EXAM #1

WEEK **#9**

Justify your answers. Cross out what is not meant to be part of your final answer.

FIRST PART: Do all of the 10 problems below. Each is worth 5 points. You are expected to PROVE your statements; stating "it is a theorem/exercise in the book" is not enough.

1. Assume that (do not prove here)

$$||x||_\infty \;\leq\; ||x||_2 \;\leq\; \sqrt{n}||x||_\infty, \;\; \forall x \in C^n.$$

   Let A be a square matrix of size n. Show that

$$||A||_\infty \;\leq\; \sqrt{n}||A||_2 \;\leq\; n||A||_\infty.$$

2. Let $A = U\Sigma V^*$ be the SVD decomposition of a matrix A of rank r. What are Range(A) and Null(A) in terms of the spaces generated by the columns of of U and V?

3. Show that every real matrix has a real SVD.

4. Let $A = U\Sigma V^*$ be the SVD decomposition of a nonsingular matrix A. What is the SVD of $A(A^*A)^{-1}$?

5. Show that

$$A^*Ax \;=\; A^*b$$

   is equivalent to

$$Pb \;=\; Ax,$$

   where P is the orthogonal projector onto the range of A.

6. Explain the choice of sign in the Householder reflectors, i.e.

$$u \;=\; x + sign(x_1)||x||e_1,$$

instead of

$$u = x - sign(x_1)||x||e_1.$$

7. In single precision IEEE, what is the smallest integer that cannot be represented exactly?

8. What can you say about the stability of the outer product xy^*, and why?

9. Let A be a hermitian positive definite matrix, and let $\kappa_2(A)$ be its condition number in norm 2. Let C be the Cholesky factor of A. Show that

$$||A|||_2 = ||C||_2^2 \quad \text{and} \quad \kappa_2(A) = \kappa_2(C).$$

10. Write down the permutation matrix that have to multiply (from left and right) a $2n$ square matrix in order to make the k row the $2n-k$ row, for $k = 1 : n$, and the k column the $2n-k$ column, for $k = 1 : n$.

<u>SECOND PART</u>: Do 3 of the following 6 problems. Specify right here which problems you solved. Each problem is worth 15 points (5 extra points are given ex–officio)

1. Let P be a nonzero projector, i.e., $P^2 = P$. Show that $||P||_2 \geq 1$, with equality if and only if P is an orthogonal projector.

2. Let $A \in C^{m\times n}$ be a rectangular matrix, with $m \geq n$. We know (do not prove here) that a linear least least squares problem, i.e., minimizing the residual $b - Ax$ given $b \in C^{m\times 1}$, reduces to solving
$$A^*Ax \;=\; A^*b.$$
This can be done using Cholesky factorization, QR factorization, or the SVD.

   For each of these three alternatives, write a pseudoalgorithm and estimate the operation count; note that, for practical purposes, the factors mn^2 and n^3 should both be taken into account. BRIEFLY comment on when you would use either algorithm.

3. The following algorithm should have been Classical Gram–Schmidt, but there a typo (or more) are lurking in the shadows. Explain what the current algorithm does, and correct it.

   for $j = 1 : n$

   $v_j = a_j;$

     for $i = 1 : (j-1)$

       $r_{ij} = q_i^* v_j;$

       $v_j = v_j - r_{ij} q_i;$

     $r_{jj} = ||v_j||_2;$

     $q_j = v_j / r_j j;$

   end

4. Write a MATLAB algorithm for the Cholesky factorization of an $n \times n$ symmetric (real) positive definite matrix which is banded of band p. Make this algorithm as efficient as possible!

   Estimate the operation count for this algorithm.

5. Assuming that back substitution and Cholesky factorization are backward stable, show that the solution of a linear system involving a symmetric positive definite matrix is backward stable.

6. Show that the growth factor for the Gaussian elimination of an $n \times n$ matrix is bounded from above by 2^{n-1}.