Singular hyperkähler quotients.

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Abstract.

Hyperkähler reduction is an analog of symplectic reduction defined for hyperkähler manifolds with isometric actions of compact groups. When the hyperkähler quotient is non-singular, it inherits a hyperkähler structure. Hyperkähler quotients include many important examples of hyperkähler manifolds.

In this paper we study the case of singular quotients. Unfortunately, there is no completely natural definition of a singular hyperkähler manifold at the moment. However, we show that some of the objects associated to a smooth hyperkähler manifold, such as a complex-analytic twistor space, exist for singular hyperkähler quotients as well.

Thesis advisor: Professor David Kazhdan.

1. Introduction.

Hyperkähler reduction is an analog of symplectic reduction defined in [HKLR] for hyperkähler manifolds with isometric actions of compact groups. When the hyperkähler quotient is non-singular, it inherits a hyperkähler structure. Hyperkähler quotients include many important examples of hyperkähler manifolds([Hi]).

In this paper we study the case of singular quotients. Unfortunately, there is no completely natural definition of a singular hyperkähler manifold at the moment. However, we show that some of the objects associated to a smooth hyperkähler manifold, such as a complex-analytic twistor space, exist for singular hyperkähler quotients as well.

The contents of the paper is as follows: in section 2 we recall the basic definitions and results about hyperkähler manifolds, twistor spaces and hyperkähler reduction. In the end we formulate our main theorem which claims that the twistor space of a hyperkähler quotient has a natural complex-analytic structure (Theorem 2.3). To prove it we use the theory of reduction for Kähler manifolds developed in [HL] and [S]. We apply this theory to the twistor space of the original hyperkähler manifold. Unfortunately twistor spaces are usually not Kähler. To circumvent this difficulty we construct an open invariant covering by Stein Kähler manifolds in section 3. After that we prove our main theorem in section 4. In section 5 we construct on the quotient some of the other structures one would expect a hyperkähler variety to possess. Throughout the paper we need some facts about relative differential forms. We have collected the definitions and proofs in the Appendix.

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2. Main definitions.

In this section we recall for the convenience of the reader main definitions and results about the hyperkähler reduction (see, e.g., [HKLR]).

Definition 2.1: A hyperkähler manifold M is a Riemannian manifold with three integrable complex structures I, J, K which are parallel with respect to the Levi-Civita connection and satisfy

$$I \circ J = -J \circ I = K. \quad \square$$

Consider a hyperkähler manifold M. For every imaginary quaternion $h = ai + bj + ck, a, b, c \in \mathbf{R}$ we can define an endomorphism H of the tangent bundle of M by the formula $H = aI + bJ + cK$. It is easy to see that when $a^2 + b^2 + c^2 = 1$ the operator H is an almost complex structure. This structure is also parallel with respect to the Levi-Civita connection. Therefore it is integrable. The set of such h is the unit sphere in the space $\mathsf{Im}\mathbf{H}$ of imaginary quaternions. We will identify it with the complex projective line $\mathbf{C}P^1$.

Consider the manifold $X = M \times \mathbf{C}P^1$. Let $\pi : X \longrightarrow \mathbf{C}P^1, \sigma : X \longrightarrow M$ be the natural projections. For every point $x \in X$ the tangent space to X at x is the sum of tangent spaces to $\sigma(x) \in M$ and to $\pi(x) \in \mathbf{C}P^1$. This decomposition allows one to introduce a natural complex structure in this tangent space: it acts as $\pi(x) \in \mathbf{C}P^1$ on the first factor and as the standard complex structure on $\mathbf{C}P^1$ the second factor.

Theorem 2.1: (see, e.g., [HKLR]). This complex structure is integrable.
∎

Thus X is a complex manifold. It is called *the twistor space* of M. By construction π is a holomorphic map. Moreover, for every point $x \in M$ the corresponding section $\tilde{x} : \mathbf{C}P^1 \longrightarrow X$ is also holomorphic, and its normal bundle is isomorphic to the sum of $dim(X) - 1$ copies of $\mathcal{O}(1)$. (See, e.g., [HKLR]).

Let $W = \mathbf{C}P^1 \times \mathsf{Im}\mathbf{H}$ be the trivial 3-dimensional real vector bundle over $\mathbf{C}P^1$ whose fibers we identify with the space of imaginary quaternions. The standard metric on $\mathsf{Im}\mathbf{H}$ gives a metric on W. Let $\nabla : W \longrightarrow \mathcal{A}^1(\mathbf{C}P^1, W)$

be the trivial connection on W. It is flat and compatible with the metric.

The identification of $\mathbb{C}P^1$ with the unit sphere in $\mathsf{Im}\mathsf{H}$ defines a unit section h of W. The orthogonal complement to h is then identified with the C^∞-tangent bundle to $\mathbb{C}P^1$, so that we have an orthogonal decomposition of metric bundles

$$W = h\mathbf{R} \oplus T(\mathbb{C}P^1).\tag{1}$$

This decomposition induces a decomposition

$$\nabla = \nabla_1 + \nabla_2 + \theta_{12} + \theta_{21},$$

where ∇_1, ∇_2 are connections induced on summands of (1) and θ_{12}, θ_{21} are 1-forms with values respectively in $Hom(h\mathbf{R}, T(\mathbb{C}P^1))$ and $Hom(T(\mathbb{C}P^1), h\mathbf{R})$.

It is easy to see that ∇_1 is trivial and ∇_2 is the standard connection on $T(\mathbb{C}P^1)$. Moreover, $\theta_{21} : T(\mathbb{C}P^1) \longrightarrow Hom(T(\mathbb{C}P^1), h\mathbf{R}) \cong T^*(\mathbb{C}P^1)$ is minus the canonical isomorphism associated to the metric tensor while $\theta_{12} : T(\mathbb{C}P^1) \longrightarrow T(\mathbb{C}P^1)$ is minus identity.

All these data can be lifted to X via π. We will use the same letters for these pullbacks to simplify notation.

For each of the complex structures I, J, K on M we have the corresponding closed Kähler 2-form

$$\omega_I(\cdot,\cdot) = l(\cdot, I\cdot).$$

$$\omega_J(\cdot,\cdot) = l(\cdot, J\cdot),$$

$$\omega_K(\cdot,\cdot) = l(\cdot, K\cdot),$$

where l is the metric tensor on M. These forms define a single 2-form ω_{H} on X with values in W by the formula

$$\omega_{\mathsf{H}} = \omega_I \cdot i + \omega_J \cdot j + \omega_K \cdot k,$$

where $i, j, k \in \mathsf{Im}\mathsf{H}$ are constant sections of W given by the standard imaginary quaternions. This form can be decomposed using (1) into a sum of a real $(1, 1)$-form ω and an $T(\mathbb{C}P^1)$-valued 2-form Ω. The standard complex structure on $T(\mathbb{C}P^1)$ allows us to consider Ω as a complex $\mathcal{O}(2)$-valued form. Then it is of type $(2, 0)$ (see, e.g., [HKLR]).

Let $\mathcal{T}_\pi$ be the relative holomorphic tangent bundle of X over $\mathbb{C}P^1$. Then the restriction of Ω onto the fibers of π defines a section Ω_{rel} of the holomorphic bundle $\Lambda^2(\mathcal{T}_\pi^*) \otimes \mathcal{O}(2)$.

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Proposition 2.1: (see. e.g.. [HKLR]). The section Ω_{rel} is holomorphic.

■

In section 3 we will need a version of the explicit formulas in [V],Appendix for the differentials of ω and Ω in local coordinates on $\mathbf{C}P^1$. We now derive it.

Every choice of a holomorphic local coordinate z defined on an open disc $V \subset \mathbf{C}P^1$ provides a holomorphic trivialization $\partial/\partial z : \mathcal{O} \longrightarrow \mathcal{O}(2)$ of $\mathcal{O}(2)$ over $\pi^{-1}(V)$. This trivialization allows us to consider Ω as a C-valued (2, 0)-form on $\pi^{-1}(V)$. Let $p(z)$ be the real-valued function on $\mathbf{C}P^1$ such that $p(z)dz \otimes d\bar{z}$ is the Kähler metric on $\mathbf{C}P^1$.

Claim 2.1: We have
(i) $\partial\omega = p(z)\Omega \wedge d\bar{z}$,
(ii) $\bar\partial\omega = p(z)\bar\Omega \wedge dz$,
(iii) $\partial\bar\partial\omega = p(z)\omega \wedge dz \wedge d\bar{z}$.

Proof. Let $s = \partial/\partial z$ be the canonical section of $\mathcal{O}(2)$ over $\pi^{-1}(V)$. We have a local frame $(h, s, \bar{s})$ in $W \otimes \mathbf{C}$. In terms of this frame

$$\omega_{\mathbf{H}} = \omega \cdot h + \Omega \cdot s + \bar\Omega \cdot \bar{s}.$$

Extend ∇ in the usual way to an operator on $W \otimes$ C-valued forms. Since $\omega_I, \omega_J, \omega_K$ are closed, $\nabla\omega_{\mathbf{H}} = 0$. Let α_1, α_2 be the orthogonal projections onto the first and second factors of (1). We have

$$0 = \alpha_1(\nabla\omega_{\mathbf{H}}) = \nabla_1(\alpha_1\omega_{\mathbf{H}}) + \theta_{21} \wedge \alpha_2\omega_{\mathbf{H}} = \nabla_1(\omega \cdot h) + \theta_{21} \wedge (\Omega \cdot s + \bar\Omega \cdot \bar{s}).$$

Since ∇_1 is trivial. $\nabla_1(\omega \cdot h) = d\omega \cdot h$. Recall that $\theta_{21} : T(\mathbf{C}P^1) \longrightarrow T^*(\mathbf{C}P^1)$ is minus the isomorphism given by the metric tensor on $\mathbf{C}P^1$. Therefore we have

$$\theta_{21}(s) = -p(z)d\bar{z} \cdot h,$$

$$\theta_{21}(\bar{s}) = -p(z)dz \cdot h.$$

This implies

$$d\omega - p(z)(\Omega \wedge d\bar{z} + \bar\Omega \wedge dz) = 0,$$

which gives (i) and (ii) by type decomposition.

We also have

$$0 = \alpha_2(\nabla\omega_\mathsf{H}) = \nabla_2(\alpha_2\omega_\mathsf{H}) + \theta_{12} \wedge \alpha_1\omega_\mathsf{H} = \nabla_2(\Omega \cdot s + \bar{\Omega} \cdot \bar{s}) + \theta_{12} \wedge (\omega \cdot h).$$

Since ∇_2 is the standard connection on $T(\mathbf{C}P^1)$, we have

$$\nabla_2 s = p^{-1}(z)(\partial p(z)/\partial z)dz \cdot s,$$

$$\nabla_2 \bar{s} = p^{-1}(z)(\partial p(z)/\partial \bar{z})d\bar{z} \cdot \bar{s}.$$

Finally,

$$\theta_{12}(h) = dz \cdot s + d\bar{z} \cdot \bar{s}.$$

Therefore we have

$$d\Omega \cdot s + \Omega \wedge p^{-1}(z)(\partial p(z)/\partial z)dz \cdot s = \omega \wedge dz \cdot s.$$

By type decomposition this implies

$$\bar{\partial}\Omega = \omega \wedge dz.$$

Together with (i) this gives

$$\partial\bar{\partial}\omega = \bar{\partial}(p(z)\Omega) \wedge d\bar{z} = p(z)\omega \wedge dz \wedge d\bar{z},$$

and we are done. ∎

Consider the real structure on $\mathbf{C}P^1$ defined by the antipodal involution. It defines an involution on $X = M \times \mathbf{C}P^1$ by action on the second factor. Since antipodal points on $\mathbf{C}P^1$ correspond to conjugate complex structures on fibers of π, this involution is antiholomorphic and defines a real structure on X.

There is the *converse theorem* which says that every 2n-dimensional complex manifold equipped with the following:

(i) A projection π onto $\mathbf{C}P^1$,
(ii) A holomorphic non-degenerate $\pi^*(\mathcal{O}(2))$−valued relative 2-form Ω_{rel},
(iii) A real structure lifting the antipodal involution on $\mathbf{C}P^1$,
(iv) A family of real holomorphic sections of π with normal bundles isomorphic to a sum of 2n copies of $\mathcal{O}(1)$ (called *twistor lines*),

satisfying certain compatibility conditions is the twistor space of a unique hyperkähler manifold. (We do not give the precise statement since we will

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never use it. See [HKLR].) Therefore the study of hyperkähler manifolds is equivalent to the study of corresponding twistor spaces.

Let a compact group K act on M preserving the hyperkähler structure. Let $\mathfrak{k}$ be its Lie algebra, $\mathfrak{k}^*$ the dual space to $\mathfrak{k}$. For every $k \in \mathfrak{k}$ we will use the same letter to denote the corresponding vector field on M. For any point $x \in M$ $k_x \in T_x(M)$ will mean the value of the vector field k at the point x.

Definition 2.2: A K-equivariant smooth map $\mu : M \longrightarrow \mathfrak{k}^* \otimes \mathrm{Im}\mathbf{H}$ is called *a hyperkähler moment map* for the action of K if for every $x \in X, k \in \mathfrak{k}$

$$d\mu(k) = i_k(\omega_{\mathbf{H}}).$$

where i_k is the contraction with the vector field k.

Definition 2.3: Let K be a compact group and G be its complexification. A *Hamiltonian hyperkähler K-manifold* $(M, \mu_{\mathbf{H}})$ is a collection of the following:
(i)A hyperkähler manifold M,
(ii)An action of K on M preserving the hyperkähler structure,
(iii)A hyperkähler moment map $\mu_{\mathbf{H}} : M \longrightarrow \mathfrak{k}^* \otimes \mathrm{Im}\mathbf{H}.\square$

A Hamiltonian hyperkähler G-manifold $(M, \mu_{\mathbf{H}})$ is called *complete* if for each of the complex structures on M the action of K extends to a holomorphic action of G. $\square$

For the rest of the paper let $(M, \mu_{\mathbf{H}})$ be a given complete Hamiltonian hyperkähler K-manifold.

Note the following easy consequence of the definitions.

Claim 2.2: For every point $m \in M$ such that $\mu_{\mathbf{H}}(m) = 0$ the restriction of the 2-form $\omega_{\mathbf{H}}$ to the orbit Km is zero.

Proof. Consider a point $y \in Km$. Since $\mu_{\mathbf{H}}(y) = \mu_{\mathbf{H}}(m) = 0$, for every $k_1, k_2 \in \mathfrak{k}$ we have by Definition 2.2

$$\omega_{\mathbf{H}}(k_1, k_2) = (d\mu_{\mathbf{H}})_y(k_{1,y})(k_2) = 0.$$

The tangent space to Km at y is spanned by vectors k_y corresponding to $k \in \mathfrak{k}$. Therefore $\omega_{\mathbf{H}}$ is zero on Km. ■

Consider the set $\mu_{reg}^{-1}(0) \subset M$ of points in the zero set of $\mu_{\mathbf{H}}$ with trivial stabilizer in K. Since $Coker(d\mu_{\mathbf{H}})_x$ is the dual to the Lie algebra of the stabilizer of x for all $x \in \mu_{\mathbf{H}}^{-1}(0)$, all points in $\mu_{reg}^{-1}(0)$ are regular values of $\mu_{\mathbf{H}}$ and $\mu_{reg}^{-1}(0)$ is a submanifold in M on which K acts freely. Therefore the quotient $\mu_{reg}^{-1}(0)/K$ is also a manifold. It inherits from M a metric and a quaternion action in its tangent bundle.

Theorem 2.2: ([HKLR]). This action is parallel with respect to the metric and all the induced complex structures are integrable. Therefore the quotient $\mu_{reg}^{-1}(0)/K$ is again a hyperkähler manifold. ■

The whole $\mu^{-1}(0)/K$ is a Hausdorff topological space but no longer a manifold. It's natural to ask in what sense can one extend the hyperkähler structure from the non-singular part. To be able to give a partial answer it is convenient to consider the corresponding twistor spaces.

For the rest of the paper let X be the twistor space of M. Let $\sigma \times \pi : X \longrightarrow M \times \mathbb{C}P^1$ be the canonical diffeomorphism. The given action of K on M and the trivial action on $\mathbb{C}P^1$ define an action of K on X. Since $(M, \mu_{\mathbf{H}})$ is complete this action extends to a holomorphic action of the complexified group G. Let $\mathfrak{g}$ be the Lie algebra of G. For any $u \in \mathfrak{g}$ we will use the same letter for the corresponding holomorphic vector field on X.

The moment map considered as a map from X to the total space of $W \otimes \mathfrak{k}^*$ can be decomposed using (1) into a sum of two factors with values in total spaces of the bundles $\mathfrak{k}^* h$ and $\mathfrak{g}^* \odot \mathcal{O}(2)$ respectively. We denote these factors by $\mu_{\mathbf{R}} : X \longrightarrow \mathfrak{k}^* \odot h\mathbb{R}$ and by $\mu_{\mathbf{C}} : X \longrightarrow Tot(\mathcal{O}(2) \otimes \mathfrak{g}^*)$.

Claim 2.3: The map $\mu_{\mathbf{C}}$ is holomorphic.
Proof. By definition of $\mu_{\mathbf{H}}$ for every $g \in \mathfrak{g}$

$$d\mu_{\mathbf{C}}(g) = i_g(\Omega_{rel}).$$

Since the form Ω_{rel} is of type (2,0), the differential of $\mu_{\mathbf{C}}$ is complex-linear. ■

Let $X_0 = \mu_{\mathbf{C}}^{-1}(0)$ be the preimage under $\mu_{\mathbf{C}}$ of the zero section in $\mathcal{O}(2) \otimes \mathfrak{g}^*$. It is an analytic G-invariant subset of X.

Definition 2.4: A point $x \in X_0$ is called *semistable* if the closure $\overline{Gx}$ of its orbit Gx contains a point from the zero set of $\mu_{\mathbf{H}}$. It is called *stable regular* if the orbit itself contains a point from $\mu_{reg}^{-1}(0)$. $\square$

We will denote the subset of all semistable, resp., regular stable points by X_{ss}, resp., X_{reg}. Both are open subsets in X_0. The action of G on X_{reg} is free, therefore the quotient X_{reg}/G is a well-defined complex manifold.

Claim 2.4: ([Hi]). The inclusion $\mu_{reg}^{-1}(0) \times \mathbf{C}P^1 \subset X_{reg}$ induces an isomorphism of the twistor space for $\mu_{reg}^{-1}(0)/K$ and X_{reg}/G. $\blacksquare$

Let $X_0//G$ be the topological space $\mathbf{C}P^1 \times \mu_{\mathbf{H}}^{-1}(0)/K$. Our main result claims that there exists a complex-analytic structure on $X_0//G$ extending the quotient one on $X_{reg}/G \subset X_0//G$. Here is the precise statement.

Theorem 2.3: (i) For every $x \in X_{ss}$ the intersection $\overline{Gx} \cap \mu_{\mathbf{H}}^{-1}(0)$ consists of a single K-orbit.

(ii) The map $q : X_{ss} \longrightarrow X_0//G$ sending a point to the class of this orbit is continuous and G-invariant.

(iii) Let $(q_*\mathcal{O}_{X_{ss}})^G$ be the subsheaf of G-invariant sections in the direct image under q of the sheaf of holomorphic functions on X_{ss}. Then the ringed space $(X_0//G, (q_*\mathcal{O}_{X_{ss}})^G)$ is a complex analytic space and q is a map of complex analytic spaces. $\square$

We will prove Theorem 2.3 in section 4.

3. Construction of an open Stein G-covering.

In this section we study the local structure of the action of G on the twistor space X. Our goal is to construct an open covering of X_{ss} by Stein G-invariant Kähler subsets.

Definition 3.1: Consider a point $x \in \mu_{\mathbf{H}}^{-1}(0) \subset X$. An *étale Kähler chart* (V, f) at $x \in X$ is a collection of the following:

(i) A Stein manifold V with an action of G and a G-equivariant map $f : V \longrightarrow X$ such that f is locally biholomorphic and $x \in f(V)$;

(ii) A K-invariant strictly plurisubharmonic function ψ on V such that $\sqrt{-1}\,\partial\bar{\partial}\psi$ is equal to $f^*(\omega)$ on fibers of $f \circ \pi$;

(iii) A holomorphic G-invariant $f^*\mathcal{O}(2)$-valued 1-form α on V such that $\partial\alpha - f^*(\Omega)$ is zero on the fibers of $f \circ \pi$.

An étale Kähler chart is called *compatible with the moment map* if in addition the following holds for every $k \in \mathfrak{k}, g \in \mathfrak{g}$ and $v \in V$:

$$f \circ \mu_{\mathbf{R}}(k) = \sqrt{-1} \, /2(\partial - \bar{\partial})\psi(k_v),$$

$$f \circ \mu_{\mathbf{C}}(g) = -\alpha(g_v).$$

An étale Kähler chart is called simply *a Kähler chart* if the map f is globally biholomorphic. □

Our goal is to construct a Kähler chart at every point in $\mu_{\mathbf{H}}^{-1}(0) \subset X$. First we construct an étale Kähler chart (Proposition 3.1). Then we prove that on an open subset the Kähler structure can be corrected so as to become compatible with the moment map (Proposition 3.2). Finally we show that on a still smaller open subset the map f is not only locally but globally biholomorphic, thus an open embedding.

Fix once and for all a point $x \in \mu_{\mathbf{H}}^{-1}(0) \subset X$. We begin with a construction of a K-invariant open neighborhood of x with suitable properties.

Definition 3.2: An open subset $U \subset X$ is called *a good neighborhood* of a point $x \in U$ if the following holds:

(i) U is Stein and K-invariant.

(ii) The subset $\pi(U) \subset \mathbf{C}P^1$ is an open disc,

(iii) There exists an open K-invariant neighborhood $V \subset M$ of the orbit $K\sigma(x)$ such that V is homotopy equivalent to $K\sigma(x)$ and

$$K\sigma(x) \subset \sigma(\pi^{-1}(y) \cap U) \subset V$$

for every point $y \in \pi(U)$, where $\sigma : X \longrightarrow M$ is the canonical projection. □

Lemma 3.1: For every point $x \in \mu_{\mathbf{H}}^{-1}(0) \subset X$ there exists a good neighborhood U of x.

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Proof. Consider the orbit $K\sigma(x) \subset M$. It is a smooth submanifold of M. Choose a K-invariant tubular neighborhood $V \subset M$ of $K\sigma(x)$. (For instance, a normal geodesic neighborhood - it is K-invariant since the metric is K-invariant.) By definition V is homotopy equivalent to $K\sigma(x)$. The subset $U_0 = \sigma^{-1}(V) \subset X$ is an open K-invariant neighborhood of the orbit Kx.

Recall that a smooth submanifold S of a complex manifold Y is called *totally real* if for every point $s \in S$ the tangent space T_s to S at s considered as a real subspace of the complex tangent space to Y satisfies $T_s \cap \sqrt{-1}T_s = \{0\}$.

The following general result is proved in [HW].

Theorem 3.1: ([HW]) Every totally real submanifold of a complex manifold has a basis of Stein open neighborhoods. ∎

Claim. The orbit Kx is a totally real submanifold of X.

Proof. By Claim 2.2 the form $\omega_\mathbf{H}$ is zero on Kx. Therefore for any two tangent vectors $a, b \in T_y(Kx)$ at a point $y \in Kx$ we have

$$l(a, \sqrt{-1}\, b) = (\omega_\mathbf{H}(a, b), \pi(x)) = 0,$$

where $(\cdot, \cdot)$ is the natural metric on the space $\mathsf{Im}\mathbf{H}$ of imaginary quaternions. Therefore the subspaces $T_y(Kx)$, $\sqrt{-1}\, T_y(Kx) \subset T_y(\pi^{-1} \circ \pi(x)) \cong T_{\sigma(y)}(M)$ are orthogonal. This implies that their intersection is zero. ∎

Therefore $Kx \in X$ has a basis of Stein open neighborhoods. Choose such a neighborhood $U'' \subset U_0$. Since K is compact, the open subset $U' = KU''$ is also Stein.

The intersection $\sigma^{-1}(\sigma(x)) \cap U'$ is an open subset of the twistor line $\sigma^{-1}(\sigma(x)) \subset X$. Therefore we can choose an open disc $D \subset \mathbf{C}P^1$ so that $\pi(x) \in D$ and $D \subset \pi(\sigma^{-1}(\sigma(x)) \cap U')$. Let U be the intersection $U' \cap \pi^{-1}(D)$. It is also a K-invariant neighborhood of the orbit Kx. Moreover, it possesses a strictly plurisubharmonic proper function, for instance, the sum of such functions for U' and D. Therefore the neighborhood U is Stein. Moreover, $\pi(U) = D$. Since $U \subset U_0 = \sigma^{-1}(V)$ it is a good neighborhood of x. ∎

Next we construct an étale Kähler chart at $x \in \mu_\mathbf{H}^{-1}(0)$. We will need some facts about relative differential forms. Precise definitions and proofs

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are collected in the Appendix. We will also need some facts about group actions on Stein spaces. We recall the appropriate definitions and results from [H].

Definition 3.3: ([H]) Consider a Stein space U on which a compact group K acts preserving the complex structure. Let G be the complexification of K. A *complexification* of U is a pair (V, i) of a Stein space V with a holomorphic G-action and a K-equivariant embedding $i : U \longrightarrow V$ with the following universal property:

For every complex space Y with a holomorphic action of G every K-equivariant map $f : U \longrightarrow Y$ factors uniquely through i. □

Note that in this definition the dimension of the complexification is not twice the dimension of the space. These dimensions are equal, contrary to what the term 'complexification' might suggest.

Definition 3.4: ([H],[GIT]) Consider a complex-analytic space Y on which a reductive complex group G acts holomorphically. The *universal categorical quotient* is pair of a complex-analytic space $Y//G$ and a G-invariant holomorphic map $\tau : Y \longrightarrow Y//G$ satisfying the following:

For any complex-analytic space U and any map $f : U \longrightarrow Y//G$ every G-invariant map $g : U \times_{Y//G} Y \longrightarrow Y$ from the fibered product to any complex space Y factors uniquely through the projection $f^*(\tau) : U \times_{Y//G} Y \longrightarrow U$.
□

Theorem 3.2: ([H], section 6). (i) For every Stein space U with an action of K preserving the complex structure there exists a unique complexification V. Moreover, the action map $G \times U \longrightarrow V$ is surjective.

(ii) For a complex space Y with an action of K call a holomorphic function on U K-*finite* if it belongs to a finite-dimensional K-invariant subspace in the space $\mathcal{O}_Y$ of all holomorphic functions. Then the space of K-finite holomorphic functions on U is dense in $\mathcal{O}_U$ and every such function extends uniquely to a K-finite holomorphic function on V. ∎

Theorem 3.3: ([H],Slice Theorem) Let K be a compact Lie group and G be its complexification. Consider a Stein space V with a holomorphic action of G and a point $v \in V$. Let $N \in G$ be the stabilizer of v in G. Assume that N is the complexification of the subgroup $N \cap K \subset K$.

Then there exists a N-invariant locally closed subset $S \subset V$ containing v with the following property:

The holomorphic map $g : S \times_N G \longrightarrow V$ is an open embedding. ∎

Theorem 3.4: ([H], section 6). For every Stein space V with a holomorphic action of a reductive complex group G there exist a unique universal categorical quotient $(V//G, \tau)$. The space $V//G$ is also Stein. ∎

Note the following corollary of the Theorem 3.2.

Lemma 3.2: Let U be a smooth Stein space with an action of a compact group K. Let V be its complexification. Then every K-finite holomorphic k-form $\theta \in \Gamma(\Omega^k_U)$ can be extended to the complexification V.

Proof. As usually, let G be the complexification of K. Since $V = GU$, it is also a smooth Stein space. Let T^*_V be the holomorphic cotangent bundle to V. Consider the total space $Y = Tot(\Lambda^k T^*_V)$ of its k-th exterior power. It has a natural action of G. Let $\rho : Y \longrightarrow V$ be the natural projection. The form θ defines a holomorphic section of $\Lambda^k T^*_V$ on U, that is, a holomorphic map $\theta : U \longrightarrow Y$ such that $\theta \circ \rho$ is the canonical embedding $\iota : U \longrightarrow V$ of U into V.

Since θ is K-finite, it belongs to a finite-dimensional K-invariant subspace $L \subset \Omega^k_U$ of the space of holomorphic k-forms on U. The K-equivariant vector bundle $\Lambda^k T^*_V \otimes L^*$ has a canonical K-invariant section f defined over U wich corresponds to the embedding of L into Ω^k_U. Again, there is the tautological K-equivariant map $g : U \longrightarrow Tot(\Lambda^k T^*_V \otimes L^*)$. It is also K-equivariant.

The space L^* is finite-dimensional, therefore the action of K on L^* extends to an action of G. By Definition 3.3 the map g can then be extended to a map $\tilde{g}$ from V to $Tot(\Lambda^k T^*_V \otimes L^*)$.

Since $\theta \in L$, it defines a linear map $\theta : L^* \longrightarrow \mathbb{C}$. Consider the composition $\tilde{\theta} = \tilde{g} \circ (\theta \otimes id) : V \longrightarrow Y$. Then $\tilde{\theta} = \theta : U \longrightarrow Y$ on U. Therefore $\tilde{\theta} \circ \rho|_U$ is the canonical embedding of U into V. Hence $\tilde{\theta} \circ \rho : V \longrightarrow V$ is the identity map.

Therefore $\tilde{\theta} : V \longrightarrow Y$ defines a holomorphic section of $\Lambda^k T^*_V$ equal to θ on U. This is the desired extension of the form θ. ∎

We can now prove the following.

Proposition 3.1: For every point $x \in \mu_{\mathbf{H}}^{-1}(0) \subset X$ there exists an étale Kähler chart (V, f) at x.

Proof. Let $U \subset X$ be a good neighborhood of x constructed in Lemma 3.1. By definition $D = \pi(U) \subset \mathbb{C}P^1$ is an open disc. Choose a local holomorphic coordinate z on D.

In the course of the proof we will use several times the following simple fact. (See Appendix for the definitions.)

Lemma 3.3: Consider a complex manifold Y and a holomorphic map $\rho : Y \longrightarrow D$ to an open disc. Assume that the differential of ρ is everywhere surjective. Let rel_Y be the canonical map from the total de Rham complex of Y to the relative de Rham complex of Y over D. Then for any differential form θ on Y we have $rel(\theta) = 0$ if and only if $\theta \wedge dz \wedge d\bar{z} = 0$.

Proof. Clear. ∎

We will use simply rel to denote the canonical maps from both the holomorphic and C^∞ de Rham complexes to their relative counterparts whenever it is clear which one is used at the moment.

First we construct a holomorphic 1-form α on U such that $rel(\partial\alpha - \Omega) = 0$.

Since $m = \sigma(x) \in M$ is in the zero set of the hyperkähler moment map, the restrictions of all the 2-forms $\omega_I, \omega_J, \omega_K$ to the orbit Km are zero by Claim 2.2. For any point $y \in D \subset \mathbb{C}P^1$ let $U_y = U \cap \pi^{-1}(y)$ be the fiber of U over y. By Definition 3.2 there exists an open set $U_0 \subset M$ such that $Km \subset \sigma(U_y) \subset U_0$ and U_0 is homotopy equivalent to Km. Since ω_I is zero on Km, it represents zero in the cohomology group $H^2(U_0)$. The same is true for ω_J and ω_K. Since the restriction of Ω to $\pi^{-1}(y) \cap \sigma^{-1}(U_0) \cong U_0$ is a linear combination of $\sigma^*(\omega_I), \sigma^*(\omega_J), \sigma^*(\omega_K)$, its cohomology class is also zero.

Therefore the relative 2-form Ω is exact on $\sigma^{-1}(U_0) \cong U_0 \times \mathbb{C}_P^1$ as a smooth form by Lemma A.4 (i). The same is true for its restriction to $U \subset \sigma^{-1}(U_0)$. Since U is Stein, it is also exact as a holomorphic form by Proposition A.1. This means that there exists a relative [1] holomorphic 1-form α_{rel} on U such that $\partial_{rel}\alpha_{rel} = \Omega_{rel}$.

Therefore we have the following.

---

[1] with respect to π

Claim. There exists a holomorphic 1-form α on U such that $rel(\partial\alpha) = rel(\Omega)$.

Proof. Since U is Stein, the surjective map of coherent sheaves

$$rel : \Omega^1_U \longrightarrow \Omega^1_{U/D}$$

admits a splitting. Therefore we can choose a holomorphic 1-form α on U so that $rel(\alpha) = \alpha_{rel}$. Then $rel(\partial\alpha - \Omega) = 0$. ■

Since Ω is K-invariant, we can assume after averaging over K that α is K-invariant as well.

Consider now the complexification V of U provided by Theorem 3.2(i). This is a Stein space with a holomorphic action of G. We have by definition a G-equivariant map $f : V \longrightarrow X$ extending the natural embedding of U. Since $V = GU$ and f is biholomorphic on U, it is biholomorphic in a neighborhood of every point in V. Therefore V is non-singular and f is locally biholomorphic.

Since the action of G preserves the fibers of π, the image $f(V) = f(GU)$ belongs to $\pi^{-1}(D)$. Therefore we can consider z as a holomorphic function on V.

The differential of the map $f \circ \pi : V \longrightarrow D$ is everywhere surjective. Therefore the relative de Rham complexes are well-defined, and Lemma 3.3 can be applied to forms on V over D.

Since α is K-invariant, it can be extended to a holomorphic 1-form on V by Lemma 3.2. Moreover, $\partial\alpha - f^*(\Omega_{rel}) = 0$ and α is K-invariant on the whole space V since both assertions hold on an open subset $U \subset V$.

Thus the pair (V, f) with the holomorphic 1-form α satisfies the requirements of Definition 3.1. In order to give the pair (V, f) a structure of an étale Kähler chart it remains to construct a strictly plurisubharmonic K-invariant function ψ on V such that $rel(\sqrt{-1}\,\partial\bar\partial\psi - f^*(\omega)) = 0$. We do it in two stages.

Consider the (1,1)-form on V given by

$$\omega' = f^*(\omega) - p(z)(\alpha \wedge d\bar z + \bar\alpha \wedge dz),$$

where $p(z)dz \otimes d\bar z$ is the Kähler metric on $D \subset \mathbb{C}P^1$.

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Step 1. There exists a $(1,0)$-form β on V such that

$$d\omega' = \beta \wedge dz \wedge d\bar{z} + \bar{\beta} \wedge dz \wedge d\bar{z}.$$

Moreover, for any such β the relative C^∞-form $rel(\beta + \bar{\beta})$ is closed and real.
Proof. By Claim 2.1 (i), (ii) we have on V

$$\partial f^*(\omega) = p(z) f^*(\Omega) \wedge d\bar{z},$$

$$\bar{\partial} f^* \omega = p(z) f^*(\bar{\Omega}) \wedge dz.$$

Therefore we have

$$\partial \omega' = \partial f^* \omega - \partial p(z) \wedge (\alpha \wedge d\bar{z} + \bar{\alpha} \wedge dz) - p(z) \partial \alpha \wedge d\bar{z} =$$

$$= \partial/\partial z (p(z)) \alpha \wedge dz \wedge d\bar{z} + p(z)(f^*\Omega - \partial \alpha) \wedge d\bar{z}.$$

Since $f^*\Omega - \partial \alpha = 0$ on fibers of $f \circ \pi$, there exist 1-forms η_1, η_2 on V such that

$$f^*\Omega - \partial \alpha = \eta_1 \wedge dz + \eta_2 \wedge d\bar{z}.$$

Let $\beta = \partial/\partial z (p(z)) \alpha + p(z) \eta_1$. This is a $(1,0)$-form. We then have

$$\partial \omega' = \beta \wedge dz \wedge d\bar{z} + \eta_2 \wedge d\bar{z} \wedge d\bar{z} = \beta \wedge dz \wedge d\bar{z}.$$

Since $\bar{\omega}' = \omega'$, we also have

$$\bar{\partial} \omega' = \bar{\beta} \wedge dz \wedge d\bar{z}.$$

Hence $d\omega' = (\beta + \bar{\beta}) \wedge dz \wedge d\bar{z}$. This in turn implies that

$$d(\beta + \bar{\beta}) \wedge dz \wedge d\bar{z} = 0.$$

Therefore $d_{rel}(rel(\beta + \bar{\beta})) = rel(d(\beta + \bar{\beta})) = 0$. ∎

Step 2. For any $(1,0)$-form β such that $rel(\beta + \bar{\beta})$ is closed there exists a smooth function ϕ on V such that $rel(\bar{\partial}\phi - \bar{\beta}) = 0$.

Proof. Choose an open covering U_i of V so that the relative cohomology sheaves $R^k(f \circ \pi)_*(f \circ \pi)^* C_D^\infty$ are trivial for $k > 0$. Since the relative form $rel(\beta + \bar{\beta})$ is closed, by Lemma A.3 (i) there exists a family of smooth real functions ϕ_i on U_i such that $rel(d\psi_i - \beta - \bar{\beta}) = 0$. For every intersection

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$U_i \cap U_j$ let $h_{i,j} = v_i - \psi_j$. Then $dh_{i,j} = 0$ on fibers of π. Therefore $h_{i,j}$ are real functions constant on fibers of π. They define a Cech 1-cocycle on V with coefficients in the sheaf of fiberwise constant functions. Consider this cocycle as a cocycle with coefficients in the sheaf $\mathcal{O}_{V/D} \cong (f \circ \pi)^* C_D^\infty \otimes_{(f \circ \pi)^* \mathcal{O}_D} \mathcal{O}_V$ of *fiberwise holomorphic* functions. Since V is Stein and the sheaf $\mathcal{O}_{V/D}$ is quasicoherent, the higher cohomology groups $H^k(V, \mathcal{O}_{V/D}), k > 0$ are trivial. Therefore our cocycle is a coboundary. This means that there exist a family of fiberwise holomorphic functions w_i such that on every intersection $U_i \cap U_j$ $w_j - w_i = h_{i,j}$. Let $\phi_i = \psi_i - w_i$. Then it is easy to see that $\phi_i = \phi_j$ on the intersection $U_i \cap U_j$. Therefore the family ϕ_i defines a single smooth C-valued function ϕ on V. We have on every U_i

$$rel(\bar{\partial}\phi) = rel(\beta - \bar{\partial}w_i) = rel(\beta).$$

■

Choose such a form β and a function ϕ. Let $v = \sqrt{-1}/2(-\phi + \bar{\phi})$. This is a real-valued smooth function on V.

Claim. We have $rel(\sqrt{-1}\,\partial\bar{\partial}v - f^*(\omega)) = 0$. [2]

Proof. Since $rel(\bar{\partial}\phi - \beta) = 0$. we have by Lemma 3.3

$$\bar{\partial}\omega' = \beta \wedge dz \wedge d\bar{z} = \bar{\partial}\phi \wedge dz \wedge d\bar{z}.$$

Recall that α is a *holomorphic* 1-form on V. This implies that $\bar{\partial}(p(z)\alpha \wedge d\bar{z}) = 0$ and $\partial(p(z)\bar{\alpha} \wedge dz) = 0$. Therefore

$$\partial\bar{\partial}f^*(\omega) = \partial\bar{\partial}\omega' = \partial\bar{\partial}\phi \wedge dz \wedge d\bar{z}.$$

On the other hand. by Claim 2.1 (iii)

$$\partial\bar{\partial}f^*(\omega) = p(z)f^*(\omega) \wedge dz \wedge d\bar{z}.$$

Comparing the right hand sides, we get

$$(\partial\bar{\partial}\phi - f^*(\omega)) \wedge dz \wedge d\bar{z} = 0.$$

Hence by Lemma 3.3 we have $rel(\partial\bar{\partial}\phi) = rel(\omega)$.

---

[2]Thus the function v is a "relative potential" for the relative Kähler form $rel(f^*(\omega))$.

Since ω is a real $(1,1)$-form. $rel(\sqrt{-1}\,\partial\bar{\partial}\psi) = 1/2rel(\partial\bar{\partial}\phi - \partial\bar{\partial}\bar{\phi}) = 1/2rel(\omega + \bar{\omega}) = rel(\omega)$. ∎

Since ω is a Kähler form on every fiber of $f \circ \pi$, the restriction of ψ to every fiber is strictly plurisubharmonic. Adding to ψ an appropriate multiple of the strictly plusubharmonic function $z\bar{z}$ and averaging over K, we can make it strictly plurisubharmonic and K-invariant on the whole of V.

The pair (V, f) equipped with the function ψ and the form α satisfies all the requirements of Definition 3.1 and is therefore an étale Kähler chart at x. This finishes the proof of Proposition 3.1. ∎

Next we show that for any étale Kähler chart (V, f) the holomorphic 1-form α and the function ψ on V can be corrected on an open subset of V so as to become compatible with the moment map in the sense of Definition 3.1.

Proposition 3.2: Let (V, f) be an étale Kähler chart at a point $x \in \mu_{\mathbf{H}}^{-1}(0) \subset X$. Then there exists a G-invariant Stein open subset $V_0 \subset V$ equipped with the structure of a étale Kähler chart at x compatible with the moment map.

Proof. Shrinking V if necessary, we may assume that the holomorphic line bundle $f^*\mathcal{O}(2)$ is trivial. We will fix a holomorphic trivialization and consider all $\mathcal{O}(2)$-valued objects as C-valued to simplify notation.

Thus we have a holomorphic 1-form α on V such that $\partial\alpha = f^*\Omega$ on fibers of $f \circ \pi$. We also have the real and complex moment maps $f \circ \mu_{\mathbf{R}} : V \longrightarrow \mathfrak{k}^*$, $f \circ \mu_{\mathbf{C}} : V \longrightarrow \mathfrak{g}$.

Consider the maps $\tilde{\mu_{\mathbf{C}}} : V \longrightarrow \mathfrak{g}^*, \tilde{\mu_{\mathbf{R}}} : V \longrightarrow \mathfrak{k}^*$ defined by

$$\tilde{\mu_{\mathbf{C}}}(g) = -\alpha(g_v).$$

$$\tilde{\mu_{\mathbf{R}}}(k) = \sqrt{-1}/2(\partial - \bar{\partial})\psi(k_v),$$

for any $v \in V, g \in \mathfrak{g}, k \in \mathfrak{k}$. Since α and ψ are K-invariant, these maps are respectively G- and K-equivariant. Since α is holomorphic, so is the map $\tilde{\mu_{\mathbf{C}}}$.

For any $v \in V$ let $r(v) = \tilde{\mu_{\mathbf{R}}}(v) - f \circ \mu_{\mathbf{R}}(v), c(v) = \tilde{\mu_{\mathbf{C}}}(v) - f \circ \mu_{\mathbf{C}}(v)$. Recall that the étale Kähler chart is called compatible with the moment map if and only if the corresponding functions r and c are zero.

Claim. The functions r and c are locally constant on fibers of $f \circ \pi$.

Proof. (i) Since the form α is G-invariant, Cartan homotopy formula gives for any $g \in \mathfrak{g}$

$$d(i_g(\alpha)) = -i_g(d\alpha),$$

where i_g is the contraction with the holomorphic vector field on V generated by g under the G-action.

Since $rel(\partial\alpha - \Omega) = 0$, this implies

$$rel(dc(g)) = rel(-di_g(\alpha) - f^*d\mu_{\mathbf{C}}(g)) = rel(i_g(d\alpha) - i_g(f^*\Omega)) = 0.$$

Hence $c(g)$ is locally constant on fibers of $f \circ \pi$.

Analogously for any $k \in \mathfrak{k}$

$$rel(dr(k)) = rel(di_k\sqrt{-1}/2(\partial - \bar\partial)\psi - f^*d\mu_{\mathbf{R}}(k)) =$$

$$= rel(i_k(-\sqrt{-1}/2d(\partial - \bar\partial)\psi - f^*\omega)) = 0.$$

Therefore $r(k)$ is also locally constant along the fibers. ∎

Still, the functions r and c need not be zero in the general situation. We will show that on an open subset of V we can correct α and ψ so as to make r and c zero.

First we construct an appropriate open subset $V_0 \subset V$.

Let $v \in V$ be a point such that $f(v) = x$. Let $N \subset G$ be its stabilizer. Note the following fact.

Claim. The complex group N is the complexification of its real subgroup $N \cap K \subset K$.

Proof. It is enough to prove that the Lie algebra $\mathfrak{n}$ of N is the complexification of the Lie algebra $\mathfrak{n} \cap \mathfrak{k} \in \mathfrak{g}$ of $N \cap K$. Since the map f is locally biholomorphic, $\mathfrak{n}$ is equal to the Lie algebra of the stabilizer of $x \in X$.

Let T_x be the complex tangent space at x to the orbit Gx. The Kähler metric on the fiber $\pi^{-1}(\pi(x)) \subset X$ induces a non-degenerate Hermitian form on T_x. The corresponding symplectic form ω on the underlying real vector space T_x is also non-degenerate. Consider the subspace T_x^K of vectors tangent to the orbit Kx of the compact group. Then by Claim 2.2 the form ω is zero on T_x^K. Therefore

$$dim_{\mathbf{R}}T_x^K \le dim_{\mathbf{C}}T_x.$$

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On the other hand. we obviously have $(\mathfrak{n} \cap \mathfrak{k}) \otimes \mathbf{C} \subset \mathfrak{n}$. Therefore

$$dim_{\mathbf{R}} T^K{}_x = dim_{\mathbf{R}} K - dim_{\mathbf{R}}(\mathfrak{n} \cap \mathfrak{k}) \geq dim_{\mathbf{C}} G - dim_{\mathbf{C}} \mathfrak{n} = dim_{\mathbf{C}} T_x.$$

Thus all the inequalities are in fact equalities. This implies that $dim_{\mathbf{R}}(\mathfrak{n} \cap \mathfrak{k}) = dim_{\mathbf{C}} \mathfrak{n}$. Hence $(\mathfrak{n} \cap \mathfrak{k}) \otimes \mathbf{C} \cong \mathfrak{n}$. $\blacksquare$

Therefore we can apply Theorem 3.3 to $v \in V$. By this theorem there exists a locally closed subset $S \subset V$ containing v such that the map $\iota : S \times_N G \longrightarrow V$ is an open embedding. Let $V' \subset V$ be the image of ι.

Let $G_0 \subset G$ be the connected component of unity in G.

Claim. There exists a Stein G_0-invariant open neighborhood U_0 of v in V' such that

(i) For every fiber $f \circ \pi^{-1}(y) \subset V$ the intersection $U_0 \cap f \circ \pi^{-1}(y)$ is either empty or connected;

(ii) If this intersection is not empty. then it contains a point u such that $f \circ \sigma(u) = f \circ \sigma(v)$, where $\sigma : X \longrightarrow M$ is the canonical projection.

Proof. We can always choose an open subset $S' \subset S$ so that $v \in S'$ and fibers of $f \circ \pi|_S$ are connected. Let $U' = \iota(S' \times_N G_0) \subset V'$. Since G_0 acts along the fibers of $f \circ \pi$, every fiber U'_y of $f \circ \pi|_{U'}$ is of the form $i(G_0 \times_N S'_y)$ for some $f \circ \pi|_{S'}$-fiber $S'_y \subset S'$. Since G_0 is connected, so are all the fibers of $f \circ \pi|_{U'}$. The intersection $D = \sigma^{-1}(\sigma(x)) \cap f(U')$ is an open Stein subset in $\sigma^{-1}(\sigma(x)) \cong \mathbf{C}P^1$. Let $U_0 = f \circ \pi^{-1}(D) \cap U'$. Then U_0 is Stein, G_0-invariant and satisfies the conditions of the claim. $\blacksquare$

Choose such a neighborhood U_0. Let $V_0 = GU_0 \subset V$. Since G/G_0 is finite and U_0 is a G_0-invariant Stein manifold. the open set $V_0 = \cup_{g \in G/G_0} gU_0$ is also a Stein manifold.

We are going to construct a holomorphic 1-form γ and a smooth real-valued function ϕ on V_0 so that $\alpha - \gamma$ and $\psi - \phi$ give a structure of an étale Kähler chart on V_0 which is compatible with the moment map. To do this we have to study the restrictions of the "error terms" r and c to V_0.

Consider first the restrictions of functions r and c to U_0. Since all the fibers of $f \circ \pi$ on U_0 are connected. both r and c are not only locally constant on the fibers but simply constant.

Consider the linear subspace $t \subset \mathfrak{k}^*$ of vectors invariant with respect to the coadjoint representation of K on $\mathfrak{k}^*$ and orthogonal to the subspace $\mathfrak{n} \cap \mathfrak{k} \subset \mathfrak{k}$ under the natural pairing $\mathfrak{k} \odot \mathfrak{k}^* \longrightarrow \mathbf{R}$.

Claim. For any point $u \in U_0$ we have $r(u) \in t \subset \mathfrak{k}$ and $c(u) \in t \otimes \mathbf{C} \subset \mathfrak{g}$.

Proof. Since both r and c are constant along the fibers of $f \circ \pi|_U$, it is enough to prove the claim for any point in each fiber. On every fiber we have a point y such that $f \circ \sigma(y) = f \circ \sigma(v)$. Then the subgroup $N \cap K$ stabilizes $f \circ \sigma(y)$. Therefore it stabilizes y itself, and so does its complexification N.

For any $n \in \mathfrak{n}$ we have

$$c(n) = -\alpha(n_y) - \mu_{\mathbf{C}}(y) = -\alpha(0) - 0 = 0.$$

Therefore $c(y)$ is orthogonal to $\mathfrak{n} \in \mathfrak{g}$. The same is obviously for $r(y)$.

Since both r and c are K-equivariant and constant on the orbit $Ky \subset V$, the orbits $c(Ky) = c(y) \subset \mathfrak{g}^*$ and $r(Ky) = r(y) \subset \mathfrak{k}^*$ consist each of a single point which are therefore fixed by the coadjoint representations. ∎

To proceed further we need the following fact from the theory of complex reductive groups.

Theorem 3.5: (see, e.g., [B]) For any reductive complex group L let $\mathfrak{X}(L)$ be the abelian group of homomorphisms from L to $\mathbf{C}^*$.

(i) The group $\mathfrak{X}(L)$ is a lattice, called the lattice of characters of L.

(ii) Consider the canonical embedding of $\mathfrak{X}(L)$ into the subspace $\mathfrak{l}^{*,L} \subset \mathfrak{l}^*$ of vectors in the dual to the Lie algebra $\mathfrak{l}$ of L fixed under the coadjoint representation. Then the associated map $\delta : \mathfrak{X}(L) \odot \mathbf{C} \longrightarrow \mathfrak{l}^{*,L}$ is an isomorphism. ∎

Consider a sublattice $\mathfrak{X}(G/N) \subset \mathfrak{X}(G)$ of characters trivial on the subgroup N. Choose a basis $\chi_1, ..., \chi_i$ of this lattice. Then the vectors $\delta\chi_1, ..., \delta\chi_i$ form a basis of the vector space $t \otimes \mathbf{C} \subset \mathfrak{g}$. Since every character $\chi : G \longrightarrow \mathbf{C}^*$ maps K into the unit circle, all the vectors $\delta\chi_1, ..., \delta\chi_i$ belong to the real subspace t. There they also form a basis.

We have already proved that our functions r and c take values in t. Therefore there exist real-valued functions $a_1, ..., a_i$ and complex-valued functions $b_1, ..., b_i$ on $\pi(U_0)$ such that for any $u \in U_0$

$$r(u) = a_1(\pi(u))\delta\chi_1 + ... + a_i(\pi(u))\delta\chi_i,$$

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$$c(u) = b_1(\pi(u))\delta\chi_1 + \ldots + b_i(\pi(u))\delta\chi_i.$$

Let $\pi_2 : S \times G \longrightarrow G$ be the projection on the second factor. For any $\chi \in X(G/N)$ the composition $\pi_2 \circ \chi : S \times G \xrightarrow{\pi_2} G \xrightarrow{\chi} \mathbf{C}^*$ can be factored though $V' \cong S \times_N G$. Then it defines a nowhere vanishing holomorphic function on V'. Let $\tilde{\chi}$ be the restriction of this function to U_0.

Claim. For any $\chi \in X(G/N), g \in \mathfrak{g}, k \in \mathfrak{k}$

$$i_g(\partial \log \tilde{\chi}) = i_g(\frac{\partial \tilde{\chi}}{\tilde{\chi}}) = \delta\chi(g).$$

$$i_k(\sqrt{-1}/2(\partial - \bar{\partial})\log|\tilde{\chi}|) = \delta\chi(k).$$

Proof. Recall that the map $\delta : \mathfrak{X}(G) \longrightarrow \mathfrak{g}^* \cong Hom(\mathfrak{g}, \mathbf{C})$ by definition sends the character χ to its differential $\delta\chi : \mathfrak{g} \longrightarrow \mathbf{C} = Lie(\mathbf{C}^*)$. Since the character χ is a group homomorphism. the Lie derivative of $\tilde{\chi}$ is

$$\mathcal{L}_g \tilde{\chi} = \delta\chi(g)\tilde{\chi}.$$

Therefore

$$i_g(\frac{\partial \tilde{\chi}}{\tilde{\chi}}) = \tilde{\chi}^{-1} i_g(\partial \tilde{\chi}) = \tilde{\chi}^{-1} \mathcal{L}_g \tilde{\chi} = \delta\chi(g).$$

We also have

$$\sqrt{-1}/2(\partial - \bar{\partial})\log|\tilde{\chi}| = 1/2\overline{d\log|\tilde{\chi}|},$$

since the function $\log|\tilde{\chi}|$ is real-valued. Therefore

$$i_k(\sqrt{-1}/2(\partial - \bar{\partial})\log|\tilde{\chi}|) = 1/2 i_k(\overline{d\log|\tilde{\chi}|}) =$$

$$= 1/2(\delta\chi(k) + \delta\chi(\bar{k})) = \delta\chi(k).$$

■

Define a holomorphic 1-form γ and a real-valued function ϕ on U_0 by the formulas

$$\gamma = -b_1(\pi(u))\partial \log \tilde{\chi}_1 - \ldots - b_i(\pi(u))\partial \log \tilde{\chi}_i,$$

$$\phi = a_1(\pi(u))\log|\tilde{\chi}_1| + \ldots + a_i(\pi(u))\log|\tilde{\chi}_i|.$$

for any $u \in U_0$. By construction both are $K \cap G_0$-invariant. Since G/G_0 is finite, γ and ϕ can be extended to $V_0 = GU_0$. Averaging over G/G_0 if necessary we can choose K-invariant extensions.

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Now we are able to finish the proof of Proposition 3.2.

Claim. The Stein open subspace $V_0 \in V$ is an étale Kähler chart at x compatible with the moment map.

Proof. On $U_0 \subset V_0$ we have for any $k \in \mathfrak{k}, g \in \mathfrak{g}$

$$r(u)(k) = a_1(\pi(u))\delta\chi_1(k) + ... + a_i(\pi(u))\delta\chi_i(k) =$$

$$= a_1(\pi(u))\sqrt{-1}/2(\partial - \bar\partial)log|\tilde\chi_1|(k)+$$

$$+... + a_i(\pi(u))\sqrt{-1}/2(\partial - \bar\partial)log|\tilde\chi_i|(k) = i_k(\sqrt{-1}/2(\partial - \bar\partial)\phi),$$

$$c(u)(g) = b_1(\pi(u))\delta\chi_1(g) + ... + b_i(\pi(u))\delta\chi_i(g) =$$

$$= b_1(\pi(u))\partial\,log\chi_1(g) + ... + b_i(\pi(u))\partial\,log\chi_i(g) = i_g(\gamma).$$

Since both sides are K-invariant, the same is true on $V_0 = KU_0$.

Therefore on V_0 for any $k \in \mathfrak{k}, g \in \mathfrak{g}$

$$\mu_{\mathbf{C}}(g) = \tilde\mu_{\mathbf{C}}(g) - c(g) = -i_g(\alpha - \gamma).$$

$$\mu_{\mathbf{R}}(k) = \tilde\mu_{\mathbf{R}}(k) - r(k) = i_k(\sqrt{-1}/2(\partial - \bar\partial)(\psi - \phi)).$$

Moreover,

$$rel(d\gamma) = 0,$$

$$rel(\partial\bar\partial\phi) = 0.$$

The space V_0 equipped with the holomorphic form $\alpha - \gamma$ and the function $\psi - \phi$ thus satisfies all the conditions of Definition 3.1 except for possibly plurisubharmonicity of $\psi - \phi$. This function is still strictly plurisubharmonic on the fibers. Thus it can be made strictly plurisubharmonic on V_0 by adding a multiple of $z\bar z$ for any holomorphic coordinate z on $f \circ \pi(U) \subset \mathbf{C}P^1$. ∎

This finishes the proof of Proposition 3.2. ∎

In order to refine this to a construction of an actual Kähler chart we will use results from [HL] on group actions on Kähler spaces.

Definition 3.5: (e.g., [HL] 2.5). A *Kähler space* X is a reduced complex analytic space X with an open covering (U_α) and a family of continuous real functions (ψ_α) called *potentials* such that

(i) On each U_α the potential ψ_α is strictly plurisubharmonic,

(ii) On each intersection $U_\alpha \cap U_\beta$ there exists a holomorphic function $f_{\alpha\beta}$ such that $\psi_\alpha - \psi_\beta = Re(f_{\alpha\beta})$. $\square$

Two Kähler spaces are considered equivalent if their underlying complex-analytic space is the same and their families of potentials are compatible, that is, the union of these families is again a family of potentials. A non-singular Kähler space is then the same as a Kähler manifold. (The Kähler form on U_α is given by $\sqrt{-1}\,\partial\bar\partial\psi_\alpha$. It is positive by (i) and independent of U_α by (ii).) An analytic subspace of a Kähler space with the induced family of potentials is a Kähler space.

Definition 3.6: ([HL] 2.5) Let K be a compact group, $\mathfrak{k}$ its Lie algebra and G its complexification. A *Hamiltonian G-space* (Y, Φ) is a pair of

(i) A Kähler space Y on which G acts holomorphically so that K preserves the Kähler structure,

(ii) A continuous K-equivariant map $\Phi : Y \longrightarrow \mathfrak{k}^*$ which is a moment map for the action of K with respect to the Kähler form on the non-singular part of Y and on every G-orbit. $\square$

Claim 3.1: Every étale Kähler chart (V, f) compatible with the moment map is a Hamiltonian G-space with the canonical function ψ as the potential and $f \circ \mu_{\mathbf{R}} : V \longrightarrow \mathfrak{k}^*$ as the moment map.

Proof. Clear. ∎

Definition 3.7: ([HL]) Let U be a open K-invariant subset in a Hamiltonian G-space (Y, μ). For any $x \in U$ and $k \in \mathfrak{k}$ let $I(k, x) \subset \mathbf{R}$ be the set of such $r \in \mathbf{R}$ that $exp(\sqrt{-1}\,rk)x \in U$.

The set U is called *orbit-convex* iff $I(k, x)$ is connected for every $k \in \mathfrak{k}, x \in X$.

The set U is called *moment-convex* iff for every connected component (a, b) of $I(k, x)$ we have

$$\mu(exp(\sqrt{-1}\,ak)x)(k) < 0,$$

$$\mu(exp(\sqrt{-1}\,bk)x)(k) > 0.$$

$\square$

The notions of orbit- and moment-convexity depend not only on the subspace but on the ambient space as well. The whole twistor space X is not a Hamiltonian Kähler space. Still, Definition 3.7 makes sense for subsets of X if we take $\mu_{\mathbf{R}}$ as the moment map.

We have $(d/dr\mu_{\mathbf{R}}(exp(\sqrt{-1}\ rk)x))(k) = l(k_x, k_x) > 0$. Therefore $\mu_{\mathbf{R}}(k)$ is strictly increasing along integral curves of $(\sqrt{-1}\ k)$. This implies that all moment-convex subsets of X are orbit-convex. The converse needn't be true. [3]

These notions of convexity allow one to formulate a convenient criterion of an étale Kähler chart being globally biholomorphic.

Lemma 3.4: Consider an étale Kähler chart (V, f) at x. Suppose there exists an open Stein K-invariant subset $U \subset V$ such that f is biholomorphic on U, $x \in f(U)$ and $f(U) \subset X$ is moment-convex. Then the pair (GU, f) is a Kähler chart.

Proof. (For an alternative proof see [HL] Section 2.) We have to show that the map f is biholomorphic on GU. Consider two points $x_1, x_2 \in GU$ such that $f(x_1) = f(x_2)$. Since f is G-equivariant we may assume that $x_1 \in U$ and $x_2 = g \cdot x_3$ for some $g \in G, x_3 \in U$.

The Polar Decomposition Theorem of Mostow ([M]) claims that every element $a \in G$ can be uniquely decomposed as a product $b \cdot exp(\sqrt{-1}\ c), b \in K, c \in \mathfrak{k}$. Let $g = k \cdot exp(\sqrt{-1}\ u)$ be the polar decomposition of g.

Since U is K-invariant, $x_0 = k \cdot x_3 \in U$. Then both 0 and 1 belong to the set $I(u, f(x_0))$. Since $f(U)$ is moment-convex, it is also orbit-convex and the whole interval $(0, 1)$ is in $I(u, f(x_0))$. Since f is locally biholomorphic, $I(u, x_0)$ is the union of a subset of the set of connected components of $I(u, f(x_0))$. It contains 0, so it contains all of $(0, 1)$.

Thus $x_2 = exp(\sqrt{-1}\ u)x_0 \in U$. Since f is biholomorphic on U, $x_1 = x_2$.

Since U is orbit-convex, GU is the complexification of U by [H] section 3. Therefore it is Stein. It is non-singular since it is an open subset of a complex manifold. Therefore GU with the induced Kähler metric is a Kähler chart. ∎

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[3] Of course, the same argument applies to subsets of an arbitrary Hamiltonian Kähler space.

Using this criterion, we can apply results of [HL] and [HHuL] to prove the following.

Proposition 3.3: For every point $x \in \mu_{\mathbf{H}}^{-1}(0) \subset X$ there exists a Kähler chart at x.

Proof. Choose an étale Kähler chart (V, f) at x compatible with the moment map. Let ψ be the potential on V. By Lemma 3.4 it is enough to show that we can choose a K-invariant open subset $U \subset V$ so that f is biholomorphic on U, $x \in f(U)$ and $f(U)$ is moment-convex.

Choose a point $v \in V$ such that $f(v) = x$. By [HHuL] Section 3, Lemma 3 we may assume, shrinking V if nesessary, that there exists a strictly plurisubharmonic K-invariant function $\tilde{v}$ on V such that

(i) $\tilde{\psi} = \psi$ in a K-invariant neighborhood $U_0 \subset V$ of v,

(ii) $\sqrt{-1}\,(\partial - \bar{\partial})\tilde{\psi}(k_u) = f \circ \mu_{\mathbf{R}}(u)(k)$ for every $u \in V, k \in \mathfrak{k}$,

(iii) $\tilde{\psi}$ is proper on every fiber of the canonical map $\tau : V \longrightarrow V//G$.

Then by [HL] 2.7, Theorem there exist a Stein subspace $V_0 \in V//G$ and a number $\epsilon > 0$ such that $U = \tilde{v}^{-1}((-\inf, \tilde{\psi}(x) + \epsilon)) \cap \tau^{-1}(V_0)$ is open and relatively compact in U_0. Then by [HL] 2.7 Lemma $f(U) \in X$ is moment convex. The set U obviously contains v and it is Stein and K-invariant. Then by Lemma 3.4 the space GU is the desired Kähler chart. ∎

4. Proof of the main theorem.

In this section we prove Theorem 1. We will use the result of [HL] in the following form.

Proposition 4.1: ([HL] 3.3 Theorem). Let (Y, Φ) be a Hamiltonian G-space, possibly singular. Consider the subspace Y_{ss} of points y for which the intersection $\overline{Gy} \cap \Phi^{-1}(0)$ is non-empty. Then there is a G-invariant continuous map $\tau : Y_{ss} \longrightarrow \Phi^{-1}(0)/K$, called *the reduction map*, such that

(i) τ restricted on $\Phi^{-1}(0) \subset Y_{ss}$ is the natural quotient map

(ii) The ringed space $(\Phi^{-1}(0)/K, (\tau_*\mathcal{O}_Y)^G)$ is a complex analytic space.

∎

In particular, this implies that for every $y \in Y$ the intersection $\overline{Gy} \cap \Phi^{-1}(0)$ consists of exactly one K-orbit, namely, $\tau(y)$.

We can now prove Theorem 2.3.

Proof of the Theorem 2.3 (i) Consider a semistable orbit Gx. For every orbit Ky in $\mu_\mathbf{H}^{-1}(0) \cap \overline{Gx}$ choose a Kähler chart $V_y \subset X$ at y. Since V_y is open, $Gx \in V_y$ for every y. For every two orbits Ky_1, Ky_2 in the closure $\overline{Gx}$ the intersection $V_{y_1} \cap V_{y_2}$ is a Hamiltonian Kähler G-space with the moment map $\mu_\mathbf{R}$ Then $Ky_1 = Ky_2$ by Proposition 4.1(i).

Therefore we have a well-defined map $q : X_{ss} \longrightarrow X_0//G$.

(ii) Consider a Kähler chart V. Let $V_0 = V \cap X_{ss}$. It is still a Hamiltonian G-space (possibly singular) and it is open in X_{ss}. By definition of X_{ss} the reduction map τ is defined on the whole of V_0 and $\tau = q$ by Proposition 4.1(i). Therefore q is continuous on V_0.

Claim 1. $V_0 = q \circ q^{-1}(V_0)$.

Proof. For every $x \in X_{ss}$ such that $q(x) = Ky \in q(V_0)$ we have by definition $Ky \subset \overline{Gx}$. On the other hand, by Proposition 4.1 $Ky \subset V_0$. Since V_0 is open and G-invariant, $Gx \in V$.

Claim 2. $q(V_0)$ is open in $X_0//G$.

Proof. By Proposition 3.1(i) $q(V_0) = q(V_0 \cap \mu_\mathbf{R}^{-1}(0))$. The restriction of q to $\mu_\mathbf{R}^{-1}(0) \cap X_{ss}$ is the quotient map with respect to an action of a compact group and is thus open.

Since we have a Kähler chart at every point in $\mu_\mathbf{H}^{-1}(0)$, the open sets $q(V_0)$ for all Kähler charts V form an open covering of $X_0//G$. Then we have a covering of X_{ss} by $q \circ q^{-1}(V_0)$. Since q is continuous on every $V_0 = q \circ q^{-1}(V_0)$, it is continuous on X_{ss}.

(iii) For every Kähler chart V the ringed space $(q(V), (q_*\mathcal{O}_{X_{ss}})^G)$ is complex-analytic by Proposition 4.1(ii). Thus we have an open covering of

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$(X_0//G, (q_*\mathcal{O}_{X_{ss}})^G)$ by ringed subspaces which are complex-analytic. Since $X_0//G$ is Hausdorff, it is complex-analytic. ∎

5. Other structures on $X_0//G$.

In this section we show that some of the structures on the twistor space X naturally descend to the quotient $X//G$.

Proposition 5.1: (i)The analytic subset $X_0 \subset X$ is preserved by the canonical real structure on X.

(ii)There exist a real structure $\iota : X_0//G \longrightarrow X_0//G$ inducing the antipodal involution on $\mathbb{C}P^1$.

Proof. Let $\iota_X : X \longrightarrow X$ be the canonical real structure on X. There is a canonical real structure $g \longmapsto \bar{g}$ on the group G so that K is the group of real points, and it induces a real strucutre on the dual to the Lie algebra $\mathfrak{g}^*$. The action of K on X obviously commutes with ι_X. Therefore $g\iota_X x = \iota_X \bar{g}x$ for any $g \in G, x \in X$. This implies that ι_X preserves orbits of G. The subset $\mu_H^{-1}(0) \in X$ is also obviously preserved by ι_X.

There is a real structure on the total space $Tot(\mathfrak{g}^* \otimes \mathcal{O}(2))$, namely, the product of the induced real structure on $\mathfrak{g}^*$ and the canonical real structure on $Tot(\mathcal{O}(2))$ which lifts the antipodal involution. It is easy to see that the complex moment map $\mu_\mathbb{C} : X \longrightarrow Tot(\mathfrak{g}^* \otimes \mathcal{O}(2))$ is a real map with respect to this complex structure and ι_X. Since the zero section is obviously a real submanifold of $Tot(\mathfrak{g}^* \otimes \mathcal{O}(2))$, the subset $X_0 = \mu_\mathbb{C}^{-1}(0)$ is preserved by ι_X. Since ι_X is continuous and preserves orbits of G and $\mu_H^{-1}(0)$, it also preserves the subspace $X_{ss} \subset X_0$.

Since ι_X preserves $\mu_H^{-1}(0)$ and commutes with the action of K, it defines a continuous map $\iota : X_0//G \longrightarrow X_0//G$ lifting the antipodal involution on $\mathbb{C}P^1$. The quotient map $q : X_{ss} \longrightarrow X_0//G$ satisfies $\iota \circ q = q \circ \iota_X$. By definition of the complex analytic structure on $X_0//G$ this implies that ι is actially a real structure on $X_0//G$. ∎

Next we show that locally the canonical relative 2-form Ω_{rel} can also be descended to $X_0//G$.

Proposition 5.2: For every point $x \in X//G$ there exists an open neighborhood U and a $\mathcal{O}(2)$-valued 1-form α on U such that $\partial q^*(\alpha)$ equals Ω_{rel} on $q^{-1}(U) \subset X_{ss}$.

Proof. Choose a point $y \in \mu_{\mathbf{H}}^{-1}(0) \subset X_{ss}$ so that $q(y) = x$ and a Kähler chart (V, f) at y compatible with the moment map. Let $U = q(f(V) \cap X_{ss}) \subset X_0//G$. By Theorem 2.3 the subset U is open and $q^{-1}(U) = f(V) \cap X_{ss} \subset X_{ss}$.

We may assume that $f \circ \pi(V)$ is an open disc $D \subset \mathbf{C}P^1$. Choose a trivialization of $\mathcal{O}(2)$ over D and consider the restriction $f^*(\Omega)$ of the canonical $\mathcal{O}(2)$-valued $(2,0)$-form as a C-valued $(2,0)$-form Ω_0 on V. Then by Definition 3.1 there exists a holomorphic G-invariant 1-form α on V such that $\partial \alpha = \Omega_0$ on fibers of $f \circ \pi$. The trivialization of $\mathcal{O}(2)$ also provides the holomorphic moment map $\mu_{\mathbf{C}} : V \longrightarrow \mathfrak{g}^*$. Since the chosen chart is compatible with the moment map, we also have $\mu_{\mathbf{C}}(g) = i_g(\alpha)$.

Claim. For every $x_1 \in U$ the restriction of α_0 to the fiber $q^{-1}(x_1)$ is zero.

Proof. By definition we have $\mu_{\mathbf{C}}(q^{-1}(x_1)) = 0$. Therefore $\alpha_1(g_y) = 0$ for any $g \in \mathfrak{g}, y \in q^{-1}(x_1)$. This implies that α_1 is zero on every G-orbit in $q^{-1}(x_1)$. The fiber $q^{-1}(x_1)$ is Stein and its universal categorical quotient by the action of G consists of one point x_1. Therefore α_1 is zero on $q^{-1}(x_1)$. ∎

Now by Lemma A.1 the relative holomorphic 1-form $rel_q(\alpha_1) \in \Omega^1_{q^{-1}(U)/U}$ is zero. This implies the existence of a holomorphic 1-form $\alpha_0 \in \Omega^1_U$ such that $q^*\alpha_0 = \alpha_1$. Multiplying α_0 by the chosen trivialization of $\mathcal{O}(2)$ we get the desired 1-form. ∎

The holomorphic relative 2-form $\partial \alpha_0 \in \Omega^2_{U/\mathbf{C}P^1}$ defined on the neighborhood U constructed in this lemma satisfies $q^*\partial \alpha_0 = \Omega$. Unfortunately, it may depend on a particular choice of U. I do not know whether there is in general a way to patch these forms into a single relative holomorphic 2-form on the whole $X_0//G$.

There is, however, the following corollary to Proposition 5.2.

Corollary 5.1: Assume that the open subset $X_{reg}/G \subset X_0//G$ is dense. Let $f : \tilde{X} \longrightarrow X_0//G$ be a resolution of singularities of $X_0//G$ in the

following sense:

$\tilde{X}$ is a smooth complex manifold, the map f is surjective everywhere and biholomorphic on a dense open subset $\tilde{X}_{reg}$, and $f(\tilde{X}_{reg}) \subset X_{reg}/G$.

Assume in addition that the map $f \circ \pi : \tilde{X} \longrightarrow \mathbf{C}P^1$ is surjective everywhere together with its differential. Then the relative holomorphic $f \circ \pi^* \mathcal{O}(2)$-valued 2-form $f|^*_{\tilde{X}_{reg}} (\Omega_{rel}$ can be extended uniquely to the whole $\tilde{X}$.

Proof. Since the space $\tilde{X}$ is non-singular, the sheaf $\Omega^2_{\tilde{X}/\mathbf{C}P^1}$ of relative holomorphic 2-forms has no torsion. Therefore on every open subset $U \subset \tilde{X}$ there is at most one 2-form extending $f|^*_{\tilde{X}_{reg}} (\Omega_{rel}$. This proves uniqueness. By Proposition 5.2 there exists an open covering of $X_0//G$, hence of $\tilde{X}$, such that on every one of elements there exists an extension. By uniqueness these extensions must patch together to give a single extention on the whole of $\tilde{X}$. ∎

This corollary might be useful in the theory of so-called shuffler spaces developed in [V].

Appendix.

In this appendix we collect some simple facts about relative differential forms for which we could not find a convenient reference.

Let X, Y be complex-analytic spaces and let $\pi : Y \longrightarrow X$ be a holomorphic map. Let $\mathcal{O}_X, \mathcal{O}_Y$ be the sheaves of holomorphic functions on X, Y and let Ω^1_X, Ω^1_Y be the sheaves of Kähler differentials.

Defintion A.1: The sheaf of *holomorphic relative 1-forms* $\Omega^1_{Y/X}$ is the sheaf of Kähler differentials of $\mathcal{O}_Y$ with respect to the pullback sheaf $\pi^*(\mathcal{O}_X)$. □

Lemma A.1: (i)There exists a natural exact sequence

$$\pi^* \Omega^1_X \otimes_{\pi^* \mathcal{O}_X} \mathcal{O}_Y \xrightarrow{\delta\pi} \Omega^1_Y \xrightarrow{rel} \Omega^1_{Y/X} \longrightarrow 0.$$

(ii)For any point $x \in X$ the restriction of $\Omega^1_{Y/X}$ to the fiber $\pi^{-1}(x) \subset Y$ is isomorphic to the sheaf of Kähler differentials $\Omega^1_{\pi^{-1}(x)}$.

Proof. These are easy consequences of the corresponding statements about Kähler differentials for rings, see [Ma]. ∎

Assume now that Y, X are complex manifolds. Then the sheaves Ω^1_X, Ω^1_Y are sheaves of holomorphic sections of holomorphic cotangent bundles T^*_X, T^*_Y. Assume in addition that both π and the differential of π at every point in Y are surjective. The map $\delta\pi$ is then a map from the vector bundle $\pi^*T^*_X$ to T^*_Y adjoint to the differential of π. Therefore it is injective. Let the *relative cotangent bundle* $T^*_{Y/X}$ be the cokernel of $\delta\pi$. This bundle possesses a canonical holomorphic structure. Then $\Omega^1_{Y/X}$ is the sheaf of holomorphic sections of $T^*_{Y/X}$. The restriction of $T^*_{Y/X}$ to every fiber of π is canonically isomorphic to the cotangent bundle to the fiber.

Note that all the exterior powers of $T_{Y/X}$ inherit a holomorphic structure. As usually, let $\overline{T}^*Y/X$ be the complex conjugate to T^*Y/X. Let C^∞_Y, C^∞_X be the sheaves of C^∞ functions on Y and X. These sheaves are naturally sheaves of algebras over $\mathcal{O}_Y$ and $\mathcal{O}_X$ respectively.

Claim A.1: The sheaves C^∞_Y and C^∞_X are quasicoherent and flat over $\mathcal{O}_Y$ and $\mathcal{O}_X$ respectively.
  Proof. Clear.

Definition A.2.: A *relative C^∞ (n, m)-form* is a C^∞ section of the vector bundle $\Lambda^n(T^*(Y/X)) \otimes \Lambda^m(\overline{T}^*(Y/X))$. The sheaf of C^∞ (n, m)-forms is denoted by $\mathcal{A}^{n,m}_{Y/X}$.
  A *relative holomorphic n-form* is a holomorphic section of the vector bundle $\Lambda^n(T^*(Y/X))$. The sheaf of holomorphic n-forms is denoted by $\Omega^n_{Y/X}$.
□

The canonical projection $rel : T^*(Y) \longrightarrow T^*(Y/X)$ from the cotangent bundle of Y onto the relative cotangent bundle of Y over X defines for every n, m an epimorphism of sheaves $rel : \mathcal{A}^{n,m}_Y \longrightarrow \mathcal{A}^{n,m}_{Y/X}$.

Claim A.2: The kernel of rel is invariant with respect to the differentials $\partial, \bar{\partial}$ on $\mathcal{A}^{\bullet,\bullet}_Y$.

Proof. Clear. ∎

Therefore we can define the relative differentials $\partial_{rel} : \mathcal{A}_{Y/X}^{\bullet,\bullet} \longrightarrow \mathcal{A}_{Y/X}^{\bullet+1,\bullet}$, $\bar{\partial}_{rel} : \mathcal{A}_{Y/X}^{\bullet,\bullet} \longrightarrow \mathcal{A}_{Y/X}^{\bullet,\bullet+1}$ and the real differential $d_{rel} = \partial_{rel} + \bar{\partial}_{rel}$. We can also define the holomorphic differential $\partial_{rel} : \Omega_{Y/X}^{\bullet} \longrightarrow \Omega_{Y/X}^{\bullet+1}$.

Claim A.3: All these differentials commute with the multiplication by a function lifted from the base manifold X.

Proof. Clear. ∎

Warning. In the absolute case the operator $\bar{\partial} : \mathcal{A}_Y^{(1,0)} \longrightarrow \mathcal{A}_Y^{(1,1)} = \mathcal{A}_Y^{(1,0)} \otimes_{C_Y^\infty} \mathcal{A}_Y^{(0,1)}$ gives the canonical holomorphic structure on the cotangent vector bundle. This is no longer true in the relative case, since the right hand side is only a subsheaf of $\mathcal{A}_{Y/X}^{(1,0)} \otimes_{C_Y^\infty} \mathcal{A}_Y^{(0,1)}$. Therefore a relative $(n,0)$-form closed with respect to $\bar{\partial}_{rel}$ need not be a relative holomorphic n-form in the sense of Definition A.2.

Definition A.3 (i) The *relative de Rham complex* $(\mathcal{A}_{Y/X}^{\bullet}, d_{rel})$ is the complex of sheaves of $\pi^* C_X^\infty$-modules on Y given by the following:

Its n-th component is the sheaf of relative $(k,l), k+l = n$, form and its differential is d_{rel};

(ii) The *relative holomorphic de Rham complex* $(\Omega_{Y/X}^{\bullet}, \partial_{rel})$ is the complex of sheaves of $\pi^* \mathcal{O}_X$-modules on Y given by the following:

Its n-th component is the sheaf of relative holomorphic n-forms and its differential is ∂_{rel};

(iii) The *relative Dolbeault complex* $(\mathcal{A}_{Y/X}^{0,\bullet}, \bar{\partial}_{rel})$ is the complex of sheaves of $\pi^* C_X^\infty$-modules on Y given by the following:

Its n-the component is the sheaf of relative $(0, n)$-forms and its differential is $\bar{\partial}_{rel}$. □

Claim A.4: Let $f : Z \longrightarrow X$ be an arbitrary holomorphic map from a complex manifold Z to X. Then the pullback under f of any of the complexes introduced in Definition A.3 is the corresponding complex for $Y \times_X Z$ over Z. (We understand pullbacks in the sense usual for sheaves of modules, that is, as a composition of the sheaf-theoretic pullback and

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tensoring with the structure sheaf: $\pi^*C_Z^\infty$ in the de Rham and Dolbeault case, $\pi^*\mathcal{O}_Z$ in the holomorphic de Rham case.)

Proof. Clear. ∎

Lemma A.2: All the complexes of sheaves on Y introduced in Definition A.3 are exact in all positive terms. The first cohomology sheaf is $\pi^*C_X^\infty$ in the de Rham case and $\pi^*\mathcal{O}_X$ in the holomorphic de Rham case.

Proof. For every point $x \in Y$ there exists a neighborhood U such that $U = D_1 \times D_2$ for some polidiscs D_1, D_2 and π restricted on U is the projection onto the first factor. Let $\sigma : U \longrightarrow D_2$ be the projection onto second factor. Then the restrictions to U of all the complexes under consideration are isomorphic by Claim A.4 to pullbacks under σ of absolute complexes for D_2. Since $\mathcal{O}_{D_1}$ and $C_{D_1}^\infty$ are flat over $\sigma^*(\mathbf{C})$, the pullback functors are exact. Then we are done by Poincare lemmas on D_2. ∎

Lemma A.3: (i) The complex $(\pi_*\mathcal{A}_{Y/X}^\bullet, d_{rel})$ of sheaves of C_X^∞-modules is quasiisomorphic to the higher direct image $R\pi_*\pi^*C_X^\infty$.

(ii) Assume that both X and Y are Stein manifolds. Then the complex $(\pi_*\Omega_{Y/X}^\bullet, \partial_{rel})$ of sheaves of $\mathcal{O}_X$-modules is quasiisomorphic to the higher direct image $R\pi_*\pi^*\mathcal{O}_X$.

Proof. The components of the complexes in question are acyclic for π_* in each case. ∎

To use this lemma we need the following statement of the "base change" type. Cosnider the sheaf $\mathcal{O}_{Y/X} \cong \pi^*C_X^\infty \otimes_{\pi^*\mathcal{O}_X} \mathcal{O}_Y$ of *fiberwise holomorphic* functions.

Lemma A.4: (i) Assume that $Y \cong X \times Z$ for some manifold Z and that the map π is the projection. Then the sheaves $R^\bullet\pi_*\pi^*C_X^\infty$ are isomorphic to $C_X^\infty \otimes H^\bullet(Z, \mathbf{C})$.

(ii) Assume that both X and Y are Stein manifolds. Then for any quasicoherent sheaf $\mathcal{E}$ of $\mathcal{O}_Y$-modules the canonical map $f : \pi^*\mathcal{E} \otimes_{\mathcal{O}_X} C_X^\infty \longrightarrow \pi_*(\mathcal{O}_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{E})$ is an isomorphism.

Proof.(i) Clear.

(ii) The map f is a map between additive functors from quasicoherent sheaves on Y to quasicoherent aheves on X. Since Y and X are Stein, the functor π_* is exact on quasicoherents sheves. Since C_X^∞ is flat over $\mathcal{O}_X$ and $\mathcal{O}_{Y/X}$ is flat over $\mathcal{O}_Y$, both functors are not only additive but exact. Therefore it is enough to prove the claim for $\mathcal{E} \cong \mathcal{O}_Y$.

For any quasicoherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules there exists a canonical map $g : \mathcal{F} \otimes_{\mathcal{O}_X} \pi_* \mathcal{O}_Y \longrightarrow \pi_*(\pi^* \mathcal{F} \otimes_{\pi^* \mathcal{O}_X} \mathcal{O}_Y)$. Since the differential of π is everywhere surjective, the sheaf $\pi_* \mathcal{O}_Y$ is flat over $\mathcal{O}_Y$ and the sheaf $\mathcal{O}_Y$ is flat over $\pi^* \mathcal{O}_X$. Therefore g is a map between exact functors from the category of quasicoherent sheaves on X to itself. It is identical when $\mathcal{F} \cong \mathcal{O}_X$, hence it is an isomorphism of functors. Since C_X^∞ is a quasicoherent sheaf, the map g is an isomorphism for $\mathcal{F} \cong C_X^\infty$. But then it coincides with the map f, and we are done. $\blacksquare$

Now we can prove the following.

Proposition A.1: Assume that X and Y are Stein manifolds. Than a closed relative holomorphic k-form θ is exact if and only if it is exact as a relative C^∞ k-form.

Proof. By Lemma A.3 (ii) the form θ is exact as a holomorphic form if and only if the corresponding section $\tilde{\theta}$ of the higher direct image sheaf $R^k \pi_* \pi^* \mathcal{O}_X$ is zero.

The complex $(\Omega^\bullet_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y/X}, \partial_{rel})$ is a resolution of $\pi^* C_X^\infty$ acyclic for π_*. Thus it can be used to compute the higher direct images $R\pi_* \pi^* C_X^\infty$.

By Lemma A.4 (ii) the sheaves $\pi_*(\Omega^\bullet_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y/X})$ are isomorphic to the sheaves $\pi_* \Omega^\bullet_{Y/X} \otimes_{\mathcal{O}_X} C_X^\infty$. These isomorphisms obviously commute with ∂_{rel}. Therefore the higher direct images $R^\bullet \pi_* \pi^* C_X^\infty$ are isomorphic to the sheaves $R^\bullet \pi_* \pi^* \mathcal{O}_Y \otimes_{\mathcal{O}_X} C_X^\infty$.

This implies that $\tilde{\theta}$ is zero is and only if the section of $R^k \pi_* \pi^* C_X^\infty$ corresponding to θ is zero. By Lemma A.3 (i) this happens if and only if θ is exact as a smooth relative k-form. $\blacksquare$

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