Stochastic Processes

\( T \) - a set, \((\Omega, \mathcal{F}, \mathbb{P})\) - probability space

A stochastic process \( X_t(\omega) = X(t, \omega): T \times \Omega \to \mathbb{R} \)

such that for each \( t \in T \), \( X_t: \Omega \to \mathbb{R} \) is a random variable

Kolmogorov’s theorem: \( F \subset T \) - finite, \( \alpha(\{X_t\}_{t \in F}) = \mathbb{P}_F \)

If \( F_1 \subset F_2, \mathbb{P}_{F_1} = \mathbb{P}_{F_2}|_{\mathbb{R}^{F_1}} \) where \( |_{\mathbb{R}^{F_1}} \) means restricted to the coordinate of \( F_1 \)

\( \mathbb{R}^T \), for a fixed \( \omega \in \Omega, X_t(\omega) \in \mathbb{R}^T \), a function \( T \to \mathbb{R} \)

\( B_T \) - cylindrical \( \sigma \)-algebra generated by cylindrical sets:

\( F \subset T \) - finite, \( B \) - Borel set in \( \mathbb{R}^F, B \times \mathbb{R}^T|_F \)

Define a map \( X: \Omega \to \mathbb{R}^T, X(\omega) + X_t(\omega) \in \mathbb{R}^T \)

\( \mathbb{P} \circ X^{-1} \) - the law of \( X_t(\omega) \) on \( \mathbb{R}^T \)

\[ \left\{ \sup_{t \in T} X_t > 1 \right\} \]

may not be a measurable set

\[ \bigcup_{t \in T} \{X_t > 1\}; T = [0, 1], \mathbb{E} \int X_t dt \]

**Definition 1.** \((T, d)\) - metric space

\( X_t \) - sample continuous if \( X_t(\omega) \in C(T) \) - continuous function on \( T \) for all \( \omega \in \Omega \)

\( X_t \) is continuous in probability of \( X_t \xrightarrow{\text{in prob}} X_{t_0} \) if \( t \to t_0 \).

**Example.**

\((\Omega, \mathbb{P}) = ([0, 1], \lambda - \text{Lebesgue measure}), T = [0, 1]\)

\( X_t(\omega) = I(t = \omega), X_0(\omega) = 0, t \text{ fixed}, X_t = I(\omega = t), \mathbb{P}(X_t = 0) = 1, \mathbb{P}(X_t^j = 0) = 1 \)

\[ \mathbb{P}(\{X_t \text{ is continuous}\}) = 0 \text{ or } 1 \]

**Definition 2.** \((T, d)\) - metric space. \( X_t \) is measurable on \((T, \mathcal{B}) \times (\Omega, \mathcal{F})\) if \( X_t(\omega): T \times \Omega \to \mathbb{R} \)

**Lemma 1.** If \((T, d)\) - separable metric space, \( X_t \) is sample continuous then \( X_t \) - measurable.

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\( (X_j)_{j \geq 1} \) - a partition of \( T \) such that diameter\( (C_j) \leq \frac{1}{n} \)

For \( t \in S_j, X_t^n(\omega) = X_{t_j}(\omega) \) for \( t_j \in S_j \)

\( X_t^n(\omega) \) is obviously measurable on \( T \times \Omega \), convergence due to the fact that \( X_t \) is sample continuous.

\( C(T) \subseteq \mathbb{R}^T; (C(T), || \cdot ||_\infty) \) - metric space

\( B \) - Borel sets in \( C(T) \), i.e. generated by open balls \( \{f \in C(T): ||f - g||_\infty < \epsilon\} \)

**Lemma 2.** \( T = [0, 1]; \mathcal{B} = S_T = \{B \cap C(T): B \in B_T \} \) - cylindrical \( \sigma \)-algebra

**Proof.**

\( S_T \subseteq \mathcal{B}; B \in \mathbb{R}^F \) - Borel set, \( B \times \mathbb{R}^T|_F = \{(x_t)_{t \in F} \in B\} = \{X(x_t) \in B\} \subseteq \mathcal{B} \)

\( X: C(T) \to \mathbb{R}^F, X(x_t) = (x_t)_{t \in F}, X \) - continuous in \( || \cdot ||_\infty \Rightarrow \text{measurable} \)
would work for separable sets

Take a closed ball \( \{ f \in C(T) : \| f - g \|_\infty \leq \epsilon \} = \bigcap_{\text{rational } t} \{ |f(t) - g(t)| \leq \epsilon \in S_T \Rightarrow \mathcal{B} \subseteq S_T \} \)

cylindrical algebra restricted to continuous functions

The law on \((C(T); \| \cdot \|_\infty)\) is entirely determined by finite dimensional distributions. \(\square\)

Brownian motion

\( X_t = 0, X_t, X_s - X_t \) is independent of \( X_t \) and \( X_s - X_t \equiv X_{s-t} \)

\( \sigma^2(t) = \text{var}(X_t), \sigma^2(nt) = n\sigma^2(t), \sigma^2\left( \frac{t}{m} \right) = \frac{1}{m^2}, \sigma^2(qt) = q\sigma^2(t) \)

\( X_t = \sum_{k=1}^{n} \left( \frac{X_{n_k}}{m} - X_{\frac{t(n-1)}{n}} \right) ; X_t \) is Gaussian; If \( \sigma^2(1) = 1, \sigma^2(t) = t \)

\( s < t, \mathbb{E}X_s X_t = \mathbb{E}X_s (X_s + (X_t - X_s)) = s = \min(t, s) \)