Nonsmooth Newton's Method and Semidefinite Optimization

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***Abstract*— We introduce basic ideas of a nonsmooth Newton's method and its application in solving semidefinite optimization (SDO) problems. In particular, the method can be used to solve both linear and nonlinear semidefinite complementarity problems. We also survey recent theoretical results in matrix functions and stability of SDO that are stemed from the research on the matrix form of the nonsmooth Newton's method.**

Index Terms—semismooth functions, semidefinite optimization, Newton's method, complementarity problems, stability, variational inequality.

I. MOTIVATION

A. Reduction of optimization problems to nonsmooth equations

The Karush-Kuhn-Tucker system of a nonlinear optimization problem is often written as *the complementarity problem*

$$\text{Find } x \geq 0, \text{ such that } F(x) \geq 0 \text{ and } x \bullet F(x) = 0, \qquad (1)$$

where $x \in \Re^n, F : \Re^n \to \Re^n$, F is continuously differentiable (smooth), and $\bullet$ stands for the inner product.

Let $\Re^n_+$ be the nonnegative orthant of $\Re^n$ and let $(\cdot)_+$ be the projection onto $\Re^n$. It is easy to show that the complementarity problem is equivalent to the following nonsmooth equation

$$\text{Find a root of } G(x) = x - (x - F(x))_+ = 0. \qquad (2)$$

Thus, solving (1) is equivalent to solving (2).

Now we generalize the above idea. Given a closed convex set $K \subseteq \Re^n$, a mapping $F : \Re^n \to \Re^n$, *the variational inequality problem*, denoted by VI (K, F), is to find a vector $x \in K$ such that

$$(\, y - x \,)^T F(x) \;\geq\; 0, \quad \forall\, y \;\in\; K.$$

The solution set of this problem is denoted $\mathrm{SOL}(K, F)$. Of fundamental importance to the VI is its normal map:

$$\mathbf{F}^{\mathrm{nor}}_K(z) \;\equiv\; F(\Pi_K(z)) + z - \Pi_K(z), \quad \forall\, z \;\in\; \Re^n, \qquad (3)$$

where Π_K denotes the Euclidean projector onto K. It is well known that if $x \in \mathrm{SOL}(K, F)$, then $z \equiv x - F(x)$ is a zero of $\mathbf{F}^{\mathrm{nor}}_K$; conversely, if z is a zero of $\mathbf{F}^{\mathrm{nor}}_K$, then $x \equiv \Pi_K(z)$ solves the VI (K, F). When K is in addition a cone, the VI (K, F) is equivalent to the CP (K, F):

$$K \;\ni\; x \;\perp\; F(x) \;\in\; K^*,$$

where K^* is the dual cone of C; i.e., $K^* \equiv \{y \in \Re^n : y^T x \geq 0,\ \forall\, x \in K\}$.

We note two points. 1. Problem (1) is a special case of CP (K, F), where $K = \Re^n_+$. 2. The functions in (2) and (3) are compositions of smooth functions and the projection function, hence are nonsmooth but Lipschitz continuous.

B. Semidefinite optimization problems

Let $\mathcal{S}^n$ denote the space of $n \times n$ symmetric matrices; let $\mathcal{S}^n_+$ and $\mathcal{S}^n_{++}$ denote the cone of $n \times n$ symmetric positive semidefinite and positive definite matrices, respectively. We could consider problem VI (K, F) and CP $(\mathcal{S}^n_+, F)$, respectively, where $K \subset \mathcal{S}^n$ and $F : \mathcal{S}^n \to \mathcal{S}^n$. These are, in a sense, the most general SDO problems. To fully understand them, more discussion ought to be made.

We write $A \succeq 0$ to mean that A is a symmetric positive semidefinite matrix. For any two matrices A and B in $\mathcal{S}^n$, we write

$$A \bullet B \;\equiv\; \sum_{i,j=1}^{n} a_{ij} b_{ij} \;=\; \mathrm{tr}(AB)$$

for the *Frobenius inner product* between A and B, where "tr" denotes the trace of a matrix. The *Frobenius norm* on $\mathcal{S}^n$ is the norm induced by the above inner product:

$$\| A \| \;\equiv\; \sqrt{A \bullet A} \;=\; \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}.$$

Under the Frobenius norm, the projection $\Pi_{\mathcal{S}^n_+}(A)$ of a matrix $A \in \mathcal{S}^n$ onto the cone $\mathcal{S}^n_+$ is the unique minimizer of the following convex program in the matrix variable B:

$$\begin{array}{ll} \text{minimize} & \| A - B \| \\ \text{subject to} & B \;\in\; \mathcal{S}^n_+. \end{array}$$

Throughout the following discussion, we let A_+ denote the (Frobenius) projection of $A \in \mathcal{S}^n$ onto $\mathcal{S}^n_+$. This projection satisfies the following complementarity condition:

$$\mathcal{S}^n_+ \;\ni\; A_+ \;\perp\; A_+ - A \;\in\; \mathcal{S}^n_+, \qquad (4)$$

where the $\perp$ notation means "perpendicular under the above matrix inner product"; i.e., $C \perp D \Leftrightarrow C \bullet D = 0$ for any two matrices C and D in $\mathcal{S}^n$. The projection A_+ has an explicit representation. Namely, if

$$A \;=\; P \Lambda P^T, \qquad (5)$$

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where Λ is the diagonal matrix of eigenvalues of A and P is a corresponding orthogonal matrix of orthonormal eigenvectors, then

$$A_+ \;=\; P\Lambda_+P^T,$$

where Λ_+ is the diagonal matrix whose diagonal entries are the nonnegative parts of the respective diagonal entries of Λ. Define three fundamental index sets associated with the matrix A:

$$\alpha \equiv \{\, i \,:\, \lambda_i \,>\, 0\,\}, \quad \beta \equiv \{\, i \,:\, \lambda_i \,=\, 0\,\},$$
$$\gamma \equiv \{\, i \,:\, \lambda_i \,<\, 0\,\};$$

these are the index sets of positive, zero, and negative eigenvalues of A, respectively. Write

$$P \;=\; [\; W_\alpha \quad W_\gamma \quad Z \;]$$

with $W_\alpha \in \Re^{n\times|\alpha|}$, $W_\gamma \in \Re^{n\times|\gamma|}$, and $Z \in \Re^{n\times|\beta|}$. Thus the columns of W_α, W_γ, and Z are the orthonormal eigenvectors corresponding to the positive, negative, and zero eigenvalues of A, respectively.

Any function from a topological vector space X to $\mathcal{S}^n$ is called a matrix function. In particular, a *scalar matrix function* is a function $F : \mathcal{S}^n \to \mathcal{S}^n$ defined through a scalar function and eigenvalues

$$F(X) = P\text{diag}\,(f(\lambda_1(X)), ..., f(\lambda_n(X)))P^T,$$

where $f : \Re \to \Re$ is a scalar function and $P^TXP = \text{diag}\,(\lambda_1, ..., \lambda_n)$. For examples, X_+ is a scalar matrix function with $f(\lambda) = \max(0, \lambda) = (\lambda)_+$ and $|X|$ is defined as a scalar function with $f(\lambda) = |\lambda|$. However, it is obvious that not all matrix functions from $\mathcal{S}^n$ to $\mathcal{S}^n$ are scalar matrix functions.

II. The semismooth Newton's method

Most of the following results can be found in [7].

Newton's Method

$$x^{k+1} = x^k - [F'(x^k)]^{-1}F(x^k) \tag{6}$$

is a classic method for solving the nonlinear equation

$$F(x) = 0, \tag{7}$$

where F is a continuously differentiable function, i.e., a *smooth function*. Many other methods for solving (7) are related to this method.

Suppose now that F is not a smooth function, but a locally Lipschitzian function. Then the formula (6) cannot be used. Let $\partial F(x^k)$ be the *generalized Jacobian* of F at x^k, defined by Clarke [3]. In this case, in stead of (6), one may use

$$x^{k+1} = x^k - V_k^{-1}F(x^k) \tag{8}$$

where $V_k \in \partial F(x^k)$, to solve (7).

What is the generalized Jacobian? Suppose F is a locally Lipschitzian function. According to Rademacher's Theorem, F is Frechét differentiable almost everywhere. Denote the set of points at which F is differentiable by D_F. We write $JF(x)$ for the usual Frechét derivative (the usual Jacobian) at x whenever x is a point at which the Frechét derivatives exist. Then the generalized Jacobian is the set

$$\partial F(x) = \text{co}\{\lim_{\substack{x_i\to x\\ x_i\in D_F}} JF(x_i)\},$$

where "co" is the convex hull [8].

We say that F is *semismooth* at x if F is locally Lipschitzian at x and

$$\lim_{\substack{V\in\partial F(x+th')\\ h'\to h, t\downarrow 0}} \{Vh'\}$$

exists for any h.

Theorem II.1: Suppose that F is a locally Lipschitzian function. The following statements are equivalent.

- F is semismooth at x;
- F is directionally differentiable and for $x + h \in D_F$ one has

$$F(x + h) - F(x) - \nabla F(x + h)h = o(\|h\|)\ \forall h \to 0;$$

- F is directionally differentiable and

$$F'(x + h, h) - F'(x, h) = o(\|h\|)\ \forall h \to 0;$$

- for any $V \in \partial F(x + h), h \to 0$,

$$Vh - F'(x; h) = o(\|h\|).$$

Semismoothness was originally introduced by Mifflin [5] for functionals. Convex functions, smooth functions and subsmooth functions are examples of semismooth functions. Linear combinations of semismooth functions are still semismooth functions.

If any of the $o(\|h\|)$ terms above is replaced by $O(\|h\|^{1+p})$, where $0 < p \leq 1$, then we say that F is *p-order semismooth* at x. Note that p-order semismoothness ($0 < p \leq 1$) implies semismoothness. In particular, 1-order semismoothness is called *strong semismoothness*.

Theorem II.2: (Global and Local Convergence) Suppose that F is locally Lipschitzian and semismooth on $S = \{x : \|x - x^0\| \leq r\}$. Also suppose that for any $V \in \partial F(x), x, y \in S$, V is nonsingular,

$$\|V^{-1}\| \leq \beta,\ \|V(y - x) - F'(x; y - x)\| \leq \gamma\|y - x\|,$$
$$\|F(y) - F(x) - F'(x; y - x)\| \leq \delta\|y - x\|,$$

where $\alpha = \beta(\gamma + \delta) < 1$ and $\beta\|F(x^0)\| \leq r(1 - \alpha)$. Then the iterates (8) remain in S and converge to the unique solution x^* of (7) in S. Moreover, (for p-order semismooth F, resp.) the error estimate

$$\|x^k - x^*\| = o(\|x^k - x^{k-1}\|) \quad \big(= O(\|x^k - x^{k-1}\|)^{1+p} \text{ resp.}\big)$$

holds for large k. Thus, the semismooth Newton method is superlinearly (quadratically) convergent.

In the above theorem, the conclusion remains to be true if the set $\partial F(x)$ is substituted by the set

$$\partial_B F(x) = \left\{ \lim_{\substack{x_i\to x\\ x_i\in D_F}} JF(x_i) \right\}.$$

This allows to extend Newton's method to solving problems like $|x| = 0$.

III. Matrix valued functions

The following results can be found in [2], [9], [6], [10]. Let $A, B \in \mathcal{S}^n$. Define a linear operator $L_A : \mathcal{S}^n \mapsto \mathcal{S}^n$ as

$$L_A(B) = AB + BA.$$

Lemma III.1: Let $F : \mathcal{S}^n \mapsto \mathcal{S}^n$ be a scalar matrix function with respect to f. Then F is (continuously) differentiable (semismooth, strongly semismooth) at X if and only if f is (continuously) differentiable (semismooth, strongly semismooth) at $\lambda_1, ..., \lambda_n$.

Lemma III.2: The matrix function $(X)_+$ is strongly semismooth and is in fact smooth at X if X is nonsingular with

$$\nabla(X)_+ = L^{-1}_{|X|} \circ L_{X_+},$$

where $L^{-1}_{|X|}$ is the inverse operator of $L_{|X|}$ and $\circ$ stands for the composition of operators.

Theorem III.1: We may apply the semismooth Newton's method to the normal equation to solve VI and CP. The method will have a local quadratic convergence rate if all $\partial_B \mathbf{F}^{\rm nor}_K(z)$ are nonsingular at z^*.

Associated with the projection problem (4) is the critical cone of $\mathcal{S}^n_+$ at $A \in \mathcal{S}^n$ defined as:

$$\mathcal{C}(A; \mathcal{S}^n_+) \equiv \mathcal{T}(A_+; \mathcal{S}^n_+) \cap (A_+ - A)^\perp,$$

where $\mathcal{T}(A_+; \mathcal{S}^n_+)$ is the tangent cone of $\mathcal{S}^n_+$ at A_+ and $(A_+ - A)^\perp$ is the subset of matrices in $\mathcal{S}^n$ that are orthogonal to $(A_+ - A)$ under the matrix inner product. The importance of the critical cone in the local analysis of constrained optimization is well known. In the present context, this cone can be completely described [1]:

$$\mathcal{C}(A; \mathcal{S}^n_+) = \{ C \in \mathcal{S}^n : W_\gamma^T C W_\gamma = 0, W_\gamma^T C Z = 0, Z^T C Z \succeq 0 \}.$$

The affine hull [8] of $\mathcal{C}(A; \mathcal{S}^n_+)$, which we denote $\mathcal{L}(A; \mathcal{S}^n_+)$, is easily seen to be the linear subspace:

$$\{ C \in \mathcal{S}^n : W_\gamma^T C W_\gamma = 0, W_\gamma^T C Z = 0 \}.$$

Theorem III.3: Consider CP $(\mathcal{S}^n_+, F)$. Suppose that for $X \in S(\mathcal{S}^n_+, F)$ the Jacobian $JF(X)$ of F at X is positive define on the linear subspace $\mathcal{L}(X - F(X); S_+)$. Then the conditions of Theorem II.2 are satisfied.

Theorems III.1 and III.3 provide a basis for a quadratically convergent Newton's method for VI and CP$(\mathcal{S}^n_+, F)$, particularly for nonlinear cases.

IV. Stability of semidefinite complementarity problems

Most of the following results can be found in [6].

How does the solution of a semidefinite variational inequality problem change if the given data has a small perturbation? By reducing the semidefinite complementarity problem to a nonsmooth equation, we can study the stability of the semidefinite complementarity problem.

Definition IV.1: A solution x^* of the VI (K, F) is said to be *strongly stable* if for every open neighborhood $\mathcal{N}$ of x^* satisfying

$$\mathrm{SOL}(K, F) \cap \mathrm{cl}\mathcal{N} = \{ x^* \}, \tag{9}$$

there exist two positive scalars c and ε such that for every continuous function G satisfying

$$\sup_{x \in K \cap \mathrm{cl}\mathcal{N}} \| G(x) - F(x) \| \le \varepsilon,$$

the set $\mathrm{SOL}(K, G) \cap \mathcal{N}$ is a singleton; moreover, for another continuous function $\tilde{G}$ satisfying the same condition as G, it holds that

$$\| x - x' \| \le c \| [F(x) - G(x)] - [F(x') - \tilde{G}(x')] \|,$$

where x and x' are the unique elements in the sets $\mathrm{SOL}(K, G) \cap \mathcal{N}$ and $\mathrm{SOL}(K, \tilde{G}) \cap \mathcal{N}$, respectively.

In essence, strong stability pertains to unique, continuous solution under small, continuous perturbations of F. Let us consider another concept.

Definition IV.2: A function $H : \Re^n \to \Re^n$ is said to be a *locally Lipschitz homeomorphism* near a vector x if there exists an open neighborhood $\mathcal{N}$ of x such that the restricted map $H|_{\mathcal{N}} : \mathcal{N} \to H(\mathcal{N})$ is Lipchitz continuous and bijective, and its inverse is also Lipschitz continuous.

We can now state the following result. The significance of this result is that the strong stability of a solution to a VI can be deduced from an inverse function theorem for the normal map.

Theorem IV.3: Let $F : K \subseteq \Re^n \to \Re^n$ be locally Lipschitz continuous on the closed convex set K. Let $x^* \in \mathrm{SOL}(K, F)$ be given. Let $z^* \equiv x^* - F(x^*)$. The following statements are equivalent:

(a) x^* is a strongly stable solution of the VI (K, F);
(b) z^* is a strongly stable zero of $\mathbf{F}^{\rm nor}_K$;
(c) $\mathbf{F}^{\rm nor}_K$ is a locally Lipschitz homeomorphism near z^*
(d) There exist an open neighborhood $\mathcal{Z}$ of z^* and a constant $c > 0$ such that

$$\| \mathbf{F}^{\rm nor}_K(z) - \mathbf{F}^{\rm nor}_K(z') \| \ge c \| z - z' \|, \quad \forall\, z, z' \in \mathcal{Z}.$$

The equivalence of statements in the above theorem remains valid for all locally Lipschitz continuous functions, of which the normal map $\mathbf{F}^{\rm nor}_K$ is a special instance.

Theorem IV.4: Let $\Phi : \Re^n \to \Re^n$ be Lipschitz continuous in an open neighborhood $\mathcal{D}$ of a vector $x^* \in \Phi^{-1}(0)$. Consider the following three statements:

(a) every matrix in $\partial\Phi(x^*)$ is nonsingular;
(b) Φ is a locally Lipschitz homeomorphism near x^*;
(c) for every $V \in \partial_B\Phi(x^*)$, $\operatorname{sgn}\det V = \pm 1$.

It holds that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Assume in addition that Φ is directionally differentiable at x^*. Consider the following two additional statements:

(d) $\Psi \equiv \Phi'(x^*;\cdot)$ is a globally Lipschitz homeomorphism;
(e) for every $V \in \partial_B\Psi(0)$, $\operatorname{sgn}\det V = \pm 1$.

It holds that (b) $\Rightarrow$ (d) $\Rightarrow$ (e). Moreover, if (b) holds and Φ is directionally differentiable at x^*, then the local inverse of Φ near x^*, denoted Φ^{-1}, is directionally differentiable at the origin; and

$$(\Phi^{-1})'(0;h) \;=\; \Psi^{-1}(h), \quad \forall\, h \,\in\, \Re^n. \tag{10}$$

If Φ is semismooth on $\mathcal{D}$ then (b) $\Leftrightarrow$ (c); in this case, the local inverse of Φ near x^* is semismooth near the origin. Finally, if Φ is semismooth on $\mathcal{D}$ and

$$\partial_B\Phi(x^*) \;\subseteq\; \partial_B\Psi(0), \tag{11}$$

then the four statements (b), (c), (d), and (e) are equivalent.

The inclusion (11) plays an essential role for the statements (b) and (c) in Theorem IV.4, which pertain to the original function Φ, to be equivalent to the corresponding statements (d) and (e), which pertain to the directional derivative Ψ.

The next technical result establishes the equality (11) that paves the way for the application of Proposition IV.4 to SDO problems.

Lemma IV.1: Let $A \in \mathcal{S}^n$ be arbitrary. Let $\Psi \equiv \Pi'_{\mathcal{S}^n_+}(A;\cdot)$. It holds that

$$\partial_B\Pi_{\mathcal{S}^n_+}(A) \;=\; \partial_B\Psi(0). \tag{12}$$

The following extends the semismooth inverse function theorem to an semismooth implicit function theorem.

Theorem IV.5: Assume that $\partial_B\Phi(x^*) \subseteq \partial_B\Phi'(x^*;\cdot)(0)$ and that $J_wG(\Phi(x^*),p^*)\circ\Phi'(x^*;\cdot)$ is a globally Lipschitz homeomorphism. There exist a neighborhood $\mathcal{U}$ of p^*, a neighborhood $\mathcal{V}$ of x^*, and a Lipschitz continuous function $x : \mathcal{U} \to \mathcal{V}$ that is semismooth at p^* such that for every $p \in \mathcal{U}$, $x(p)$ is the unique vector in $\mathcal{V}$ satisfying $G(\Phi(x(p)),p) = 0$. Moreover, for every vector $dp \in \Re^m$, $x'(p^*,dp)$ is the unique solution dx of the following equation:

$$J_wG(\Phi(x^*),p^*)\Phi'(x^*;dx) + J_pG(\Phi(x^*),p^*)dp \;=\; 0. \tag{13}$$

Applying this theorem to the normal map of CP $(\mathcal{S}^n_+, F)$, the condition of $J_wG(\Phi(x^*),p^*)\circ\Phi'(x^*;\cdot)$ being a globally Lipschitz homeomorphism can be interpreted as the uniqueness of the solution to the following linear complementarity problem (details omitted)

$$\mathcal{C} \;\ni\; S^* \;\perp\; -Q + (\,JF(X^*) + \mathcal{A}_{Z^*}\,)(S^*) \;\in\; \mathcal{C}^*, \tag{14}$$

where $\mathcal{A}_{Z^*}$ is a certain computable operator at Z^*. We may conclude that

Theorem IV.6: Let $F : \mathcal{S}^n \to \mathcal{S}^n$ be continuously differentiable in a neighborhood of a solution X^* of the problem CP$(\mathcal{S}^n_+, F)$. The following three statements are equivalent.

(a) X^* is strongly stable for CP$(\mathcal{S}^n_+, F)$;
(b) for every $Q \in \mathcal{S}^n$, the problem (14) has a unique solution that is Lipschitz continuous in Q;
(c) for every $V \in \partial_B\Pi_{\mathcal{S}^n_+}(Z^*)$, $\operatorname{sgn}\det((JF(X^*) + \mathcal{A}_{Z^*})\circ V + I - V) = \pm 1$.

We may apply Theorem IV.5 to a parametric CP in SDO:

$$\mathcal{S}^n_+ \;\ni\; X \;\perp\; F(X,p) \;\in\; \mathcal{S}^n_+, \tag{15}$$

where $F : \mathcal{S}^n \times \Re^m \to \mathcal{S}^n$ is a given mapping. In what follows, we show how to calculate the directional derivative of an implicit solution function of the above problem at a base parameter vector $p^* \in \Re^m$. For this purpose, let X^* be a strongly stable solution of the above problem at p^*. Assume that F is continuously differentiable in a neighborhood of the pair (X^*,p^*). It follows that there exist open neighborhoods $\mathcal{V} \subseteq \mathcal{S}^n_+$ of X^* and $\mathcal{P} \subseteq \Re^m$ of p^* and a locally Lipschitz continuous function $X : \mathcal{P} \to \mathcal{V}$ such that for every $p \in \mathcal{P}$, $X(p)$ is the unique matrix in $\mathcal{V}$ that solves (15); moreover, the implicit solution function X is semismooth at p^*. We wish to compute $X'(p^*;dp)$ for $dp \in \Re^m$. For each $p \in \mathcal{P}$, let $Z(p) \equiv X(p) - F(X(p),p)$. We have $X(p) = \Pi_{\mathcal{S}^n_+}(Z(p))$ and

$$F(\Pi_{\mathcal{S}^n_+}(Z(p)),p) + Z(p) - -\Pi_{\mathcal{S}^n_+}(Z(p)) \;=\; 0.$$

Taking the directional derivative of the above normal equation at p^* along the direction dp and writing $dZ \equiv Z'(p^*;dp)$, we obtain

$$\begin{aligned} J_xF(X^*,p^*)\Pi'_{\mathcal{S}^n_+}(Z^*;dZ) + J_pF(X^*,p^*)dp \\ +dZ - \Pi'_{\mathcal{S}^n_+}(Z^*;dZ) \;=\; 0. \end{aligned}$$

Note that $X'(p^*;dp) = \Pi'_{\mathcal{S}^n_+}(Z^*;dZ)$. By the previous derivation, we deduce that $X'(p^*;dp)$ is the unique solution S^* of the linear complementarity problem:

$$\mathcal{C} \;\ni\; S^* \;\perp\; J_pF(X^*,p^*)dp + (\,J_xF(X^*,p^*) + \mathcal{A}_{Z^*}\,)(S^*) \;\in\; \mathcal{C}^*,$$

where $\mathcal{C} \equiv \mathcal{T}(X^*;\mathcal{S}^n_+) \cap F(X^*,p^*)^\perp$ is the critical cone of the CP $(\mathcal{S}^n_+, F(\cdot,p^*))$ at the solution X^*.

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