Lecture 11: Isolated Singularities

(Text 126-130)

Remarks on Lecture 11

Singularities: Let \( f(z) \) be holomorphic in a disk \( 0 < |z - a| < \delta \) with the center \( a \) removed.

(i) If

\[
\lim_{z \to a} f(z)
\]

exist or if just

\[
\lim_{z \to a} f(z)(z - a) = 0,
\]

then \( a \) is a removable singularity and \( f \) extends to a holomorphic function on the whole disk \( |z - a| < \delta \).

(ii) If

\[
\lim_{z \to a} f(z) = \infty,
\]

\( a \) is said to be a pole. In this case

\[
f(z) = (z - a)^{-h}f_h(z),
\]

where \( h \) is a positive integer and \( f_h(z) \) is holomorphic at \( a \) and \( f_h(a) \neq 0 \). We also have the polar development

\[
f(z) = B_h(z - a)^{-h} + \cdots + B_1(z - a)^{-1} + \varphi(z),
\]

where \( \varphi(z) \) is holomorphic at \( a \).

If neither (i) nor (ii) holds, \( a \) is said to be an essential singularity.
**Theorem 9** A holomorphic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.

**Simplified Proof:** Suppose statement false. Then \( \exists A \in \mathbb{C} \) and \( \delta > 0 \) and \( \epsilon > 0 \) such that

\[
|f(z) - A| < \delta \quad \text{for } |z - a| < \epsilon.
\]

Then

\[
\lim_{z \to a} (z - a)^{-1}(f(z) - A) = \infty.
\]

So

\[
(z - a)^{-1}(f(z) - A)
\]

has a pole at \( z = a \). Thus

\[
f(z) - A = (z - a)(z - a)^{-h} g(z),
\]

where \( h \in \mathbb{Z}^+ \) and \( g(z) \) is holomorphic at \( z = a \).

If \( h = 1 \), \( f(z) \) has a removable singularity at \( z = a \). If \( h > 1 \), \( f(z) - A \) has a pole at \( z = a \) and so does \( f(z) \). Both possibilities are excluded by assumption, so the proof is complete.

Q.E.D.

**Exercise 4 on p.130.**

Suppose \( f \) is meromorphic in \( \mathbb{C} \cup \{\infty\} \). We shall prove \( f \) is a rational function. If \( \infty \) is a pole, we work with \( g = 1/f \), so we may assume \( \infty \) is not a pole. It is not an essential singularity, so \( \infty \) is a removable singularity. Thus for some \( R > 0 \), \( f(z) \) is bounded for \( |z| \geq R \). Since the poles of \( f(z) \) are isolated, there are just finitely many poles in the disk \( |z| < R \). (Poles of \( f(z) \) are zeroes of \( 1/f(z) \).) At a pole \( a \), use the polar development near \( a \)

\[
f(z) = B_h(z - a)^{-h} + \cdots + B_1(z - a)^{-1} + \varphi(z).
\]

The equation shows that \( \varphi \) extends to a meromorphic function on \( \mathbb{C} \cup \infty \) with one less pole than \( f(z) \). We can then do this argument with \( \varphi(z) \) and after iteration we obtain

\[
f(z) = \sum_{i=1}^{n} P_i \left( \frac{1}{z - a_i} \right) + g(z),
\]

where \( P_i \) are polynomials and \( g \) is holomorphic in \( \mathbb{C} \). The formula shows that \( g \) is bounded for \( |z| > geR \) and being analytic on \( |z| \leq R \), it thus must be bounded on \( \mathbb{C} \). By Liouville’s theorem, it is constant. So \( f \) is a rational function.