EXAM #2

WEEK #15

Do all of the 7 problems below. Problems 3, 5, and 7 are worth 20 points each. The other problems are worth 10 points each.
You are expected to PROVE your statements; stating "it is a theorem in the book" will not be enough.

1. Let A be a square matrix of size n and P be the permutaiton matrix having 1 on hte positions $(k, n-k+1)$, for $k = 1 : n$. What is
$$PAP'?$$

2. Show that every real matrix has a real SVD.

3. Let P be a nonzero projector, i.e., $P^2 = P$. Show that $||P||_2 \geq 1$, with equality if and only if P is an orthogonal projector.

4. Let $A \in C^{m \times n}$ be a rectangular matrix, with $m \geq n$. We know (do not prove here) that a linear least least squares problem, i.e., minimizing the residual $b - Ax$ given $b \in C^{m \times 1}$, reduces to solving
$$A^*Ax = A^*b.$$
   This can be done using Cholesky factorization, QR factorization, or the SVD.

   For each of these three alternatives, write a pseudoalgorithm and estimate the operation count; note that, for practical purposes, the factors mn^2 and n^3 should both be taken into account. BRIEFLY comment on when you would use either algorithm.

5. Show that the growth factor for the Gaussian elimination of an $n \times n$ matrix is bounded from above by 2^{n-1}.

6. Assume that: Any QR algorithm starts by reducing a matrix to its Hessenberg form. If one starts with a symmetric matrix, the Hessenberg form will be tridiagonal.

   Show that the tridiagonal structure is preserved for the unshifted QR algorithm.

7. Explain the relationships between SOR, Gauss-Siedel, and Jacobi, and give the intuition behind which algorithm are better than the others.

8. Assume that, in the Arnoldi iteration, at step n, the entry $H(n+1, n) = 0$.

   (i) Show that each eigenvalue of $H(1 : n, 1 : n)$ is also an eigenvalue of A;

   (ii) Show that the GMRES algorithm with an n-step Arnoldi iteration will no longer involve solving a least squares problem, but rather solving a linear system;

   (iii) Thinking in terms of operation count for that linear system, what would be an effective way of solving the system?