Lecture 9: **Convergence of Laws**

\((X, \mathcal{B}), \{\mathbb{P}_n\}_{n \geq 1}, \mathbb{P}\) - probability distribution on \(\mathcal{B}\)

\(X\): topological space of open sets

\((X, d)\) - metric space, \(d\) - metric, \(C_b(X) = \{f : X \to \mathbb{R} \text{ continuous and bounded}\}\)

**Definition 1.** \(\mathbb{P}_n \to \mathbb{P}\) in distribution in law weakly if \(\int f d\mathbb{P}_n \to \int f d\mathbb{P} \; \forall f \in C_b(X)\)

**Theorem 1.** \(X = \mathbb{R}, \mathbb{P}_n \to \mathbb{P} \iff F_n(t) = \mathbb{P}_n((\infty, t]) \to \mathbb{P}((\infty, t]) \; \forall t\)

Point of continuity of \(F(t) = \mathbb{P}((\infty, t])\)

**Proof.**

"\(\Rightarrow\)" Probability as expectation of indicator:

\[
\begin{align*}
\varphi_1(X) \leq I(X \leq t) \leq \varphi_2(X), \quad \varphi_1, \varphi_2 \in C_b(\mathbb{R}), \ X_n \sim \mathbb{P}_n \\
\mathbb{E}\varphi_1(X_n) \leq F_n(t) = \mathbb{P}_n(X \leq t) \leq \mathbb{E}\varphi_2(X_n)
\end{align*}
\]

\[
\mathbb{P}(X \leq t - \epsilon) \leq \mathbb{E}\varphi_1(X) \leq \mathbb{E}\varphi_2(X) \leq \mathbb{P}(X \leq t + \epsilon) \\
\left\|egin{array}{c}
F(t - \epsilon) \\
\downarrow
\end{array}\right\| \leq \limsup \mathbb{P}_n(X \leq t) \leq \limsup \mathbb{P}_n(X \leq t) \leq F(t + \epsilon) \\
\left\|egin{array}{c}
\downarrow
\end{array}\right\|
\]

"\(\Leftarrow\)" \(F_n(t) \to F(t), f \in C_b(\mathbb{R})\)

The set of points of continuity of \(F\) is dense in \(\mathbb{R} = PC(F)\)

\((PC = \text{points of continuity})\)

Take \(M\) - large, \(\mathbb{P}(X \in [-M, M]) \leq \epsilon, \mathbb{P}_n \to \mathbb{P}\)

On a large enough compact, probability of being outside is small.

\(\Rightarrow \mathbb{P}_n(X \in [-M, M]) \leq \delta\)

Take sequence of points such that distance between them is getting small uniformly.

\(X_1 \leq X_2 \leq \cdots \leq X_n \in M\), such that \(\max|X_{i+1} - X_i| \leq \epsilon'\)

\((*)\) Consider an approximating function \(f_n(x) = \sum f(x_i)I(x \in (X_{i-1}, X_i]) + 0 \cdot I(x \notin [-M, M])\)

\(|f_n(x) - f(x)| \leq \delta' \forall x \in [-M, M], f \in C_b(\mathbb{R})\)

\[
P_1: \quad \mathbb{E}f_n(X_i) = \sum f(X_j)(F_i(X_j) - F_i(X_{j-1})) \xrightarrow{\text{prob as difference of cdf}} \sum f(X_j)(F(X_j) - F(X_{j-1})) = \mathbb{E}f_n(X)
\]

\[
|\mathbb{E}f(X_i) - \mathbb{E}f_n(X_i)| \leq \sup_{x \in \mathbb{R}} |f(x)|\mathbb{P}(X_i \notin [-M, M]) + \delta'
\]

\[
|\mathbb{E}f(X) - \mathbb{E}f_n(X_i)| \leq \sup_{x \in \mathbb{R}} |f(x)|\mathbb{P}(X_i \notin [-M, M]) + \delta'
\]

\[
X = \mathbb{R}^k, F(t) = \mathbb{P}(X_1 \leq t_1, \ldots, X_k \leq t_k), t = (t_1, \ldots, t_k)
\]

\[
\int f d\mathbb{P}_n \to \int f d\mathbb{P} \iff F_n(t) \to F(t) \text{ at the points of continuity.}
\]

Harder to define points of continuity. Need to redefined (*) over rectangular spaces.

**Theorem (Selection theorem).** If \(\{\mathbb{P}_n\}_{n \geq 1}\) is uniformly tight = \(\forall \epsilon > 0, \exists K \text{ - compact, } \mathbb{P}_n(K) > 1 - \epsilon \forall n \Rightarrow \exists\) subsequence \(n(K), \mathbb{P}_{n(k)} \to \mathbb{P}\) - probability distribution.
Lemma 2. If \( A \)-countable \( \subseteq X, f_n : X \to \mathbb{R}, m \geq 1, \exists n(K) \) such that \( f_{n(k)}(a) \) converges (if \( f_{n(k)} \) not bounded, maybe to \( \pm \infty \)) for all \( a \in A \).

Diagonalization method
\( A = \{a_1, a_2, \ldots\}, \exists n^1(k), f_{n^1(k)}(a_1) \) converges
\( \exists n^2(k) \subseteq n^1(k) \) such that \( f_{n^2(k)}(a_2) \) converges
\( \exists n^l(k) \subseteq n^{l-1}(k) \) such that \( f_{n^l(k)}(a_l) \) converges
\( n^k(k), f_{n^k(k)}(a_l) \) - converges for any \( l, k \geq l, n^k(k) \in \{n^l(k)\} \)

\( X = \mathbb{R}^k, F_{n(k)}(t) \to F(t) \)?

Let \( A \) be a dense set of points in \( \mathbb{R}^k \)
\( \exists n(i), F_{n(i)}(a) \xrightarrow{i \to \infty} F(a) \forall a \in A \)
x \( \in \mathbb{R}^k, x \notin A \), extend \( F \) to \( x \) by \( F(x) = \inf \{F(a) : a \in A, x_i < a_i\} \) where \( F(x) \) - joint cdf on \( \mathbb{R}^k \)
If \( x \) - point of continuity of \( F(x) \)
\( \exists a, b \in A \), \( a_i < x_i < b_i \forall i \) dense set coordinate
\( F_n(a) \leq F_n(x) \leq F_n(b), x \) - point of continuity, \( F(a) \xrightarrow{a \to x} F(x), F(b) \xrightarrow{b \to x} F(x) \)

\( \int f d\mathbb{P}_n \to \int f d\mathbb{P} \) proved

next time