^2}} \int_0^\infty q^2 dq \; \theta(\frac{\Omega}{2} + \frac{qPu}{\Omega}) \\
&\quad \times \theta(\frac{\Omega}{2} - \frac{qPu}{\Omega}) \;\; \delta(q^2 - \frac{\frac{\Omega^2}{4} - \frac{P^2}{4} - \bar{m}^2}{1 - \frac{P^2u^2}{\Omega^2}})
\end{aligned}$$

The δ function above kills the dq integral and we get,

$$
\begin{aligned}
Im(Int) &= -\frac{\pi^2}{(2\pi)^3\Omega}\sqrt{\frac{\Omega^2}{4}-\frac{P^2}{4}-\bar{m}^2}\int_{-1}^{+1}\frac{du}{(1-\frac{P^2u^2}{\Omega^2})^{\frac{3}{2}}} \\
&\quad\times\theta\left(\frac{Pu}{\Omega}\sqrt{\frac{\frac{\Omega^2}{4}-\frac{P^2}{4}-\bar{m}^2}{1-\frac{P^2u^2}{\Omega^2}}}+\frac{\Omega}{2}\right)\theta\left(-\frac{Pu}{\Omega}\sqrt{\frac{\frac{\Omega^2}{4}-\frac{P^2}{4}-\bar{m}^2}{1-\frac{P^2u^2}{\Omega^2}}}+\frac{\Omega}{2}\right) \\
&= -\frac{\pi^2}{(2\pi)^3\Omega}\sqrt{\frac{\Omega^2}{4}-\frac{P^2}{4}-\bar{m}^2}\int_{-\frac{P}{\Omega}}^{\frac{P}{\Omega}}dx\frac{\Omega}{P}\frac{1}{(1-x^2)^{\frac{3}{2}}} \\
&\quad\times\theta\left(\sqrt{\frac{\Omega^2}{4}-\frac{P^2}{4}-\bar{m}^2}\frac{x}{\sqrt{1-x^2}}+\frac{\Omega}{2}\right)\theta\left(-\sqrt{\frac{\Omega^2}{4}-\frac{P^2}{4}-\bar{m}^2}\frac{x}{\sqrt{1-x^2}}+\frac{\Omega}{2}\right)
\end{aligned}
$$

where we have defined a new variable $x \equiv \frac{Pu}{\Omega}$.

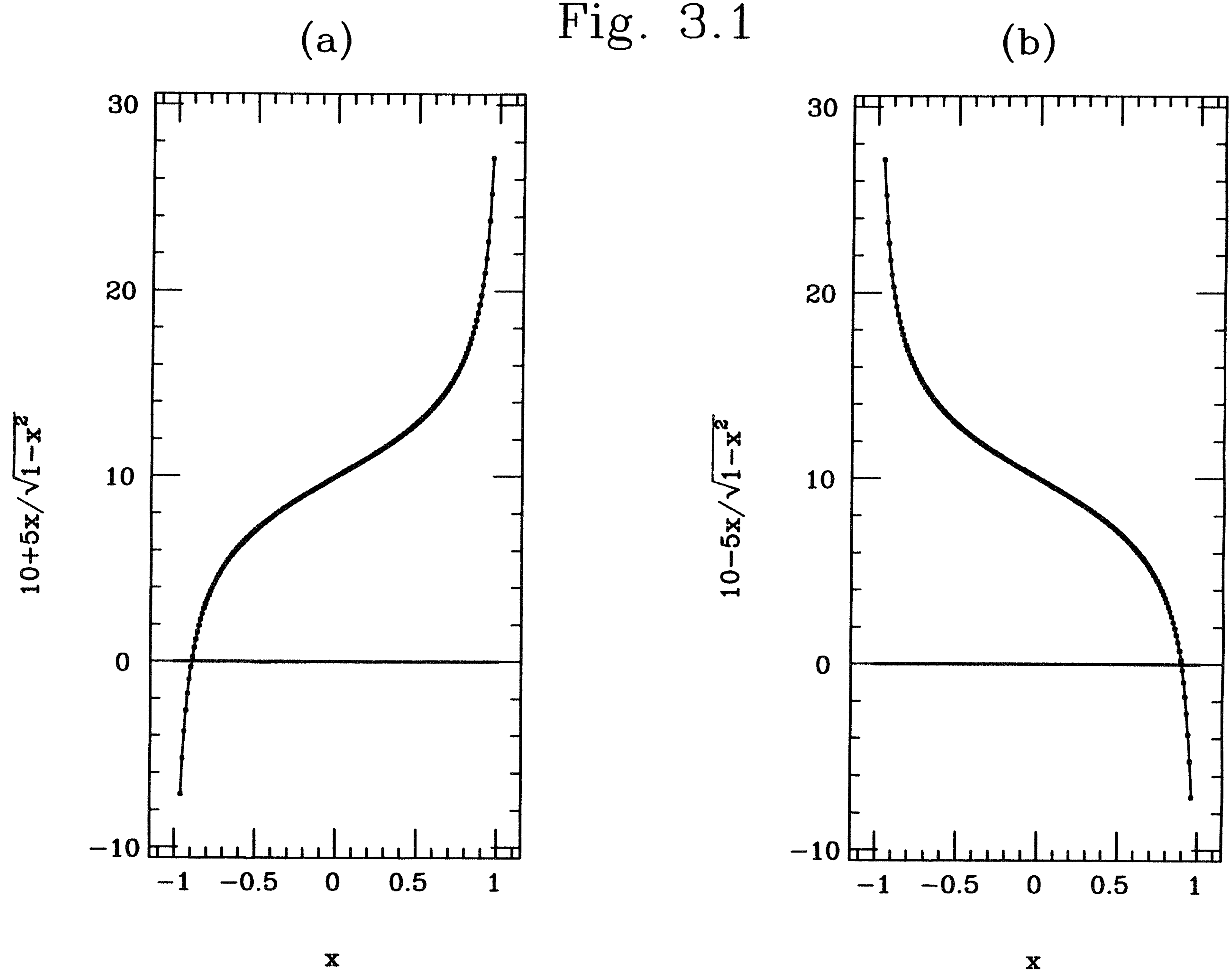
Scaling Ω and P by $\bar{m}$, and denoting $\Omega/\bar{m}$ and $P/\bar{m}$ by $\tilde{\Omega}$ and $\tilde{P}$, we can eliminate one degree of freedom in the argument of θ functions through the replacement of $\bar{m}$ by 1,

$$
\begin{aligned}
Im(Int) &= -\frac{\pi^2}{(2\pi)^3\Omega}\sqrt{\frac{\Omega^2}{4}-\frac{P^2}{4}-\bar{m}^2}\int_{-\frac{\tilde{P}}{\tilde{\Omega}}}^{\frac{\tilde{P}}{\tilde{\Omega}}}dx\frac{\tilde{\Omega}}{\tilde{P}}\frac{1}{(1-x^2)^{\frac{3}{2}}} \\
&\quad\times\theta\left(\sqrt{\frac{\tilde{\Omega}^2}{4}-\frac{\tilde{P}^2}{4}-1}\frac{x}{\sqrt{1-x^2}}+\frac{\tilde{\Omega}}{2}\right)\theta\left(-\sqrt{\frac{\tilde{\Omega}^2}{4}-\frac{\tilde{P}^2}{4}-1}\frac{x}{\sqrt{1-x^2}}+\frac{\tilde{\Omega}}{2}\right)
\end{aligned}
$$

As a function of x, the argument in the first θ function behaves as Fig. 3.1(a); and the argument of the second θ function behaves as Fig. 3.1(b). In the figures we choose $\tilde{\Omega}/2 = 10$ and $\sqrt{\tilde{\Omega}^2/4 - \tilde{P}^2/4 - 1} = 5$. Let's look at the argument of the first θ function. The minimum value of x is $-\tilde{P}/\tilde{\Omega}$. However, we can show that the root of the argument as a function of x lies below this minimum value of x. In other words, the argument is always positive and the θ function is nothing but one. Similarly, the root of the argument of the second θ function lies above the maximum value of x: $\tilde{P}/\tilde{\Omega}$. Therefore, the second θ function is nothing but one again. Let's solve for the root of the first argument only. The root for the second argument can be solved similarly.

$$
x_0 = -\frac{\tilde{\Omega}}{\sqrt{2\tilde{\Omega}^2 - \tilde{P}^2 - 4}}
$$

Fig. 3.1



The following statement,

$$-\frac{\tilde{\Omega}}{\sqrt{2\tilde{\Omega}^2-\tilde{P}^2-4}} < -\frac{\tilde{P}}{\tilde{\Omega}}$$

after several reversible steps, can be shown to be equivalent to the following inequality, which obviously holds.

$$(\frac{\tilde{P}}{\tilde{\Omega}}-\frac{\tilde{\Omega}}{\tilde{P}})^2+\frac{4}{\tilde{\Omega}^2} > 0$$

Thus our claim is proved. Now we can do the x integral and get the following answer,

$$\begin{aligned} Im(Int(s)) &= -\frac{2\pi^2}{(2\pi)^3}\frac{1}{\sqrt{\Omega^2-\mathbf{P}^2}}\sqrt{\frac{\Omega^2}{4}-\frac{\mathbf{P}^2}{4}-\bar{m}^2} \\ &= -\frac{\pi^2}{(2\pi)^3}\sqrt{\frac{s}{s+4\bar{m}^2}} \end{aligned}$$

where $s \equiv \Omega^2 - \mathbf{P}^2 - 4\bar{m}^2$. In the above expressions we explicitly change P^2 back to $\mathbf{P}^2$ to avoid confusions. Now we can calculate the real part through the following dispersion relation,

$$Ref(x_0) = \frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{Imf(x)}{x-x_0}dx$$

We get,

$$Re(Int(s)) = -\frac{\pi}{(2\pi)^3}P\int_0^{+\infty}\frac{ds'}{s'-s}\sqrt{\frac{s'}{s'+4\bar{m}^2}} \tag{3.13}$$

where we changed the lower integral limit to 0 since imaginary part is 0 if s' is negative, as can be seen from the δ function in equation (3.9). The analytical form of the indefinite integral above can be easily obtained,

$$\begin{aligned} \int\frac{ds'}{s'-s}\sqrt{\frac{s'}{s'+4\bar{m}^2}} &= \frac{\sqrt{s}\ln(s'-s)}{\sqrt{s+4\bar{m}^2}} + \ln\left[2\bar{m}^2+s'+\sqrt{s'(s'+4\bar{m}^2)}\right] \\ &\quad -\sqrt{\frac{s}{s+4\bar{m}^2}}\ln\left[2\bar{m}^2 s+2\bar{m}^2 s'+ss'+\sqrt{ss'(s+4\bar{m}^2)(s'+4\bar{m}^2)}\right] \end{aligned} \tag{3.14}$$

For the integral in equation (3.6), we have $\Omega^2 = -\omega_n^2$, thus we have $s < -4\bar{m}^2$ according to its definition. This means we can ignore the principal value operator in equation (3.13) and plug in $\Lambda_{s'}$, the cutoff of s', and 0 directly into equation (3.14) and we get,

$$Int(s) = Re(Int(s)) = -\frac{1}{8\pi^2}\left[\ln 2\Lambda_{s'} - \ln 2\bar{m}^2 + \sqrt{\frac{s}{s+4\bar{m}^2}}\ln\frac{\sqrt{s}-\sqrt{s+4\bar{m}^2}}{\sqrt{s}+\sqrt{s+4\bar{m}^2}}\right] + O(\frac{1}{\Lambda_{s'}})$$
$$s < -4\bar{m}^2 \tag{3.15}$$

If we renormalize the λ as in chapter II (equation (2.24)) and make use of the following relationship $\Lambda_{s'} = (2\Lambda)^2$ for on energy shell s', then

$$\begin{aligned}
&\int \frac{d^3q}{(2\pi)^3}\left[\frac{\omega_+ + \omega_-}{[-\omega_n^2 - (\omega_+ + \omega_-)^2]\omega_+\omega_-}\right] - \frac{1}{2\lambda} \\
&= -\frac{1}{4\pi^2\lambda_R} + \frac{1}{8\pi^2}\left[\ln\left(\frac{\alpha\Lambda^2}{\mu^2}\right) + \ln\bar{m}^2 - \ln(2\Lambda)^2\right] - \frac{1}{8\pi^2}\sqrt{\frac{s}{s+4\bar{m}^2}}\ln\frac{\sqrt{s}-\sqrt{s+4\bar{m}^2}}{\sqrt{s}+\sqrt{s+4\bar{m}^2}} \\
&= -\frac{1}{4\pi^2\lambda_R} - \frac{1}{8\pi^2}\left[1 + \sqrt{\frac{s}{s+4\bar{m}^2}}\ln\frac{\sqrt{s}-\sqrt{s+4\bar{m}^2}}{\sqrt{s}+\sqrt{s+4\bar{m}^2}} - \ln\frac{\bar{m}^2}{\mu^2}\right]
\end{aligned} \tag{3.16}$$

In the last step we used $\ln\alpha = 2\ln 2 - 1$. Here $s = -\omega_n^2 - \mathbf{P}^2 - 4\bar{m}^2$. For $Im(Int(s'))$ to exist, we know from equation (3.9) that s' must always be on energy shell in the sense that

$$s' \equiv \Omega^2 - \mathbf{P}^2 - 4\bar{m}^2 = (\omega'_+ + \omega'_-)^2 - \mathbf{P}^2 - 4\bar{m}^2$$

with $\omega'_\pm$ defined in equation (3.7) except for the replacement of $\mathbf{q}$ by $\mathbf{p}'$. Then in any reference system of fixed $\mathbf{P}$, we can establish the relationship between the two cutoffs above as $\mathbf{p}'$ approaches infinity.

Finally, after renormalization, the second order contribution becomes:

$$\begin{aligned}
\ln Z_2 = -\frac{1}{2}V\sum_n\int\frac{d^3P}{(2\pi)^3}\Bigg\{&\ln\frac{-1}{4\pi^2\lambda_R} \\
+&\ln\left[1 + \frac{\lambda_R}{2}\left(1 + \sqrt{\frac{s}{s+4\bar{m}^2}}\ln\frac{\sqrt{s}-\sqrt{s+4\bar{m}^2}}{\sqrt{s}+\sqrt{s+4\bar{m}^2}} - \ln\frac{\bar{m}^2}{\mu^2}\right) - 4\pi^2\lambda_R\int\frac{d^3q}{(2\pi)^3}(F_1 + F_2)\right]\Bigg\}
\end{aligned} \tag{3.17}$$

If we look at the second term in the parenthesis, which depends on s only, we can realize that the sum over n and the integration over P do not give us a finite result as they become large. The term containing F_1 and F_2 is harmless because F_1 and F_2 decay exponentially as n and P get larger and larger. In addition, the infinities due to the s dependent term is temperature dependent through the temperature dependence of $\bar{m}$. However, we can reasonably anticipate that this temperature dependence can be removed by the $\ln(\bar{m}^2/\mu^2)$ term. So let's expand the s dependent term around large $-s$ (recall s is negative). Then indeed the leading divergence from the s dependent term is transformed into a temperature independent term $\ln(\mu^2/-s)$ by $\ln(\bar{m}^2/\mu^2)$. However, the other terms which are of higher power of $1/-s$ are divergent and temperature dependent. To be explicit, let's present the result here:

$$\ln Z_2 = -\frac{1}{2}V\sum_n \int \frac{d^3P}{(2\pi)^3}\left\{\ln\frac{-1}{4\pi^2\lambda_R} + \ln\left[1+\frac{\lambda_R}{2}\left(1+\ln\frac{\mu^2}{-s}\right)\right]\right.$$
$$\left. +\ln\left(1+\frac{\frac{\lambda_R}{2}\left[\frac{2\bar{m}^2}{-s}\left(1+\ln\frac{\bar{m}^2}{-s}\right)+\frac{7\bar{m}^4}{s^2}+\frac{6\bar{m}^4}{s^2}\ln\frac{\bar{m}^2}{-s}\right]-4\pi^2\lambda_R\left[F_{s^{-3}}+\int\frac{d^3q}{(2\pi)^3}(F_1+F_2)\right]}{1+\frac{\lambda_R}{2}\left(1+\ln\frac{\mu^2}{-s}\right)}\right)\right\} \tag{3.18}$$

Here $F_{s^{-3}}$ denotes the term of order $1/(-s)^3$ and higher, which is convergent. We can readily realize that the terms of order $1/(-s)$ and $1/(-s)^2$ are not convergent and they are temperature dependent.

Before we continue with our discussion of renormalizability, we need to examine whether at the point $\bar{m}$, the point we identified as the saddle point from the constraint equation, really gives us a maximum of the partition function. In obtaining the above result (3.18), we have only formally done the functional integral of $D[\delta\sigma]$ in partition function (3.1). We have not proved that every diagonal matrix element in the exponent after diagonalization is negative. However, we can study this question now by examining equation (3.18). If under certain circumstance we get an overall negative argument in the logarithm function of equation (3.18), that means there are direction(s) at our saddle point along which the partition function is increasing. This will disqualify our calculation in last chapter as a valid approximation. However, as we can see, all the terms multiplying λ_R in the last logarithm function have upper and lower bounds once the saddle point $\bar{m}$ is fixed except for the term $1 + \ln(\mu^2 / - s)$. Therefore, as long as $\lambda_R < 0$ and it is sufficiently small, we will never encounter the case of an overall negative argument for our logarithm function (recall that $\lambda \to 0^-$). However, we can not rule out the case of $\lambda_R > 0$ since there are occasions when a large and positive λ_R can guarantee the overall nonnegativity of the logarithm function.

Coming back to renormalizability, it seems that to make the second order contribution renormalizable, we have to choose $\lambda_R = 0$ (and 0^- regarding discussion above), in which case $\ln Z_2$ reduces to a pure infinite constant, which can be dropped. This signifies that the theory becomes free or $\lambda\phi^4$ theory is trivial. However, we may need to renormalize our theory in a different way which could renormalize leading and second order contributions as a whole rather than separately. In more than four space-time dimensions there are rigorous proofs of triviality. However, the proofs fail for four dimensions ([9] and [10]). The lattice calculation suggests that although the theory is trivial, namely, $\lambda_R = 0$, it is not entirely free in the sense that spontaneous symmetry can occur ([11]). With all these in mind, we will keep all the calculations up to first order and perform similar calculations in the following chapters. The question of whether the theory is free is left open at this moment.

Chapter IV

Effective Potential of $\lambda\phi^4$ Theory

IV.1 Effective Potential

We start with the generating functional for the temperature Green's function

$$\begin{aligned} Z[J] &= N' \int_{peri} D[\phi] exp \left[\int_0^\beta d\tau \int d^3x (\mathcal{L}_E + J\phi) \right] \\ &= N' \int_{peri} D[\phi] exp \left\{ - \int d^4x \left[\frac{1}{2} (\frac{\partial \phi}{\partial \tau})^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 + \lambda \phi^4 - J(x)\phi(x) \right] \right\} \end{aligned} \tag{4.1}$$

As before, we can remove the $\lambda\phi^4$ term by introducing an auxiliary variable $\sigma(x)$. In the following I assume the $\lambda < 0$, since the case $\lambda > 0$ can be dealt with similarly, as discussed at the end of section II.1.

$$\begin{aligned} Z[J] = N \int_{peri} D[\phi] D[\sigma] exp \Bigg\{ \int d^4x \Bigg[&-\phi(x) \left(-\frac{1}{2} \frac{\partial^2}{\partial \tau^2} - \frac{1}{2} \nabla^2 + \frac{1}{2} m^2 - \sigma(x) \right) \phi(x) \\ &+ J(x)\phi(x) + \frac{1}{4\lambda} \sigma^2(x) \Bigg] \Bigg\} \end{aligned} \tag{4.2}$$

Here $N \equiv N' (\det A)^{-\frac{1}{2}}$ as in Chapter II (see equation (2.6)).

After doing the $\phi(x)$ functional integral, which is now quadratic in $\phi(x)$, we get,

$$\begin{aligned} Z[J] &= N \int_{peri} D[\sigma] exp \left[-\frac{1}{2} tr\ln[(-\frac{\partial^2}{\partial \tau^2} - \nabla^2 + m^2 - 2\sigma(x))\delta(x - x')] + \int d^4x \frac{1}{4\lambda} \sigma^2(x) \right] \\ &\quad \times exp \left[\frac{1}{2} \int d^4x \int d^4x' J(x') \frac{1}{(-\frac{\partial^2}{\partial \tau^2} - \nabla^2 + m^2 - 2\sigma(x))\delta(x - x')} J(x) \right] \\ &= N \int_{peri} D[\sigma] exp \left[-\frac{1}{2} tr\ln[\hat{p}^2 + m^2 - 2\hat{\sigma}] + \int d^4x \frac{1}{4\lambda} \sigma^2(x) \right] \\ &\quad \times exp \left[\frac{1}{2} \int d^4x d^4x' < x | \hat{J} \frac{1}{\hat{p}^2 + m^2 - 2\hat{\sigma}} \hat{J} | x' > \right] \end{aligned} \tag{4.3}$$

Again, we have adopted the operator notation in order to facilitate the derivations later. Expanding the exponent around σ_0, the saddle point, we get,

$$\begin{aligned} Z[J] = N \int_{peri} D[\delta\sigma] exp\left[-\frac{1}{2}tr\ln[\hat{p}^2 + m^2 - 2\hat{\sigma}_0] + \int d^4x \frac{1}{4\lambda}\sigma_0^2(x)\right] \\ \times\, exp\left[\frac{1}{2}\int d^4x d^4x' < x|\hat{J}\frac{1}{\hat{p}^2 + m^2 - 2\hat{\sigma}_0}\hat{J}|x' > + tr[\frac{\delta\hat{\sigma}}{\hat{p}^2 + m^2 - 2\hat{\sigma}_0}]\right] \\ \times\, exp\left[\frac{1}{2\lambda}\int d^4\sigma_0(x)\delta\sigma(x) + \frac{1}{2}\int d^4x d^4x' < x|\hat{J}\frac{2}{(\hat{p}^2 + m^2 - 2\hat{\sigma}_0)^2}\hat{\delta\sigma}\hat{J}|x' >\right] \\ \times\, exp\left[\frac{1}{4\lambda}\int d^4x \delta\sigma^2(x) + tr[\frac{1}{\hat{p}^2 + m^2 - 2\hat{\sigma}_0}\delta\hat{\sigma}\frac{1}{\hat{p}^2 + m^2 - 2\hat{\sigma}_0}\delta\hat{\sigma}]\right] \\ \times\, exp\left[2\int d^4x d^4x' < x|\hat{J}\frac{1}{(\hat{p}^2 + m^2 - 2\hat{\sigma}_0)^2}\delta\hat{\sigma}\frac{1}{\hat{p}^2 + m^2 - 2\hat{\sigma}_0}\delta\hat{\sigma}\hat{J}|x' >\right] \end{aligned} \tag{4.4}$$

Let's first look at the terms which are of first order in $\delta\sigma$.

$$\begin{aligned} &\frac{1}{2}\int d^4x d^4x' < x|\hat{J}\frac{2}{(\hat{p}^2 + m^2 - 2\hat{\sigma}_0)^2}\hat{\delta\sigma}\hat{J}|x' > \\ &= \frac{1}{2}\int d^4x d^4x' d^4p J(x) < x|p > \frac{2}{(p^2 + m^2 - 2\sigma_0)^2} < p|x' > \delta\sigma(x')J(x') \end{aligned}$$

Where we have assumed translational invariance for σ_0 by assuming it to be a constant and $\delta\hat{\sigma}$ and $\hat{J}$ are local operators in configuration space. If we further assume J is x independent, then we can simplify the above expression to

$$\begin{aligned} &\frac{J^2}{2}\int d^4x d^4x' d^4p \frac{1}{(2\pi)^3\beta}e^{ip(x'-x)}\frac{2}{(p^2 + m^2 - 2\sigma_0)^2}\delta\sigma(x') \\ &= \frac{J^2}{2}\int d^4x' d^4p \delta^4(p) e^{ipx'}\frac{2}{(p^2 + m^2 - 2\sigma_0)^2}\delta\sigma(x') \\ &= \int d^4x \frac{J^2}{(m^2 - 2\sigma_0)^2}\delta\sigma(x) \end{aligned}$$

By definition, the terms of first order in $\delta\sigma$ should sum to zero, thus we get,

$$< x|\frac{1}{\hat{p}^2 + m^2 - 2\hat{\sigma}_0}|x > + \frac{1}{2\lambda}\sigma_0 + \frac{J^2}{(m^2 - 2\sigma_0)^2} = 0 \tag{4.5}$$

Upon evaluating the matrix element, we obtain

$$\sigma_0 = \int \frac{d^3p}{(2\pi)^3}\frac{-2\lambda}{\omega}(\frac{1}{2} + \frac{1}{e^{\beta\omega} - 1}) + \frac{-2\lambda}{\bar{m}^4}J^2 \tag{4.6}$$

With

$$\omega \equiv \sqrt{\mathbf{p}^2 + \bar{m}^2} \equiv \sqrt{\mathbf{p}^2 + m^2 - 2\sigma_0} \tag{4.7}$$

This equation implicitly gives us the stationary point of σ_0 for a given external source J.

In the same spirit of calculating the first order term, we can calculate the zeroth and second order terms. In the end we get,

$$\begin{aligned} Z[J] = exp&\left[V\int\frac{d^3p}{(2\pi)^3}[-\frac{1}{2}\beta\omega - \ln(1-e^{-\beta\omega})] + \frac{1}{4\lambda}\beta V\sigma_0^2 + \frac{1}{2}\beta V\frac{J^2}{\bar{m}^2}\right] \\ &\times \int_{peri} D[\delta\sigma]exp\left[tr[\frac{1}{\hat{p}^2+m^2-2\hat{\sigma}_0}\delta\hat{\sigma}\frac{1}{\hat{p}^2+m^2-2\hat{\sigma}_0}\delta\hat{\sigma}]\right] \\ &\times exp\left[\frac{1}{4\lambda}\int d^4x\delta\sigma^2(x) + \frac{2J^2}{\bar{m}^4}\int d^4x d^4x' < x|\delta\hat{\sigma}\frac{1}{\hat{p}^2+m^2-2\hat{\sigma}_0}\delta\hat{\sigma}|x'>\right] \end{aligned} \tag{4.8}$$

We approximate the generating functional by keeping only the zeroth order term and get the following expression for the exponent of the $Z[J]$, or the generating functional for the connected Green's function.

$$W[J] = V\int\frac{d^3p}{(2\pi)^3}[-\frac{1}{2}\beta\omega - \ln(1-e^{-\beta\omega})] + \frac{1}{4\lambda}\beta V\sigma_0^2 + \frac{1}{2}\beta V\frac{J^2}{\bar{m}^2} \tag{4.9}$$

Note the expression above differs from equation (2.13) only by the last term. Defining renormalized mass μ and coupling constant λ_R as in equation (2.24), we get the renormalized expression for $W[J]$,

$$W[J] = -\beta V\left[F(\beta,\bar{m}^2) + \frac{1}{64\pi^2}\bar{m}^4(\ln\frac{\bar{m}^2}{\mu^2} - \frac{1}{2}) - \frac{1}{32\pi^2}\frac{(\bar{m}^2-\mu^2)^2}{\lambda_R} - \frac{J^2}{2\bar{m}^2}\right] \tag{4.10}$$

As for the case without external source, the first order constraint equation can also be renormalized since it differs from equation (2.16) in II.2 only by a harmless source term. The explicit expression for the renormalized constraint equation is,

$$\bar{m}^2 = \mu^2 + \frac{1}{2}\lambda_R\bar{m}^2\ln\frac{\bar{m}^2}{\mu^2} - 4\pi^2\lambda_R f(\beta,\bar{m}^2) + 8\pi^2\lambda_R\frac{J^2}{\bar{m}^4} \tag{4.11}$$

As in Chapter II, taking partial derivative to equation (4.10) with respect to $\bar{m}^2$ gives us equation (4.11). In the above expression for $W[J]$, $\bar{m}$ implicitly depends on J through equation (4.11). This equation solves for σ_0 for a given J, and $\bar{m}$ depends on σ_0 through the defining equation (4.7).

The classical field $\bar{\phi}$ is defined as

$$\bar{\phi} \equiv \frac{\delta W[J]}{\delta J} = \frac{\partial W}{\partial \bar{m}^2}\frac{d\bar{m}^2}{dJ} + \frac{\partial W}{\partial J} \tag{4.12}$$

Since the $\frac{\partial W}{\partial \bar{m}^2}$ gives us the constraint equation, the first term above vanishes. Consequently,

$$\bar{\phi} = \frac{J}{\bar{m}^2} \tag{4.13}$$

and we can rewrite the constraint equation as,

$$\bar{m}^2 = \mu^2 + \frac{1}{2}\lambda_R \bar{m}^2 \ln\frac{\bar{m}^2}{\mu^2} - 4\pi^2\lambda_R f(\beta, \bar{m}^2) + 8\pi^2\lambda_R\bar{\phi}^2 \tag{4.14}$$

From this relationship between $\bar{\phi}$ and J we can get effective action defined by

$$\Gamma[\bar{\phi}] \equiv W[J] - \int d^4x J\bar{\phi} \tag{4.15}$$

and the effective potential,

$$\begin{aligned} V(\bar{\phi}) &= -\frac{1}{\beta V}\Gamma(\bar{\phi}) \\ &= \frac{1}{2}\bar{m}^2\bar{\phi}^2 + F(\beta, \bar{m}^2) + \frac{1}{64\pi^2}\bar{m}^4(\ln\frac{\bar{m}^2}{\mu^2} - \frac{1}{2}) - \frac{1}{32\pi^2}\frac{(\bar{m}^2 - \mu^2)^2}{\lambda_R} \end{aligned} \tag{4.16}$$

It is worth noting now that if we set $\frac{\partial V(\bar{\phi})}{\partial \bar{m}^2} = 0$, then we can recover the constraint equation (4.14). Therefore, solving the constraint equation is nothing but finding the extrema of $V(\bar{\phi})$ with respect to $\bar{m}^2$.

If we want to numerically plot the effective potential as a function of $\bar{\phi}$, then we need first fix a temperature, then we solve for $\bar{m}$ for all possible values of J through the constraint equation (4.11). Then using equation (4.13) we can relate $\bar{m}$ to $\bar{\phi}$. With this relationship equation (4.16) gives the effective potential as a function of $\bar{\phi}$.

However, the constraint equation (4.11) does not have solutions to $\bar{m}$ when J is above a certain value. We have known that taking partial derivative to equation (4.10) with respect to $\bar{m}^2$ gives us equation (4.11). Therefore, if we plot W[J] as a function of $\bar{m}^2$, the extrema on the plot are solutions to constraint equation (4.11). As an example, in the case of $T = 1$, $\lambda_R = 1$ and $\mu^2 = 10$, the constraint equation has solution(s) only when J is between -6.9 and 6.9 (Fig. 4.1). The corresponding range for $\bar{\phi}$ can be obtained by the relationship (4.13) (Fig. 4.2). If we plot $W[J]$ as a function of $\bar{m}$, then we can realize that when J is outside the above range, we do not have any nonzero maxima to the $W[J]$ curve (Fig. 4.3). Namely, we do not have a point around which to do the saddle point integration except for the $\bar{m} = 0$ peak. When $J \neq 0$, it seems that we should always expand our generating functional around this $\bar{m} = 0$ peak since it is absolutely the global maximum (actually infinity), even when J is within the range of $[-6.9, 6.9]$. Although this peak is a singular point of $W[J]$, suppose we can, nevertheless, make a saddle point expansion around $\bar{m} = 0$, then we have $d\bar{m}^2/dJ = 0$ for $\bar{m}$ identically being zero, and consequently we have $\bar{\phi} = J/\bar{m}^2$ as before. In other words, all the formal derivations from (4.10) to (4.16) go through even for $\bar{m} = 0$. However, if we insert $\bar{\phi} = J/\bar{m}^2$ into effective potential (4.16), then in the limit $\bar{m} \to 0$, $\bar{\phi} \to \infty$ and $V(\bar{\phi}) \to +\infty$, which is certainly not interesting to us. Therefore, we can safely ignore the $\bar{m} = 0$ peak on the $W[J]$ curve when $J \neq 0$. When $J = 0$, this analysis no longer holds and we will come back to $J = 0$ case later.

Besides the $\bar{m} = 0$ peak, we still have two extrema in Fig. 4.3 when J is between -6.9 and 6.9. Ideally, we want to expand our generating functional around its maximum. Nevertheless, we can keep both extrema (meaning two solutions to $\bar{m}$) and plot the effective potential from both roots. The result is shown in Fig. 4.4. The dashed curve is from the minimum in Fig. 4.3 and the solid curve is from the maximum. The result confirmed our intuition that we should keep the root corresponding to the maximum because this branch of solutions result in a lower effective potential.

Now the question which needs to be answered is how we construct the effective potential for J outside the above range. Note that J is actually the total derivative of $V(\bar{\phi})$ with respect to $\bar{\phi}$ because of the Legendre transformation. Therefore, we conclude that when J is too large, there are no points on the effective potential curve $V(\bar{\phi})$ where the derivative with respect to $\bar{\phi}$ equals to J. Namely, there exists an upper bound to the steepness of the effective potential curve. That is why the constraint equation (4.11) has no solutions to $\bar{m}$ when J is larger than a certain critical value, which is the maximum steepness of the effective potential $V(\bar{\phi})$. Recall that the constraint equations (4.11) can only give us the nonzero $\bar{m}$ solution to the saddle point. Thus, when J becomes larger than the maximum steepness of the effective potential, the only saddle point we are left with is $\bar{m} = 0$. We have shown that the relationship (4.13) still holds even when $\bar{m} = 0$. However, this relationship should be interpreted as $J(\bar{\phi}) = \bar{\phi}\bar{m}^2(\bar{\phi}) = 0$ when $\bar{m} = 0$ since in order to obtain the effective action $\Gamma(\bar{\phi})$, J (and $\bar{m}$) in the Legendre Transformation (4.15) should be regarded as a function of $\bar{\phi}$. In other words, when $\bar{m} = 0$, both $\bar{m}$ and J are independent of $\bar{\phi}$ and remain zero for any $\bar{\phi}$ when they are interpreted as a function of $\bar{\phi}$. In fact $\bar{\phi}$ can be all the values outside the J region where (4.11) has nonzero solution to $\bar{m}$. With $\bar{m} = 0$ and $\bar{\phi}$ any finite value, equation

(4.16) gives a flat effective potential. Because J is the steepness of the effective potential, this flat effective potential means $J = 0$, which agrees with our interpretation above.

In summary, the effective potential is given by (4.16) when J is within the region where (4.11) has nonzero solutions to $\bar{m}$. Outside this J region, the effective potential is given by the constant $V(\bar{m} = 0)$.

The above conclusion is equivalent to saying that the effective potential is given by the closed system of two equations: (4.14) and (4.16). Note that equation (4.14) is essentially the same as equation (4.11) with the only difference that $\bar{\phi}$ is replaced by J through the relationship (4.13). Thus, for any J that gives us nonzero solutions to $\bar{m}$ from equation (4.11), we have a corresponding $\bar{\phi}$ that gives us nonzero solutions to $\bar{m}$ from equation (4.14), and vice versa. For a given temperature, the equation (4.14) has solutions to $\bar{m}$ for only a finite interval of $\bar{\phi}$. This property can be phrased the other way around: for any fixed $\bar{\phi}$, there is only a finite interval of temperature during which the constraint equation has a solution to $\bar{m}$. But this is not surprising at all considering the discussion at the end of II.3. Here we have an effective potential (4.16) which is even in $\bar{m}$, thus for any finite $\bar{\phi}$, $\bar{m} = 0$ is an extrema on the V vs. $\bar{m}$ plot. The constraint equation (4.14) only keeps track of the non-zero extrema. Therefore, as in chapter II, for $\bar{\phi}$ outside the above range, we can simply set $\bar{m} = 0$ in equation (4.16) and get the effective potential.

At a given temperature, the constraint equation (4.14) usually has two solutions to the $\bar{m}$ for each value of $\bar{\phi}$ (This is always true in the case of $\lambda_R > 0$ and true for a certain interval of temperatures in the case of $\lambda_R < 0$). This can also be understood the other way around: for a given $\bar{\phi}$, the constraint equation has two solutions for $\bar{m}$ at every temperature. Again, this is what we have encountered in II.3. Admittedly, here we generally have $\bar{\phi} \neq 0$. But qualitatively, the discussions in II.3 are unchanged.

Now if we look at Fig. 4.3, an interesting question arises: W[J] certainly attains a higher maximum at $\bar{m} = 0$ (although it is a singular point) than at the solution(s) to the constraint equation (4.11) when $J \neq 0$, why should we bother to consider the solution(s) to the constraint equation in the first place? Should we always pick the $\bar{m} = 0$ point as our saddle point as long as $J \neq 0$? We have several arguments to deal with this question. First of all, as we have pointed out earlier, if we literally let $\bar{m} \to 0$ for fixed nonzero J, we will obtain a single point for the effective potential: $V \to \infty$ at $\bar{\phi} \to \infty$, which is completely uninteresting. Secondly, if we want to set $\bar{m} = 0$, then the relationship $J(\bar{\phi}) = \bar{\phi}\bar{m}^2(\bar{\phi})$ means J is independent of $\bar{\phi}$ and remains zero for any value of $\bar{\phi}$. This line of arguments means $J = 0$ must hold when $\bar{m} = 0$ and it leads to the flat part of our effective potential. Indeed, if $J = 0$ when $\bar{m} = 0$, for a certain $\bar{\phi}$, the $W[J]$ curve will not blow up at $\bar{m} = 0$. In addition, according to this line of arguments, we have already considered the case of $\bar{m} = 0$ and we have already obtained all the effective potential curves by regarding all the extrema in Fig. 4.3 as potential candidates for saddle points. We take the point of view that the "true" saddle point should be the one that gives a lower effective potential. In other words, the effective potential curve has more "determining power" than the $W[J]$ curve in Fig. 4.3.

However, if one insists that we should consider the combination of $J \neq 0$ and $\bar{m} \to 0$, then our "soft" line of arguments is that the curves in Fig. 4.3 give us an incomplete picture. In Fig. 4.3, the $\bar{m}$ is assumed to be independent of positions in the configuration space. Therefore, as we move toward $\bar{m} = 0$, the $\bar{m}(x)$ has to be changed uniformly across all positions. In terms of functional integral (4.4), this means we are looking at one direction in an infinite dimensional space. Although the exponent $W[J]$ is growing in this direction as we move toward $\bar{m} = 0$, it could be a very sharp "ridge" in the whole infinite dimensional "terrain" and consequently contributes little to the whole functional integral. In short, they could have very small measure. To supplement our "soft" arguments, we present our "hard" line of calculations here. First of all, as $\bar{m}$ moves toward 0, the linear term in the functional integral (4.4) does not vanish any more. If we keep this term and complete a square with the quadratic term, then we can obtain an additional contribution to the leading order $W[J]$. To be specific, the functional integral (4.4) can be rewritten as (referring to III.1):

$$Z[J] = N \int_{peri} D[\delta\sigma] exp\left[W_0[J] + \int d^4x \delta\sigma(x) M_1 - \frac{1}{2}\int d^4x d^4x' \delta\sigma(x) M_2 \delta\sigma(x')\right] \quad (4.17)$$

with $W_0[J]$ defined as the leading order contribution given previously by (4.9) and

$$M_1 =< x|\frac{1}{L(\hat{p}}|x> + \frac{\sigma_0}{2\lambda} + \frac{J^2}{\bar{m}^4} \quad (4.18)$$

$$M_2 =< x|\left[\frac{-2}{(2\pi)^4} f(\hat{P}) - \frac{4J^2}{\bar{m}^4}\frac{1}{L(\hat{P})} - \frac{1}{2\lambda}\right]|x'> \quad (4.19)$$

Upon completing the square of the linear and quadratic terms, we obtain an additional term which is essentially of the nature of $M_1^2/(2M_2)$. As $\bar{m}$ approaches zero,

$$\frac{M_1^2}{M_2} \sim \frac{J^4/\bar{m}^8}{-J^2/\bar{m}^4} \sim -\frac{J^2}{\bar{m}^4}$$

Since $W_0[J] \sim J^2/\bar{m}^2$ as $\bar{m}$ approaches zero, $M_1^2/(2M_2)$ will dominate $W_0[J]$ and consequently their sum decreases as $\bar{m} \to 0$. Therefore, the inclusion of this additional term will make $\bar{m} = 0$ a global minimum rather than a maximum. In addition, we do not have a "ridge" at all as we move $\bar{m}$ toward zero uniformly across all positions in configuration space. However, as $\bar{m} \to 0$, $M_2 \to -\infty$, meaning the quadratic term in (4.17) will not give us a convergent functional integral. To ensure convergence, it seems that we have to let J go to zero as $\bar{m}$

approaches zero, while maintaining the ratio $J/\bar{m}^2 = \bar{\phi}$ fixed. Then this comes back to our previous arguments that J must be zero when $\bar{m}$ is zero.

According to equation (2.24), where the renormalized parameters are defined, when $\lambda_R \neq 0$, $\lambda \to 0^-$. This seems to be the only scenario when we can obtain a nontrivial, bounded (from below) renormalized effective potential. However, intuition suggests that the theory in this case is unstable if we look at the bare classical potential. In addition, the initial functional integral (2.3) defining the partition function is ill defined in this case. However, we can overcome this difficulty by first study the case when $\lambda > 0$. As discussed in section II.1, when $\lambda > 0$, all our derivations so far can be carried out similarly with the same renormalization prescription. Then we can regard what we got for $\lambda \to 0^-$ as an analytic continuation of the $\lambda > 0$ case. Then a question naturally arises as to what this continuation represents. In other words, whether it represents the true $\lambda\phi^4$ theory or something else. However, if we adopt the language of the effective field theories, then a negative λ is perfectly legitimate ([3]). Since we will be always ignorant of the physics at arbitrarily small distances, we are allowed to treat $\lambda\phi^4$ as an effective theory at a certain low energy scale characterized by M. Below energy scale M, we have $\lambda\phi^4$ theory; Above that scale, we have another unknown, underlying theory. To be specific, suppose the unknown, underlying theory is fermionic and has the following general Lagrangian,

$$\mathcal{L} = \mathcal{L}(\psi, \bar{\psi}, M)$$

As temperature drops, the fermionic fields get coupled and $\bar{\psi}\psi$ plays the role of our ϕ field. Then effectively, the underlying theory can have the following Lagrangian below the characteristic energy scale:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 + A_1\phi^6 + A_2\phi^8 + \dots$$

On dimensional grounds, $A_1 \sim M^{-2}$ and $A_2 \sim M^{-4}$. When $\phi \lesssim M$, we obtain the $\lambda\phi^4$ theory. When $\phi \gtrsim M$, the otherwise suppressed terms in low energy case may dominate the $\lambda\phi^4$ term and bends over the potential curve back up again for sufficiently large ϕ. Consequently, our theory can be stable when $\lambda < 0$. In addition, if this is the case, then the functional integral (2.3) is well defined. In summary, the energy scale M introduced in the language of the effective field theories serves as the cutoff both for the momentum and for the ϕ field. We only have $\lambda\phi^4$ theory if our energy scale is below M and $\phi \lesssim M$. In this case, the $\phi = 0$ minimum corresponds to a metastable state. The decay of this metastable state to the true vacuum is hindered by a large potential barrier. As M becomes sufficiently large, this barrier becomes so wide that the decay width approaches zero. Namely, it takes an infinite amount of time for this metastable state to decay. In other words, this metastable state effectively becomes stable. It is in this spirit that we consider the $\phi = 0$ minimum as our vacuum when $\lambda < 0$.

Finally, we need to confirm that when $\lambda > 0$, we get a free theory. Recall that the bare parameters are related to the renormalized ones through

$$\lambda(\Lambda) = \frac{\lambda_R}{\frac{1}{2\pi^2} - \frac{\lambda_R}{4\pi^2}\ln\frac{\alpha\Lambda^2}{\mu^2}} \qquad m^2(\Lambda) = \frac{\mu^2 - \lambda_R\Lambda^2/2\pi^2}{\frac{1}{2\pi^2} - \frac{\lambda_R}{4\pi^2}\ln\frac{\alpha\Lambda^2}{\mu^2}} \tag{4.20}$$

It is easy to see that only $\lambda_R = 0$ allows the possibility of $\lambda > 0$. If we denote λ by $f(\Lambda)$, which is a general function of cutoff Λ, then our renormalization prescription,

$$\begin{aligned} \lambda_R &\equiv \frac{f(\Lambda)/(2\pi^2)}{1 + \frac{f(\Lambda)}{4\pi^2}\ln\frac{\alpha\Lambda^2}{\mu^2}} \\ \mu^2 &\equiv \frac{m^2 + f(\Lambda)\Lambda^2/(2\pi^2)}{1 + \frac{f(\Lambda)}{4\pi^2}\ln\frac{\alpha\Lambda^2}{\mu^2}} \end{aligned} \tag{4.21}$$

says that when $f(\Lambda) > 0$, either finite or not, $\lambda_R \to 0^+$. This corresponds to a free theory and it is easy to realize that the corresponding effective potential is given by a flat curve

$$V(\bar{\phi}) = \mathcal{F}(\beta, 0)$$

which corresponds to the endpoint $\bar{m} = 0$. The solution from constraint equation (4.14) is $\bar{m}^2 = \mu^2$ when $\lambda_R = 0$, which always gives a higher effective potential than the $\bar{m} = 0$ endpoint.

It is also interesting to note that when λ is negative and finite, we also have $\lambda_R \to 0^+$, which gives us a free theory again.

As a comparison with the usual perturbative approach, we write down the result of one-loop effective potential ([8])

$$V(\bar{\phi}) = \frac{1}{2}\mu^2\bar{\phi}^2 + \lambda_R\bar{\phi}^4 + \frac{1}{64\pi^2}\left[\bar{m}^4 \ln\frac{\bar{m}^2}{\mu^2} - \frac{3}{2}\left(\bar{m}^2 - \frac{2}{3}\mu^2\right)^2\right] + F(\beta, \bar{m}^2) \tag{4.22}$$

with

$$\bar{m}^2 = \mu^2 + 12\lambda_R\bar{\phi}^2 \tag{4.23}$$

The first two terms in this one-loop effective potential is the tree level effective potential. The third term and last term are due to quantum and thermal fluctuations respectively around the classical field. Note the $\bar{m}^2$ does not depend on temperature in the one-loop effective potential. Therefore, thermal fluctuations are separated from quantum fluctuations. In contrast, the effective potential (4.16) does not exhibit the tree level expression explicitly and the quantum and thermal fluctuations are no longer segmented because the effective mass $\bar{m}^2$ depends on temperature.

We want to study the weak coupling limit of our nonperturbative effective potential and compare this limit with the perturbative result. We will only study the zero temperature case for simplicity. In the next section we will see that at zero temperature, the range of $\bar{\phi}$ over which the constraint equation (4.14) has solution(s) extends to infinity as $\lambda_R \to 0^-$. In addition, essentially all these solutions are lower than the flat part of the effective potential given by $\bar{m} = 0$. Therefore, we can concentrate on the constraint equation (4.14) and ignore the special case of $\bar{m} = 0$. In the weak coupling limit, $\bar{m}^2 \to \mu^2$ in the constraint equation (4.14). Therefore, we can express the difference $\bar{m}^2 - \mu^2$ as a power series of λ_R:

$$\bar{m}^2 - \mu^2 = \sum_{n=1}^{\infty} a_n \lambda_R^n \tag{4.24}$$

with

$$a_1 = 8\pi^2\bar{\phi}^2 \qquad a_2 = \frac{a_1}{2} \qquad a_3 = 2\pi^2\bar{\phi}^2 + \frac{16\pi^4\bar{\phi}^4}{\mu^2} \qquad a_4 = \pi^2\bar{\phi}^2 + \frac{24\pi^4\bar{\phi}^4}{\mu^2} - \frac{128\pi^6\bar{\phi}^6}{3\mu^4}$$

With this power series expansion, we can easily get the weak coupling expansion of our effective potential (4.16):

$$V(\bar{\phi}) = \frac{1}{2}\mu^2\bar{\phi}^2 + 2\pi^2\lambda_R\bar{\phi}^4 + \pi^2\bar{\phi}^4\lambda_R^2 + \left[\frac{\pi^2}{2}\bar{\phi}^4 + \frac{8\pi^4}{3\mu^2}\bar{\phi}^6\right]\lambda_R^3 + O(\lambda_R^4) \tag{4.25}$$

In our definition of the renormalized coupling constant (2.24), we could have absorbed a factor of $2\pi^2$ into λ_R, then our effective potential in the weak coupling limit will be:

$$V(\bar{\phi}) = \frac{1}{2}\mu^2\bar{\phi}^2 + \lambda_R\bar{\phi}^4 + \frac{\lambda_R^2}{4\pi^2}\bar{\phi}^4 + \left[\frac{\bar{\phi}^4}{16\pi^4} + \frac{\bar{\phi}^6}{3\pi^2\mu^2}\right]\lambda_R^3 + O(\lambda_R^4) \tag{4.26}$$

Therefore, this way of defining the renormalized coupling constant can recover the functional form of the classical potential at the tree level.

The weak coupling limit of the one-loop effective potential is:

$$V(\bar{\phi}) = \frac{1}{2}\mu^2\bar{\phi}^2 + \lambda_R\bar{\phi}^4 + \frac{9\lambda_R^3\bar{\phi}^6}{\pi^2\mu^2} + O(\lambda_R^4) \tag{4.27}$$

Clearly, our effective potential differs from the one-loop result. The main reason for this difference is that in obtaining the one-loop effective potential, an unjustified interchange of limits is taken. In other words, an ultraviolet cutoff is first introduced to regularize the theory and then the limit $\lambda_R \to 0$ is taken for a fixed cutoff. This cutoff is then sent to infinity after the limit $\lambda_R \to 0$ has already be taken. However, the true $\lambda\phi^4$ theory has no intrinsic cutoff in it. If a cutoff is introduced temporarily to regularize the theory, it must first be removed before any physical interpretations can be made (including any assertions about weak coupling limit). Clearly, the way that the one-loop effective potential is obtained violated this principle ([3]). In addition, like the free energy density in Chapter II, our effective potential (4.16) has the desired property of renormalization group invariance, which means that we are always dealing with one single $\lambda\phi^4$ theory, no matter how we choose to parametrize it. In contrast, the one-loop effective potential breaks this invariance and consequently leads to the complication of the "renormalization-scheme-dependence", which suggests the physical content of the $\lambda\phi^4$ theory is different if we parametrize it differently.

The nonperturbative effective potential from the Functional Schroedinger Picture approach is given by ([6]):

$$V(\bar{\phi}) = \frac{1}{2}\bar{m}^2\bar{\phi}^2 + \frac{1}{64\pi^2}\bar{m}^4\left(\ln\frac{\bar{m}^2}{\mu^2} - \frac{1}{2}\right) - \frac{(\bar{m}^2 - \mu^2)^2}{2\lambda_R} \tag{4.28}$$

with

$$\bar{m}^2 = \mu^2 + \frac{\lambda_R}{2}\left(\bar{\phi}^2 + \frac{1}{16\pi^2}\bar{m}^2\ln\frac{\bar{m}^2}{\mu^2}\right) \tag{4.29}$$

The above results agree perfectly with our effective potential at zero temperature by reparametrizing our λ_R by a factor of $1/16\pi^2$.

The Gaussian Effective Potential is ([3]):

$$V(\bar{\phi}) = \frac{1}{2}\bar{m}^2\bar{\phi}^2 + \frac{1}{64\pi^2}\bar{m}^4 \ln\frac{\bar{m}^2}{\mu^2} - \frac{(\bar{m}^2 - \mu^2)\mu^2}{64\pi^2} - \left(3 + \frac{8\pi^2}{\lambda_R}\right)(\bar{m}^2 - \mu^2)^2 \tag{4.30}$$

with

$$\bar{m}^2 = \mu^2 + \frac{\lambda_R}{\lambda_R + 4\pi^2}\left[\bar{m}^2 \ln\frac{\bar{m}^2}{\mu^2} + 16\pi^2\bar{\phi}^2\right] \tag{4.31}$$

This effective potential does not exhibit as striking a similarity to our effective potential as that of the Functional Schroedinger Picture. However, this effective potential reproduces all the qualitative features of our effective potential ([3]). The Gaussian Effective Potential reached the same conclusion that $\lambda \to 0^-$ is the only surviving scenario of the $\lambda\phi^4$ theory. Its potential looks exactly like ours in Fig. 4.4. However, for $\lambda > 0$, we conclude that the theory is free, whereas the Gaussian Effective Potential concludes that the theory is unbounded from below and not necessarily free.

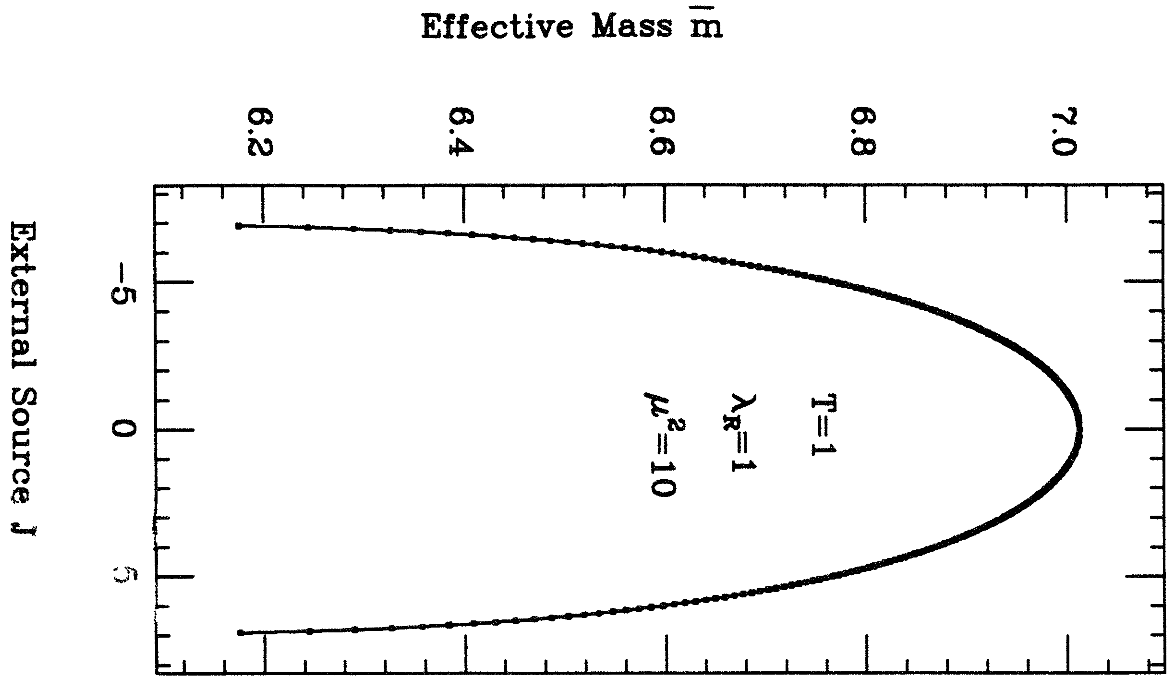

Effective Mass $\bar{m}$
7.0
6.8
6.6
6.4
6.2
T=1
λ_R=1
μ^2=10
−5
0
5
External Source J

Fig. 4.1

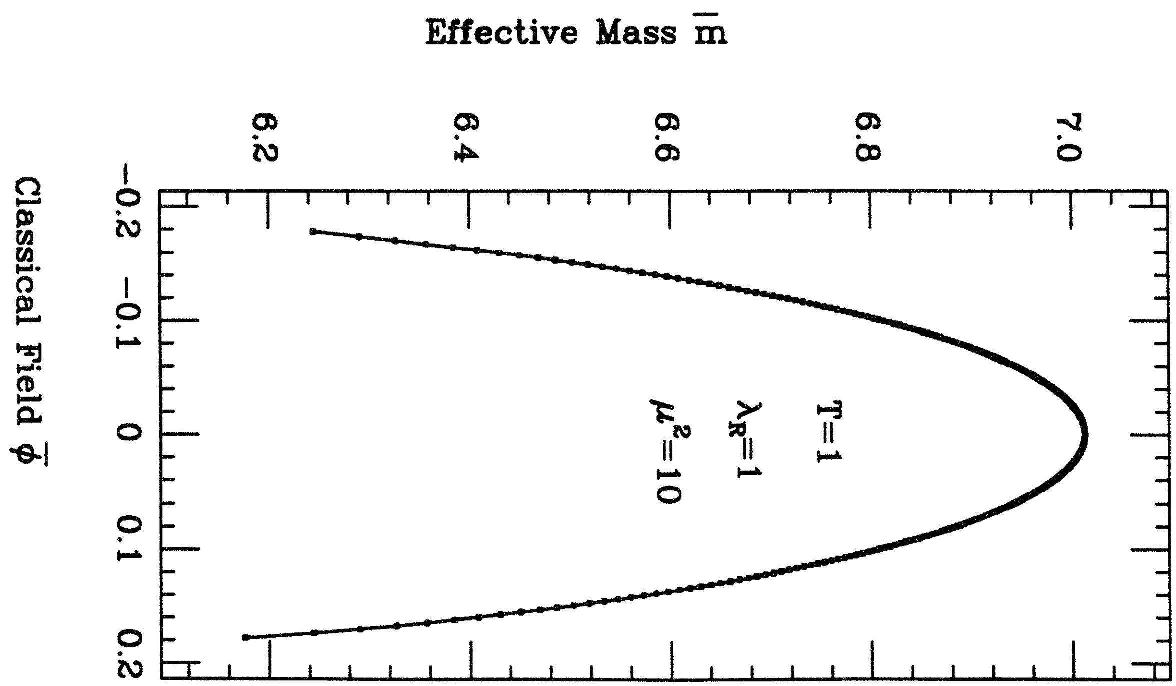

Effective Mass $\bar{m}$
7.0
6.8
6.6
6.4
6.2
T=1
λ_R=1
μ^2=10
−0.2
−0.1
0
0.1
0.2
Classical Field $\bar{\phi}$

Fig. 4.2

Fig. 4.3

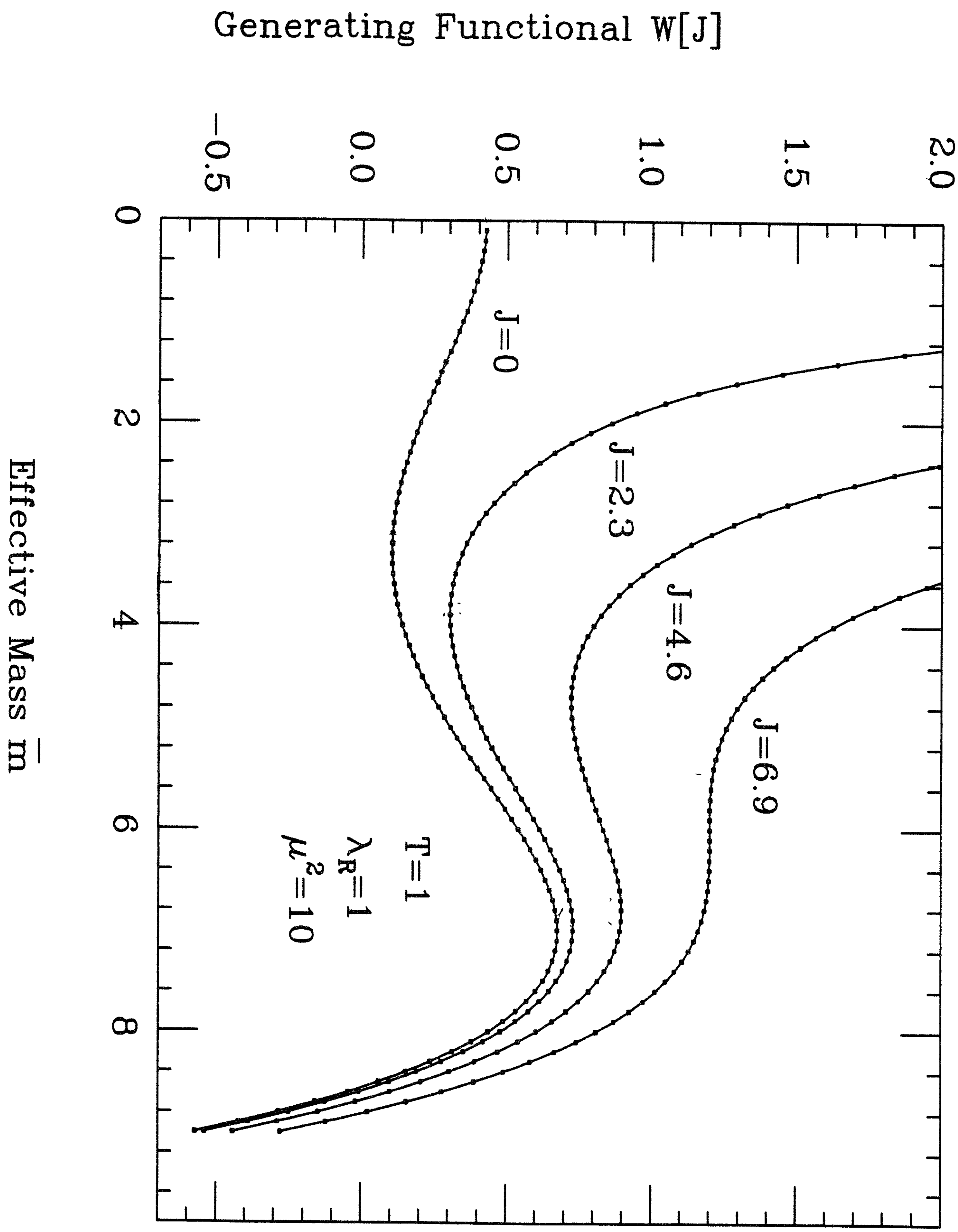
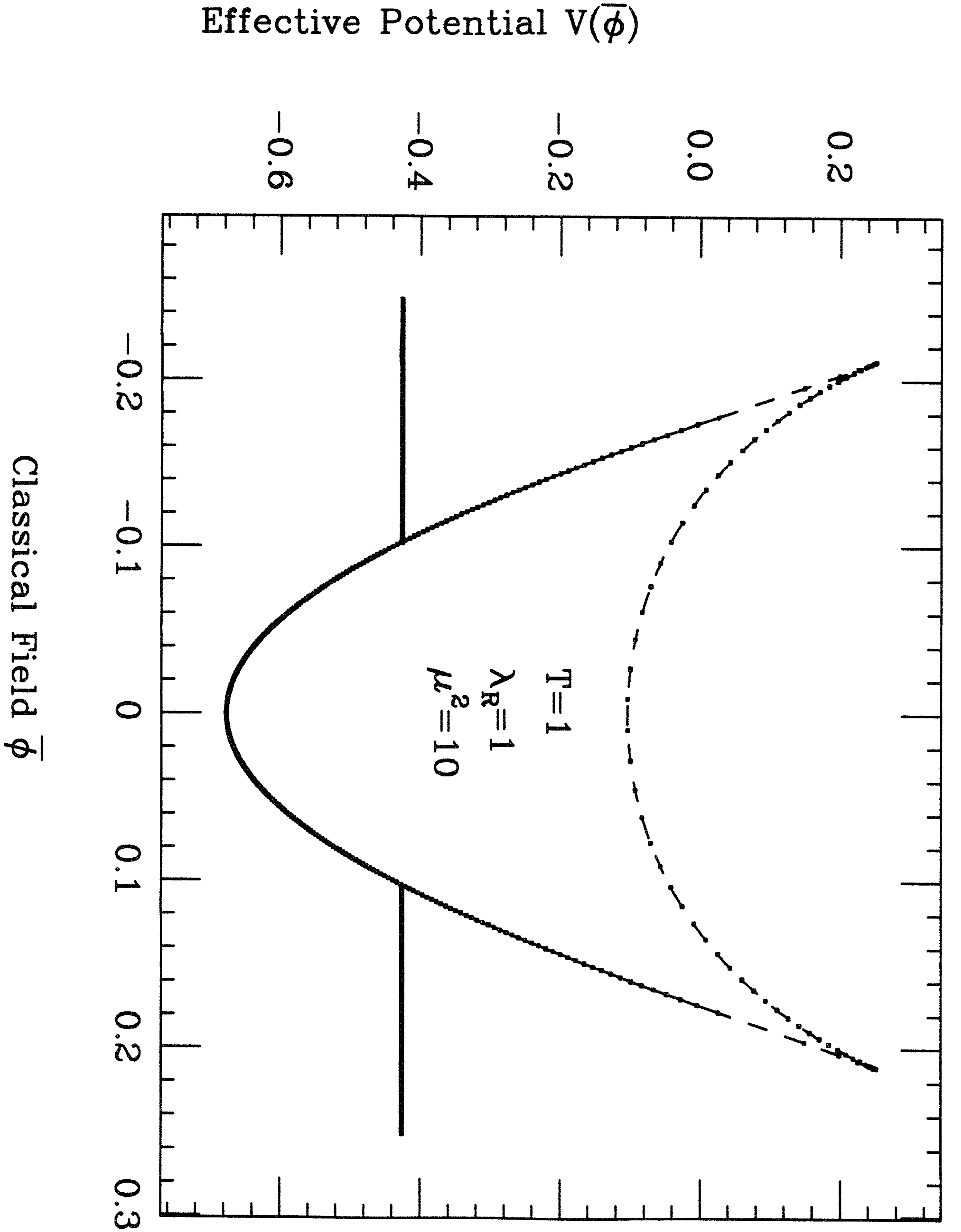

Generating Functional W[J]
Effective Mass $\bar{m}$
J=0
J=2.3
J=4.6
J=6.9
T=1
λ_R=1
μ^2=10
2.0
1.5
1.0
0.5
0.0
−0.5
0
2
4
6
8

Fig. 4.4


Effective Potential $V(\bar{\phi})$
0.2
0.0
−0.2
−0.4
−0.6
−0.2
−0.1
0
0.1
0.2
0.3
T=1
λ_R=1
μ^2=10
Classical Field $\bar{\phi}$

IV.2 Symmetry Breaking

Now we want to study symmetry breaking by examining the extrema of the effective potential (4.16). The total derivative of the effective potential with respect to $\bar{\phi}$ is:

$$\frac{dV(\bar{\phi})}{d\bar{\phi}} = \frac{\partial V}{\partial \bar{m}^2}\frac{d\bar{m}^2}{d\bar{\phi}} + \frac{\partial V}{\partial \bar{\phi}} = \bar{m}^2\bar{\phi} \tag{4.32}$$

where we have used the fact that $\frac{\partial V}{\partial \bar{m}^2} = 0$ again. Thus the extremum of the effective potential occurs either at $\bar{\phi} = 0$ or $\bar{m}^2 = 0$ and only the second case allows the possibility of spontaneous symmetry breaking of the reflection symmetry. By taking the second derivative of the effective potential at $\bar{\phi} = 0$, we can easily realize that the $\bar{\phi} = 0$ extremum is always a minimum.

If we insert $\bar{m}^2 = 0$ into the constraint equation (4.14), then we obtain a very general equation between $\bar{\phi}$ and temperature T.

$$0 = \mu^2 - 4\pi^2\lambda_R f(\beta, 0) + 8\pi^2\lambda_R\bar{\phi}^2 \tag{4.33}$$

The above equation does not always have a solution to $\bar{\phi}$. If it does have one, then it has only one solution: $\bar{\phi}_0$ (except for the other one connected by reflection symmetry). However, corresponding to this $\bar{\phi}_0$, there exists another $\bar{m}^2$ which satisfies the constraint equation (4.14). This $\bar{m}^2$ is:

$$\bar{m}^2 = \mu^2 e^{2/\lambda_R} \tag{4.34}$$

We are interested in knowing whether the $\bar{m} = 0$ extremum is a minimum of the effective. To this end, we calculate the second derivative of the effective potential:

$$V'' = \bar{m}^2 + A\bar{\phi}^2 \tag{4.35}$$

with

$$A \equiv \frac{16\pi^2\lambda_R}{1 - \lambda_R/2 - (\lambda_R/2)\ln(\bar{m}^2/\mu^2) + 4\pi^2\lambda_R(\partial f(\bar{m}^2, \beta)/\partial \bar{m}^2)} \tag{4.36}$$

It is easy to realize that as $\bar{m}^2 \to 0$, A becomes positive and therefore, the $\bar{m} = 0$ extremum is always a minimum. As an example, see Fig. 4.5 for the case $\lambda_R = -2$, $\mu^2 = 10$ and $T = 0.1$. Fig. 4.5(b) is the blowup of the upper-left corner of the Fig. 4.5(a). The point C corresponds to the point where $\bar{m} = 0$ and straight down below is point A, where $\bar{m}$ has the value as given by equation (4.34). At point B, the flat part of the effective potential meets the curve part. The value of the effective potential at point C must be the same as that of flat part of the effective potential. Therefore, the curve part of the effective potential must merge smoothly with the flat part (not shown in Fig. 4.5(b)) at point C. However, the minimum at point C is above the lower "branch" of the effective potential. Therefore, it is at best unstable if not fictitious.

Nevertheless, we want to know under what circumstances this unstable state exists, or when equation (4.33) has a solution to $\bar{\phi}$. First let's consider the high temperature regime, namely, the special case of $T >> \bar{m}$ and $T >> \mu$. The high temperature expansion of $F(\beta, \bar{m}^2)$ is given in appendix C,

$$F(\beta, \bar{m}^2) = -\frac{2T^4}{\pi^2} H_5(\bar{m}, r = 0) \tag{4.37}$$

Thus, we have,

$$F(\beta, \bar{m}^2) = -\frac{\pi^2 T^4}{90} + \frac{T^2 \bar{m}^2}{24} - \frac{T}{12\pi}(\bar{m}^2)^{3/2} + \frac{\bar{m}^4}{64\pi^2}\left[\ln\frac{16\pi^2 T^2}{\bar{m}^2} - 2\gamma + \frac{3}{2}\right] \tag{4.38}$$

and consequently,

$$\begin{aligned} f(\beta, \bar{m}^2) &= -4\frac{\partial F}{\partial \bar{m}^2} \\ &= -\frac{T^2}{6} + \frac{T\bar{m}}{2\pi} + \frac{\bar{m}^2}{16\pi^2} - \frac{\bar{m}^2}{8\pi^2}\left[\ln\frac{16\pi^2 T^2}{\bar{m}^2} - 2\gamma + \frac{3}{2}\right] \end{aligned} \tag{4.39}$$

Inserting the above expansion into the constraint equation (4.33), we obtain,

$$\bar{\phi}^2 = -\frac{\mu^2}{8\pi^2\lambda_R} - \frac{1}{12}T^2 \tag{4.40}$$

We have a solution to the equation above only if

$$-\frac{3\mu^2}{2\pi^2} < T^2\lambda_R < 0 \tag{4.41}$$

In other words, only in the case of $\lambda_R < 0$ can we obtain a unstable state with $\bar{\phi} \neq 0$. Put in another way, for given renormalized mass and coupling constant, regardless of the sign of the renormalized coupling constant, no $\bar{\phi} \neq 0$ solution exists at high enough temperature. The critical temperature below which the unstable solution starts to exist is,

$$-\lambda_R T_c^2 = \frac{3\mu^2}{2\pi^2} \tag{4.42}$$

In this case, the effective potential has a value of,

$$V(\beta, \bar{m}^2 = 0) = -\frac{\pi^2 T^4}{90} - \frac{1}{32\pi^2}\frac{\mu^4}{\lambda_R} \tag{4.43}$$

Now let's consider the low temperature case. From appendix C we can get $F(\beta, \bar{m}^2)$ at low temperature,

$$F(\beta, \bar{m}^2) = -\frac{2T^4}{\pi^2}H_5(\bar{m}, r=0) = -\frac{4T^4}{\pi^2}\frac{\Gamma(5/2)}{\Gamma(5)}\left(\frac{2\bar{m}}{T}\right)^{3/2} e^{-\bar{m}/T} \tag{4.44}$$

and consequently,

$$f(\beta, \bar{m}^2) = -\frac{8(2T)^{3/2}\bar{m}^{1/2}}{\pi^2}\frac{\Gamma(5/2)}{\Gamma(5)}e^{-\bar{m}/T} \tag{4.45}$$

It is clear that as $T \to 0$, $f(\beta, \bar{m}^2) \to 0$, therefore, at zero temperature, the constraint equation (4.33) gives us,

$$\bar{\phi}^2 = -\mu^2/8\pi^2\lambda_R \tag{4.46}$$

which means we can only get a unstable state when $\lambda_R < 0$ at zero temperature. This result agrees with our conclusion in the high temperature regime.

With the $\bar{\phi}^2$ given by equation (4.46), we can easily get the corresponding value of the effective potential at zero temperature,

$$V(\bar{m} = 0) = -\frac{1}{32\pi^2}\frac{\mu^4}{\lambda_R} > 0 \tag{4.47}$$

When $\bar{\phi} = 0$, the constraint equation (4.14) at zero temperature gives us the solution $\bar{m}^2 = \mu^2$. Then we can get the value of the effective potential at $\bar{\phi} = 0$,

$$V(\bar{\phi} = 0) = -\frac{1}{128\pi^2}\mu^4 < 0 \tag{4.48}$$

Comparing the two values from equation (4.47) and (4.48), we confirmed that at zero temperature, the unstable state is indeed unstable relative to the $\bar{\phi} = 0$ minimum.

In summary, only when $\lambda_R < 0$ and $T < T_c$ can we get a unstable state in addition to the $\bar{\phi} = 0$ minimum of the effective potential. The existence of this unstable state means that the curve part of the effective potential merges smoothly with the flat part, although it is above another lower "branch" of the effective potential. The merging point corresponds to the unstable state. As we increase the temperature above T_c, the unstable state disappears. If we further increase the temperature, the curve part rises completely above the flat part. The effective potentials at different temperatures described above are shown in Fig. 4.6.

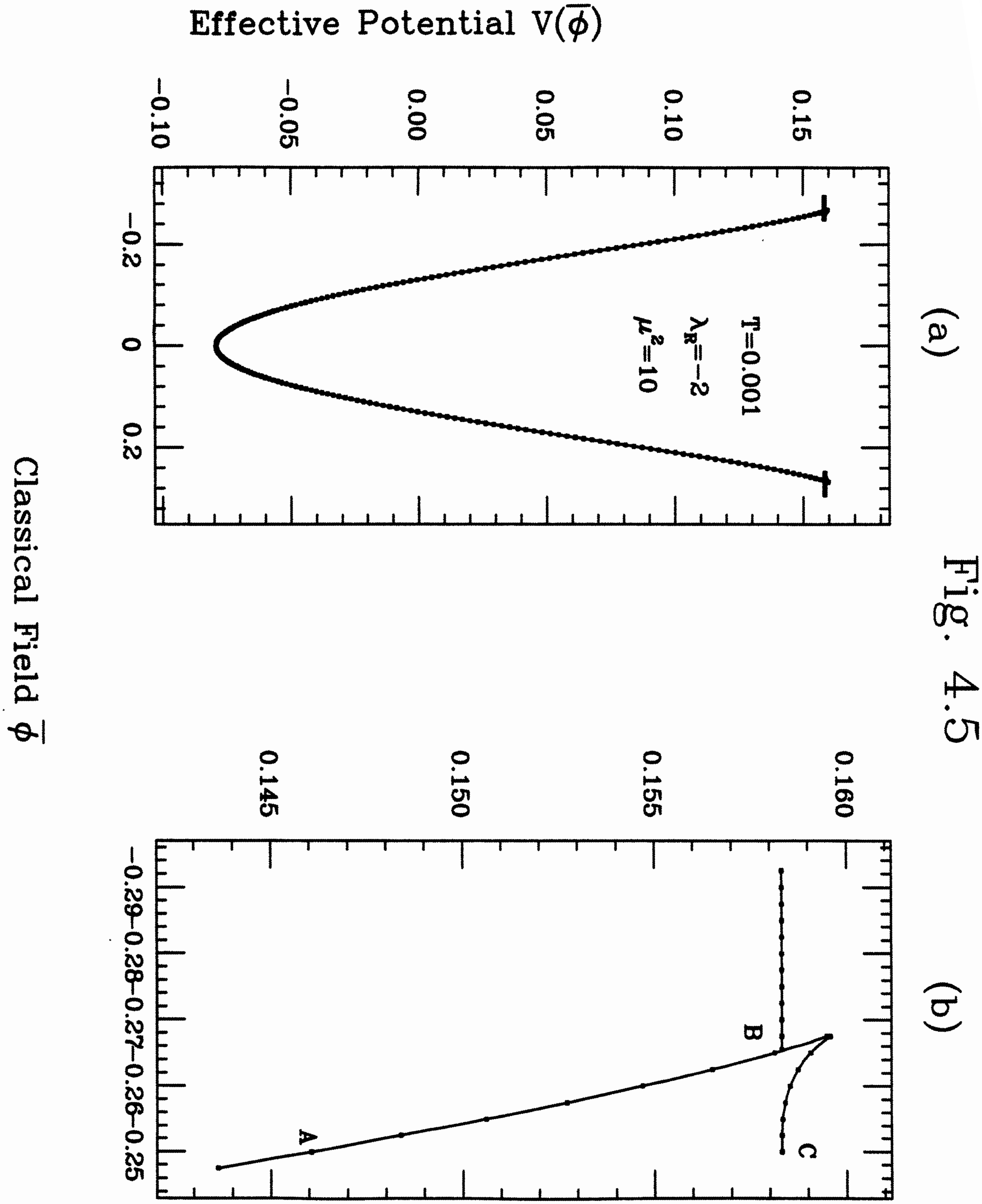

Effective Potential V($\overline{\phi}$)
Classical Field $\overline{\phi}$
(a)
T=0.001
λ_R=−2
μ^2=10
(b)
A
B
C
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.2
0
0.2
0.160
0.155
0.150
0.145
−0.29
−0.28
−0.27
−0.26
−0.25

Fig. 4.5

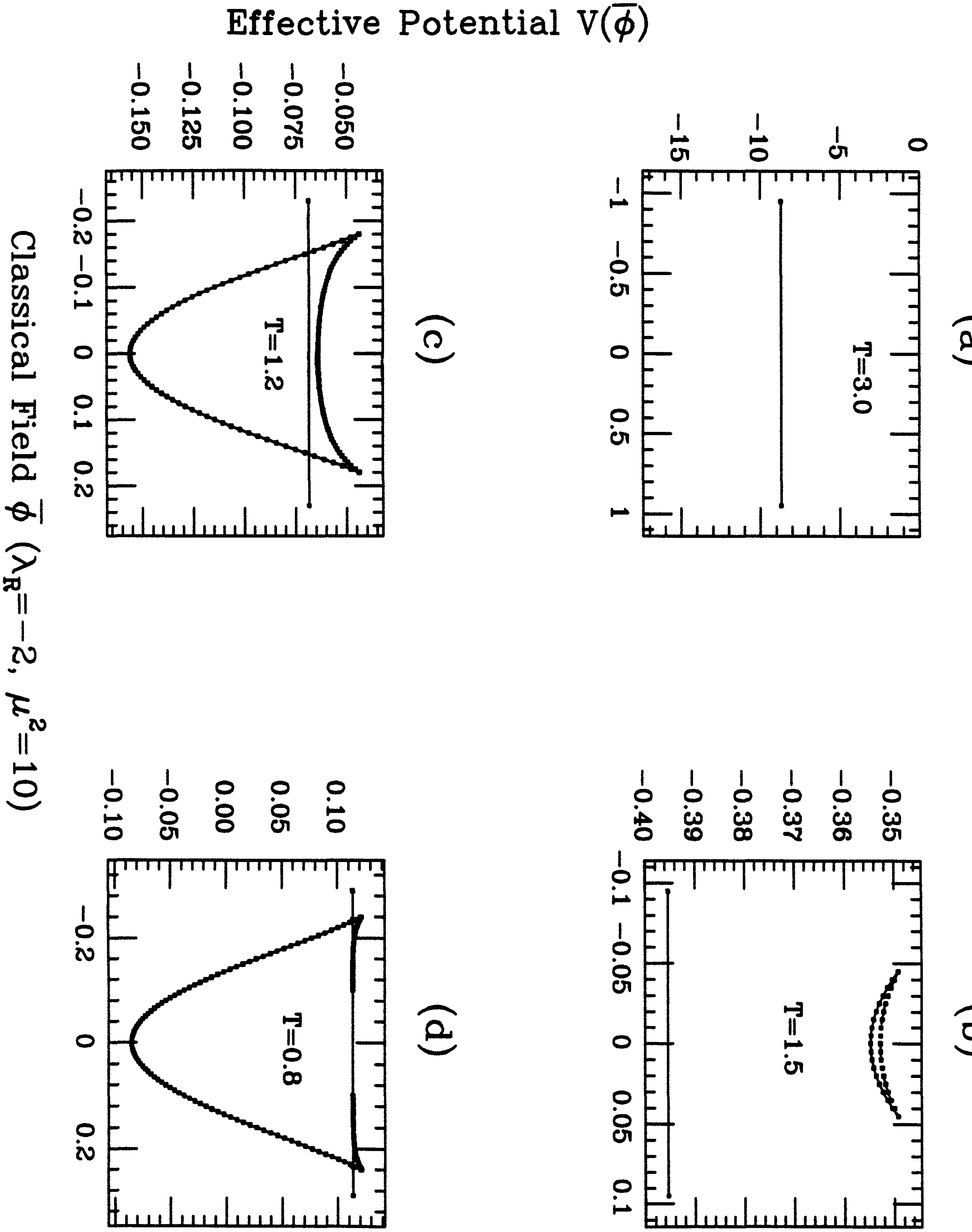

Effective Potential V($\bar{\phi}$)
(a)
T=3.0
(b)
T=1.5
(c)
T=1.2
(d)
T=0.8
Classical Field $\bar{\phi}$ (λ_R=−2, μ^2=10)

Fig. 4.6

As the conclusion to this section, we want to present our preliminary calculations on the second order contribution to the generating functional for the connected Green's function. From equation (4.8), following the procedures similar to those in section III.1, we get,

$$W_2 = -\frac{1}{2}\beta V \int \frac{d^4P}{(2\pi)^4} \ln\left[\frac{-2}{(2\pi)^4} f(P) + \frac{-1}{2\lambda} - \frac{4J^2}{\bar{m}^4}\frac{1}{L(P)}\right] \tag{4.49}$$

with $L(P)$ and $f(P)$ defined by equations (3.2) and (3.4) respectively in section III.1. Then the classical field becomes

$$\bar{\phi} = \frac{\delta W_0}{\delta J} + \frac{\delta W_2}{\delta J} = \frac{J}{\bar{m}^2} + \frac{\partial W_2}{\partial \bar{m}^2}\frac{d\bar{m}^2}{dJ} + \frac{\partial W_2}{\partial J} \tag{4.50}$$

Here W_0 is just the zeroth order contribution to the generating functional, which is already given by equation (4.9). The individual terms in equation (4.50) can be calculated and they are,

$$\frac{\partial W_2}{\partial J} = \beta V \int \frac{d^4P}{(2\pi)^4} \frac{-8J}{\frac{4\bar{m}^4}{(2\pi)^4} f(P)L(P) + \frac{\bar{m}^4 L(P)}{\lambda} + 8J^2} \tag{4.51}$$

$$\frac{\partial W_2}{\partial \bar{m}^2} = \beta V \int \frac{d^4P}{(2\pi)^4}\left[\frac{1}{\frac{4\bar{m}^4}{(2\pi)^4} f(P)L(P) + \frac{\bar{m}^4 L(P)}{\lambda} + 8J^2}\right] \times \left[\frac{-2\bar{m}^4 L(P)}{(2\pi)^4}\frac{\partial f(P)}{\partial \bar{m}^2} + \frac{8J^2}{\bar{m}^2} + \frac{4J^2}{L(P)}\right] \tag{4.52}$$

with

$$\frac{\partial f}{\partial \bar{m}^2} = -\frac{1}{16}\int d^4q \frac{1}{L(\frac{P+q}{2})L(\frac{P-q}{2})}\left[\frac{1}{L(\frac{P+q}{2})} + \frac{1}{L(\frac{P-q}{2})}\right] \tag{4.53}$$

and from constraint equation (4.11) we can obtain

$$\frac{d\bar{m}^2}{dJ} = \frac{8J/\bar{m}^4}{\frac{1}{2\pi^2\lambda_R} + \frac{8J^2}{\bar{m}^6} + \frac{1}{2\pi^2}\ln\frac{2}{\bar{m}} - \frac{1}{4\pi^2}\ln\frac{\alpha}{\mu^2} + \bar{f}} \tag{4.54}$$

with $\ln\alpha = 2\ln 2 - 1$ and $\bar{f} \equiv \int \frac{d^3p}{(2\pi)^3}\left[\frac{2}{\omega^3(e^{\beta\omega}-1)} + \frac{2\beta e^{\beta\omega}}{\omega^2(e^{\beta\omega}-1)^2} - \frac{\bar{m}^2}{p^2\omega^3}\right]$, which is finite. Therefore, $\frac{d\bar{m}^2}{dJ}$ is finite. From the above expressions, we can easily find out that $\frac{\partial W_2}{\partial J}$ and $\frac{\partial W_2}{\partial \bar{m}^2}\frac{d\bar{m}^2}{dJ}$

are divergent and they have similar leading divergence structures with opposite signs. However, the integrations in the above expressions can not be performed. Even if we can perform the integrations, it is unlikely that all the divergences in $\frac{\partial W_2}{\partial J}$ and $\frac{\partial W_2}{\partial \bar{m}^2}\frac{d\bar{m}^2}{dJ}$ will cancel since the later certainly contains other divergences besides the leading divergence. This can be confirmed by the fact that the second order contribution to the free energy in last chapter can not be renormalized when $J = 0$.

Aside from renormalizability, the discussions at the end of section III.2 regarding the validity of saddle point integral remains true in this case. In other words, we have similar ranges of λ_R over which our saddle point becomes a maximum of $W[J]$. It is easy to realize this since equation (4.49) differs from (3.5) only by the term $-4J^2/(\bar{m}^4 L(P))$, which is bounded above and below for fixed J and $\bar{m}(\neq 0)$.

Chapter V

Implications to Inflation

V.1 General Picture

The inflationary universe scenario was first proposed by Guth ([17]). Guth suggested that if the phase transition associated with the spontaneous symmetry breaking of the Grand Unified Theories (GUT) is first order, then the isotropy, homogeneity, flatness and monopoles problems, which the standard big bang model failed to elucidate, could be explained. In this scenario, the early universe was trapped in a metastable symmetric phase (false vacuum) through supercooling. The false vacuum would decay by the process of Coleman-Callan bubble nucleation. The bubbles of true vacuum have to coalesce uniformly in order to achieve the observed mass homogeneity of the universe. However, it was later shown that the bubbles would never merge and consequently the desired smooth coalescence can not be achieved. This "graceful exit" problem was later overcome by the new inflationary scenario where the effective potential near the false vacuum is assumed to be very flat ([18], [19]). It was shown that with the Coleman-Weinberg effective potential, the symmetry breaking of the SU(5) theory could follow the desired "slow-rollover" process and a single bubble could undergo enough inflation to encompass all the observed entropy in the universe. Since the observed universe has inflated by about a factor of 10^{28}, a small patch of the early universe can become the observed universe today. Under the assumption that the small patch is homogeneous due to the thermal equilibrium before inflation, this homogeneity would be preserved during inflation. Consequently, new inflation scenario can generate enough inflation as well as solve the homogeneity problem.

Except for the contribution from the vector mesons which acquired masses through symmetry breaking, the Coleman-Weinberg effective potential is essentially due to the Higgs sector of the SU(5) theory. However, in order for the Higgs field to drive inflation successfully, the field must be extremely weakly coupled (fine structure constant $\alpha \approx 3 \times 10^{-8}$). This is because the observed mass density fluctuations in the universe is due to the quantum fluctuations of the Higgs field as it rolls down the false vacuum. And this mass density fluctuation sets an upper limit on the relevant coupling constant in the theory. The necessary extreme weak coupling in the theory poses a severe problem to the SU(5) model since this extreme weak coupling is far too small to fit into any framework of unification.This suggests that the inflation driving field may not be the Higgs field in the SU(5) theory. In other words, the Higgs field may not break the symmetry of SU(5) theory and drive inflation at the same time. Therefore, we are allowed to consider a model which drives inflation only and ignore the constraints due to the symmetry breaking of SU(5) theory for the moment.

In light of the above discussion, we want to study whether our effective potential can drive inflation although it can not break reflection symmetry. We will concentrate on the zero temperature case partially because of its analytical tractability and partially because we believe that the qualitative features of the low temperature case can be captured. Since low temperature means the temperature is low relative to the mass scale, in our case $\bar{m}$ and μ, which could be extremely large, the temperature actually has a great deal of latitude in the absolute sense.

From section IV.2, we have qualitatively presented the effective potentials as temperature changes. Our calculations done in section IV.2 can be easily repeated for zero temperature case. From now on in this chapter, we will simplify our notation by omitting the bar on top of $\bar{\phi}$ and $\bar{m}$. Referring to Fig.4.5(b), I will present the relevant results which will be useful later. The ϕ and m of points A,B,C are given respectively:

$$\phi_A^2 = \phi_C^2 = \frac{-\mu^2}{8\pi^2\lambda_R} \qquad \phi_B^2 = \frac{m_B^2}{32\pi^2} - \frac{\mu^2}{8\pi^2\lambda_R} \tag{5.1}$$

$$\ln\frac{m_A^2}{\mu^2} = \frac{2}{\lambda_R} \qquad \ln\frac{m_B^2}{\mu^2} = \frac{2}{\lambda_R} - \frac{1}{2} \qquad m_C^2 = 0 \tag{5.2}$$

From equation (4.35), we can easily get the second derivative of the effective potential at point B:

$$V_B'' = m_B^2 - 64\pi^2\phi_B^2 \tag{5.3}$$

It is easy to see that as $\lambda_R \to 0^-$, m_B^2 approaches 0 exponentially and we are left with a negative second derivative at point B. In addition, in this limit, $V_B' = m_B^2\phi_B = 0$. Therefore, we can see that at zero temperature, as $\lambda_R \to 0^-$, the curve part of the effective potential at point B could merge smoothly with the flat part. Although the effective potentials as shown in Fig. 4.5 and Fig. 4.6 are often double valued, we should always look at the lower "branch". Then the above interesting limit is suggestive of the potentials that can drive inflation in the new inflationary scenario. Of course, at any finite λ_R, there will be a kink at the merging point. Nevertheless, we are interested in knowing whether our effective potential allows a picture similar to the one in the new inflationary scenario. We will only study the case when λ_R is negative and extremely small partly because of this "level off" property of the effective potential and partly because of our discussions at the ends of sections III.2 and IV.2 regarding the validity of saddle point approximation.

The general picture of inflation, according to our effective potential is the following: At high temperature, the effective potential is essentially flat, therefore, the vacuum expectation value of ϕ can be any value. This is similar to the starting point of the universe in Linde's chaotic inflation ([20]). The ϕ can be any value with equal probability. However, there should be such values of ϕ that as it rolls down the potential curve, it can generate enough inflation to encompass all the entropies of the observed universe. As in the chaotic inflation scenario, there could be many such inflationary universes and our observed universe is just a tiny part of one of them.

Suppose initially the vacuum expectation value of the effective potential is ϕ_i. As temperature drops, the curve part of the effective potential becomes lower than the flat part. As the curve part lowers down when temperature drops, it becomes wider in its range also, as can be seen on Fig. 4.6. For sufficiently weak coupling constant, equation (5.1) says the range of curve part of the effective potential gets wider and wider. Actually its width attains infinity in the limit of $\lambda_R \to 0^-$. Therefore, for sufficiently weakly coupled fields, the curve part of the effective potential will always hit the ϕ_i as temperature drops. After that, we hope ϕ will roll down the hill and generate enough inflation.

Now we want to review the underlying equations in the new inflationary scenario and the main requirements for a workable model ([21]). The key equation governing the time evolution of ϕ as it rolls down the hill is:

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + V'(\phi) = 0 \tag{5.4}$$

Here the Hubble parameter $H \equiv \dot{R}/R$ is determined by the Friedman equation:

$$H^2 = (8\pi/3m_{pl}^2)[V(\phi) + \frac{1}{2}\dot{\phi}^2 + \rho_r] \tag{5.5}$$

with plank mass $m_{pl} = 1.22 \times 10^{19} Gev$. ρ_r is the energy density in radiation. Here Γ accounts for particle creation due to the time evolution of ϕ and is only important when the time evolution of ϕ is large compared with the expansion rate. Actually Γ is the energy density per unit time that is drained from the ϕ field through particle creation. Its value is determined by the fields which couple to ϕ and the strength of the coupling. Since we do not know what our ϕ is and to what fields it is coupled, we can not determine this Γ. Actually this Γ is the inverse of the lifetime of the ϕ field during the radiation phase in the end of exponential inflation. Since this Γ is related to the reheating of the universe in the radiation phase, our lack of knowledge about it means we can not study reheating process and we will not be able to determine reheating temperature and related phenomena. We need to imbed our model into a sensible larger model which gives us a complete picture of inflation. The energy in radiation ρ_r and the $\Gamma\dot{\phi}$ term are both negligible during the slow rollover phase, therefore we can ignore them in equations (5.4) and (5.5).

The key assumption during the slow rollover phase is that $\ddot{\phi}$ term is negligible compared to the friction term $3H\dot{\phi}$ and the kinetic energy $(1/2)\dot{\phi}^2$ is much smaller compared to the potential energy $V(\phi)$. Under these assumptions, the Hubble parameter is determined only by the absolute value of the effective potential and becomes

$$H^2 \simeq 8\pi V(\phi)/3m_{pl}^2 \tag{5.6}$$

and equation (5.4) becomes

$$\dot{\phi} \simeq -V'(\phi)/3H \tag{5.7}$$

Futhermore, using this $\dot{\phi}$ to calculate $\ddot{\phi}$ and making use of equation (5.5), we get

$$\ddot{\phi}/3H\dot{\phi} \simeq -V''/9H^2 + (V'm_{pl}/V)^2/48\pi \tag{5.8}$$

Therefore, it is self consistent to neglect $\ddot{\phi}$ term when

$$|V''(\phi)| \lesssim 9H^2 \tag{5.9}$$

$$|V' m_{pl}/V| \lesssim (48\pi)^{1/2} \tag{5.10}$$

For polynomial potentials, it is generally true that inequality (5.10) follows directly from (5.9). It can be shown later that for our potential, this is true. Therefore, (5.10) is redundant. It is easy to verify that if (5.10) is true, then $(1/2)\dot{\phi}^2 \lesssim V(\phi)$ follows directly also. Therefore, in the most general sense, if equations (5.9) and (5.10) are true, then our assumptions ($\ddot{\phi} \lesssim 3H\dot{\phi}$ and $(1/2)\dot{\phi}^2 \lesssim V(\phi)$) are self consistent and the universe begins the de Sitter phase when it expands exponentially and cosmic scale factor will grow by a factor of $exp\left[\int H dt\right]$.

V.2 Parameters Fitting

To fit our effective potential into this inflation picture, we need to find a range of ϕ where equations (5.9) and (5.10) are satisfied. Suppose the range is from ϕ_b till ϕ_e (see Fig. 5.1). It is tempting to believe that point B is the beginning point. Form equations (5.1) and (5.2), we can easily obtain the values of the effective potential at point A, B and C.

$$V_B = V_C = -\frac{1}{32\pi^2}\frac{\mu^4}{\lambda_R} \approx V_A \tag{5.11}$$

If we assume the cosmological constant is zero, then we need to set the value of the effective potential to zero at its true vacuum. From equations (4.47) and (4.48), we can see that all the values of the effective potential need to be shifted upward by a constant amount of $(1/128\pi^2)\mu^4$. However, in the limit of extremely weak coupling, this constant amount is negligible compared to the values in equation (5.11). Therefore, from now on, we will neglect this upward shift. If we substitute the appropriate values corresponding to point B into equation (5.9), then we get,

$$\mu^2 \gtrsim \frac{32\pi}{3} m_{pl}^2 \tag{5.12}$$

This is not good since the this means we have to consider quantum gravity effect (We can repeat the same calculation to point A and we will get a similar answer). In other words, if we do not want to consider quantum gravity effects, we have to choose a μ^2 less than m_{pl}^2. This can only be achieved by choosing a beginning point ϕ_b somewhere further down the potential curve.

It is useful to present the second derivative of the effective potential at zero temperature at this moment:

$$\begin{aligned} V'' &= m^2 + \left[\frac{16\pi^2\lambda_R}{1-\lambda_R/2-(\lambda_R/2)\ln(m^2/\mu^2)}\right]\phi^2 \\ &= \frac{3m^2-\lambda_R m^2/2-3\lambda_R m^2\ln(m^2/\mu^2)/2-2\mu^2}{1-\lambda_R/2-(\lambda_R/2)\ln(m^2/\mu^2)} \end{aligned} \tag{5.13}$$

If we insert the value of m_B^2 or m_A^2 into the above equation, we can realize that the dominant contribution to the denominator of the second term is of order λ_R. This makes the second derivative relatively large and consequently violates inequality (5.9). From the fact that $m_A^2 = \mu^2 e^{2/\lambda_R}$ and $m^2 = \mu^2$ when $\phi = 0$, we can parametrize the m^2 in the interval between ϕ_A and $\phi = 0$ as follows:

$$m^2 = \mu^2 e^{a/\lambda_R} \tag{5.14}$$

with a changing from 2 to 0. With this parametrization, inequality (5.9) becomes:

$$|3m_b^2 - 3am_b^2/2 - 2\mu^2| \lesssim \frac{3(1-a/2)}{4\pi\lambda_R m_{pl}^2}\left[2m_b^2(m_b^2-\mu^2-\frac{1}{2}am_b^2) - \frac{1}{2}am_b^4 - (m_b^2-\mu^2)^2\right] \tag{5.15}$$

in the limit of negative and extremely weak coupling, as is assumed throughout this chapter. If we are interested in an a such that m_b^2 is negligible compared to μ^2, then the above inequality becomes:

$$-\lambda_R m_{pl}^2 \lesssim \frac{3}{8\pi}\mu^2\left(1-\frac{a}{2}\right) \tag{5.16}$$

If we choose $\mu^2 \sim -\lambda_R m_{pl}^2$, then we can realize that there is a wide range of small a which we can choose to satisfy the above inequality. In summary, if we choose our renormalized parameters such that $\mu^2 \sim -\lambda_R m_{pl}^2$, then we have a wide range of a to make inequality (5.9) satisfied as long as $a \gg -\lambda_R$. This final requirement for a is to make sure $\mu^2 \gg m_b^2$. For definiteness, we will choose $a_b \sim -100\lambda_R$ for reasons which will become apparent later. So far we have only specified the ratio of μ^2 to $-\lambda_R$. Later we will see that we need to choose $\mu^2 \sim 10^{26} Gev^2$ and $-\lambda_R \sim 10^{-12}$. With these numbers, we know $m_b^2 \sim 10^{-20} Gev^2$, which is very small. Now we should check with the above range of parameters, whether inequality (5.10) can be satisfied also. At the beginning point ϕ_b, we have as an order of magnitude estimation: $V' = m_b^2 \phi_b \sim \mu^2 e^{a_b/\lambda_R} [\mu^2/(-\lambda_R)]^{1/2}$ and $V \sim \mu^4/(-\lambda_R)$. These estimation means inequality (5.10) is equivalent to

$$e^{a/\lambda_R} m_{pl} \left(\frac{-\lambda_R}{\mu^2} \right) \lesssim (48\pi)^{1/2} \tag{5.17}$$

which is apparently true due to the exponential decay. Indeed, for the beginning point, we proved that (5.10) is a less stringent requirement then (5.9).

After identifying the beginning point, we need to identify the ending point ϕ_e also. First we want to point out that inequality (5.9) says that the point ϕ_m, where $V''(\phi_m) = 0$, should be encompassed by the slow rollover range: from ϕ_b to the ϕ_e. Indeed, as we go from the top of the hill down to the bottom, V'' changes from a large negative number (of the order μ^2/λ_R) to a large positive number (of the order μ^2). For later convenience, we present the relevant results for point ϕ_m:

$$m_m^2 \simeq \frac{2\mu^2}{3} \qquad \phi_m^2 \simeq \frac{1}{3} \left(-\frac{\mu^2}{8\pi^2 \lambda_R} \right) \tag{5.18}$$

Now we look for such ϕ_e that satisfies $m_e^2 \sim \mu^2$. If we look at equation (5.13), then the dominant contribution to V'' in this case will be $3m_e^2 - 2\mu^2$. We can similarly look at the dominant contribution to H^2, then inequality (5.9) becomes:

$$(3m_e^2 - 2\mu^2)\mu^2 \simeq -\frac{3\mu^2}{4\pi\lambda_R m_{pl}^2}(\mu^4 - m_e^4) \tag{5.19}$$

where we have used the fact that m_e should be larger than $m_m^2 \simeq 2\mu^2/3$. From equation (5.16) we know it has to be true that $-\frac{3\mu^2}{4\pi\lambda_R m_{pl}^2} \gtrsim 2$, if we choose it to be 3, for example, we can get a value $m_e^2 \approx 5\mu^2/6$. We have to check further whether inequality (5.10) can be satisfied at the same time. The dominant contribution to the effective potential V at $\phi_e \approx (1/6)(-\mu^2/(8\pi^2\lambda_R)$ is $V(\phi_e) \approx (11/36)(-\mu^4/32\pi^2\lambda_R)$. Remember also that we have chosen $-\frac{3\mu^2}{4\pi\lambda_R m_{pl}^2} = 3$. With all these values given, we can easily verify that inequality (5.10) is satisfied also. Actually this inequality is almost saturated. Nevertheless, we have shown that the inequality (5.10) is less stringent than inequality (5.9) again. In conclusion, we have identified the range of ϕ for the slow rollover phase so far.

Now we need to see how reliable our effective potential over the slow rollover range is. The scalar field is subject to quantum fluctuations during the slow rollover process. The scale for this quantum fluctuations is set by Hawking temperature $H/2\pi$. In other words, the fuzziness of the effective potential is of the order of H, therefore, the range of the slow rollover process must be many times larger than the scale H. Namely, we need to see whether

$$|\phi_b| - |\phi_e| \gg H \tag{5.20}$$

We can do an order of magnitude estimation here: $\Delta\phi \sim (-\mu^2/\lambda_R)^{1/2}$ and $H \sim V^{1/2}/m_{pl} \sim (-\mu^4/(\lambda_R m_{pl}^2))^{1/2}$, remember $\mu^2 \sim -\lambda_R m_{pl}^2$, then we can see the $\Delta\phi \sim 10^6 H$ if we choose $\lambda_R \sim -10^{-12}$, as before. Therefore, the quantum fuzziness can be safely ignored.

With the above preparation, we want to answer the following two questions: (**a**) Can this slow rollover process generate enough inflation? (**b**) Can it produce the right amount of small scale density fluctuation?

(**a**) During the slow rollover phase, the cosmic scale factor is enlarged by a factor of

$$exp\left[\int H dt\right] \simeq \int_{\phi_b}^{\phi_e} \frac{3H^2}{V'(\phi)} d\phi \equiv N \tag{5.21}$$

For a phase transition at GUT energy scale, about 60 e-fold of inflation is required in order to solve the large scale homogeneity and the flatness puzzles. In addition, this number is very insensitive to the energy scale. As the energy scale changes from $1Gev$ to $10^{19}Gev$, it only changes from 24 to 68. Therefore, we can roughly say $N \sim 10^2$ (Chapter 8 of [22]).

With our previous parametrization of $m^2 = \mu^2 e^{a/\lambda_R}$, the number of e-fold can be written as

$$N \simeq \int_{\phi_b}^{\phi_e} \frac{1}{m_{pl}^2 \mu^2 e^{a/\lambda_R}} \frac{-\mu^4}{\lambda_R} \frac{d\phi}{\phi} \simeq \frac{1}{e^{a/\lambda_R}} \frac{d\phi}{\phi} \tag{5.22}$$

Here we have made the order of magnitude estimation for $V \sim -\mu^4/\lambda_R$ over the range of slow rollover phase. Making use of the constraint equation (4.14), we can derive the following relationship:

$$\frac{d\phi}{\phi} = -\frac{1}{2} e^{a/\lambda_R} \frac{1/\lambda_R - 1/2 - a/(2\lambda_R)}{e^{a/\lambda_R} - 1 - (a/2)e^{a/\lambda_R}} da \tag{5.23}$$

Then the number of e-fold becomes

$$N \simeq \int_{a_b}^{a_e} \frac{1/\lambda_R - 1/2 - a/(2\lambda_R)}{1 - e^{a/\lambda_R} + (a/2)e^{a/\lambda_R}} da \tag{5.24}$$

Recall that we have chosen $a_b \sim -100\lambda_R \sim 10^{-10}$. In addition, because $m_e^2 \approx (5/6)\mu^2$, We must have $a_e \sim -\lambda_R$. For this range of a, the above equation gives us the following number of e-fold:

$$N \simeq \int_{a_b}^{a_e} \frac{1}{\lambda_R} da \simeq 100 \tag{5.25}$$

Therefore, we can reproduce the desired number of e-fold to solve the homogeneity problem and flatness problem. Recall that we had a great deal of freedom in choosing a_b. In addition, we can just change a_b itself without the need of adjusting other parameters in the model. Therefore, this genuine degree of freedom gives us a lot of latitude in fitting any desired number of e-fold.

(**b**) In order to achieve the small-scale density fluctuation of the universe, the density fluctuation $(\delta\rho/\rho)_H$ must have a value smaller than 10^{-5} when the mass scales relevant for galaxy formation reenter the horizon. At the GUT energy scale, it can be proved that this requirement means at about 50 e-fold of inflation, $H^2/\dot{\phi} \sim 10^{-5}$. Again, this number of e-fold, 50, is not very sensitive to the energy scale (Chapter 8 of [22]). Recall that we have $N \sim 100$ at the end of slow rollover phase. Therefore, as an order of magnitude estimation, we can calculation whether $H^2/\dot{\phi} \sim 10^{-5}$ is satisfied at the end of slow rollover phase. According to equation (5.7), the above requirement means

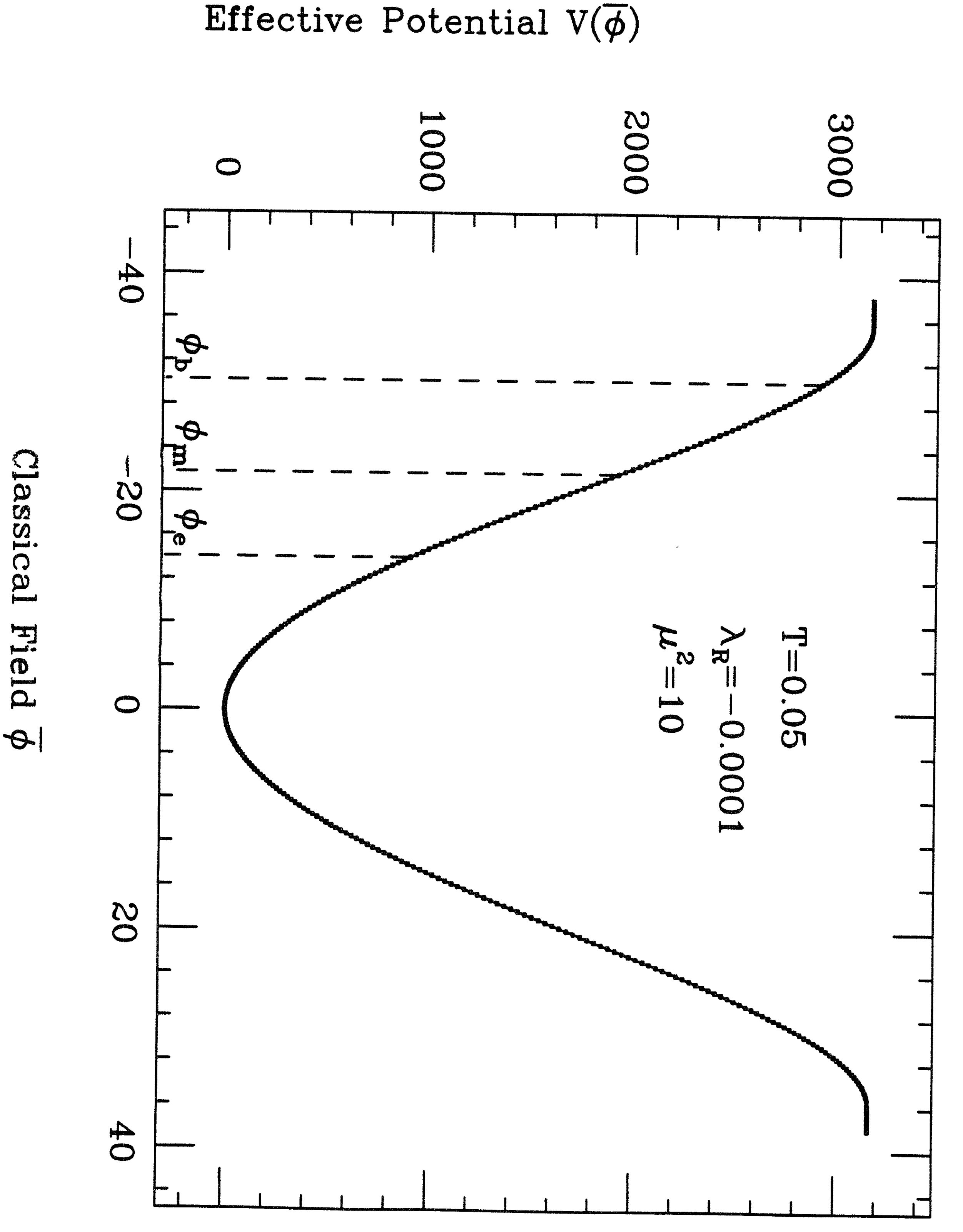
$$-\frac{3H^3}{V'} \simeq 10^{-5} \tag{5.26}$$

However, at the end of inflation, the inequality (5.9) is saturated. This means $H^2 \simeq V''/9 \simeq 8\pi V/(3m_{pl}^2)$. Then it follows readily from equation (5.13) that the above equation becomes

$$(3m_e^2 - 2\mu^2)^{3/2} \simeq 10^{-6} m_e^2 \left[\frac{1}{\lambda_R}(m_e^2 - \mu^2)\right]^{1/2} \tag{5.27}$$

From this we can see why we have chosen $\lambda_R \sim -10^{-12}$. With this order of magnitude for λ_R, the above equation reduces to a very simple one which relates m_e^2 to μ^2. It is easy to see that with our previously determined $m_e^2 \approx (5/6)\mu^2$, the above equation is almost exactly satisfied. The small residual discrepancy can be absorbed into a factor on the right hand side of the equation. Since this factor is of the order of one, we retain our initial 10^{-5} for $H^2/\dot{\phi}$.

In conclusion, it seems that our effective potential satisfies the generic requirement for driving inflation in the new inflationary scenario. With carefully chosen parameters, it also seems that it can solve the homogeneity and flatness problem with sufficient inflation and with the right amount of small scale density perturbations. Of course, as said earlier, we do not know what our scalar field is in the sense we do not have a complete picture of inflation and symmetry breaking as a whole. Therefore, we have difficulties to extend our calculations to the radiation phase after the slow rollover phase and we will not be able to determine the reheating temperature and study the related phenomena. Nevertheless, it seems we can extend our calculations to low (nonzero) temperature case. An apparent problem with the low temperature case is that the temperature has to be lower than both μ and m in order to make low temperature expansions. However, as we have seen, m^2 can be extremely small at zero temperature, for example, $m_b^2 \sim 10^{-20} Gev^2$. Therefore, as temperature rises from zero, we have to enlarge our m_b^2 in order to make any sensible low temperature expansion. This can be accomplished by moving the beginning point of the slow rollover phase transition closer to the true vacuum. Therefore, in principle, low temperature case can be studied similarly and with carefully chosen parameters, the required amount of inflation and small scale density perturbation should be achievable.



Effective Potential V($\bar{\phi}$)
0
1000
2000
3000
−40
−20
0
20
40
ϕ_p
ϕ_m
ϕ_e
T=0.05
λ_R=−0.0001
μ^2=10
Classical Field $\bar{\phi}$

Fig. 5.1

Chapter VI

Relativistic Bose-Einstein Condensation

VI.1 Free Field Case

Now we study a more interesting system with the auxiliary field approach. The theory we consider is the two component, or charged scalar $\lambda\phi^4$ theory. It is well known that when $\lambda = 0$, this free field theory exhibits the phenomenon of Bose-Einstein condensation. However, the Bose-Einstein condensation of ideal gases has never been observed. The presence of interactions, however week they are, can qualitatively alter many features of this phenomenon. Much work has been done in this respect and the possible link between spontaneous symmetry breaking and Bose-Einstein condensation has also been extensively studied. Here we first review Bose-Einstein condensation in free field theory and study the interacting system with auxiliary field approach and compare our results with those of previous studies.

The Lagrangian of this theory is,

$$\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) - \lambda(\phi_1^2 + \phi_2^2)^2 \tag{6.1}$$

Associated with the obvious $U(1)$ symmetry is a conserved charge density,

$$\phi_2\pi_1 - \phi_1\pi_2 \tag{6.2}$$

Where $\pi_i = \frac{\partial\phi_i}{\partial t}$ and consequently we get the Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)] + \lambda(\phi_1^2 + \phi_2^2)^2 \tag{6.3}$$

Then the partition function for the grand canonical ensemble is

$$Z = \int [d\pi_1][d\pi_2] \int_{periodic} [d\phi_1][d\phi_2] exp[\int d^4x (i\pi_1 \frac{\partial \phi_1}{\partial \tau} + i\pi_2 \frac{\partial \phi_2}{\partial \tau} \\ - \mathcal{H}(\pi_1, \pi_2, \phi_1, \phi_2) + \mu(\phi_2 \pi_1 - \phi_1 \pi_2))] \tag{6.4}$$

where we have inserted a chemical potential μ associated with the conserved charge. After doing the integration of the field momenta, we obtain

$$Z = N' \int_{peri} D[\phi_1] D[\phi_2] exp\{\int_0^\beta d\tau \int d^3x [-\frac{1}{2}(\frac{\partial \phi_1}{\partial \tau} + i\mu\phi_2)^2 - \frac{1}{2}(\frac{\partial \phi_2}{\partial \tau} - i\mu\phi_1)^2 \\ - \frac{1}{2}(\nabla\phi_1)^2 - \frac{1}{2}(\nabla\phi_2)^2 - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - \lambda(\phi_1^2 + \phi_2^2)^2]\} \tag{6.5}$$

where $N' = \exp\left(-2V \int \frac{d^3p}{(2\pi)^3} \sum_n \ln\beta\right)$ is a β dependent, dimensional constant resulting from the functional integral of the momenta. When $\lambda = 0$ we are left with terms quadratic in ϕ_1 and ϕ_2 only. Completing the Gaussian functional integral, we obtain

$$\ln Z = -\frac{1}{2} \ln \det \hat{A}$$

with

$$\hat{A} \equiv \begin{pmatrix} \hat{p}^2 + m^2 - \mu^2 & 2\mu\hat{p}_0 \\ -2\mu\hat{p}_0 & \hat{p}^2 + m^2 - \mu^2 \end{pmatrix}$$

The calculation of $\ln Z$ here is a simplified version of the interacting case considered later, thus the details are omitted here. We finally get the thermodynamic potential as a function of μ,V and T.

$$\Omega \equiv -\ln Z = V \int \frac{d^3p}{(2\pi)^3} [\omega + \frac{1}{\beta} \ln(1 - e^{-\beta(\omega-\mu)}) + \frac{1}{\beta} \ln(1 - e^{-\beta(\omega+\mu)})] \tag{6.6}$$

In calculating the Gaussian functional integral, we must have $|\mu| \leq m$ in order to ensure the exponent is positive definite, which guarantees that the final result from the functional integral is convergent. Thus, we must require $|\mu| \leq m$ if we expect to obtain sensible results later.

From this thermodynamic potential we can calculate pressure P, charge density $\rho = Q/V$,entropy S and energy U as given by,

$$P = -\left[\frac{\partial\Omega}{\partial V}\right]_{T,\mu}, \rho = -\frac{1}{V}\left[\frac{\partial\Omega}{\partial \mu}\right]_{T,V}, S = -\left[\frac{\partial\Omega}{\partial T}\right]_{V,\mu}, \tag{6.7}$$

and $U = TS - PV + \mu\rho V$.

Here we only want to concentrate on the charge density ρ,

$$\rho = \int \frac{d^3p}{(2\pi)^3}(n_p - \bar{n}_p) \tag{6.8}$$

with $n_p = 1/exp[\beta(w-\mu)] - 1$ and $\bar{n}_p = 1/exp[\beta(w+\mu)] - 1$. Here n_p and $\bar{n}_p$ can be interpreted as number density of particles and antiparticles. Note the requirement $|\mu| \leq m$ means the number densities have to be positive in this context. The sign of μ determines whether particles outnumber antiparticles or vice versa.

The above equation gives us a function of ρ in terms of temperature and μ, however, since ρ is a physical quantity, for a system with fixed number of net charges, the above equation actually gives us a implicit solution to μ for any given ρ and at any temperature. However, for any given ρ, only for T above some critical temperature T_c can we always find a $\mu < |m|$ satisfying the above equation. For T below T_c no such μ can be found and we interpret the expression above as the charge of excited states. The rest of the charges (particles and antiparticles) stay in the ground state and form a Bose-Einstein condensate. Thus the critical temperature corresponds to $|\mu| = m$.

Here I only quote the results for Bose-Einstein condensation in the case of a relativistic Bose gas ($T \geq T_c >> m$). The high temperature expansion of ρ as given in equation (6.8) can be found in Appendix C. For $T \geq T_c$, we have,

$$\rho = \frac{1}{3}\mu T^2 \tag{6.9}$$

Note for ρ fixed, as temperature decreases, μ increases. Hence T_c is given by

$$\rho = \frac{1}{3} m T_c^2 \tag{6.10}$$

As temperature drops below T_c, the charge density in excited states are given by

$$\rho(p > 0) = \rho[T/T_c]^2 \tag{6.11}$$

The rest of the charges stay in the $p = 0$ state.

What we had written down earlier in the partition function (6.4) was effectively the following Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)] + \lambda(\phi_1^2 + \phi_2^2)^2 - \mu(\phi_2\pi_1 - \phi_1\pi_2) \tag{6.12}$$

It is easy to verify that the following Lagrangian density gives us the above Hamiltonian density through the standard procedures,

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}_1 + \mu\phi_2)^2 + \frac{1}{2}(\dot{\phi}_2 - \mu\phi_1)^2 - \frac{1}{2}[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)] - \lambda(\phi_1^2 + \phi_2^2)^2 \tag{6.13}$$

First we have,

$$\pi_1^0 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = \dot{\phi}_1 + \mu\phi_2 \qquad \pi_2^0 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = \dot{\phi}_2 + \mu\phi_1 \tag{6.14}$$

substitute above expressions of momenta for π_1^0 and π_2^0 in $\mathcal{H} = \pi_1^0\dot{\phi}_1 + \pi_2^0\dot{\phi}_2 - \mathcal{L}$, we will recover the above expression of Hamiltonian.

A brief excursion to the conservation laws associated with symmetries in classical field theory is in order here. For a given Lagrangian $\mathcal{L}(\phi_a)$, if we perform a transformation to the fields such that $\phi_a \to \phi_a(\lambda)$, where λ characterize the trans formation, then we have,

$$\begin{aligned} D\mathcal{L} &= \sum_a \left(\frac{\partial\mathcal{L}}{\partial\phi_a} D\phi_a + \pi_a^\mu D\partial_\mu\phi_a \right) \\ &= \sum_a (\partial_\mu\pi_a^\mu D\phi_a + \pi_a^\mu D\partial_\mu\phi_a) \\ &= \partial_\mu \left(\sum_a \pi_a^\mu D\phi_a \right) \end{aligned}$$

where $D\phi_a$ is defined as,

$$D\phi_a \equiv \left.\frac{d\phi_a}{d\lambda}\right|_{\lambda=0}$$

In the derivations above we have used the definition of the conjugate momenta and the Lagrangian equations of motion. If the Lagrangian is invariant under this transformation, then, we end up with,

$$\partial_\mu J^\mu = \partial_\mu \left(\sum_a \pi_a^\mu D\phi_a \right) = 0$$

Now we consider the rotation transformation,

$$\phi_1 \to \phi_1 \cos\lambda + \phi_2 \sin\lambda \qquad\qquad \phi_2 \to -\phi_1 \sin\lambda + \phi_2 \cos\lambda$$

The Lagrangian (6.13) consists of three kinds of terms: $\vec{\phi}\cdot\vec{\phi}$, $\dot{\vec{\phi}}\cdot\dot{\vec{\phi}}$ and $\vec{\phi}\times\dot{\vec{\phi}}$, they are all invariant under rotations of $\vec{\phi}$. In addition, we have

$$D\phi_1 = \phi_2 \qquad\qquad D\phi_2 = -\phi_1$$

Thus, we obtain the conserved current density,

$$J^\mu = (\partial^\mu\phi_1)\phi_2 - (\partial^\mu\phi_2)\phi_1 + \delta^{\mu 0}\mu(\phi_1^2 + \phi_2^2) \tag{6.15}$$

The corresponding conserved charge density is

$$\rho = \pi_1^0\phi_2 - \pi_2^0\phi_1 \tag{6.16}$$

with π_a^0 given in equations (6.14). Note the conserved charge above has the same formal expression as (6.2).

We will find later that at low temperatures the ground state has the property

$$\bar{\phi} \equiv < 0|\hat{\phi}|0 > \neq 0$$

Then the rotational symmetry associated with Q is broken. In this case if we shift the field operator by $\hat{\phi}_a = \hat{\phi}'_a + \bar{\phi}_a$, then the charge operator contains a c-number piece,

$$\hat{Q} = V\mu(\bar{\phi}_1^2 + \bar{\phi}_2^2) + \hat{Q}'$$

where $\hat{Q}'$ contains the terms linear and quadratic in the filed operators ϕ'_a. Here $\hat{Q}'$ characterize the quantum fluctuations. Thus the thermal average charge density is

$$\begin{aligned}\rho &= \frac{1}{V}\frac{Tr\{Q\exp[-\beta(H-\mu Q)]\}}{Tr\{\exp[-\beta(H-\mu Q)]\}} \\ &= \mu(\bar{\phi}_1^2 + \bar{\phi}_2^2) + \frac{1}{V}\frac{Tr\{Q'\exp[-\beta(H-\mu Q)]\}}{Tr\{\exp[-\beta(H-\mu Q)]\}}\end{aligned} \tag{6.17}$$

Note the expectation value of the charges in ground state is,

$$< 0|\hat{Q}|0 >= V\mu(\bar{\phi}_1^2 + \bar{\phi}_2^2)$$

Later we will prove that the transition to the broken symmetric phase is just the onset of the Bose-Einstein condensation in the ultrarelativistic regime, therefore the first term above is the charge density in the condensed (zero momentum) ground state, and the second term the charge density in the excited states ([12]). For a system confined in a finite volume, the total charge is conserved. The distribution of the charges between the condensed ground state and the excited states depends on temperature.

VI.2 Interacting Field Case

We start with the generating functional:

$$\begin{aligned}Z[J] = N' \int_{peri} D[\phi_1]D[\phi_2]exp\{\int_0^\beta d\tau \int d^3x[-\frac{1}{2}(\frac{\partial\phi_1}{\partial\tau} + i\mu\phi_2)^2 - \frac{1}{2}(\frac{\partial\phi_2}{\partial\tau} - i\mu\phi_1)^2 \\ + \sum_{i=1}^{2}(-\frac{1}{2}(\nabla\phi_i)^2 - \frac{m^2}{2}\phi_i^2 + J_i(x)\phi_i(x)) - \lambda(\phi_1^2 + \phi_2^2)^2]\}\end{aligned}$$

Removing the $-\lambda(\phi_1^2+\phi_2^2)^2$ term by introducing an additional functional integral of the auxiliary field $\sigma(x)$, we obtain,

$$Z[J] = N' \int_{peri} D[\phi_1]D[\phi_2]D[\sigma]exp\{\frac{-1}{2}\int d^4x[\sum_{i=1}^{2}((\frac{\partial\phi_i}{\partial\tau})^2 + (\nabla\phi_i)^2 + (m^2-\mu^2)\phi_i^2 - 2\phi_i J_i)$$
$$+ 2i\mu\phi_2(\frac{\partial\phi_1}{\partial\tau}) - 2i\mu\phi_1(\frac{\partial\phi_2}{\partial\tau}) - 2\sigma(\phi_1^2+\phi_2^2) - \frac{1}{2\lambda}\sigma^2]\}$$
$$\equiv N' \int_{peri} D[\phi_1]D[\phi_2]D[\sigma]exp\{\frac{-1}{2}\int d^4x[\Phi^T\hat{A}\Phi - 2\Phi^T J - \frac{1}{2\lambda}\sigma^2]\}$$

Where we have defined

$$\Phi \equiv \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \qquad J \equiv \begin{pmatrix}J_1\\ J_2\end{pmatrix}$$
$$\hat{A} \equiv \begin{pmatrix} -\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2 - \mu^2 - 2\sigma & -2i\mu\frac{\partial}{\partial\tau} \\ 2i\mu\frac{\partial}{\partial\tau} & -\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2 - \mu^2 - 2\sigma \end{pmatrix}$$

Now we can do the functional integrals of ϕ_1 and ϕ_2 and we obtain,

$$Z[J] = N' \int_{peri} D[\sigma]exp\{-\frac{1}{2}tr\ln\hat{A} + \frac{1}{2}\int d^4x \int d^4x' J(x)^T A^{-1}(x,x')J(x') + \frac{1}{4\lambda}\int d^4x\sigma^2(x)\} \tag{6.18}$$

We have dropped some irrelevant constants when we do the functional integrals. Here the trace is taken over the configuration space as well as the 2 by 2 matrix $\hat{A}$. $A(x,x')$ is understood as

$$A(x,x') = \hat{A}\begin{pmatrix}\delta(x-x') & \\ & \delta(x-x')\end{pmatrix}$$

Note the term with double integral of x and x' in (6.18) can be rewritten in operator form,

$$\frac{1}{2}\int d^4x \int d^4x' < x|\hat{J}^T\hat{A}^{-1}\hat{J}|x' >$$

Expanding around a uniform σ_0 with $\sigma(x) = \sigma_0 + \delta\sigma(x)$, and keeping only terms up to first order in $\delta\sigma$, we obtain,

$$Z[J] = N' \int_{peri} D[\delta\sigma]exp\{-\frac{1}{2}tr\ln\hat{A}_0 + \frac{1}{2}\int d^4x \int d^4x' < x|\hat{J}^T\hat{A}_0^{-1}\hat{J}|x' >$$
$$+ \frac{1}{4\lambda}\int d^4x\sigma_0 - \frac{1}{2}tr\left[\hat{A}_0^{-1}\begin{pmatrix}-2\delta\hat{\sigma} & 0\\ 0 & -2\delta\hat{\sigma}\end{pmatrix}\right] + \frac{1}{2\lambda}\int d^4x\sigma_0\delta\sigma \tag{6.19}$$
$$+ \frac{1}{2}\int d^4x \int d^4x' < x|\hat{J}^T\hat{A}_0^{-1}\begin{pmatrix}2\delta\hat{\sigma} & 0\\ 0 & 2\delta\hat{\sigma}\end{pmatrix}\hat{A}_0^{-1}\hat{J}|x' >\}$$

Where

$$\hat{A}_0 \equiv \begin{pmatrix} \hat{p}^2 + m^2 - \mu^2 - 2\sigma_0 & 2\mu\hat{p}_0 \\ -2\mu\hat{p}_0 & \hat{p}^2 + m^2 - \mu^2 - 2\sigma_0 \end{pmatrix}$$

$$\hat{A}_0^{-1} = \frac{1}{(\hat{p}^2 + m^2 - \mu^2 - 2\sigma_0)^2 + 4\mu^2\hat{p}_0^2} \begin{pmatrix} \hat{p}^2 + m^2 - \mu^2 - 2\sigma_0 & -2\mu\hat{p}_0 \\ 2\mu\hat{p}_0 & \hat{p}^2 + m^2 - \mu^2 - 2\sigma_0 \end{pmatrix}$$

First we look at the zero order terms in $\delta\sigma$. Assuming J_1 and J_2 are x independent and inserting complete set of states we obtain,

$$\begin{aligned}
&\frac{1}{2}\int d^4x \int d^4x' < x|\hat{J}^T\hat{A}_0^{-1}\hat{J}|x' > \\
&= \frac{1}{2}\int d^4x \int d^4x' \int d^4p(J_1, J_2) < x|p > A_0^{-1}(p) < p|x' > \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \\
&= \frac{1}{2}\int d^4x \int d^4p(J_1, J_2) < x|p > A_0^{-1}(p)\sqrt{\beta(2\pi)^3}\delta^4(p) \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \\
&= \frac{1}{2}\int d^4x \frac{1}{\bar{m}^2 - \mu^2}[J_1^2 + J_2^2]
\end{aligned}$$

Where $\bar{m}^2 \equiv m^2 - 2\sigma_0$ and

$$tr\ln\hat{A}_0 = \ln\det\hat{A}_0 = \ln\det'[(\hat{p}^2 + \bar{m}^2 - \mu^2)^2 + 4\mu^2\hat{p}_0^2] = tr'\ln[(\hat{p}^2 + \bar{m}^2 - \mu^2)^2 + 4\mu^2\hat{p}_0^2]$$

Note we first calculated the determinant of the 2 by 2 matrix, thus $\det'$ denotes taking the determinant of the remaining matrix in configuration space only. Similarly tr' means the trace in configuration space. Thus we have,

$$\begin{aligned}
-\frac{1}{2}tr\ln\hat{A}_0 &= -\frac{1}{2}\frac{V}{(2\pi)^3}\sum_n \int d^3p\ln[\beta^4(\omega_n^2 + p^2 + \bar{m}^2 - \mu^2)^2 + 4\beta^4\mu^2\omega_n^2] \\
&= -\frac{1}{2}\frac{V}{(2\pi)^3}\sum_n \int d^3p\ln\{[\beta^2(\omega_n^2 + (\omega + \mu)^2)][\beta^2(\omega_n^2 + (\omega - \mu)^2)]\} \\
&= -V\int \frac{d^3p}{(2\pi)^3}[\beta\omega + \ln(1 - e^{-\beta(\omega-\mu)}) + \ln(1 - e^{-\beta(\omega+\mu)})] + C
\end{aligned}$$

where C is a dimensional constant which cancels $\ln N'$ in equation (6.19).

Now let's look at the first order terms. As before, it is straight forward to obtain,

$$\frac{1}{2}\int d^4x \int d^4x' < x|\hat{J}^T\hat{A}_0^{-1}\begin{pmatrix} 2\delta\hat{\sigma} & 0 \\ 0 & 2\delta\hat{\sigma} \end{pmatrix}\hat{A}_0^{-1}\hat{J}|x' >= \int d^4x \frac{[J_1^2 + J_2^2]}{(\bar{m}^2 - \mu^2)^2}\delta\sigma(x)$$

and

$$-\frac{1}{2}tr\left[\hat{A}_0^{-1}\begin{pmatrix}-2\delta\hat{\sigma} & 0\\ 0 & -2\delta\hat{\sigma}\end{pmatrix}\right] = \frac{\beta}{2}\int d^3x\int\frac{d^3p}{(2\pi)^3}\sum_n\frac{\omega_n^2+p^2+\bar{m}^2-\mu^2}{(\omega_n^2+p^2+\bar{m}^2-\mu^2)^2+4\mu^2\omega_n^2}\delta\sigma(x)$$

$$= \int d^3x\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\left[1+\frac{1}{e^{\beta(\omega-\mu)}-1}+\frac{1}{e^{\beta(\omega+\mu)}-1}\right]\delta\sigma(x)$$

The sum over n is done in appendix A with $\omega \equiv \sqrt{p^2+\bar{m}^2}$.

The first order terms should sum to zero, thus we get the constraint equation,

$$\bar{m}^2 = \frac{\lambda}{\pi^2}\{\Lambda^2+\frac{1}{2}\bar{m}^2\ln\frac{\bar{m}^2}{\alpha\Lambda^2}\}+m^2-2\lambda f(\beta,\bar{m}^2,\mu)+4\lambda\frac{(J_1^2+J_2^2)}{(\bar{m}^2-\mu^2)^2}$$

Here we have done the divergent integral in momentum p explicitly and Λ is the momentum cutoff. We have also defined a finite function $f(\beta,\bar{m}^2,\mu)$,

$$f(\beta,\bar{m}^2,\mu) \equiv \int\frac{d^3p}{(2\pi)^3}\frac{-2}{\omega}\left(\frac{1}{e^{\beta(\omega-\mu)}-1}+\frac{1}{e^{\beta(\omega+\mu)}-1}\right)$$

If we define renormalized mass and renormalized coupling constant as follows,

$$\begin{aligned}\lambda_R &\equiv \frac{\lambda/\pi^2}{1-\frac{\lambda}{2\pi^2}\ln\frac{\Sigma^2}{\alpha\Lambda^2}}\\ \Sigma^2 &\equiv \frac{m^2+\lambda\Lambda^2/\pi^2}{1-\frac{\lambda}{2\pi^2}\ln\frac{\Sigma^2}{\alpha\Lambda^2}}\end{aligned} \tag{6.20}$$

then we can obtain the renormalized version of the constraint equation,

$$\bar{m}^2 = \Sigma^2+\frac{1}{2}\lambda_R\bar{m}^2\ln\frac{\bar{m}^2}{\Sigma^2}-2\pi^2\lambda_R f(\beta,\bar{m}^2,\mu)+4\pi^2\lambda_R\frac{(J_1^2+J_2^2)}{(\bar{m}^2-\mu^2)^2} \tag{6.21}$$

Keeping only the zeroth order result as an approximation for the generating functional, we get the generating functional for the connected Green's function

$$W[J] = -V\int\frac{d^3p}{(2\pi)^3}[\beta\omega+\ln(1-e^{-\beta(\omega-\mu)})+\ln(1-e^{-\beta(\omega+\mu)})]+\frac{1}{4\lambda}\beta V\sigma_0^2+\frac{1}{2}\beta V\frac{J_1^2+J_2^2}{\bar{m}^2-\mu^2} \tag{6.22}$$

For notational simplicity, we define the following finite function

$$F(\beta,\bar{m}^2,\mu) \equiv \int \frac{d^3p}{(2\pi)^3}\frac{1}{\beta}[\ln(1-e^{-\beta(\omega-\mu)}) + \ln(1-e^{-\beta(\omega+\mu)})]$$

The divergent part of the expression (6.22) can be renormalized as before. We first invert the equations defining the renormalized mass and coupling constant,

$$\lambda = \lambda_R/A, \qquad m^2 = \frac{1}{\pi^2}(\Sigma^2 - \lambda_R\Lambda^2)/A \tag{6.23}$$

with $A \equiv \frac{1}{\pi^2} - \frac{\lambda_R}{2\pi^2}\ln\frac{\alpha\Lambda^2}{\Sigma^2}$. Then we substitute equation (6.23) for the bare mass and bare coupling constant in equation (6.22). Apart from some irrelevant constant, the divergences in equation (6.22) all cancel out and we finally obtain the renormalized version of the generating functional of the connected Green's function,

$$W(J) = -\beta V\left[F(\beta,\bar{m}^2,\mu) + \frac{1}{32\pi^2}\bar{m}^4(\ln\frac{\bar{m}^2}{\Sigma^2} - \frac{1}{2}) - \frac{1}{16\pi^2\lambda_R}(\bar{m}^2-\Sigma^2)^2 - \frac{1}{2}\frac{J_1^2+J_2^2}{\bar{m}^2-\mu^2}\right] \tag{6.24}$$

As before, if we set $\frac{\partial W[J]}{\partial \bar{m}^2} = 0$, then we obtain the renormalized constraint equation (6.21). Making use of this fact, we can relate source J to classical field $\bar{\phi}$,

$$\bar{\phi}_i \equiv \frac{\delta W}{\delta J_i} = \frac{\partial W}{\partial \bar{m}^2}\frac{d\bar{m}^2}{dJ_i} + \frac{\partial W}{\partial J_i} = \frac{\partial W}{\partial J_i} = \frac{J_i}{\bar{m}^2-\mu^2} \qquad i = 1,2$$

Thus the standard Legendre transformation leads to the following renormalized effective potential

$$V(\bar{\phi}) = \frac{1}{2}(\bar{m}^2-\mu^2)(\bar{\phi}_1^2+\bar{\phi}_2^2) + F(\beta,\bar{m}^2,\mu) + \frac{1}{32\pi^2}\bar{m}^4(\ln\frac{\bar{m}^2}{\Sigma^2}-\frac{1}{2}) - \frac{1}{16\pi^2\lambda_R}(\bar{m}^2-\Sigma^2)^2 \tag{6.25}$$

and the renormalized constraint equation can be rewritten as,

$$\bar{m}^2 = \Sigma^2 + \frac{1}{2}\lambda_R\bar{m}^2\ln\frac{\bar{m}^2}{\Sigma^2} - 2\pi^2\lambda_R f(\beta,\bar{m}^2,\mu) + 4\pi^2\lambda_R(\bar{\phi}_1^2+\bar{\phi}_2^2) \tag{6.26}$$

which can also be obtained by setting $\frac{\partial V(\bar{\phi})}{\partial \bar{m}^2} = 0$. This fact immediately gives us,

$$\frac{dV(\bar{\phi})}{d\bar{\phi}_i} = \frac{\partial V(\bar{\phi})}{\partial \bar{\phi}_i} + \frac{\partial V(\bar{\phi})}{\partial \bar{m}^2}\frac{d\bar{m}^2}{d\bar{\phi}_i} = \frac{\partial V(\bar{\phi})}{\partial \bar{\phi}_i} = (\bar{m}^2 - \mu^2)\bar{\phi}_i \qquad i = 1,2 \tag{6.27}$$

It is easy to establish the relationship between the thermodynamic potential and the effective potential,

$$\Omega(T, V, \mu) = V \cdot V(\bar{\phi})\Big|_{dV/d\bar{\phi}=0}$$

In other words, the thermodynamic potential is the value of the effective potential at its minimum. To study the thermodynamics of the system, we must differentiate the minimum value of $V(\bar{\phi})$ with respect to T, V and μ. Without the knowledge of the minimum of the effective potential at this moment, we can write down the following general expressions, where V depends on μ (or T) via $\bar{\phi}$, $\bar{m}^2$ and $F(\beta, \bar{m}^2, \mu)$.

$$\begin{aligned}\frac{\partial V(\bar{\phi})}{\partial \mu}\Big|_{V,T} &= \frac{\partial V}{\partial \bar{m}^2}\frac{d\bar{m}^2}{d\mu} + \frac{\partial V}{\partial \bar{\phi}^2}\frac{d\bar{\phi}^2}{d\mu} + \frac{\partial V}{\partial \mu} \\ &= \frac{\partial F}{\partial \mu} + \frac{1}{2}\bar{m}^2\frac{d(\bar{\phi}_1^2 + \bar{\phi}_2^2)}{d\mu} - \frac{1}{2}\frac{d}{d\mu}[\mu^2(\bar{\phi}_1^2 + \bar{\phi}_2^2)]\end{aligned} \tag{6.28}$$

and similarly,

$$\frac{\partial V(\bar{\phi})}{\partial T}\Big|_{V,\mu} = \frac{\partial F}{\partial T} + \frac{1}{2}(\bar{m}^2 - \mu^2)\frac{d(\bar{\phi}_1^2 + \bar{\phi}_2^2)}{dT} \tag{6.29}$$

We are interested in the relativistic regime where $T >> \bar{m}$ and $T >> \mu$. In this region we have the following expansions of $F(\beta, \bar{m}^2, \mu)$ and $f(\beta, \bar{m}^2, \mu)$ according to Appendix C,

$$\begin{aligned}F = &-\frac{\pi^2 T^4}{45} + \frac{T^2(\bar{m}^2 - 2\mu^2)}{12} - \frac{T(\bar{m}^2 - \mu^2)^{3/2}}{6\pi} - \frac{\mu^2(3\bar{m}^2 - \mu^2)}{24\pi^2} \\ &+ \frac{\bar{m}^4}{16\pi^2}\left[\ln\frac{4\pi T}{\bar{m}} - \gamma + \frac{3}{4}\right] + O(\frac{\bar{m}^6}{T^2}, \frac{\bar{m}^4\mu^2}{T^2})\end{aligned} \tag{6.30}$$

and

$$f = -\frac{T^2}{3} + \frac{T(\bar{m}^2 - \mu^2)^{1/2}}{\pi} + \frac{\mu^2}{2\pi^2} - \frac{\bar{m}^2}{2\pi^2}\left[\ln\frac{4\pi T}{\bar{m}} - \gamma + \frac{1}{2}\right] \tag{6.31}$$

We will show later that at high temperatures, the system is in the symmetric phase $\bar{\phi} = 0$. We first summarize some of the common features in this phase in order to facilitate the discussions later. We can obtain the charge density and entropy by setting $\bar{\phi} = 0$ in equations (6.28) and (6.29).

$$\rho = -\frac{\partial F}{\partial \mu} \qquad S = -V\frac{\partial F}{\partial T} \tag{6.32}$$

The ideal gas form of the F shows that all the interactions among the bosons are incorporated into the effective mass $\bar{m}^2$. However, the interactions appear nontrivially in the pressure and energy,

$$P = -V(\bar{\phi} = 0) \qquad U = V[V(\bar{\phi} = 0) + TS/V + \mu\rho]$$

since all the complicated terms in effective potential contribute. With the help of the expansion for F, we obtain the following leading contributions to charge density and entropy,

$$\rho = \frac{1}{3}\mu T^2 \qquad S = \frac{4\pi^2 T^3 V}{45}$$

If the system is cooled with charge Q and volume V fixed, then the above equation requires $\mu(T)$ increases with $1/T^2$. If we take the quotient of the above two equation, then

$$\frac{Q}{S} = \frac{15}{4\pi^2}\frac{\mu}{T}$$

From this we see if the system is cooled with Q and S fixed, as in the cosmological applications, then μ has to decrease linearly with T.

From this high temperature symmetric phase, we want to see what happens to it as we lower the temperature if we fix the total charge and volume. At high temperature, the constraint equation (6.26) gives us the following $\bar{m}$ as a function of T if we keep only the leading order term in the expansion of $f(\beta, \bar{m}^2, \mu)$.

$$\bar{m}(T) = \left[\Sigma^2 + \frac{2\pi^2 \lambda_R}{3} T^2\right]^{1/2}$$

comparing with the high temperature behaviour of chemical potential $\mu(T)$,

$$\mu(T) = \frac{3\rho}{T^2} \tag{6.33}$$

we see in this phase $\mu(T) < \bar{m}(T)$, thus the minimum has to occur at $\bar{\phi} = 0$, namely, in this high temperature region the system is in the symmetric phase. As we lower the temperature, there will be a temperature T_c at which $\mu(T) = \bar{m}(T)$, at this temperature, the system will jump into the broken symmetric phase in which the system develops a nonzero $\bar{\phi}$ and maintains the relationship $\mu(T) = \bar{m}(T)$ throughout the temperature region of $T \geq T_c$. Since the symmetry broken is a $U(1)$ rotational symmetry, we can set $\bar{\phi}_2 = 0$ and $\bar{\phi}_1$ takes the nonzero value. The critical temperature satisfies,

$$\frac{3\rho}{T_c^2} = \left(\Sigma^2 + \frac{2\pi^2 \lambda_R}{3} T_c^2\right)^{1/2} \tag{6.34}$$

The value of the chemical potential at critical temperature is

$$\mu_c = \left(\Sigma^2 + \frac{2\pi^2 \lambda_R}{3} T_c^2\right)^{1/2}$$

making use of the equation (6.33), we can find this value of chemical potential satisfies the following cubic equation,

$$\frac{\mu_c}{2\pi^2 \lambda_R} (\mu_c^2 - \Sigma^2) = \rho$$

At temperatures lower than T_c, the system develops nonzero values of $\bar{\phi}$ which can be determined by the constraint equation,

$$\bar{\phi}^2(T) = \frac{1}{4\pi^2\lambda_R}\left[\mu^2 - \Sigma^2 - \frac{1}{2}\lambda_R\mu^2\ln\frac{\mu^2}{\Sigma^2} + 2\pi^2\lambda_R f(\beta,\mu^2,\mu)\right] \tag{6.35}$$

To connect the spontaneous symmetry breaking with relativistic Bose-Einstein condensation, we computer charge density ρ. With $\bar{m}(T) = \mu(T)$ and nonzero $\bar{\phi}$, we obtain ρ from equation (6.28).

$$\rho = \mu\bar{\phi}^2 - \left.\frac{\partial F}{\partial\mu}\right|_{\mu(T)=\bar{m}(T)} \tag{6.36}$$

Comparing with equation (6.17), we can see that the first term is the charge density in the condensed ground state and the second term is the charge density in the excited states. Note

$$-\left.\frac{\partial F}{\partial\mu}\right|_{\mu(T)=\bar{m}(T)} = \int\frac{d^3p}{(2\pi)^3}\left[\frac{1}{\exp[\beta(\omega-\mu)]-1} - \frac{1}{\exp[\beta(\omega+\mu)]-1}\right]_{\bar{m}(T)=\mu(T)} \tag{6.37}$$

From our earlier interpretation of the above two term as number densities of particles and antiparticles in the states with momentum p, we see that this integration over p is indeed the charge density in excited states ($p > 0$).

Here we can raise an interesting question: we have seen in Chapter IV that the state $\bar{\phi} \neq 0$ is at best an unstable state. Therefore, how can we identify it with a physical, symmetry breaking state? The answer lies in the fact that for fixed total charges, the amount of charges in excited states, which is determined by equation (6.37), reach their maximum when $\mu(T) = \bar{m}(T)$. Thus, charge conservation forces the scalar field to develop a nonzero vacuum expectation value to absorb some charges in this otherwise unstable state. However, for a system with zero net charge, the above argument breaks down. Therefore, in contrast to the conclusion from the usual perturbative approach ([12]), it seems to us that for this kind of system, relativistic Bose-Einstein condensation should not occur.

Making use of the equations (6.35), (6.36) and high temperature expansion of f, equation (6.31), we obtain the explicit expressions for charge densities in ground and excited states,

$$\rho(p=0) = \frac{\mu}{4\pi^2\lambda_R}\left[\mu^2 - \Sigma^2 - \frac{2\pi^2\lambda_R}{3}T^2\right]$$

$$\rho(p>0) = \frac{1}{3}T^2\mu$$

The sum of them should give us the total net charge, which provides us with a cubic equation for $\mu(T)$. For temperatures just below T_c, we can expand μ around μ_c and solve the cubic equation,

$$\mu(T) = \mu_c + \frac{2\pi^2\lambda_R\mu_c(T_c^2 - T^2)}{3(4\mu_c^2 - 2\Sigma^2)}$$

Inserting this back to the expression for $\rho(p=0)$, we obtain the ground state charge densities at temperatures just below T_c,

$$\rho(p=0) = \rho\left(1 - \frac{T^2}{T_c^2}\right)\frac{3\mu_c^2 - \Sigma^2}{4\mu_c^2 - 2\Sigma^2}$$

This result is certainly invalid for the region $T \to 0$, otherwise we would get an absurd result that some particles still stay in the excited states when $T \to 0$.

Finally, we mention in passing that the entropy in the broken symmetric phase can be similarly calculated.

$$\frac{S}{V} = -\frac{\partial F}{\partial T}\bigg|_{\mu(T)=\bar{m}(T)}$$

So far we fixed charge and volume, we can also consider the case of fixed charge and entropy. In the high temperature symmetric phase, we quote the results obtained previously,

$$\mu(T) = \frac{4\pi^2 T}{15}\frac{Q}{S} \qquad \bar{m}(T) = \left[\Sigma^2 + \frac{2\pi^2\lambda_R}{3}T^2\right]^{1/2}$$

In this phase, from equation (6.32) we can see if we are to get an physical , real entropy, F has to be real, namely, $\mu(T) < \bar{m}(T)$. Here we assume this is the case, which means, in high temperature,

$$\left[\frac{2\pi^2\lambda_R}{3}\right]^{1/2} > \frac{4\pi^2}{15}\frac{Q}{S}$$

However, if this is true, then as temperature decreases, the relationship $\bar{m}(T) > \mu(T)$ remains true, consequently, in contrast to the conclusions from the usual perturbative approach ([12]), no symmetry breaking occurs.

Chapter VII

Gross-Neveu Model

VII.1 Dynamical Symmetry Breaking

In most theories where spontaneous symmetry breaking plays an essential role, such as in $SU(2) \times U(1)$ electroweak theory and the $SU(5)$ grand unified theory, we explicitly introduce an elementary scalar field in order to generate masses for the gauge vector bosons by means of Higgs Mechanism in order to preserve the renormalizability of the theories. However, the introduction of a scalar field is not necessary for a theory to exhibit the phenomenon of spontaneous symmetry breaking. The fields in a theory, either composite or elementary, can develop nonzero vacuum expectation values by themselves and thus break the intrinsic symmetry of the theory. Interests in this dynamical symmetry breaking can be explained in part by the difficulties encountered in the non-abelian gauge theories of strong interactions. In these theories, it is impossible to break the gauge symmetry by explicitly introducing Higgs particles without destroying the asymptotic freedom. However, if the gauge symmetry remains unbroken, then infrared singularities associated with the masslessness of the gauge bosons prevent the appearance of the charged gauge bosons and quarks in physical states.

Here we study the Gross-Neveu model since it is the only known physical, asymptotically free theory besides the non-abelian gauge theory of the strong interactions. It was first introduced by Heisenberg and then studied by Nambu and Jona-Lasinio who demonstrated that a fermion mass can be generated by the dynamical symmetry breaking of the discrete chiral symmetry in this theory, in analogue with the energy gap in the theory of superconductivity. The Lagrangian of the Gross-Neveu Model is

$$\mathcal{L} = \bar{\psi}_i(i\partial\!\!\!/)\psi_i + \frac{1}{2}g(\bar{\psi}_i\psi_i)^2 \tag{7.1}$$

with $i = 1, \ldots, N$, and summation over repeated indices. We will omit the explicit indices from now on. This theory is perturbatively renormalizable in $1+1$ dimensions and we need to perform coupling constant and wave function renormalizations. The masslessness of the fermion fields ensure the discrete chiral symmetry

$$\psi \to \gamma_5\psi, \qquad \bar{\psi} \to -\gamma_5\bar{\psi}$$

We want to construct the effective potential for the composite field $\bar{\psi}_R\psi_R$, where ψ_R denotes the renormalized wave function. In order to obtain a self-consistent result from the auxiliary field approach, we need to perform wave function renormalization explicitly, as opposed to the case in the $\lambda\phi^4$ theory of chapter II. Here we demand that the renormalized wave function be defined as $\bar{\psi}_R\psi_R = g\bar{\psi}\psi$. Thus the generating functional for temperature Green's function is

$$Z[J] = N' \int_{antiperi} D[\bar{\psi}]D[\psi] \exp\left\{\int d^2x \left[\mathcal{L} + Jg\bar{\psi}\psi\right]\right\} \tag{7.2}$$

With $\int d^2x = \int dx d\tau$. Associated with the antiperiodicity of the ψ field in the Euclidean time direction, we have a Matsubara frequency for fermion field $\omega_n \equiv \frac{(2n+1)\pi}{\beta}$.

Now we can introduce the auxiliary field as before,

$$Z[J] = N' \int_{antiperi} D[\bar{\psi}]D[\psi]D[\sigma] \exp\left\{\int d^2x \left[\bar{\psi}(i\partial\!\!\!/ + \sigma + gJ)\psi - \frac{\sigma^2}{2g}\right]\right\} \tag{7.3}$$

Gaussian functional integral of Grassman variables

$$\int D[\bar{\eta}]D[\eta]e^{-\bar{\eta}_i A_{ij}\eta_j+\bar{\eta}_i\rho_i+\bar{\rho}_i\eta_i} = \det A e^{\bar{\rho}_i A_{ij}^{-1}\rho_j} \tag{7.4}$$

leads to

$$Z[J] = N' \int D[\sigma] \exp\left\{N tr \ln[-i\partial\!\!\!/ - \sigma - gJ] - \int d^2x \frac{\sigma^2}{2g}\right\} \tag{7.5}$$

where the trace is with respect to x as well as Dirac indices. Expanding around σ_0 and omitting second order terms, we get the generating functional for the connected Green's function

$$W[J] = N tr_2 \ln[(\sigma_0 + gJ)^2 - p^2] - \int d^2x \frac{\sigma_0^2}{2g} + \ln N' \tag{7.6}$$

where we have used

$$tr_D \ln(\pm p\!\!\!/ + m) = \ln(m^2 - p^2) \tag{7.7}$$

in $1+1$ dimensions, where tr_2 means trace with respect to x and tr_D trace with respect to Dirac indices.

The linear constraint equation is

$$-N < x|tr_D \frac{1}{-i\partial\!\!\!/ - \bar{m}}|x >= \frac{\sigma_0}{g} \tag{7.8}$$

with $\bar{m} \equiv \sigma_0 + gJ$, or equivalently,

$$2N < x|\frac{\bar{m}}{\omega_n^2 + p^2 + \bar{m}^2}|x >= \frac{\sigma_0}{g} \tag{7.9}$$

Taking the matrix element and summing over n, we obtain

$$N \int \frac{dp}{2\pi}\frac{1}{\omega} + N f(\beta, \bar{m}^2) - \frac{\sigma_0}{g\bar{m}} = 0 \tag{7.10}$$

with $f(\beta,\bar{m}^2) \equiv \int \frac{dp}{2\pi}\frac{1}{\omega}\frac{-2}{e^{\beta\omega}-1}$, which is a finite, temperature dependent function. Use have been made of the following relation

$$\sum_n \frac{1}{\omega_n^2+\omega^2} = \frac{\beta}{2\omega}\left(1-\frac{2}{e^{\beta\omega}+1}\right) \tag{7.11}$$

With the help of

$$\int \frac{dp}{2\pi}\frac{1}{\sqrt{p^2+\bar{m}^2}} = \frac{1}{2\pi}\ln(p+\sqrt{p^2+\bar{m}^2})\Big|_{-\Lambda}^{\Lambda} = \frac{1}{\pi}\ln\frac{2\Lambda}{\bar{m}} + O\left(\frac{1}{\Lambda^2}\right) \tag{7.12}$$

we obtain the following expression for the constraint equation,

$$Ng\frac{\bar{m}}{\pi}\ln\frac{2\Lambda}{\bar{m}} + Ng\bar{m}f(\beta,\bar{m}^2) - \bar{m} + gJ = 0 \tag{7.13}$$

As before, if we split the logarithm term into a sum of two terms by introducing an arbitrary mass scale Σ, then we can obtain the following renormalized version of the constraint equation

$$\bar{m} = -g_R\bar{m}\ln\frac{\bar{m}}{\Sigma} + \pi g_R\bar{m}f(\beta,\bar{m}^2) + \frac{g_R J\pi}{N} \tag{7.14}$$

with

$$g_R = \frac{Ng}{\pi - Ng\ln\frac{2\Lambda}{\Sigma}} \tag{7.15}$$

Inverting this equation and we get,

$$g = \frac{\pi g_R}{N + Ng_R\ln\frac{2\Lambda}{\Sigma}} \tag{7.16}$$

Had we started with a term $J\bar{\psi}\psi$ in (7.2), then we would not have been able to accomplish renormalization for the constraint equation. That is why we want to study the vacuum expectation value of $g\bar{\psi}\psi$.

we can see in the limit of $\Lambda \to 0$, $g \to 0^+$. From the Lagrangian (7.1), it is clear that the positive sign of g makes the theory unstable at the classical level. However, like the $\lambda\phi^4$

theory, this Lagrangian can be regarded as the low energy effective theory of the Yukawa interaction and the positivity of g can be substantiated ([23]).

Now we can come back to renormalize the $W[J]$,

$$W[J] = NL\sum_n \int \frac{dp}{2\pi}\ln[\omega_n^2 + p^2 + \bar{m}^2] - \beta L\frac{\sigma_0^2}{2g} + \ln N' \tag{7.17}$$

where L is the one dimensional volume to which the system of fermions are confined. For fermions, we have

$$\sum_n \ln(\omega_n^2 + x^2) = \beta x + 2\ln(1 + e^{-\beta x}) + C \tag{7.18}$$

where C is an x independent, temperature dependent constant. In our case, it cancels the $\ln N'$ in $W[J]$. Thus, we have

$$W[J] = NL\beta\int \frac{dp}{2\pi}\omega + 2NL\beta F(\beta, \bar{m}^2) - \beta L\frac{\sigma_0^2}{2g} \tag{7.19}$$

with $F(\beta, \bar{m}^2) \equiv \int \frac{dp}{2\pi}\frac{1}{\beta}\ln(1 + e^{-\beta\omega})$, which is a finite, temperature dependent function. It is easy to obtain the following result

$$\begin{aligned}
&\int \frac{dp}{2\pi}\omega \\
&= \frac{1}{2\pi}\left\{\frac{p\omega}{2} + \frac{\bar{m}^2\ln(p+\omega)}{2}\right\}\Big|_{-\Lambda}^{\Lambda} \\
&= \frac{1}{2\pi}\left\{\Lambda^2 + \bar{m}^2\ln\frac{\alpha\Lambda}{\bar{m}}\right\} + O\left(\frac{1}{\Lambda^2}\right)
\end{aligned} \tag{7.20}$$

with $\ln\alpha = \ln 2 + \frac{1}{2}$. Substituting this in $W[J]$, we get

$$W[J] = \beta L\left\{\frac{N\bar{m}^2}{2\pi}\ln\frac{\alpha\Lambda}{\bar{m}} + 2NF(\beta, \bar{m}^2) - \frac{\sigma_0^2}{2g}\right\} \tag{7.21}$$

For notational simplicity, we denote $\langle g\bar{\psi}\psi\rangle$ by $\bar{\phi}$ from now on.

$$\bar{\phi} = \frac{\delta W[J]}{\delta J} = \frac{\partial W}{\partial \bar{m}}\frac{\delta \bar{m}}{\delta J} + \frac{\partial W}{\partial J} = \bar{m} - gJ \tag{7.22}$$

In the limit $\Lambda \to 0$, we have $\bar{\phi} = \bar{m}$ and $\bar{m} = \sigma_0$. Thus, the effective potential is

$$\begin{aligned} V[\bar{\phi}] &= -\frac{1}{\beta L}(W[J] - \bar{\phi} J) \\ &= -\frac{N\bar{m}^2}{2\pi}\ln\frac{\alpha\Lambda}{\bar{m}} - 2NF(\beta, \bar{m}^2) + \frac{1}{2}\frac{N\bar{m}^2\left(1 + g_R \ln\frac{2\Lambda}{\Sigma}\right)}{\pi g_R} \\ &= N\left[\frac{\bar{\phi}^2}{2\pi g_R} + \frac{\bar{\phi}^2}{2\pi}\ln\frac{2\bar{\phi}}{\alpha\Sigma} - 2F(\beta, \bar{\phi}^2)\right] \end{aligned} \tag{7.23}$$

This is the final, renormalized expression for the effective potential. We have dropped a term $\frac{1}{2}gJ^2$ which approaches zero for any finite J in the limit of large Λ. The constraint equation (7.14) relates $\bar{\phi} = \bar{m}$ to J.

It is interesting to note that regardless of the sign of λ_R, this effective potential is always bounded from below and consequently the system is always stable.

At $T = 0$, $F(\beta, \bar{\phi}^2) = 0$, we have

$$\frac{dV}{d\bar{\phi}} = \left(\frac{1}{\pi g_R} + \frac{1}{\pi}\ln\frac{2\bar{\phi}}{\alpha\Sigma} + \frac{1}{2\pi}\right)\bar{\phi}N \tag{7.24}$$

and

$$\frac{d^2V}{d\bar{\phi}^2} = \left(\frac{3}{2\pi} + \frac{1}{\pi g_R} + \frac{1}{\pi}\ln\frac{2\bar{\phi}}{\alpha\Sigma}\right)N \tag{7.25}$$

Thus, the $\bar{\phi} = 0$ solution is never a minimum, whereas it is easy to check the $\bar{\phi} \neq 0$ solution $\bar{\phi}_0$ is always a minimum and the fermion field obtains a mass $M = \bar{\phi}_0$.

We can similarly do the renormalization group analysis as we change Σ, the β function associated with it is

$$\beta(g_R) = \Sigma\frac{\partial g_R}{\partial\Sigma} = -g_R^2 \tag{7.26}$$

the minus sign indicates that we recovered the property of asymptotic freedom in this theory.

Now we want to study the temperature dependence of the effective potential. Here we study high temperature regime only. If we define

$$h_l^F(\bar{m}) = \frac{1}{\Gamma(l)}\int_0^\infty dx\frac{x^{l-1}}{\omega}\frac{1}{1+e^\omega}$$

with $\omega = \sqrt{x^2 + \bar{m}^2}$, then it is easy the establish the following algebraic relation,

$$h_l^F(\bar{m}) = h_l(\bar{m}, 0) - \frac{1}{2^{l-2}} h_l(2\bar{m}, 0)$$

where $h_l(\bar{m}, r = 0)$ is defined in Appendix C. Furthermore, we can relate the finite, temperature dependent function $F(\beta, \bar{m}^2)$ to $h_3^F(\bar{m})$ by the following relationship,

$$F(\beta, \bar{m}^2) = \frac{2}{\pi\beta^2} h_3^F(\bar{m})$$

Therefore, we can easily get the following high temperature expansion of $F(\beta, \bar{m}^2)$ from the high temperature expansion of $H_3(\bar{m}, 0) = 2h_3(\bar{m}, 0)$ in Appendix C,

$$F(\beta, \bar{m}^2) = \frac{\pi}{12}T^2 + \frac{\bar{m}^2}{4\pi}\ln\left(\frac{\bar{m}}{\pi T}\right) + \left(\frac{1}{2} + \frac{\gamma}{4\pi} - \frac{1}{8\pi}\right)\bar{m}^2 - \frac{7}{64\pi^3}\zeta(3)\bar{m}^4\frac{1}{T^2} + O\left(\frac{1}{T^4}\right)$$

Dropping a pure constant $-\frac{\pi}{6}T^2$ and keeping only leading order contribution from the high temperature expansion, we obtain the high temperature effective potential,

$$V(\bar{\phi}) = N\left[\frac{\bar{\phi}^2}{2\pi g_R} - \left(1 + \frac{\gamma}{2\pi} - \frac{1}{4\pi}\right)\bar{\phi}^2 + \frac{\bar{\phi}^2}{2\pi}\ln\frac{2\pi T}{\alpha\Sigma}\right]$$

with the Euler's constant $\gamma \approx 0.577$.

Thus, we see that at high enough temperature, the sign in front of the quadratic term $\bar{\phi}^2$ is always positive, which indicates the system is always in symmetric state at high enough temperature.

In summary, our non-perturbative effective potential confirmed the picture from the usual perturbative calculations that the symmetry of the system is dynamically broken at low temperature. Our effective potential also exhibited the expected phenomenon of high temperature symmetry restoration and recovered the asymptotic freedom of the theory. Admittedly, this phenomenon of dynamical symmetry breaking as exhibited here is unrealistically simple. However, it is believed that this is indicative of what one would expect in realistic theories of strong interactions.

VII.2 Nonuniform σ_0

The extension to the nonuniform case in space is straight forward. The approach here parallels that in II.4. The derivations up to (7.6) and (7.9) are general. We can do the sum on the Euclidean time component as in VII.1. In the end we obtain the following results for the constraint equation and generating functional for the connected Green's function

$$-N < \mathbf{x}|\frac{1}{\hat{\omega}}\left(1+\frac{2}{e^{\beta\hat{\omega}}-1}\right)|\mathbf{x}> = \frac{\sigma_0}{g} \tag{7.27}$$

$$W[J] = N\beta \int dx < \mathbf{x}|\hat{\omega}|\mathbf{x}> + 2N\int d^2x \tilde{F}(\beta,\mathbf{x}) - \int d^2x \frac{\sigma_0^2(\mathbf{x})}{2g} \tag{7.28}$$

with $\tilde{F}(\beta,\mathbf{x}) \equiv \frac{1}{\beta} < \mathbf{x}|\ln(1+e^{-\beta\hat{\omega}})|\mathbf{x}>$.

Expanding $\bar{m}(\mathbf{x})$ around an arbitrary mass scale Σ^2, and retain the divergent terms explicitly, we obtain

$$\frac{Ng\bar{m}(\mathbf{x})}{\pi}\ln\frac{2\Lambda}{\Sigma} + Ng\bar{m}(\mathbf{x})r_f(\beta,\mathbf{x}) + Ng\bar{m}(\mathbf{x})\tilde{f}(\beta,\mathbf{x}) - \bar{m}(\mathbf{x}) + gJ(\mathbf{x}) = 0 \tag{7.29}$$

for the constraint equation, where $\tilde{f}(\beta,\mathbf{x}) \equiv \left\langle \mathbf{x}|\frac{1}{\hat{\omega}}\frac{-2}{e^{\beta\hat{\omega}}-1}|\mathbf{x}\right\rangle$, which is a finite, temperature dependent function. Here r_f denotes the finite part of the expansion of $\langle \mathbf{x}|\frac{1}{\hat{\omega}}|\mathbf{x}\rangle$. Defining the renormalized coupling constant as in equation (7.15), we obtain the renormalized version of the constraint equation

$$\bar{m}(\mathbf{x}) = \pi g_R \bar{m}(\mathbf{x})(r_f + \tilde{f}) + \frac{\pi g_R J(\mathbf{x})}{N} \tag{7.30}$$

We can similarly expand $\bar{m}(\mathbf{x})$ around Σ for $W[J]$:

$$
\begin{aligned}
W[J] = & N\beta L\frac{\Sigma^2}{2\pi}\ln\frac{\alpha\Lambda}{\Sigma} + N\beta\int dx(\bar{m}^2(\mathbf{x}) - \Sigma^2)\frac{1}{2\pi}\ln\frac{2\Lambda}{\Sigma} \\
& + N\int d^2x(R_f(\beta,\mathbf{x}) + 2\tilde{F}(\beta,\mathbf{x})) - \int d^2x\frac{\sigma_0^2(\mathbf{x})}{2g}
\end{aligned}
\tag{7.31}
$$

where $R_f(\beta, \mathbf{x})$ denotes the finite part in the expansion of $\langle\mathbf{x}|\hat{\omega}|\mathbf{x}\rangle$. As before, we have

$$
\bar{\phi}(\mathbf{x}) = \bar{m}(\mathbf{x}) - gJ(\mathbf{x}) \tag{7.32}
$$

which goes to $\bar{m}(\mathbf{x})$ as $\Lambda \to 0$. Thus, replacing the bare coupling constant by the renormalized one through equation (7.16), we get the renormalized effective action

$$
\begin{aligned}
\Gamma[\mathbf{x}] &= W[J(\mathbf{x})] - \int d^2x J(\mathbf{x})\bar{\phi}(\mathbf{x}) \\
&= N\left[\frac{\beta L\Sigma^2}{2\pi}\ln\frac{\alpha}{2} + \int d^2x(R_f(\beta,\mathbf{x}) + 2\tilde{F}(\beta,\mathbf{x})) - \int d^2x\frac{\bar{m}^2(\mathbf{x})}{2\pi g_R}\right]
\end{aligned}
\tag{7.33}
$$

Chapter VIII

Scalar QED

In this chapter we want to extend our auxiliary field approach to a gauge theory: scalar electrodynamics (scalar QED). The Lagrangian for the scalar electrodynamics is,

$$\mathcal{L} = (\partial^\nu - ieA^\nu)\Phi^* \cdot (\partial^\nu + ieA^\nu)\Phi - m^2\Phi^*\Phi - 4\lambda(\Phi^*\Phi)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{8.1}$$

with $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, where ϕ_1 and ϕ_2 are two real fields and Φ^* is the hermitian conjugate of Φ. We also have

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}A^\mu \left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right) A^\nu$$

This Lagrangian is expressed in Minkowski space, but we will be secretly working in Euclidean space. The generating functional for temperature Green's function from this Lagrangian is,

$$Z[J] = [N'(\beta)]^4 \int_{peri} [dA][d\Phi][d\Phi^*] \exp(\int d^4x\mathcal{L} + \phi_i J_i) \det(\frac{\partial F}{\partial \omega})\delta(F)$$

where ω is the variable that parameterizes the local $U(1)$ gauge, and F is an arbitrary function chosen to fix the gauge. Without this gauge fixing, the functional integral will sum over paths in space-time which are connected by gauge transformations. When we study partition functions, this means we are taking the trace over both physical and unphysical states, therefore, in order to exclude unphysical states, gauge fixing is necessary.

For later convenience, we choose

$$F = \partial_\mu A^\mu - f \equiv \partial_\mu A^\mu - \alpha e \phi_1 \phi_2$$

This choice of F was first suggested by 't Hooft ([14] and [15]). As usual, we will also multiply the above generating functional by a function $G(f) = \exp\left[-\frac{1}{2\alpha}\int d^4x f\right]$. The effect of this function is adding an extra gauge fixing term $-\frac{1}{2\alpha}(\partial_\mu A^\mu - \alpha e\phi_1\phi_2)^2$ to the Lagrangian.

The advantage of 't Hooft gauge fixing is that in the limit of $\alpha \to 0$, the cross term, $-e(\partial_\mu A^\mu)\phi_1\phi_2$, from the Lagrangian (8.1) is canceled by the gauge fixing term. This cancellation greatly simplifies the algebra later. In this limit, we also obtain a temperature dependent constant $\det(-\partial^2)$ from $\frac{\partial F}{\partial \omega}$.

As before, we replace the original $(\Phi^*\Phi)^2$ term by an auxiliary field σ. We are going to calculate the effective potential for this theory, and it is known that effective potentials can be gauge dependent for gauge theories since it is not a physical observable. With the chosen gauge, we get,

$$
\begin{aligned}
Z[J] = {} & [N'(\beta)]^4 \det[-\partial^2] \int_{peri} [dA][d\Phi][d\Phi^*][d\sigma] \\
& \exp\left\{ \int d^4x \left[-\frac{1}{2}(-(\partial_\mu\phi_i)^2 - e^2A^2\phi_i^2 + m^2\phi_i^2 + 4e\phi_2 A_\mu\partial^\mu\phi_1 - 2\sigma\phi_i^2) + \phi_i J_i \right] \right\} \\
& \times \exp\left\{ \int d^4x \left[\frac{1}{4\lambda}\sigma^2(x) + \frac{1}{2}A_\mu(x)\left(\delta_{\mu\nu}\partial^2 - (1-\frac{1}{\alpha})\partial^\mu\partial_\nu\right)A^\nu(x) \right] \right\}
\end{aligned}
$$

Now the exponent is quadratic in ϕ_i and after we do the functional integral of them, we obtain,

$$
\begin{aligned}
Z[J] = {} & [N'(\beta)]^4 \det[-\partial^2] \int_{peri} [dA][d\sigma] \exp\left\{ \int d^4x d^4x' < x|\frac{1}{2}\hat{\mathbf{J}}^T\hat{B}^{-1}\hat{\mathbf{J}}|x' > \right\} \\
& \times \exp\left\{ -\frac{1}{2}tr\ln\hat{B} + \int d^4x \left[\frac{1}{4\lambda}\sigma^2(x) + \frac{1}{2}A_\mu(x)\left(\delta_{\mu\nu}\partial^2 - (1-\frac{1}{\alpha})\partial^\mu\partial_\nu\right)A^\nu(x) \right] \right\}
\end{aligned}
$$

with

$$
\hat{B} \equiv \left\{ \begin{matrix} \partial^2 - e^2\hat{A}^2 + \hat{m}^2 - 2\hat{\sigma} & 0 \\ 4e\hat{A}_\mu\partial^\mu & \partial^2 - e^2\hat{A}^2 + \hat{m}^2 - 2\hat{\sigma} \end{matrix} \right\}
$$

and $\hat{\mathbf{J}}$ is the column vector of $\hat{J}_1$ and $\hat{J}_2$. Now we define a new matrix $\hat{B}_0$ around which we are going to expand matrix $\hat{B}$,

$$
\hat{B}_0 \equiv \left\{ \begin{matrix} \partial^2 + \hat{m}^2 - 2\hat{\sigma}_0 & 0 \\ 0 & \partial^2 + \hat{m}^2 - 2\hat{\sigma}_0 \end{matrix} \right\}
$$

with $\hat{\sigma} = \hat{\sigma}_0 + \delta\hat{\sigma}$. Keeping terms up to $\delta\sigma^2$ or A^2 (dropping cross product terms of $\delta\sigma^2$ and A^2), we have,

$$-\frac{1}{2}tr\ln\hat{B} = -\frac{1}{2}tr\ln\hat{B}_0 + tr[\hat{L}^{-1}(e^2\hat{A}^2 + 2\delta\hat{\sigma})] + \frac{1}{2}tr[\hat{L}^{-1}2\delta\hat{\sigma}\hat{L}^{-1}2\delta\hat{\sigma}] \tag{8.2}$$

with $\hat{L} \equiv \partial^2 + \hat{m}^2 - 2\hat{\sigma}_0$. Now we need to calculate the inverse of $\hat{B}$

$$\hat{B}^{-1} = \hat{B}_0^{-1} - \hat{B}_0^{-1}\delta\hat{B}\hat{B}_0^{-1} + \hat{B}_0^{-1}\delta\hat{B}\hat{B}_0^{-1}\delta\hat{B}\hat{B}_0^{-1} - \ldots \tag{8.3}$$

with

$$\delta\hat{B} \equiv \begin{Bmatrix} -e^2\hat{A}^2 - 2\delta\hat{\sigma} & 0 \\ 4e\hat{A}_\mu\partial^\mu & -e^2\hat{A}^2 - 2\delta\hat{\sigma} \end{Bmatrix}$$

The insertion of $\delta\hat{B}$ into equation (8.3) leads to

$$\hat{B}^{-1} = \hat{B}_0^{-1} + \hat{B}_1 + \hat{B}_2 + \ldots \tag{8.4}$$

with

$$\hat{B}_1 \equiv \begin{pmatrix} \hat{L}^{-1}(2\delta\hat{\sigma} + e^2\hat{A}^2)\hat{L}^{-1} & 0 \\ -\hat{L}^{-1}4e\hat{A}_\mu\partial^\mu\hat{L}^{-1} & \hat{L}^{-1}(2\delta\hat{\sigma} + e^2\hat{A}^2)\hat{L}^{-1} \end{pmatrix}$$

and

$$\hat{B}_2 \equiv \begin{pmatrix} \hat{L}^{-1}2\delta\hat{\sigma}\hat{L}^{-1}2\delta\hat{\sigma}\hat{L}^{-1} & 0 \\ \hat{B}_{2e} & \hat{L}^{-1}2\delta\hat{\sigma}\hat{L}^{-1}2\delta\hat{\sigma}\hat{L}^{-1} \end{pmatrix}$$

where the matrix element

$$\hat{B}_{2e} \equiv \hat{L}^{-1}2e\hat{A}_\mu\partial^\mu\hat{L}^{-1}(-2\delta\hat{\sigma})\hat{L}^{-1} + \hat{L}^{-1}(-2\delta\hat{\sigma})\hat{L}^{-1}2e\hat{A}_\mu\partial^\mu\hat{L}^{-1}$$

So far we kept terms up to $\delta\sigma^2$ or A^2 and dropped cross product terms of $\delta\sigma^2$ and A^2.

By substituting $tr\ln\hat{B}$ and $\hat{B}^{-1}$ with equations (8.2) and (8.4), we obtain the following expression for the exponent in the generating functional,

$$\begin{aligned}
&\int d^4x \frac{1}{4\lambda}\sigma_0^2 - \frac{1}{2}tr\ln\hat{B}_0 + \int d^4x d^4x' < x|\frac{1}{2}\hat{\mathbf{J}}^T\hat{B}_0^{-1}\hat{\mathbf{J}}|x' > \\
&+ \int d^4x \frac{1}{2}A_\mu(x)\left(\delta_{\mu\nu}\partial^2 - (1-\frac{1}{\alpha})\partial^\mu\partial_\nu\right)A^\nu(x) + \int d^4x \frac{1}{2\lambda}\delta\sigma\sigma_0 \\
&+ tr[\hat{L}^{-1}(2\delta\hat{\sigma} + e^2\hat{A}^2)] + \int d^4x d^4x' < x|\frac{1}{2}\hat{\mathbf{J}}^T(\hat{B}_1 + \hat{B}_2)\hat{\mathbf{J}}|x' > \\
&+ \frac{1}{2}tr[\hat{L}^{-1}2\delta\hat{\sigma}\hat{L}^{-1}2\delta\hat{\sigma}] + \int d^4x \frac{1}{4\lambda}\delta\sigma^2
\end{aligned} \tag{8.5}$$

In the case of $\mathbf{J}(x)$ being a constant, the nondiagonal element of $\hat{B}_1$ doesn't contribute to the generating functional since

$$\begin{aligned}
&\int d^4x d^4x' J_2 < x| - \hat{L}^{-1}4e\hat{A}_\mu\partial^\mu\hat{L}^{-1}|x' > J_1 \\
&= \int d^4x d^4q J_2 < x| - \hat{L}^{-1}4e\hat{A}_\mu|q > \frac{q^\mu}{-q^2 + m^2 - 2\sigma_0}\delta^4(q)J_1 \\
&= 0
\end{aligned}$$

Similarly, the second term in $\hat{B}_{2e}$, $\hat{L}^{-1}(-2\delta\hat{\sigma})\hat{L}^{-1}2e\hat{A}_\mu\partial^\mu\hat{L}^{-1}$, doesn't contribute.

Now we want to do the [dA] functional integral. All the terms containing A field in the above generating functional are quadratic in A except the term due to the nondiagonal element of $\hat{B}_2$, or the first term in $\hat{B}_{2e}$. However, we can do the functional integral of [dA] and the only effect of this first term in $\hat{B}_{2e}$ is that it generates a extra term which is of second order in $\delta\sigma$. From now on, we will only consider leading contribution to generating functional from the expansion of σ around σ_0, thus, effectively, $\hat{B}_2$ can be considered as a diagonal matrix. Consequently, only the three quadratic terms in A are relevant to the $[dA]$ functional integral. Collecting all of them, we get

$$-\frac{1}{2}\int d^4x d^4x' A^\mu(x')\left[-[g_{\mu\nu}\partial^2 - (1-\frac{1}{\alpha})\partial_\mu\partial\nu + e^2 g_{\mu\nu}(2d+h)]\delta(x-x')\right]A^\nu(x)$$

with

$$h \equiv \frac{J_1^2 + J_2^2}{(m^2 - 2\sigma_0)^2} \equiv \frac{J_1^2 + J_2^2}{\bar{m}^4}$$

which results from the term $\int d^4x d^4x' < x|\frac{1}{2}\hat{\mathbf{J}}^T \hat{B}_1 \hat{\mathbf{J}}|x' >$ in (8.5) and

$$d \equiv \int d^4q \frac{1}{-q^2 + \bar{m}^2}$$

which is due to the term $tr[\hat{L}^{-1}e^2\hat{A}^2]$ in (8.5). In Euclidean space d becomes

$$d = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} \left(1 + \frac{2}{e^{\beta\omega} - 1}\right)$$

with $\omega \equiv \sqrt{p^2 + \bar{m}^2}$

In momentum space the matrix is

$$C^{\mu\nu} = \frac{1}{\alpha}\left(k^2 - 2\alpha e^2 d - \alpha e^2 h\right)\frac{k^\mu k^\nu}{k^2} + \left(k^2 - 2e^2 d - e^2 h\right)\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$$

The functional integral of [dA] replaces all the terms quadratic in A in equation (8.5) by the term $\exp\left(-\frac{1}{2}tr\ln C\right)$. Using the formula

$$\det\left[A\frac{k^\mu k^\nu}{k^2} + B\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\right] = -AB^3$$

we obtain the following result for the generating functional in the limit of $\alpha \to 0$

$$\begin{aligned}
Z[J] = [N'(\beta)]^4 \det[-\partial^2] \int_{peri} [d\delta\sigma] \exp\left\{\int d^4x \frac{1}{4\lambda}\sigma_0^2 - \frac{1}{2}tr\ln\hat{B}_0\right\} \\
\times \exp\left\{\int d^4x d^4x' < x|\frac{1}{2}\hat{\mathbf{J}}^T\hat{B}_0^{-1}\hat{\mathbf{J}}|x' > + \int d^4x \frac{1}{2\lambda}\delta\sigma\sigma_0 + tr[\hat{L}^{-1}2\delta\hat{\sigma}]\right\} \\
\times \exp\left\{\int d^4x d^4x' \frac{1}{2}(J_1^2 + J_2^2) < x|\hat{L}^{-1}2\delta\hat{\sigma}\hat{L}^{-1}|x' >\right\} \\
\times \exp\left\{-\frac{1}{2}\frac{V}{(2\pi)^3}\sum_n \int d^3k[-\ln(\omega_n^2 + k^2) + 3\ln(\omega_n^2 + k^2 + 2e^2 d + e^2 h)]\right\}
\end{aligned}$$

Keeping only the leading order contribution in $\delta\sigma$ we obtain the generating functional for connected Green's function $W[J]$

$$
\begin{aligned}
W[J] = \frac{1}{4\lambda}\beta V\sigma_0^2 - V\int\frac{d^3p}{(2\pi)^3}\left[\beta\omega + 2\ln\left(1-e^{-\beta\omega}\right)\right] + \frac{1}{2}\beta V\frac{J_1^2+J_2^2}{\bar{m}^2} \\
- V\int\frac{d^3p}{(2\pi)^3}\left[\frac{3}{2}\beta\omega_M + 3\ln\left(1-e^{-\beta\omega_M}\right) - \frac{1}{2}\beta|p| - \ln\left(1-e^{-\beta|p|}\right)\right]
\end{aligned}
\tag{8.6}
$$

with $\omega_M \equiv \sqrt{p^2+M^2} \equiv \sqrt{p^2+e^2(2d+h)}$. The first three terms are due to the scalar fields and the last four terms are due to the gauge fields. Note the structure from the gauge fields are the same as that from the one loop calculation. This is not surprising since for the gauge fields sector, all we did was a Gaussian functional integral, as in the one loop calculation.

It is easy to obtain the first order constraint equation

$$
\frac{1}{2\lambda}\sigma_0 + 2d + h = 0 \tag{8.7}
$$

This constraint equation and the scalar fields sector from equation (8.6) are identical to those in chapter IV, thus, the same renormalized mass and coupling constant from equation (6.20) will produce the renormalized versions for both $W[J]$ in the scalar sector and the constraint equation. However, the new effective mass M is temperature dependent, as can be seen from the constraint equation. The gauge coupling constant e in our calculation so far is the bare gauge coupling constant. However, this will not change the temperature dependence of M. If we treat the gauge coupling constant as the renormalized one, as in most of the one loop calculations, then we have in our minds some counter terms, which is not explicitly written so far. However, these counter terms in principle should not have temperature dependence either. In addition, treating e as the renormalized gauge coupling constant no longer sets the scalar fields and the gauge fields on an equal footing in terms of renormalization. The temperature dependence of M prevents us from dropping the divergent $\int d^3p\beta\omega_M$ term, which is in our way in attaining a renormalized version of $W[J]$. We believe the problem here lies in the fact that we are trying to renormalize this theory in a segmented fashion: scalar fields sector and the gauge fields sector. The possible renormalization involving both the gauge fields and scalar fields is left for further studies.

II. APPENDIX A

For a function $f(p_0 = i\omega_n = 2\pi nTi)$, if it has no singularities along the imaginary p_0 axis, then we have the following general formula for calculating sums over Matsubara frequency (See page 40 of reference [16])

$$\sum_{n=-\infty}^{\infty} f(p_0 = i\omega_n) = \frac{\beta}{2\pi i}\int_{-i\infty}^{i\infty} dp_0 \frac{1}{2}[f(p_0)+f(-p_0)] + \frac{\beta}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 [f(p_0)+f(-p_0)]\frac{1}{e^{\beta p_0}-1} \tag{A.1}$$

Now for the sum

$$\sum_n \frac{\omega_n^2 + p^2 + \bar{m}^2 - \mu^2}{(\omega_n^2 + p^2 + \bar{m}^2 - \mu^2)^2 + 4\mu^2\omega_n^2}$$

we have

$$f(p_0) = \frac{-p_0^2 + p^2 + \bar{m}^2 - \mu^2}{(-p_0^2 + p^2 + \bar{m}^2 - \mu^2)^2 - 4\mu^2 p_0^2} \tag{A.2}$$

Inserting this $f(p_0)$ into (A.1), we get two contour integrals, which we call f_1 and f_2 respectively. We can rewrite the first contour integral if we define $q \equiv ip_0$,

$$\begin{aligned} f_1 &= \frac{\beta}{2\pi}\int_{-\infty}^{\infty} dq \frac{q^2 + p^2 + \bar{m}^2 - \mu^2}{(q^2 + p^2 + \bar{m}^2 - \mu^2)^2 + 4\mu^2 q^2} \\ &= \frac{\beta}{2\pi}\int_{-\infty}^{\infty} dq \left[\frac{q^2 + p^2 + \bar{m}^2 - \mu^2}{(q - i\mu - i\omega)(q - i\mu + i\omega)(q + i\mu - i\omega)(q + i\mu + i\omega)}\right] \end{aligned}$$

Where $\omega \equiv \sqrt{p^2 + \bar{m}^2}$. To proceed with the calculation, we can assume that $\omega > \mu > 0$. It can be proved that the result is independent of the relative values of ω and μ or their signs. Under this assumption, we can close the contour of integration by a semicircle in the upper half plane. Then the contour encircled two poles: $q = i\mu + i\omega$ and $q = -i\mu + i\omega$. The residues give us the result for the integration,

$$\begin{aligned} f_1 &= i\beta\left[\frac{-(\mu+\omega)^2 + \omega^2 - \mu^2}{2i\omega \cdot 2i\mu \cdot 2i(\mu+\omega)} + \frac{-(\mu-\omega)^2 + \omega^2 - \mu^2}{2i\omega \cdot 2i\mu \cdot 2i(\mu-\omega)}\right] \\ &= \frac{\beta}{2\omega} \end{aligned}$$

Similarly, we have

$$f_2 = \frac{\beta}{i\pi} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 \left[\frac{-p_0^2 + \omega^2 - \mu^2}{(p_0 - \mu - \omega)(p_0 - \mu + \omega)(p_0 + \mu - \omega)(p_0 + \mu + \omega)} \right] \frac{1}{e^{\beta p_0} - 1}$$

Note because of the presence of the infinitesimal ϵ, we have to close the contour of integral on the right half plane of p_0. If we close on the left half plane, then we will enclose an infinite number of poles due to the exponential term in the integral. Under the assumption of $\omega > \mu > 0$, we have the following two poles: $p_0 = \omega - \mu$ and $p_0 = \omega + \mu$. The residues give us,

$$f_2 = \frac{\beta}{2\omega} \left[\frac{1}{e^{\beta(\omega-\mu)} - 1} + \frac{1}{e^{\beta(\omega+\mu)} - 1} \right]$$

Thus, finally we obtain,

$$\sum_n \frac{\omega_n^2 + p^2 + \bar{m}^2 - \mu^2}{(\omega_n^2 + p^2 + \bar{m}^2 - \mu^2)^2 + 4\mu^2\omega_n^2} = \frac{\beta}{2\omega} \left[1 + \frac{1}{e^{\beta(\omega-\mu)} - 1} + \frac{1}{e^{\beta(\omega+\mu)} - 1} \right]$$

II. APPENDIX B

For a function $f(p_0 = i\omega_n = 2\pi nTi)$, if it has no singularities along the imaginary p_0 axis, then we have the following general formula for calculating sums over Matsubara frequency (See page 40 of reference [16])

$$\sum_{n=-\infty}^{\infty} f(p_0 = i\omega_n) = \frac{\beta}{2\pi i}\int_{-i\infty}^{i\infty} dp_0 \frac{1}{2}[f(p_0)+f(-p_0)] + \frac{\beta}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 [f(p_0)+f(-p_0)]\frac{1}{e^{\beta p_0}-1} \tag{B.1}$$

Now for the sum

$$\sum_{m=-\infty}^{+\infty} \frac{1}{\left(\frac{\omega_m+\omega_n}{2}\right)^2+\omega_+^2}\frac{1}{\left(\frac{\omega_m-\omega_n}{2}\right)^2+\omega_-^2}$$

with

$$\omega_\pm \equiv \sqrt{\bar{m}^2 + \left(\frac{\mathbf{P}\pm\mathbf{q}}{2}\right)^2}$$

we can shift m by n since it goes from $-\infty$ to ∞, therefore we have,

$$f(p_0) = \frac{1}{\left(\frac{-ip_0+2\omega_n}{2}\right)^2+\omega_+^2}\frac{1}{\frac{-p_0^2}{4}+\omega_-^2} \tag{B.2}$$

Inserting this $f(p_0)$ into (B.1), we get two contour integrals, which we call f_1 and f_2 respectively. We can rewrite the first contour integral if we define $q \equiv ip_0$,

$$f_1 = \frac{\beta}{4\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} dq \frac{\frac{q^2}{2}+2\omega_n^2+2\omega_+^2}{\left(\frac{-q+2\omega_n}{2}+i\omega_+\right)\left(\frac{-q+2\omega_n}{2}-i\omega_+\right)}$$
$$\times\frac{1}{\left(\frac{q+2\omega_n}{2}+i\omega_+\right)\left(\frac{q+2\omega_n}{2}-i\omega_+\right)\left(\frac{q}{2}+i\omega_-\right)\left(\frac{q}{2}-i\omega_-\right)}$$

We can close the contour of integration by a semicircle in the upper half plane. Then the contour encircled three poles: $q = 2\omega_n + 2i\omega_+$, $q = -2\omega_n + 2i\omega_+$ and $q = 2i\omega_-$. The residues give us the result for the integration. We obtain, after simplification,

$$f_1 = \beta\frac{\omega_+ + \omega_-}{\omega_+\omega_-[(\omega_+ + \omega_-)^2 + \omega_n^2]} \tag{B.3}$$

We can similarly calculate f_2. First, we factorize its denominator,

$$f_2 = \frac{\beta}{2i\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 \frac{-\frac{p_0^2}{2}+2\omega_n^2+2\omega_+^2}{\left(-i\frac{p_0}{2}+\omega_n+i\omega_+\right)\left(-i\frac{p_0}{2}+\omega_n-i\omega_+\right)}$$
$$\times \frac{1}{\left(i\frac{p_0}{2}+\omega_n+i\omega_+\right)\left(i\frac{p_0}{2}+\omega_n-i\omega_+\right)\left(-i\frac{p_0}{2}+\omega_-\right)\left(i\frac{p_0}{2}+\omega_-\right)\left(e^{\beta p_0}-1\right)}$$

Note because of the presence of an infinite number of poles from the exponential term in the integral, we have to close the contour of integration on the right half plane of p_0. The residues from the following three simple poles: $p_0 = 2i\omega_n + 2\omega_+$, $p_0 = -2i\omega_n + 2\omega_+$ and $p_0 = 2\omega_-$ give us, after simplification, the following result,

$$f_2 = \beta\left[\frac{(2\omega_n^2+2\omega_-^2-2\omega_+^2)(\cos 2\beta\omega_n\cdot e^{2\beta\omega_+}-1)+4\omega_+\omega_n\cdot\sin 2\beta\omega_n\cdot e^{2\beta\omega_+}}{\omega_+[(\omega_++\omega_-)^2+\omega_n^2][(\omega_+-\omega_-)^2+\omega_n^2][e^{4\beta\omega_+}-2\cos 2\beta\omega_n\cdot e^{2\beta\omega_+}+1]}\right]$$
$$+\beta\left[\frac{2\omega_n^2+2\omega_+^2-2\omega_-^2}{\omega_-[(\omega_++\omega_-)^2+\omega_n^2][(\omega_+-\omega_-)^2+\omega_n^2][e^{2\beta\omega_-}-1]}\right]$$

Thus, combining the above result with (B.3), we finally obtain,

$$\frac{1}{\beta}\left[\sum_{m=-\infty}^{+\infty}\frac{1}{\left(\frac{\omega_m+\omega_n}{2}\right)^2+\omega_+^2}\frac{1}{\left(\frac{\omega_m-\omega_n}{2}\right)^2+\omega_-^2}\right]$$
$$=\frac{\omega_++\omega_-}{\omega_+\omega_-[(\omega_++\omega_-)^2+\omega_n^2]}$$
$$+\frac{(2\omega_n^2+2\omega_-^2-2\omega_+^2)(\cos 2\beta\omega_n\cdot e^{2\beta\omega_+}-1)+4\omega_+\omega_n\cdot\sin 2\beta\omega_n\cdot e^{2\beta\omega_+}}{\omega_+[(\omega_++\omega_-)^2+\omega_n^2][(\omega_+-\omega_-)^2+\omega_n^2][e^{4\beta\omega_+}-2\cos 2\beta\omega_n\cdot e^{2\beta\omega_+}+1]}$$
$$+\frac{2\omega_n^2+2\omega_+^2-2\omega_-^2}{\omega_-[(\omega_++\omega_-)^2+\omega_n^2][(\omega_+-\omega_-)^2+\omega_n^2][e^{2\beta\omega_-}-1]}$$
$$\equiv\frac{\omega_++\omega_-}{\omega_+\omega_-[(\omega_++\omega_-)^2+\omega_n^2]}+F_1+F_2$$

In the last line above we introduced two functions F_1 and F_2 for notational simplicity.

II. APPENDIX C

We begin with the definition of two dimensionless variables: $\bar{m} = m/T$ and $r = \mu/m$. The physical region is $\bar{m} \geq 0$ and $|r| \leq 1$. With the above definition, we define,

$$g_l(\bar{m}, r) \equiv \frac{1}{\Gamma(l)} \int_0^\infty x^{l-1} dx \left[\frac{1}{exp[(x^2 + \bar{m}^2)^{1/2} - r\bar{m}] - 1} \right] \tag{C.1}$$

$$h_l(\bar{m}, r) \equiv \frac{1}{\Gamma(l)} \int_0^\infty \frac{x^{l-1} dx}{(x^2 + \bar{m}^2)^{1/2}} \left[\frac{1}{exp[(x^2 + \bar{m}^2)^{1/2} - r\bar{m}] - 1} \right] \tag{C.2}$$

The functions of physical interest are

$$G_l(\bar{m}, r) = g_l(\bar{m}, r) - g_l(\bar{m}, -r) \tag{C.3}$$

$$H_l(\bar{m}, r) = h_l(\bar{m}, r) + h_l(\bar{m}, -r) \tag{C.4}$$

We want to obtain the high temperature expansion of the above functions, namely, the expansion around $\bar{m} = 0$. If we just naively expand the above functions around $\bar{m} = 0$, then the coefficients in front of powers of $\bar{m}$ will be divergent integrals, this is due to a branch cut at $\bar{m} = 0$ in above functions. Two methods have been introduced to get around this problem ([12] and [13]). Here we follow closely the derivations from [12].

Using the definition of Riemann's Zeta function,

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = \sum_{k=1}^\infty \frac{1}{k^z} \tag{C.5}$$

we can easily obtain the following result,

$$G_l(0,0) = 0 \qquad (l > 0), \tag{C.6}$$

$$H_l(0,0) = \frac{2\zeta(l-1)}{l-1} \qquad (l > 2). \tag{C.7}$$

By taking derivatives of $G_l(\bar{m},r)$ and $H_l(\bar{m},r)$ with respect to $\bar{m}$ and r respectively, we can obtain the following recursion relations,

$$\frac{dG_{l+1}}{d\bar{m}} = lrH_{l+1} - \frac{\bar{m}}{l}G_{l-1} + \frac{\bar{m}^2 r}{l}H_{l-1}, \tag{C.8}$$

$$\frac{dH_{l+1}}{d\bar{m}} = \frac{r}{l}G_{l-1} - \frac{\bar{m}}{l}H_{l-1}, \tag{C.9}$$

$$\frac{dG_{l+1}}{dr} = l\bar{m}H_{l+1} + \frac{\bar{m}^3}{l}H_{l-1}, \tag{C.10}$$

$$\frac{dH_{l+1}}{dr} = \frac{\bar{m}}{l}G_{l-1}. \tag{C.11}$$

Consequently, if we know $G_1(\bar{m},r)$ and $H_1(\bar{m},r)$, then, together with the initial conditions of equations (C.6) and (C.7), we can derive all the $G_l(\bar{m},r)$ and $H_l(\bar{m},r)$ for all positive odd l, similarly, with the knowledge of $G_2(\bar{m},r)$ and $H_2(\bar{m},r)$, we can obtain all the $G_l(\bar{m},r)$ and $H_l(\bar{m},r)$ for all positive even l. However, for physical systems in an odd number of spatial dimensions, only $G_l(\bar{m},r)$ and $H_l(\bar{m},r)$ with positive odd l are relevant for their thermodynamics. Thus, we will confine our calculation to the odd l case.

We start with the equation (2.12) we used earlier, but we rewrite it as follows,

$$\frac{1}{exp(y)-1} = \frac{1}{y} - \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{y}{y^2 + (2\pi n)^2} \tag{C.12}$$

We substitute this identity into the integrand of $G_l(\bar{m},r)$ and $H_l(\bar{m},r)$ and then expand around $\bar{m} = 0$. After that we collect all the terms of the same order of $\bar{m}$ and integrate term by term. As we will see later, as we do the integration of individual terms, we will encounter divergent integrals. However, if we regularize those integrals by multiplying the integrand with a convergent factor $x^{-\epsilon}$ $(0 < \epsilon < 1)$, we will be able to keep track of the divergence structures of the individual terms. This is in the same spirit of dimensional regularization. Not surprisingly, in the end the divergences of the individual terms cancel when we sum them up.

Inserting the identity (C.12) into $G_1(\bar{m}, r)$, we obtain,

$$G_1(\bar{m}, r) = I + 2\sum_{n=1}^{\infty} L_n \tag{C.13}$$

with

$$I = 2r\bar{m}\int_0^{\infty} \frac{x^{-\epsilon}dx}{x^2 + \bar{m}^2(1-r^2)} = \frac{\pi r}{(1-r^2)^{1/2}} + O(\epsilon) \tag{C.14}$$

$$L_n = 2r\bar{m}\int_0^{\infty} x^{-\epsilon}dx \frac{x^2 + \bar{m}^2(1-r^2) - (2\pi n)^2}{[x^2 + \bar{m}^2(1-r^2) + (2\pi n)^2]^2 + (4\pi n r\bar{m})^2} \tag{C.15}$$

Note in obtaining the result for I, we have made use of the assumption of $\bar{m} > 0$, otherwise we would have obtained the opposite sign for I. Now we expand equation (C.15) in power series of $\bar{m}$ and integrate term by term. We integrate terms linear in $\bar{m}$ for the purpose of illustration,

$$\begin{aligned} L_n^{(1)} &= 2r\bar{m}\int_0^{\infty} x^{-\epsilon}dx \frac{x^2 - (2\pi n)^2}{[x^2 + (2\pi n)^2]^2} \\ &= \frac{r\bar{m}}{(2\pi n)^{1+\epsilon}}\left[\Gamma\left[\frac{3-\epsilon}{2}\right]\Gamma\left[\frac{1+\epsilon}{2}\right] - \Gamma\left[\frac{3+\epsilon}{2}\right]\Gamma\left[\frac{1-\epsilon}{2}\right]\right] \\ &= \frac{-r\bar{m}\pi\epsilon}{(2\pi n)^{1+\epsilon}} + O(\epsilon^2). \end{aligned} \tag{C.16}$$

Summing over n gives us the zeta function,

$$2\sum_{n=1}^{\infty} L_n^{(1)} = \frac{-r\bar{m}2\pi\epsilon}{(2\pi)^{(1+\epsilon)}} - O(\epsilon^2) = -r\bar{m} + O(\epsilon). \tag{C.17}$$

where we have used the following property of the zeta function,

$$\zeta(1+\epsilon) = \frac{1}{\epsilon} + \gamma + O(\epsilon). \tag{C.18}$$

where γ is the Euler's constant.

We can similarly obtain the results for higher orders of $\bar{m}$. The integration on x is straightforward and the sum over n just gives us a zeta function. In these integrations ϵ doesn't play any role any more, therefore we can safely set $\epsilon = 0$. We just quote the result here,

$$G_l(\bar{m}, r) = \frac{\pi r}{(1-r^2)^{1/2}} - r\bar{m} + 2\pi r \times \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(2k+1) a_k \left[\frac{\bar{m}}{2\pi}\right]^{2k+1}, \tag{C.19}$$

where $a_1 = 1, \quad a_2 = 2r^2 + 3/2, \quad a_3 = 3r^4 + 15r^2/2 + 15/8.$

Now if we insert the identity (C.12) into the expression for $H_1(\bar{m}, r)$, then,

$$H_1(\bar{m}, r) = I_1 + I_2 + 2\sum_{n=1}^{\infty} M_n \tag{C.20}$$

with

$$I_1 = 2\int_1^{\infty} \frac{x^{-\epsilon} dx}{x^2 + \bar{m}^2(1-r^2)} = \frac{\pi}{\bar{m}(1-r^2)^{1/2}} + O(\epsilon), \tag{C.21}$$

$$I_2 = -\int_0^{\infty} \frac{x^{-\epsilon} dx}{(x^2 + \bar{m}^2)^{1/2}} = -\frac{1}{\epsilon} + \ln\left[\frac{\bar{m}}{2}\right] + O(\epsilon), \tag{C.22}$$

$$M_n = 2\int_0^{\infty} x^{-\epsilon} dx \frac{x^2 + \bar{m}^2(1-r^2) + (2\pi n)^2}{[x^2 + \bar{m}^2(1-r^2) + (2\pi n)^2]^2 + (4\pi n r \bar{m})^2}. \tag{C.23}$$

As before, we expand the equation (C.23) in powers of $\bar{m}$, integrate term by term and sum over n in the end. Here we can also set $\epsilon = 0$ except for the leading terms in $\bar{m} = 0$. For illustration, we calculate the leading term here,

$$M_n^{(1)} = 2\int_0^{\infty} \frac{x^{-\epsilon} dx}{x^2 + (2\pi n)^2} = \frac{1}{2n^{1+\epsilon}}[1 - \epsilon \ln 2\pi + O(\epsilon^2)]. \tag{C.24}$$

Summing over n and use equation (C.18), we get,

$$2\sum_{n=1}^{\infty} M_n^{(1)} = \frac{1}{\epsilon} + \gamma - \ln(2\pi) + O(\epsilon). \tag{C.25}$$

As mentioned, the $1/\epsilon$ terms cancel as we sum the above term with equation (C.22). The higher order terms in $\bar{m}$ can be calculated similarly and we quote the result here, which is only valid for $\bar{m} > 0$.

$$H_1(\bar{m}, r) = \frac{\pi}{\bar{m}(1-r^2)^{1/2}} + \ln\left[\frac{\bar{m}}{4\pi}\right] + \gamma + \sum_{k=1}^{\infty}(-1)^k\zeta(2k+1)b_k\left[\frac{\bar{m}}{2\pi}\right]^{2k}. \tag{C.26}$$

where $b_1 = r^2 + 1/2, \quad b_2 = r^4 + 3r^2 + 3/8, \quad b_3 = r^6 + 15r^4/2 + 45r^2/8 + 5/16.$

From $G_1(\bar{m}, r)$ and $H_1(\bar{m}, r)$ we can get all $G_l(\bar{m}, r)$ and $H_l(\bar{m}, r)$ for positive odd $\bar{m}$, here we list some of them,

$$\begin{aligned} H_3(\bar{m}, r) = {} & \zeta(2) - \frac{\bar{m}^2}{2}\pi(1-r^2)^{1/2} - \frac{r^2\bar{m}^2}{4} - \frac{\bar{m}^2}{4}\gamma + \frac{\bar{m}^2}{8} - \frac{\bar{m}^2}{4}\ln\frac{\bar{m}}{4\pi} \\ & + \sum_{k=1}^{\infty}(-1)^{k+1}\zeta(2k+1)c_k\frac{\bar{m}^{2k+2}}{(2\pi)^{2k}} \end{aligned} \tag{C.27}$$

with $c_1 = r^2/4 + 1/16, \quad c_2 = r^4/4 + 3r^2/8 + 1/32, \quad c_3 = r^6/4 + 15r^4/16 + 15r^2/32 + 5/256.$

$$\begin{aligned} G_3(\bar{m}, r) = {} & \frac{1}{4}r\bar{m}^3 + 2r\bar{m}\zeta(2) - \frac{r^3\bar{m}^3}{6} - \pi\bar{m}^2 r(1-r^2)^{1/2} \\ & + \sum_{k=1}^{\infty}(-1)^k\zeta(2k+1)d_k\frac{\bar{m}^{2k+3}}{(2\pi)^{2k}} \end{aligned} \tag{C.28}$$

with $d_1 = r/8, \quad d_2 = r^3/4 + r/8, \quad d_3 = 3r^5/8 + 5r^3/8 + 15r/128.$

$$\begin{aligned} H_5(\bar{m}, r) = {} & \frac{1}{32}r^2\bar{m}^4 + \frac{1}{4}r^2\bar{m}^2\zeta(2) - \frac{r^4\bar{m}^4}{96} + \frac{\pi\bar{m}^3}{24}(1-r^2)^{3/2} \\ & + \frac{\bar{m}^4}{64}\gamma - \frac{3\bar{m}^4}{256} - \frac{\bar{m}^2}{8}\zeta(2) + \frac{\bar{m}^4}{64}\ln\frac{\bar{m}}{4\pi} + \frac{1}{2}\zeta(4) \\ & + \sum_{k=1}^{\infty}(-1)^k\zeta(2k+1)e_k\frac{\bar{m}^{2k+4}}{(2\pi)^{2k}} \end{aligned} \tag{C.29}$$

with $e_1 = r^2/64+1/384$, $\quad e_2 = r^4/64+r^2/64+1/1024$, $\quad e_3 = r^6/64+5r^4/128+15r^2/1024+1/2048$.

For later convenience, we quote,

$$\zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^4}{90}.$$

Now let's consider the low temperature case. The derivation in the following is given by H. E. Haber and H. A. Weldon ([13]). Low temperature regime is equivalent to the case of $\bar{m} \to \infty$ at fixed r. If we make the substitution, $\omega = exp[\bar{m} - (x^2 + \bar{m}^2)^{1/2}]$ in (C.1) and (C.2), then the results are

$$g_l(\bar{m}, r) = \frac{1}{\Gamma(l)} \int_0^1 d\omega \frac{(-\ln\omega)^{l/2-1}(2\bar{m} - \ln\omega)^{l/2-1}(y - \ln\omega)}{exp[(1-r)\bar{m}] - \omega} \tag{C.30}$$

$$h_l(\bar{m}, r) = \frac{1}{\Gamma(l)} \int_0^1 d\omega \frac{(-\ln\omega)^{l/2-1}(2\bar{m} - \ln\omega)^{l/2-1}}{exp[(1-r)\bar{m}] - \omega} \tag{C.31}$$

Expanding the numerators under the assumption that $|\ln\omega/2\bar{m}| < 1$, we may use the following definition of the polylogarithm function $Li_l(x)$ (for $l > 0$)

$$Li_l(x) = \frac{-1}{\Gamma(l)} \int_0^1 dt \frac{(-\ln t)^{l-1}}{t - x^{-1}} = \sum_{p=1}^{\infty} \frac{x^p}{p^l} \tag{C.32}$$

to integrate term by term and obtain,

$$\begin{aligned} g_l(\bar{m}, r) = & \frac{\Gamma(l/2)}{\Gamma(l)} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(l/2-k)} \left(\frac{1}{2\bar{m}}\right)^{k+1-l/2} \\ & \times \left\{\bar{m}\Gamma(l/2+k) Li_{k+l/2}(e^{(r-1)\bar{m}}) + \Gamma(l/2+k+1) Li_{k+l/2+1}(e^{(r-1)\bar{m}})\right\} \end{aligned} \tag{C.33}$$

$$h_l(\bar{m}, r) = \frac{\Gamma(l/2)}{\Gamma(l)} \sum_{k=0}^{\infty} \frac{\Gamma(l/2+k)}{\Gamma(k+1)\Gamma(l/2-k)} \left(\frac{1}{2\bar{m}}\right)^{k+1-l/2} Li_{k+l/2}(e^{(r-1)\bar{m}}) \tag{C.34}$$

The above equations give us the low temperature expansions.

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