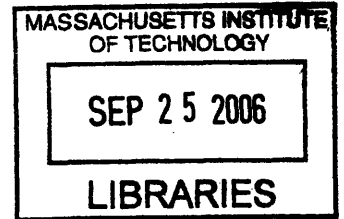# Three Essays in Macroeconomics

by

Emmanuel Farhi

Submitted to the Department of Economics
in partial fulfillment of the requirements for the degree of

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Abstract

Chapter 1 analyzes the theoretical and quantitative implications of optimal fiscal policy in a business cycle model with incomplete markets. I first consider the problem of a government facing expenditure shocks in an economy where the only asset is a real risk-free bond. The model features a representative-agent economy with proportional taxes on labor and capital. Taxes on capital must be set one period in advance, reflecting inertia in tax codes. This rules out replication of the complete markets allocation. In the model, capital taxation and capital ownership provide a state contingent source of revenues - creating a new potential role for capital taxation and ownership as risk sharing instruments between the government and private agents. For a baseline case, I show that the optimal policy features a zero tax on capital. Numerical simulations show that the baseline case provides an excellent benchmark: the average tax on capital, while not theoretically zero, turns out to be small. The volatility of capital taxes decreases sharply as the period length is increased. I then allow the government to hold a non trivial position in capital. Capital ownership allows the government to realize about 90% of the welfare gains from moving to complete markets. Large positions are typically required for optimality. But smaller positions achieve substantial benefits. In a business-cycle simulation, I show that a 15% short equity position achieves over 40% of the welfare gains from completing markets.

Chapter 2 is the product of joint work with Ivan Werning and analyzes how estate taxes should optimally be set.For an economy with altruistic parents facing productivity shocks, the optimal estate taxation is progressive: fortunate parents should face lower net returns on their inheritances. This progressivity reflects optimal mean reversion in consumption, which ensures that a long-run steady state exists with bounded inequality—avoiding immiseration.

Chapter 3 is the product of joint work with Stavros Panageas. We study optimal consumption and portfolio choice in a framework where investors save for early retirement and assume that agents can adjust their labor supply only through an irreversible choice of their retirement time. We obtain closed form solutions and analyze the joint behavior of retirement time, portfolio choice, and consumption. Investing for early retirement tends to increase savings and stock market exposure,

and reduce the marginal propensity to consume out of accumulated personal wealth. Contrary to common intuition, prior to retirement an investor might find it optimal to increase the proportion of financial wealth held in stocks as she ages, even when she receives a constant income stream and the investment opportunity set is also constant. This is particularly true

when the wealth of the investor increases rapidly due to strong stock market performance, as was the case in the late 1990's. We also show that the model can potentially provide a rational explanation for the paradoxical fact that some investors saving for retirement chose to increase their allocation to stocks as the market was booming and reduce it thereafter.

Thesis Supervisor: Ricardo Caballero
Title: Ford International Professor of Economics

Thesis Supervisor: Iván Werning
Title: Assistant Professor of Economics

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I am deeply indebted to Ricardo Caballero for his constant guidance, support and advice. His enthusiastic passion for research, his desire for understanding the world, his intelectual openness and his willingness to chalenge existing theories have been an invaluable example throughout my Ph.D. experience. Collaborating with him has been, and I hope, will continue to be, a fascinating experience. I will never forget that on my first day back to MIT after a one year leave in France, he was the first one to tell me, in his inimitable style: "welcome back, come to my office".

I cannot overstate my debt to Iván Werning. Iván is one of the deepest minds I know in economics. Above all, I think I learnt from him some great princinples about economic research which will stay with me forever: always listen to the model, never settle for half-baked intuitions, always reach for parsimony, and most importantly, always keep pushing. I learnt an immense deal from him, without ever taking a class from him, through discussions and arguments at a coffee place or at the blackboard. I cannot thank him enough for his generosity with his time. This learning experience has evolved into a fruitful collaboration which has been one of the great intelectual experiences of my life, and also, a warm friendship.

I owe a great deal to Marios Angeletos. Marios was the first one to introduce me to Chicago-Minnesotta macroeconomics. His engaging teaching got me started on research. His ability to draw connections, his curiosity, his tenacity, his patience, his intelectual honnesty and his passion for economics were great examples. Many times, he also took the time to seat and read what I had written line by line. I cannot thank him enough for that.

I also want to thank Jean Tirole, who was the first to take my interest for Industrial Organisation seriously. Jean's generosity is famous: He invited me many times to Toulouse.

A mon père,

Chapter 1

Capital Taxation and Ownership When Markets are Incomplete[1]

1.1 Introduction

In this paper, I explore the issue of capital taxation and government ownership when fiscal expenditures and aggregate income are random, and the government can only trade a limited number of assets. Markets are incomplete, and hedging of the government budget is limited in scope. This creates a new potential role for capital taxation and ownership as risk sharing instruments between the government and private agents. In the model, capital taxation and capital ownership provide a state contingent source of revenues. For example, if the marginal product of capital is positively correlated with adverse shocks to the government budget, positive capital taxes, or a long position in the capital stock, might provide the government with a good hedging instrument.

I approach these issues by taking a minimal step away from the complete markets case: I consider the stochastic neoclassical growth model with homogenous consumers and a benevolent government facing fiscal expenditure shocks. I allow the government to trade a risk free bond,

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to levy fully state contingent linear taxes on labor, and to levy linear taxes on capital that are *not* fully state contingent. Specifically, I assume that taxes on capital are set one period in advance, at the time investment decisions are made. This assumption is meant to capture inertia in fiscal policy. With fully state contingent taxes on capital prevents the government would be able to replicate the complete markets allocation, as shown by Chari, Christiano and Kehoe (1994).[2] With a lag in capital taxes, the complete markets allocation is not achievable.

An important insight in this paper is that when contemplating the hedging consequences of a marginal increase in capital taxes, the government needs to take two effects into consideration. First is the direct effect in the form of increased revenues in proportion to the marginal product of capital. Second is an opposing indirect effect through the adjustment of capital: Lower capital accumulation reduces the revenues from labor and capital taxes. The hedging benefits of capital taxation depend only on the covariance of these two effects with the government's need for funds across states of the world on a given date. An important benchmark is the baseline case where the production function is Cobb Douglas and preferences are quasi-linear. I show in this case that taxes on capital should be set to zero from the first period on. With these specifications, the covariances of the direct and indirect effects with the government's need for funds exactly cancel out, leaving no role for capital taxes.

For general preferences, I show that optimal taxes on capital can be decomposed into two terms. The first "hedging" term – the only one present when preferences are quasi-linear – reflects the hedging role of capital taxes, and would be zero if markets were complete. The second "intertemporal" term corresponds to the motive for capital taxation when markets are complete: It might be optimal for the government to distort capital accumulation to smooth available resources in the economy across time. Echoing the result in the previous paragraph, I show that the hedging term is zero when the production function is Cobb Douglas. Regarding the intertemporal term, the optimal policy prescribes a one time capital tax (respectively subsidy) following a high (respectively low) government expenditure shock in order to reduce the variability of the net present value of labor tax surpluses across states.

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[2] An intuition for this result can be given along the following lines: Since investment only reacts to average taxes on capital, the government can vary capital taxes across states while keeping the average constant, leaving investment in physical capital unaffected. This endows the government with enough degrees of freedom to perfectly shift the tax burden across states and to replicate the complete markets outcome as long as long as it can also trade a risk free bond.

Numerical simulations show that the baseline case provides a good benchmark: the hedging term, while not theoretically zero, turns out to be negligible. Incomplete markets do not seem to make a case for using capital taxation as a hedging instrument. The intertemporal term is approximately zero mean. Its volatility decreases sharply with the period length.

In contrast to capital taxes, capital ownership may provide the government with a powerful hedging instrument. The reason is that unlike taxing, trading does not introduce additional distortions: The indirect effect on labor and capital tax bases arising with capital taxation is absent for capital trading. Indeed, when preferences are quasi-linear and the only disturbance in the economy takes the form of government expenditures shocks, I show that the government can perfectly approximate the complete markets allocation by taking a very large position in capital, counterbalanced by an equally large opposite position in the risk free bond.

Outside of this benchmark case, numerical simulations show that capital ownership allows to realize about 90% of the welfare gains from moving to complete markets. Government expenditure shocks tend to call for a long position, while productivity shocks typically require a short position. In business cycle simulations, productivity shocks dwarf government expenditure shocks as a source of variation in the government's budget, calling for a short position.

Optimality typically requires large position. The magnitude of the position decreases sharply with the period length. Moreover, smaller positions allow to reap substantial benefits: In a business cycle simulation with a five years period length, I show that a short position of 15% of the capital stock achieves more than 40% of the gains from completing markets.

I then characterize the optimal holding of capital by the government in a more general portfolio problem with additional assets. I derive the government's optimal liability structure in a unified framework, that I term the GCAPM – Government Capital Asset Pricing Model – and that resembles the CCAPM: Assets that covary with the government's need for funds command a lower expected return. I am able to explicitly derive the pricing kernel of the government. This can be used to price non-traded assets, and provides theoretical foundations for capital budgeting rules in public, non traded companies.[3]

Related Literature.

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[3]This is also potentially useful to compute the value of non-traded government liabilities, as for example dispensing with future nuclear waste, honoring implicit social contracts etc.

An extensive literature on capital taxation with complete markets has emerged from the celebrated zero long run capital tax result established by Chamley (1986) and Judd (1985). This paper adds to this literature by studying the case of incomplete markets.

Chamely (1986) and Judd (1985) showed that in all steady states of the economy, taxes on capital are optimally set to zero. Zhu (1992) proved a stochastic analog of the Chamley-Judd result: In every stochastic steady state of the neoclassical growth model, depending on the underlying parameters of the model, taxes on capital are either zero or take both signs with positive probability. I prove that in my model, an analogous result holds for the intertemporal term. Moreover, as Zhu (1992) and Chari, Christiano and Kehoe (1994) pointed out, if preferences are homogenous and separable between consumption and leisure, a stronger version of the zero capital tax result applies: Taxes on capital should be zero from the second period on. Echoing this result, I show that if preferences are quasi-linear, the intertemporal term is zero. However outside of this case, homogenous preferences do not imply that the intertemporal term should be zero in my model.

The paper also contributes to the literature on fiscal policy in incomplete markets pioneered by Barro (1979). Barro considered a deterministic, partial equilibrium environment and associated an exogenous convex deadweight cost to taxes. Variations in the deadweight costs are costly; a benevolent government should therefore seek to smooth taxes across time. Barro showed that just like consumption smoothing problems, tax smoothing by the government imparts a random walk component to taxes and public debt. Most closely related to this paper is Aiyagari, Marcet, Sargent and Seppälä (2002) and, more recently, Werning (2005). They studied fiscal policy in general equilibrium under incomplete markets. They analyzed a version of the no capital economy in Lucas and Stokey (1983) with only risk free debt. They showed that Ramsey outcomes display features of Barro's model. In particular, labor taxes inherit a near unit root component. I extend this line of research by introducing capital along with a more general asset structure, and studying capital taxation in addition to labor taxation. I demonstrate that the results of Aiyagari, Marcet, Sargent and Seppälä (2002) carry through when capital is introduced: Labor taxes and government debt inherit a random walk component.

Finally, this paper is related to literature studying the optimal liability structure of the government under incomplete markets. The foundational paper is Bohn (1990). Bohn considered

a stochastic version of Barro's model with risk neutral consumers and an ad hoc convex cost for distortionary taxes. The literature on the optimal portfolio of the government under incomplete markets has entirely focused Bohn's model, maintaining the assumption of risk neutrality and adopting an ad hoc deadweight cost for taxes. My model provides microfoundations for Bohn's findings, and shows that important caveats to his theory need to be introduced. I also analyze explicitly the situation where consumers are risk averse. This goes beyond existing contributions.

Angeletos (2002) and Buera and Nicolini (2004) assume that markets are incomplete: the government can only trade risk free debt with multiple maturities. They show that generically, the government can replicate the complete market allocation when enough maturities are traded, and characterize the optimal maturity structure of government debt in such instances.

The central idea in this paper that non-state contingent taxes have important risk sharing implications can be traced back to Stiglitz (1969)[4]. Stiglitz considered the partial equilibrium problem of a consumer that must decide how to allocate a given amount of wealth between a riskless asset and a risky asset. He examined how the government can affect the allocation of wealth and welfare using a variety of linear taxes. My model can be seen as a general equilibrium risk sharing model between the government and the agents. As in Stiglitz (1969), linear taxes on assets – here capital – have hedging consequences. However, I show that taking into account the distortionary consequences of linear taxes by analyzing the savings margin – taken as inelastic in Stiglitz's analysis – is crucial.

On the methodological side, this paper builds on Werning (2005). Existing models either adopt a Lagrangian approach – as for example Aiyagari, Marcet, Sargent and Seppälä (2002) – or develop a recursive representation by incorporating some multipliers in the state space – an approach initiated by Marcet and Marimon (1998). Werning (2005) revisits Aiyagari, Marcet, Sargent and Seppälä's model and develops a recursive representation with three state variables: debt, expected marginal utility one period ahead, and the state of the Markov shock process. I develop a parsimonious recursive representation of the Ramsey problem with only four state variables, which are directly related to the allocation: the present value of government liabilities, capital, past marginal utility and the state of the Markov shock process. In the

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[4] I thank Abhijit Banerjee for pointing this analogy to me.

particular case where preferences are quasi-linear, I am able to reduce the state space to two state variables: the present value of government liabilities and the state of the Markov shock process. Developing a recursive approach using only variables directly linked to the allocation allows a better understanding of the properties of Ramsey outcomes, is useful for developing intuition, simplifies calculations, and in some cases, permits easier numerical simulations.

The rest of the paper is organized as follows. Section 2 introduces the economic environment, sets up the Ramsey problem, and develops a recursive representation. Section 3 presents the properties of debt and taxes in the quasi-linear preferences case. I analyze the general case in Section 4. In Section 5, I study capital ownership by the government and characterize the optimal liability structure of the government. All proofs omitted in the main text are contained in the Appendix.

1.2   The Economy

The model is a neoclassical, stochastic production economy. The economy is populated by a continuum of identical, infinite-lived individuals and a government.

Time is discrete, indexed by $t \in \{0, 1, ...\}$. The exogenous stochastic disturbances in period t are summarized by a discrete random variable $s_t \in \mathbb{S} \equiv \{1, 2, ..., S\}$: the state at date t. I let $s^t \equiv \{s_0, s_1, ..., s_t\} \in \mathbb{S}^t$ denote the history of events at date t. I assume that s_t follows a Markov process with transition density $P(s'|s)$ and initial distribution $\pi_0 = P(.|s_{-1})$.

In each period t, the economy has two goods: a consumption-capital good and labor. Households have access to an identical constant returns to scale technology to transform capital k_{t-1} and labor l_t into output via the production function $k_{t-1} + F(k_{t-1}, l_t, s_t)$[5]. The production function is smooth in (k_{t-1}, l_t) and satisfies the standard Inada conditions. Notice that this formulation incorporates a stochastic productivity shock. The output can be used for private consumption c_t, government consumption g_t, and new capital k_t. Throughout, I will take government consumption $g_t = g(s_t)$ to be exogenously specified. Therefore, the resource constraints in the economy are

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[5] This formulation allows for capital depreciation. The depreciation of capital is subsumed in the production function $F(k_{t-1}, l_t, s_t)$.

$$c_t + g_t + k_t \leq F(k_{t-1}, l_t, s_t) + k_{t-1}, \qquad \forall t \geq 0 \text{ and } s^t \in \mathbb{S}^t \tag{1.1}$$

Households rank consumption streams according to

$$\mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t, s_t) \tag{1.2}$$

where $\beta \in (0,1)$ and u is smooth and concave in (c_t, l_t), increasing in consumption, decreasing in labor, and satisfies the standard Inada conditions. Note that this formulation incorporates a stochastic preference shock.

The government raises all revenues through a tax on labor income τ_t^l and a tax on capital income τ_t^k. Except for taxes on capital τ_t^k, households and the government make decisions whose time t components are functions of the history of shocks s^t up up to t. By contrast, I assume that taxes on capital are predetermined: the government makes decisions on τ_t^k one period in advance. Hence τ_t^k is a function of the history of shocks up to $t-1$, s^{t-1}. The capital stock k_0 is inelastic, hence providing a non distortionary source of revenues to the government. In order to limit the amount of revenues the government can extract at no cost, I assume that the date 0 tax rate on capital τ_0^k is exogenously fixed.

The assumption that taxes on capital are set one period in advance deserves some discussion. It is more efficient to postpone this discussion below after some definitions have been introduced.

Incomplete markets and debt limits. Households and the government borrow and lend only in the form of risk-free one period bonds paying interest $r_t(s^{t-1})$ in every state at date t. The government budget and debt limit constraints are[6]:

$$(1 + r_t)b_{t-1}^g \leq \tau_t^l l_t F_l(k_{t-1}, l_t, s_t) + \tau_t^k k_{t-1} F_k(k_{t-1}, l_t, s_t) + b_t^g, \qquad \forall t \geq 0 \text{ and } s^t \in \mathbb{S}^t \tag{1.3}$$

$$\underline{M}(k_t, u_c(c_t, l_t, s_t), s_t) \leq u_c(c_t, l_t, s_t)b_t^g \leq \overline{M}(k_t, u_c(c_t, l_t, s_t), s_t), \qquad t \geq 0 \text{ and } s^t \in \mathbb{S}^t \tag{1.4}$$

Here b_t^g is the amount of government debt outstanding at date t. When (1.3) holds with

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[6]Note that in a specification with capital depreciation, this formulation supposes that capital depreciation is deductible.

strict inequality, I let the difference between the right hand side and the left hand side be a nonnegative level of lump sum transfers T_t to the households. The upper and lower debt limits $\underline{M}(k_t, u_c(c_t, l_t, s_t), s_t)$, $\overline{M}(k_t, u_c(c_t, l_t, s_t), s_t)$ in (1.4) influence the optimal government plan. In full generality, I allow the debt limits to depend on the capital stock of the economy and the current marginal utility of consumption. I discuss alternative possible settings for $\underline{M}(k_t, u_c(c_t, l_t, s_t), s_t)$, $\overline{M}(k_t, u_c(c_t, l_t, s_t), s_t)$ below. Note that I define debt and asset limits on $u_c(c_t, l_t, s_t) b_t^g$ instead of b_t^g. This is natural given my definition of debt: b_t^g is the amount of debt issued at the end of period t. The quantity $u_c(c_t, l_t, s_t) b_t^g$ is therefore just debt weighed by the state price density.

The representative household operates a firm and supplies and hires labor at wage w_t on a competitive market. The household's problem is to choose stochastic processes $\{c_t, l_t, l_t^d, k_t\}$ to maximize (1.2) subject to the sequence of budget constraints

$$c_t + b_t^g \le \left(1 - \tau_t^l\right) w_t l_t + \left(1 - \tau_t^k\right) \left(F(k_{t-1}, l_t^d, s_t) - w_t l_t^d\right) + k_{t-1} - k_t + (1 + r_t) b_{t-1}^g + T_t \quad (1.5)$$

taking wages, interest rates and taxes $\{w_t, r_t, \tau_t^l, \tau_t^k\}_{t \ge 0}$ as given. Here b_t^g represents the household's holding of government debt, l_t^d the household's labor demand and l_t the household's labor supply. The stochastic processes $\{c_t, l_t, l_t^d, k_t\}_{t \ge 0}$ must be measurable with respect to s^t. The labor market clears if $l_t = l_t^d$.

The household also faces debt limits analogous to (1.4), which I assume are less stringent than those faced by the government. Therefore, in equilibrium, the household's problem always has an interior solution. The household's first order conditions that a stochastic $\{c_t, l_t, l_t^d, k_t\}_{t \ge 0}$ require that two Euler equations hold, one for the risk free rate and the other for the net return on capital, in addition to a labor-leisure arbitrage condition and a the condition that labor is paid its marginal product.

$$1 = (1 + r_t) \mathbb{E}_t \left\{ \beta \frac{u_{c,t+1}}{u_{c,t}} \right\}, \qquad \forall t \ge 0 \text{ and } s^t \in \mathbb{S}^t \quad (1.6)$$

$$1 = \mathbb{E}_t \left\{ \beta \frac{u_{c,t+1}}{u_{c,t}} \left[1 + \left(1 - \tau_t^k\right) F_{k,t+1} \right] \right\}, \qquad \forall t \ge 0 \text{ and } s^t \in \mathbb{S}^t \quad (1.7)$$

$$\tau_t^l = 1 + \frac{u_{l,t}}{u_{c,t} w_t}, \qquad \forall t \ge 0 \text{ and } s^t \in \mathbb{S}^t \quad (1.8)$$

$$w_t = F(k_{t-1}, l_t^d, s_t), \qquad \forall t \ge 0 \text{ and } s^t \in \mathbb{S}^t \quad (1.9)$$

Definition 1 *Given b^g_{-1}, r_0, k_0, τ^k_0 and a stochastic process $\{s_t\}_{t \geq 0}$, a **feasible allocation** is a stochastic process $\{c_t, l_t, k_t\}_{t \geq 0}$ satisfying (1.1) whose time t elements are measurable with respect to s^t. A **risk free rate process** $\{r_t\}_{t \geq 0}$, a **wage process** $\{r_t\}_{t \geq 0}$ and a **government policy** $\{\tau^l_t, \tau^k_t, b^g_t\}_{t \geq 0}$ is a set of stochastic processes such that w_t, τ^l_t and b^g_t are measurable with respect to s^t, τ^k_t and r_t are measurable with respect to s^{t-1}.*

Definition 2 *Given b^g_{-1}, r_0, k_0, τ^k_0 and a stochastic process $\{s_t\}$, a **competitive equilibrium** is a feasible allocation, a risk free rate process, a wage process and a government policy that solve the household's optimization problem, clear the labor market and satisfy the government budget constraints (1.3) and (1.4)*

Definition 3 *The **Ramsey problem** is to maximize (1.2) over competitive equilibria. A **Ramsey outcome** is a competitive equilibrium that attains the maximum.*

Discussion of debt limits. By analogy with Aiyagari, Marcet, Sargent and Seppälä (2002), henceforth AMSS, I shall study two kinds of debt limits, called *natural* and *ad hoc*. Natural debt limits amount to imposing that debt be less than the maximum debt that could be repaid almost surely under an optimal tax policy. Following AMSS, I call a debt or asset limit ad hoc if it is more stringent than the natural one. In this model, natural debt limits, which depend on the capital stock k_t in the economy, are in general difficult to compute. But as mentioned above, it is easy to see that they are of the form $\underline{M}^n(k_t, u_c(c_t, l_t, s_t), s_t) \leq u_c(c_t, l_t, s_t) b^g_t \leq \overline{M}^n(k_t, u_c(c_t, l_t, s_t), s_t)$. Imposing that debt limits are weakly tighter than the natural ones rules out Ponzi schemes.

Discussion of the measurability assumption for τ^k_t. It is well known since the work of Chari, Christiano and Kehoe (1994) that with complete markets, taxes on capital are indeterminate: the government faces an embarrassment of riches, with too many instruments to implement the Ramsey outcome. In fact, the complete markets optimum can be implemented with complete markets and taxes on capital set one period in advance, or under incomplete markets with only a risk free bond but fully adjustable taxes on capital. The reason for this is that investment depends only on the average tax rate on capital, and not on the particular way it is spread between states. When only a risk free bond is traded, the government can

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take advantage of this and adjust taxes on capital to hedge its burden across states – thereby replicating the complete markets outcome.

Inertia in fiscal policy – captured here by the assumption that taxes have to be set one period in advance – restricts the state-contingency of capital taxes, prevents replication of the complete markets allocation, and requires analyzing optimal taxes on capital in a truly incomplete markets environment.

Because they are not the focus of this paper, taxes on labor, on the other hand, are left fully adjustable. One may wonder whether this asymmetric treatment of labor and capital taxes doesn't bias the results in favor of labor taxation. This intuition is wrong. In fact, most of the insights are still valid if additional restrictions were put on labor taxes. Of course, the exact results depend on the particular form of these restrictions. But for example, if the production function is Cobb Douglas (with or without depreciation), the formulas for taxes on capital (1.22) and (1.28) are still valid if taxes on labor are also restricted to be set one period in advance. The reason is that in this case, capital can be factored out from the additional restrictions imposed on the planing problem.

1.3   A Recursive Representation for the Ramsey Problem

The following lemma characterizes the restrictions that the government budget and behavior of households place on competitive equilibrium feasible allocations, risk free rate processes and government policies.

Lemma 4 *A feasible allocation $\{c_t, l_t, k_t\}_{t \geq 0}$, a risk free rate process $\{r_t\}_{t \geq 0}$, a wage process $\{w_t\}_{t \geq 0}$, and a government policy $\{\tau_t^l, \tau_t^k, b_t^g\}_{t \geq 0}$ constitute a competitive equilibrium if and only if (1.1), (1.3), (1.4), (1.6), (1.7), (1.8) and (1.9), hold, with $l_t = l_t^d$.*

I develop a recursive representation of the Ramsey problem from $t = 1$ on. This recursive representation uses four state variables: the value of the capital stock k inherited from the previous period, the value of government debt from the previous period b, the marginal utility of consumption in the previous period $\theta \equiv u_c(c_-, l_-, s_-)$ and the shock that hit the economy in the previous period, s_-. It is easy to see that the planning problem is recursive in these four

state variables, and that the value function of the government satisfies the following Bellman equation.

Bellman equation 1

$$V(k, b, \theta, s_-) = \max \mathbb{E} \left\{ u(c_s, l_s, s) + \beta V(k'_s, b'_s, u_c(c_s, l_s, s), s) | s_- \right\}$$

subject to

$$(1 + r)\mathbb{E} \left\{ \beta u_c(c_s, l_s, s) | s_- \right\} = \theta$$

$$\mathbb{E} \left\{ \beta u_c(c_s, l_s, s) \left[1 + \left(1 - \tau^k \right) F_k(k, l_s, s) \right] | s_- \right\} = \theta$$

$$\tau_s^l = 1 + \frac{u_l(c_s, l_s, s)}{u_c(c_s, l_s, s) F_l(k, l_s, s)}, \qquad \forall s \in \mathbb{S}$$

$$(1 + r)b + g_s \leq \tau_s^l l_s F_l(k, l_s, s) + \tau^k k F_k(k, l_s, s) + b'_s, \qquad \forall s \in \mathbb{S}$$

$$c_s + g_s + k'_s \leq F(k, l_s, s) + k, \qquad \forall s \in \mathbb{S}$$

$$\underline{M}(k'_s, u_c(c_s, l_s, s), s) \leq u_c(c_s, l_s, s) b'_s \leq \overline{M}(k'_s, u_c(c_s, l_s, s), s), \qquad \forall s \in \mathbb{S}$$

The constraints on the problem are, in order of appearance: (i) that the risk free rate satisfies the usual Euler equation; (ii) that the net return on capital satisfies the usual Euler equation; (iii) that agents equalize their marginal rates of substitutions between leisure and consumption to the net real wage; (iv) that the budget constraint of the government is satisfied in each state $s \in \mathbb{S}$; (v) that the resource constraint holds in each state $s \in \mathbb{S}$ and (vi) that the amount of government debt issued in each state $s \in \mathbb{S}$ satisfies the debt and asset limits.

The initial period must be treated in isolation since there marginal utility of consumption in the previous period is not defined. Equivalently, one can think of the problem at date $t = 0$ as solving $V(k_0, b^g_{-1}, \theta_0, s_{-1})$ with the additional constraint that the date 0 tax on capital is given by τ^k_0 and that θ_0 is such that the implied date 0 risk free rate is equal to r_0. Hence it is straightforward to obtain the entire solution for the Ramsey problem once the solution to the Bellman equation above has been found.

It will prove convenient to replace b by a new state variable $\tilde{b} = b\theta$ representing debt weighed by the state price density. I can then define the corresponding value function $\tilde{V}(k, \tilde{b}, \theta, s_-) =$

$V(k, b\theta, \theta, s_-)$. In order to write the Bellman equation satisfied by $\tilde{V}$, I first rearrange the constraints. I use the first constraint to substitute r, the third to substitute τ_s^l, and I multiply the fourth constraint by $u_c(c_s, l_s, s)$. To save on notation I write X_s for any function $X(k, k_s', \tilde{b}_s', l_s, s)$ in state s.

Bellman equation 2

$$\tilde{V}(k, \tilde{b}, \theta, s_-) = \max \mathbb{E}\left\{u_s + \beta\tilde{V}(k_s', \tilde{b}_s', u_{c,s}, s)|s_-\right\} \tag{1.10}$$

subject to

$$\mathbb{E}\left\{\beta u_{c,s}\left[1 + \left(1 - \tau^k\right)F_{k,s}\right]|s_-\right\} = \theta \tag{1.11}$$

$$\tilde{b}\frac{u_{c,s}}{\beta\mathbb{E}\left\{u_{c,s}|s_-\right\}} + g_s u_{c,s} \leq l_s F_{l,s} u_{c,s} + l_s u_{l,s} + \tau^k k F_{k,s} u_{c,s} + \tilde{b}_s', \qquad \forall s \in \mathbb{S} \tag{1.12}$$

$$c_s + g_s + k_s' \leq F_s + k, \qquad \forall s \in \mathbb{S} \tag{1.13}$$

$$\underline{M}(k_s', u_{c,s}, s) \leq \tilde{b}_s' \leq \overline{M}(k_s', u_{c,s}, s), \qquad \forall s \in \mathbb{S} \tag{1.14}$$

The presence of capital, capital taxes, and marginal utility make the constraint set in (1.10) non-convex. This poses two kinds of problems. First, first order conditions are necessary but not sufficient for characterizing the solution. Second, it considerably complicates the task of establishing the differentiability of the value function $\tilde{V}$ – which is required to partially characterize the solution by a set of necessary first order conditions.

All the properties of Ramsey outcomes I derive can be established using either a Lagrangian approach, or expanding the Bellman equation (1.10) over two periods – hence bypassing this technical difficulty, but at the cost of heavier notations and poorer intuition. I therefore proceed assuming the value function $\tilde{V}$ is differentiable in $(k, \tilde{b}, \theta)$, and refer the reader to the appendix for an approach that does not rely on this assumption.

1.4 The Quasi-Linear Case

In the Ramsey problem, the government simultaneously chooses taxes and manipulates intertemporal prices. Manipulating prices substantially complicates the problem, especially with incomplete markets. Here I simplify by adopting a specification of preferences that eliminates

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the government's ability to manipulate prices. As in AMSS, this brings the model into the form of a consumption smoothing model and allows me to adapt results for that model to the Ramsey problem.

I assume that $u(c, l, s) = c + H(l, s)$, where H is a smooth, decreasing and concave function. I assume $H'(0) = \infty$ in order for labor supply to be interior. Making preferences linear pins down intertemporal prices. This allows a drastic simplification of (1.10), as two state variables b (which is equal to $\tilde{b}$ in this case) and s_- are now sufficient to describe the state of the economy. Intuitively, the reasons for this simplification are twofold. The first reason is that θ is now fixed and equal to 1 – hence θ can be dropped as a state variable. The other reason is that, intertemporal prices being entirely pinned down, I can perform a change of timing in the recursive approach: the optimal investment in capital k can now be thought as being chosen simultaneously with the the tax rate on capital τ^k – which allows to turn the state variable k into a control variable.

Under this specification, natural debt limits are independent of the capital stock, and marginal utility is constant: $\underline{M}^n(k, u_c, s) = \underline{M}^n_s$ and $\overline{M}^n(k, u_c, s) = \overline{M}^n_s$. Consistent with this property, in this setup, I consider only fixed debt limits $\underline{M}_s$ and $\overline{M}_s$. After further simplifying the constraints by using the resource constraint to substitute c_s, I derive a new Bellman equation.

Bellman equation 3

$$\tilde{V}(\tilde{b}, s_-) = \max \mathbb{E} \left\{ F_s + k \left(1 - \frac{1}{\beta} \right) - g_s + \beta \tilde{V}(\tilde{b}'_s, s) | s_- \right\} \tag{1.15}$$

subject to

$$\mathbb{E} \left\{ \beta \left[1 + \left(1 - \tau^k \right) F_{k,s} \right] | s_- \right\} = 1 \tag{1.16}$$

$$\tilde{b} \frac{1}{\beta} + g_s \le l_s F_{l,s} + l_s H_{l,s} + \tau^k k F_{k,s} + \tilde{b}'_s, \qquad \forall s \in \mathbb{S} \tag{1.17}$$

$$\underline{M}_s \le \tilde{b}'_s \le \overline{M}_s, \qquad \forall s \in \mathbb{S} \tag{1.18}$$

1.4.1 Stochastic properties of Ramsey outcomes

I attach a multiplier μ to (1.16), ν_s to (1.17) and $\nu_{2,s}$ and $\nu_{1,s}$ to the two constraints in (1.18). I can then form a Lagrangian associated with the right hand side of (1.15)

$$L(\tilde{b}, s_-) = \mathbb{E}\left\{F_s + k\left(1 - \frac{1}{\beta}\right) - g_s + \beta\tilde{V}(\tilde{b}'_s, s) + \mu\left[1 - \beta\left[1 + \left(1 - \tau^k\right)F_{k,s}\right]\right]\middle|s_-\right\}$$

$$+ \mathbb{E}\left\{\nu_s\left[l_s F_{l,s} + l_s H_{l,s} + \tau^k k F_{k,s} + \tilde{b}'_s - \tilde{b}\frac{1}{\beta} - g_s\right] + \nu_{2,s}\left[\tilde{b}'_s - \underline{M}\right] + \nu_{1,s}\left[\overline{M} - \tilde{b}'_s\right]\middle|s_-\right\}$$

where μ, ν_s, $\nu_{1,s}$ and $\nu_{2,s}$ are functions of $\tilde{b}$ and s_-.

The Envelope condition delivers

$$\beta\tilde{V}_b(\tilde{b}, s_-) = -\mathbb{E}\{\nu_s|s_-\} \tag{1.19}$$

and the first order condition for $\tilde{b}'_s$ gives

$$\beta\tilde{V}_b(\tilde{b}'_s, s) = -\nu_s + \nu_{1,s} - \nu_{2,s} \tag{1.20}$$

Combining these two equations, I find that, denoting by ν_{s-}, $\nu_{1,s-}$ and $\nu_{2,s-}$ the corresponding multipliers in the previous period, the following martingale equation hold

$$\nu_{s-} = \mathbb{E}\{\nu_s|s_-\} + \nu_{1,s-} - \nu_{2,s-} \tag{1.21}$$

In the rest of the this section, I will often, with some abuse of notation, switch from recursive notations to sequential notations.

Therefore, off debt-limits, $\nu_{s-} = \mathbb{E}\{\nu_s|s_-\}$. Using sequential notations, the process $\{\nu_t\}$ is a positive martingale. Equation (1.20) then shows that debt $\tilde{b}_t$ is a non-linear invariant function of $-\nu_t + \nu_{1,t} - \nu_{2,t}$ and s_t, and hence inherits a near random walk component. The policy functions $\{l_s, k, \tau^k\}$ associated with (1.15) show that for a Ramsey outcome, c_t, l_t and τ^l_t are invariant functions of $\tilde{b}_{t-1}$, s_{t-1} and s_t, while k_{t-1} and τ^k_t are invariant functions of $\tilde{b}_{t-1}$ and s_{t-1}.

Debt and taxes – on both labor and capital – therefore inherit a random walk component, reflecting the desire to smooth distortionary taxes across states and time. This tax smoothing

intuition is familiar in incomplete markets environments since the work of Barro (1979) and AMSS. Note that under complete markets, a similar Bellman equation would hold, but ν_t would be constant across time and states, and not a mere martingale. Therefore debt and taxes would depend only on the current shock s_t affecting the economy as well as on s_{t-1}, and would hence inherit the serial correlation properties of $\{s_t\}$, as in Lucas and Stokey (1983).

The fact that capital taxes, when set optimally, have a random walk component – and hence are persistent – is new, and might come as a surprise when confronted with the results of Chamley (1986) and Judd (1985). This reflects the fact that hedging needs of the government depend on the level of government debt, which has a random walk component. When public debt is low, the government is free to raise debt when confronted with an adverse shock : debt is then a good shock absorber. By contrast, when public debt is close to the debt limit, the ability of the government to shift the tax burden to the future is restricted. Hedging through capital taxes is then more attractive. As the simulations below will show, τ_t^k is an extremely non-linear function of $\tilde{b}_{t-1}$– in all simulations increasing and convex. This reflects the non-linearity in government hedging needs, and is a contrast with labor taxes.

1.4.2 Taxes on capital

Manipulating the first order conditions, it is possible to derive a formula to characterize taxes on capital

$$\tau^k = \frac{\frac{\mathbb{E}\{-\left(1-\tau^k\right)kF_{kk,s}|s_-\}}{\mathbb{E}\{F_{k,s}|s_-\}}}{1 + \frac{\mathbb{E}\{F_{k,s}\nu_s|s_-\}}{\mathbb{E}\{F_{k,s}|s_-\}}} \left[\frac{\text{Cov}\left\{kF_{k,s}, \nu_s|s_-\right\}}{\mathbb{E}\{kF_{k,s}|s_-\}} - \frac{\text{Cov}\left\{kF_{kk,s}, \nu_s|s_-\right\}}{\mathbb{E}\{kF_{kk,s}|s_-\}} \right] \tag{1.22}$$

The left hand side of (1.22) is increasing in τ^k. The right hand side comprises three terms. The first term

$$\frac{\frac{\mathbb{E}\{-\left(1-\tau^k\right)kF_{kk,s}|s_-\}}{\mathbb{E}\{F_{k,s}|s_-\}}}{1 + \frac{\mathbb{E}\{F_{k,s}\nu_s|s_-\}}{\mathbb{E}\{F_{k,s}|s_-\}}} \tag{1.23}$$

has as its numerator the inverse of the elasticity of capital k to taxes on capital τ^k. This inverse elasticity factor is standard in the taxation literature. The higher the elasticity, the lower the absolute value of the tax rate.

The second term

$$\frac{\text{Cov}\{kF_{k,s}, \nu_s | s_-\}}{\mathbb{E}\{kF_{k,s} | s_-\}} \tag{1.24}$$

represents the direct effect of an increase in τ^k : it relaxes the budget constraint of the government (1.17) in state s in proportion to the tax base of τ^k, $kF_{k,s}$. The more $kF_{k,s}$ is correlated with ν_s, the higher the optimal τ^k, as taxes on capital pay better in states where the budget constraint of the government is more binding, i.e. where the need for funds is higher.

The third term

$$-\frac{\text{Cov}\{kF_{kk,s}, \nu_s | s_-\}}{\mathbb{E}\{kF_{kk,s} | s_-\}} \tag{1.25}$$

reflects the indirect effect of an increase in τ^k. Increasing τ^k affects investment k and hence the capital tax base $kF_{k,s}$ and the revenues from labor taxation $l_sF_{l,s} + l_sH_{l,s}$ in each state $s \in \mathbb{S}$. The formula makes use of the constant returns to scale assumption to replace $l_sF_{kl,s}$ by $-kF_{kk,s}$. How adverse these effects are depends on the correlation between $kF_{kk,s}$ and ν_s. The higher the correlation, the bigger the effects in states where the need for funds is high, and hence the lower τ^k.

Hence, (1.22) brings together a standard inverse elasticity factor with two terms that have an asset pricing feel, and reflect the government's desire to use taxes on capital set one period in advance to hedge its need for funds across states. This illustrates that the government uses capital taxes not to levy funds on average, but only to smooth its need for funds across states – a stark difference with labor taxes.

In a complete market environment, (1.22) still holds, but ν_s is constant across states, so that τ^k is equal to 0. This outcome is then a particular case of the classical uniform taxation result by Atkinson and Stilgitz (1972), transposed to this Ramsey setup by Zhu (1992) and Chari, Christiano and Kehoe (1994), which holds more generally for preferences which are CRRA and separable between consumption and leisure.

As the following proposition shows, this zero tax result carries through in a particular case.

Proposition 5 *If F is Cobb Douglas, $\tau_t^k = 0$ for all $t \geq 1$.*

Therefore, for the strong Cobb Douglas benchmark, taxes on capital are 0 from period 1 on. In this case where the elasticity of substitution between capital and labor σ is equal to 1,

the hedging benefits from the direct effect of a marginal increase in τ^k at $\tau^k = 0$ are exactly offset by the marginal hedging cost from the indirect effect through the capital tax base and labor revenues.

Remark 6 *Equation (1.22) and Proposition 2 are valid irrespective of the asset structure of the economy. It holds for example if the government is required to balance its budget in every period, if it is restricted to trade only a perpetuity etc.*

In the general non Cobb Douglas case, the sign of τ^k is ambiguous – it may be optimal to tax or subsidize capital. The sign of τ^k will in general depend on the way productivity shocks and preference shocks interact with government consumption shocks and on the particular functional form for the production function. Under technical conditions, It is possible to characterize the sign of τ^k in special cases.

Proposition 7 *1) Assume that F is CES with elasticity of substitution σ. Consider the case where productivity shocks are Hicks neutral. The following holds: (i) if $\sigma > 1$, and if $l_s > l_{s'}$ if and only if $\nu_s < \nu_{s'}$, then $\tau^k > 0$; (ii) if $\sigma > 1$, and if $l_s < l_{s'}$ if and only if $\nu_s < \nu_{s'}$, then $\tau^k < 0$; (iii) if $\sigma < 1$, and if $l_s > l_{s'}$ if and only if $\nu_s < \nu_{s'}$, then $\tau^k < 0$; (iv) if $\sigma < 1$, and if $l_s < l_{s'}$ if and only if $\nu_s < \nu_{s'}$, then $\tau^k > 0$. 2) Assume that $F(k,l) = A(s)k^\alpha l^{1-a} - \delta k$: then τ^k has the same sign as $\mathrm{Cov}\{F_{k,s}, \nu_s | s_-\}$.*

This proposition is partly unsatisfactory as it relies on an assumption on endogenous objects l_s and ν_s. It is natural, for example, to expect case (i) in the absence of productivity or preference shocks: it is reasonable to expect that in this environment with no wealth effects, a higher need for funds calls for higher taxes on labor resulting in lower labor supply. Non-convexities in (1.15) considerably complicate the task of establishing how l_s co-varies with ν_s.[7]

Remark 8 *Note that taxes on capital are zero when $\sigma = 1$ and when $\sigma = \infty$: absent productivity shocks, the marginal product of capital is fixed if capital and labor are perfect substitutes, and hence capital taxation provides no hedging benefits.*

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[7]In the case where F is CES with only government expenditure shocks or only technological shocks, for all the numerical simulations that I have performed, only cases (i) and (iii) ever occured. In the case where $F(k,l) = A(s)k^\alpha l^{1-a} - \delta k$, with only technological shocks or only government expenditure shocks, for the numerical simulations that I have performed, $\mathrm{Cov}\{F_{k,s}, \nu_s | s_-\}$ was always negative.

1.4.3 Long run behavior

The long run behavior of Ramsey outcomes is similar to AMSS. I refer the reader to this paper for an extensive discussion, and sketch the principal properties. Here the difference between natural and ad hoc debt limits is marked.

Under natural asset limits, the multiplier $\nu_{2,t}$ is zero throughout. The natural asset limit $-\underline{M}_s^n$ is the amount of assets that allows the government to withstand any sequence of shocks with zero taxes. It makes no sense for the government to accumulate more assets than $-\underline{M}_s^n$. When favorable shocks cause government assets to grow beyond $-\underline{M}_n$, it is optimal for the government to rebate consumers the difference via a lump sum rebate. Therefore (1.21) becomes

$$\nu_{s-} = \mathbb{E}\{\nu_s | s_-\} + \nu_{1,s_-}$$

so that the stochastic process $\{\nu_t\}$ is a nonnegative supermartingale. Therefore, the supermartingale convergence theorem (see Lòeve (1977)) asserts that ν_t converges almost surely to a nonnegative random variable. As in AMSS, there are two possibilities:

(i) If the Markov process $\{s_t\}_{t \geq 0}$ is ergodic – so that in particular $\underline{M}_s^n = \underline{M}^n$ and $\overline{M}_s^n = \overline{M}^n$ do not depend on s for every s in the ergodic class of $\{s_t\}_{t \geq 0}$– then the lemma below shows that under the condition that $\tilde{V}$ is concave in $\tilde{b}$ and that the policy functions in (1.15) are continuous, ν_t converges almost surely to zero. In that case, taxes τ_t^k and τ_t^l converge to the first best levels $\tau_t^k = 0$ and $\tau_t^l = 0$. The level of government assets converges to $-\underline{M}^n$ sufficient to finance the worst possible sequence of shocks forever from interest earnings.

(ii) If the Markov process $\{s_t\}_{t \geq 0}$ has an absorbing state, then ν_t can converge to a strictly positive value; ν_t converges when s_t enters the absorbing state. From then on, taxes and all other variables in the model are constant. Taxes on capital are zero.

Lemma 9 *Consider the case of natural debt and asset limits. Assume that the Markov process $\{s_t\}_{t \geq 0}$ is ergodic, that the value function $\tilde{V}$ is continuously differentiable and concave in $\tilde{b}$, and that the policy functions in (1.15) are continuous. Then ν_t converges to zero almost surely.*

When the asset limit is more stringent than the natural one, convergence to the first best can be ruled out. In this case, the lower debt limit occasionally binds. This puts a nonnegative

multiplier $\nu_{2,t}$ in (1.21), and $\{\nu_t\}$ ceases to be a supermartingale. This fundamentally alters the limiting behavior of the model in the case where the Markov process $\{s_t\}_{t\geq 0}$ has a unique invariant distribution. In particular, rather than converging almost surely, ν_t continues to fluctuate randomly. Off debt limits, ν_t behaves like a martingale, and capital taxes do not converge to 0.

In addition, if the range of the policy functions $\tilde{b}'_s$ can be restricted to a compact set, one can show that an invariant distribution for government debt exists.

1.5 The general case

The insights from the simple example examined in the previous section largely carry through to the case where preferences are not risk neutral and separable. But the possibility to manipulate intertemporal prices brings about the traditional motive for taxes on capital, which interacts with the motive uncovered in the previous section.

1.5.1 Stochastic properties of Ramsey outcomes

Let us attach a multiplier μ to (1.11), ν_s to (1.12), $\nu_{2,s}$ and $\nu_{1,s}$ to the two constraints in (1.14), and ψ_s to (1.13). I can then form a Lagrangian associated with the right hand side of (1.10)

$$
\begin{aligned}
L(k, \tilde{b}, \theta, s_-) \;=\; & \mathbb{E}\left\{ u_s + \beta \tilde{V}(k'_s, \tilde{b}'_s, u_{c,s}, s) + \mu \left[\beta u_{c,s} \left[1 + \left(1 - \tau^k\right) F_{k,s} \right] - \theta \right] | s_- \right\} \\
& + \mathbb{E}\left\{ \nu_s \left[-\tilde{b} \frac{u_{c,s}}{\beta \mathbb{E}\{u_{c,s}\}} - g_s u_{c,s} + l_s F_{l,s} u_{c,s} + l_s u_{l,s} + \tau^k k F_{k,s} u_{c,s} + \tilde{b}'_s \right] | s_- \right\} \\
& + \mathbb{E}\left\{ \psi_s \left[F_s + k - c_s - g_s - k'_s \right] + \nu_{2,s} \left[\tilde{b}'_s - \underline{M}(k'_s, u_{c,s}, s) \right] + \nu_{1,s} \left[\overline{M}(k'_s, u_{c,s}, s) - \tilde{b}'_s \right] | s_- \right\}
\end{aligned}
$$

where μ, ν_s, $\nu_{1,s}$, $\nu_{2,s}$ and ψ_s are functions of $k, \tilde{b}, \theta$ and s_-.

Using the Envelope condition for $\tilde{b}$, a martingale equation similar to (1.21) can be derived

$$
\nu_{s-} = \frac{\mathbb{E}\{\nu_s u_{c,s}|s_-\}}{\mathbb{E}\{u_{c,s}|s_-\}} + \nu_{1,s-} - \nu_{2,s-} \tag{1.26}
$$

Off debt limits, the multiplier ν_t is a now a risk adjusted martingale, imparting, as in the simple example in the previous section, a unit root component to the solution of (1.10). The

condition

$$\beta \tilde{V}_b(k'_s, \tilde{b}'_s, u_{c,s}, s)) = -\nu_s + \nu_{1,s} - \nu_{2,s} \qquad (1.27)$$

shows that the vector of endogenous state variables $\{k_{t-1}, \tilde{b}_t, u_{c,t-1}\}$ inherits a near random walk component. The policy functions $\{c_s, l_s, k, \tau^k\}$ associated with (1.10) imply that for a Ramsey outcome, k_t, $\tilde{b}_t$, c_t, l_t and τ^l_t are invariant functions of $k_{t-1}, \tilde{b}_{t-1}, u_{c,t-1}$, s_{t-1} and s_t, while τ^k_t is invariant function of $k_{t-1}, \tilde{b}_t, u_{c,t-1}$ and s_{t-1}. Therefore, k_t, $\tilde{b}_t$, c_t, l_t, τ^l_t and τ^k_t inherit a unit root component.

As in the risk neutral case, debt and taxes display a random walk component, reflecting the desire to smooth distortionary taxes across time, a sharp contrast with the complete markets Ramsey outcome, where a similar Bellman equation holds, but with ν_t constant across time and not a mere martingale.

Hence, the stochastic properties of Ramsey outcomes are similar to those discussed in the quasi-linear example. The analysis is only made more difficult by the need to keep track of additional state variables k_{t-1} and $u_{c,t-1}$.

1.5.2   Taxes on capital

A formula similar to (1.22) can be derived

$$\tau^k = T^h(k, \tilde{b}, \theta, s_-) + T^i(k, \tilde{b}, \theta, s_-) + T^b(k, \tilde{b}, \theta, s_-) \qquad (1.28)$$

To save on space, I report the formulas for T^h, T^i and T^b in the first part of the Appendix. It should be noted that these are valid from $t = 1$ on. There are three motives for taxing capital, corresponding to the three terms T^h, T^i and T^b on the right hand side of (1.28).[8]

The first term, T^h is the "hedging" term and reflects the hedging motive discussed in the previous section. Two differences with (1.22) should be emphasized. First, the formula is adjusted for risk through $u_{c,s}$. Second, the multiplier ψ_s on the resource constraint (1.13) appears. In the quasi-linear case, ψ_s is equal to one. When risk aversion is introduced, the stochastic process $\{\beta^t \psi_t\}_{t \geq 0}$ represents the intertemporal prices the government would be

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[8] The superscrpits "h", "i" and "b" stand respectively for "hedging", "intertemporal" and "bounds".

willing to pay for additional resources at different dates. The process $\left\{\frac{\psi_t}{u_{c,t}}\right\}_{t\geq 0}$ converts these prices in consumption equivalent units.The presence of ψ_s is natural since taxes on capital affect capital accumulation and hence resources available.

The second term T^i is the "intertemporal" term and corresponds to the traditional motive for capital taxation: The government can induce intertemporal resource transfers by affecting capital accumulation through capital taxation to reduce the burden taxation. The formula calls for subsidizing capital between t and $t+1$ when resources are expected to be scarcer at $t+1$ than at t – i.e. when $\frac{\psi_{t+1}}{u_{c,t+1}}$is expected to be larger on average than $\frac{\psi_t}{u_{c,t}}$ – especially if the net marginal product of capital

$$\left[1 + \left(1 - \tau^k_{t+1}\right) F_{k,t+1}\right]$$

or marginal utility $u_{c,t+1}$ is positively correlated with $\frac{\psi_{t+1}}{u_{c,t+1}}$.

This formula (1.28) for τ^k is valid under complete markets, with the only difference that ν_s is constant. Under complete markets, T^h is equal to zero, but not T^i in general. The well known case where taxes on capital are zero under complete markets is the case of CRRA preferences, separable between consumption and leisure. Indeed, under complete markets,

$$\frac{\psi_s}{u_{c,s}} = 1 + \nu \frac{c_s u_{cc,s}}{u_{c,s}} - \frac{u_{cl,s}}{u_{c,s}} l_s$$

so that if $u(c,l,s) = \frac{c^{1-\sigma}-1}{1-\sigma} + H(l,s)$,

$$\frac{\psi_s}{u_{c,s}} = 1 - \nu\sigma$$

is constant along the optimal path, and T^i is equal to zero from period 2 on.

When markets are incomplete, $\frac{\psi_s}{u_{c,s}}$ is not constant anymore at a Ramsey outcome when preferences are CRRA and separable, so that T^i is not equal to zero even in this particular case – at the Ramsey outcome, the government would be willing to pay a different price – in consumption equivalent units – for additional resources at different dates and in different states, so that the traditional motive for capital taxation remains. In fact, as already mentioned, the zero capital taxation result under CRRA and separable preferences is an application of the uniform commodity taxation result of Atkinson and Stigitz (1972). This result relies crucially

on the assumption of complete markets.

The last term T^b imparts a role for relaxing debt limits to capital taxes. For example, if the maximum debt limit is increasing in the capital stock in the economy, it is optimal to subsidize capital when the limit is binding to relax this constraint and allow for more debt accumulation. This term is zero if the imposed debt limits do not depend on capital. Note that this term would have been present in (1.22) if I had allowed debt limits to depend on capital.

The case where the production function is Cobb Douglas still provides a useful benchmark.

Proposition 10 *If F is Cobb Douglas, then $T_t^h = 0$. for $t \geq 1$.*

As in the quasi-linear case, T^h is zero as soon as the need for hedging disappears.

Remark 11 *If the Markov process $\{s_t\}_{t \geq 0}$ enters an absorbing state at t_0, then $T_t^h = 0$ for $t \geq t_0 + 1$.*

Remark 12 *As in the quasi-linear case, equation (1.28) and Proposition 3 are valid irrespective of the asset structure of the economy.*

It is also possible to give a characterization of T^i along the lines of Zhu (1992).

The stochastic process $x_t = \{k_t, \tilde{b}_t, u_{c,t}, s_t\}$ is a stationary, ergodic, first order Markov process: there exists a probability measure P^∞ such that for all t, and measurable set A,

$$P^\infty \{x_t \in A\} = P^\infty \{A\}$$

$$P^\infty \left\{ \lim_{j \to \infty} \Pr\{x_{t+j} \in A | x_t\} = P^\infty \{A\} \right\} = 1$$

The policy functions in (1.10) are continuous.

$$P^\infty \{1 + \left(1 - \tau^k\right) F_{k,t} > 0\} = 1.$$

Proposition 13 *If assumptions 1, 2 and 3 hold, then one of the following holds: (i) $P^\infty \left\{T_t^i = 0\right\} = 1$; (ii) $P^\infty \left\{T_t^i > 0\right\} > 0$ and $P^\infty \left\{T_t^i < 0\right\} > 0$.*

This proposition shows, that the insights of Zhu (1992) are still valid for the traditional motive for capital taxation embodied in T^i. At a stochastic steady state, T^i cannot be always positive or always negative.

1.6 Capital Ownership and the Structure of Government Liabilities

So far, I have restricted the government to trade only a risk free bond with consumers, whereas consumers faced a non-trivial portfolio decision in allocating their savings between government debt and capital. Preventing the government from trading on capital is without loss of generality under complete markets. But in environments with incomplete markets, this arbitrary restriction regains bite. Allowing the government to trade on the capital gives the government more ways to smooth taxes across time and states, and to hedge government expenditure shocks.

1.6.1 The Government Capital Asset Pricing Model

I now remove this restriction. To illustrate that physical capital is an asset among others that the government can trade with consumers, I also introduce exogenous assets indexed by $i \in \mathbb{I}_{s_-}$. The subscript s_- allows for the possibility to let the investment opportunity set formed by the exogenous assets to vary with the state variable s.

I now allow the government and consumers to trade three kinds of assets: (i) a risk free bond; (ii) capital – an asset whose return in state s is $1 + \left(1 - \tau^k\right) F_{k,s}$; (iii) and $\#\mathbb{I}_{s_-}$ assets in zero net supply indexed by $i \in \mathbb{I}_{s_-}$ – whose return in state s when the shock in the previous period was s_- is R_s^{i,s_-}.

Generically, if the number of traded assets is less than the number of shocks, markets are truly incomplete and the complete markets Ramsey outcome is not attainable. I will maintain this assumption throughout.

It is easy to see that the planning problem is still recursive with the same state variables $k, \tilde{b}, \theta, s_-$, where $\tilde{b}$ is now the value of the government's net financial position vis-a-vis the private sector. Denoting by x_i government's holdings of asset $i \in \mathbb{I}_{s_-}$ and by k_g government's holdings of capital, the government's value function satisfies a modified version of (1.10):

Bellman equation 2'

$$\tilde{V}(k, \tilde{b}, \theta, s_-) = \max \mathbb{E}\left\{ u_s + \beta \tilde{V}(k'_s, \tilde{b}'_s, u_{c,s}, s) | s_- \right\} \tag{1.29}$$

subject to

$$\mathbb{E}\left\{\beta u_{c,s}\left[1+\left(1-\tau^k\right)F_{k,s}\right]|s_-\right\}=\theta \tag{1.30}$$

$$\mathbb{E}\left\{\beta u_{c,s}R_s^{i,s-}|s_-\right\}=\theta \tag{1.31}$$

$$\sum_{i\in\mathbb{I}_{s_-}}x_i(R_s^{i,s-}-\frac{\theta}{\beta\mathbb{E}\{u_{c,s}\}})u_{c,s}+k_g(1+\left(1-\tau^k\right)F_{k,s}-\frac{\theta}{\beta\mathbb{E}\{u_{c,s}\}})u_{c,s}+\tilde{b}\frac{u_{c,s}}{\beta\mathbb{E}\{u_{c,s}\}}+g_su_{c,s}$$
$$\leq\quad l_sF_{l,s}u_{c,s}+l_su_{l,s}+\tau^kkF_{k,s}u_{c,s}+\tilde{b}'_s,\qquad\forall s\in\mathbb{S} \tag{1.32}$$

$$c_s+g_s+k'_s\leq F_s+k,\qquad\forall s\in\mathbb{S} \tag{1.33}$$

$$\underline{M}(k'_s,u_{c,s},s)\leq\tilde{b}'_s\leq\overline{M}(k'_s,u_{c,s},s),\qquad\forall s\in\mathbb{S} \tag{1.34}$$

There are two differences between (1.10) and (1.29). First, there is now one Euler equation for each exogenous asset (1.31). The second difference is in the budget constraint of the government (1.32) , where the total liability the government has to repay or refinance in state s is now

$$\sum_{i\in\mathbb{I}_{s_-}}x_i(R_s^{i,s-}-\frac{\theta}{\beta\mathbb{E}\{u_{c,s}|s_-\}})+k_g(1+\left(1-\tau^k\right)F_{k,s}-\frac{\theta}{\beta\mathbb{E}\{u_{c,s}|s_-\}})+\tilde{b}\frac{1}{\beta\mathbb{E}\{u_{c,s}|s_-\}}$$

The government therefore faces a non-trivial portfolio decision. It must decide not only the level but also the composition of its liabilities.

As is clear from (1.29), introducing more assets only relaxes the constraints in the planning problem. Therefore, as in the quasi-linear case, introducing more traded assets, or allowing the government to trade capital, improves the Ramsey outcome.

Remark 14 *Expanding the set of assets the government can trade with consumers improves the Ramsey outcome*

Here, more traded assets only give the government more flexibility to smooth distortionary taxes. This result hinges crucially on the absence of consumer heterogeneity. In this case, the introduction of additional traded assets would impose new constraints since heterogenous agents will generally engage in trades between themselves – which could potentially render the

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task of the planner more difficult.

Assuming an interior solution exists and that debt limits do not bind in state s_-, the following set of first order conditions characterize the optimal asset and liability structure of the government. For this, it is convenient to label the risk free rate $R^{s-} \equiv \frac{\theta}{\beta \mathbb{E}\{u_{c,s}|s_-\}}$.

$$\mathbb{E}\left\{\beta R^{s-} u_{c,s}\nu_s | s_-\right\} = \theta \nu_{s_-} \tag{1.35}$$

$$\mathbb{E}\left\{\beta \left[1 + \left(1 - \tau^k\right) F_{k,s}\right] u_{c,s}\nu_s | s_-\right\} = \theta \nu_{s_-} \tag{1.36}$$

$$\mathbb{E}\left\{\beta R_s^{i,s-} u_{c,s}\nu_s | s_-\right\} = \theta \nu_{s_-} \tag{1.37}$$

These equations form the government counterpart of the standard CCAPM Euler equations: I refer to them as the Government Capital Asset Pricing Model (GCAPM). Assets that pay well in states of the world where government funds are scarce require a lower expected rate of return, as can be seen for example from rewriting (1.37) in the following way

$$\mathbb{E}\left\{R_s^{i,s-} | s_-\right\} = R^{s-} - \frac{\mathrm{Cov}\left\{R_s^{i,s-}, u_{c,s}\nu_s | s_-\right\}}{\mathbb{E}\left\{u_{c,s}\nu_s | s_-\right\}}$$

Equations (1.35), (1.36) and (1.37) hold together with the standard CCAPM Euler equations

$$\mathbb{E}\left\{\beta R^{s-} u_{c,s} | s_-\right\} = \theta$$

$$\mathbb{E}\left\{\beta \left[1 + \left(1 - \tau^k\right) F_{k,s}\right] u_{c,s} | s_-\right\} = \theta$$

$$\mathbb{E}\left\{\beta R_s^{i,s-} u_{c,s} | s_-\right\} = \theta$$

Equations (1.35), (1.36) and (1.37) show that $\{\beta^t u_{c,t}\nu_t\}_{t\geq 0}$ can be thought of a pricing kernel for the government. This pricing kernel applies to assets traded with consumers – which is the case for all assets considered so far. It characterizes the price the government is willing to pay for an asset trade with consumers. Note that the result that ν_s is a risk adjusted martingale is simply a restatement of (1.35).

If consumers are not on the other side of the trade, the pricing kernel $\{\beta^t u_{c,t}(\nu_t + \frac{\psi_t}{u_{c,t}})\}_{t\geq 0}$

is different and involves the shadow costs of resources ψ_t. Indeed, the price the government would be willing to pay for a non traded investment project paying out X_s in state s is

$$\frac{\mathbb{E}\left\{X_s u_{c,s}(\nu_s + \frac{\psi_s}{u_{c,s}})|s_-\right\}}{\nu_{s-} + \frac{\psi_{s-}}{\theta}}$$

It should also be emphasized that the results on capital taxation still hold when more traded assets are introduced. In particular, taxes are still given by (1.28). The only difference is in the hedging term: the elasticity of capital to the tax rate has to be replaced by the elasticity of the part of the capital stock held by the private sector to the tax rate

$$\frac{\mathbb{E}\{-(k - k_g)(1 - \tau^k)F_{kk,s}u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}$$

The GCAPM, together with the CCAPM, provides a simple and powerful framework for addressing the optimal liability structure of the government in various contexts. In ongoing work, Farhi and Werning (2005) study the optimal maturity structure of government expenditures when the government is restricted to issue only risk-free debt of a limited number of maturities.

1.6.2 The GCAPM in The Quasi-Linear Case

I consider here the case where preferences are quasi-linear. This stripped down setup will allow me to derive stark results on the optimal structure of government liabilities.

As in the previous subsection, I allow the government and consumers to trade three kinds of assets: (i) a risk free bond; (ii) capital – an asset whose return in state s is $1 + \left(1 - \tau^k\right) F_{k,s}$; (iii) and $\#\mathbb{I}_{s_-}$ assets in zero net supply indexed by $i \in \mathbb{I}_{s_-}$ – whose return in state s when the shock in the previous period was s_- is R_s^{i,s_-}. The definitions of feasible allocations, competitive equilibria and Ramsey outcomes can be straightforwardly extended to this new setup.

It is easy to see that the planning problem is still recursive with the same state variables $\tilde{b}$ and s_-, where $\tilde{b}$ is now the value of the government's net financial position vis-a-vis the private sector. Denoting by x_i government's holdings of asset $i \in \mathbb{I}_{s_-}$ and by k_g government's holdings of capital, the government's value function satisfies a modified version of (1.15) analogous to

(1.32).

Bellman equation 3'

$$\tilde{V}(\tilde{b}, s_-) = \max \mathbb{E} \left\{ F_s + k \left(1 - \frac{1}{\beta} \right) - g_s + \beta \tilde{V}(\tilde{b}'_s, s) | s_- \right\} \tag{1.38}$$

subject to

$$\mathbb{E} \left\{ \beta \left[1 + \left(1 - \tau^k \right) F_{k,s} \right] | s_- \right\} = 1 \tag{1.39}$$

$$\mathbb{E} \left\{ \beta R_s^{i,s_-} | s_- \right\} = 1 \qquad \forall i \in \mathbb{I}_{s_-} \tag{1.40}$$

$$\sum_{i \in \mathbb{I}_{s_-}} x_i (R_s^{i,s_-} - \frac{1}{\beta}) + k_g (1 + \left(1 - \tau^k \right) F_{k,s} - \frac{1}{\beta}) + \tilde{b} \frac{1}{\beta} + g_s \leq l_s F_{l,s} + l_s H_{l,s} + \tau^k k F_{k,s} + \tilde{b}'_s, \qquad \forall s \in \mathbb{S} \tag{1.41}$$

$$\underline{M}_s \leq \tilde{b}'_s \leq \overline{M}_s, \qquad \forall s \in \mathbb{S} \tag{1.42}$$

In a particular case, allowing the government to trade physical capital actually allows its to reach the complete markets Ramsey outcome.

Proposition 15 *Absent productivity and preference shocks, the government can perfectly approximate the complete markets Ramsey outcome with a larger and larger long or short position in capital.*

The intuition for this proposition is that absent productivity and preference shocks, the complete markets Ramsey outcome features constant labor supply across states. Hence the return on physical capital is risk free: physical capital and the risk free bond are colinear assets. By commanding small deviations from the constant level labor supply of the complete markets allocation, the government can align the variations of the returns on capital with its need for funds. By taking extreme positions in capital, compensated by opposite positions on the risk free bond, the government can then leverage these variations and smooth perfectly its need for funds across states.

The extreme positions required for replicating the complete markets allocation are reminiscent of the findings of Angeletos (2002) and Buera and Nicolini (2004). Both contributions analyze how the government can use different maturities of risk free debt to implement complete markets Ramsey outcome. They find that generically, if the number of maturities is larger

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than the number of shocks, then the complete markets allocation can be implemented, but that typically, very large positions in the different maturities are required. It is also worth noting that in this example with quasi-linear preferences, different maturities of risk free debt would not permit the government to replicate the complete markets Ramsey outcome since the entire term structure is entirely pinned down by preferences.

With productivity and preferences shocks, the government can typically no longer replicate the complete markets Ramsey outcome if fewer assets than states of the world are traded. The GCAPM takes the particularly simple form

$$\mathbb{E}\left\{\left[1 + \left(1 - \tau^k\right) F_{k,s} - R\right] \nu_s | s_-\right\} = 0 \qquad (1.43)$$

$$\mathbb{E}\left\{\left[R_s^{i,s_-} - R\right] \nu_s | s_-\right\} = 0 \qquad (1.44)$$

where $R \equiv \frac{1}{\beta}$ is the risk free rate.

Equations (1.43) and (1.44) can be compared to the standard CCAPM Euler equations

$$\mathbb{E}\left\{1 + \left(1 - \tau^k\right) F_{k,s} - R | s_-\right\} = 0$$

$$\mathbb{E}\left\{R_s^{i,s_-} - R | s_-\right\} = 0$$

which characterize the optimal portfolio for consumers. The only difference between the two sets of equations is that the marginal utility of consumption, 1, is replaced by ν_s – the multiplier measuring the severity of the government budget constraint in state s. The stochastic process $\{\beta^t (1 + \nu_t)\}_{t \geq 0}$ can be interpreted as the pricing kernel of the government. A way to see this is to introduce, instead of assuming an exogenous process for government expenditures, a standard utility for government funds $v(g_t, s_t)$. In this case, the first order condition for government expenditures is $v_g(g_t, s_t) = 1 + \nu_t$.

Note that the pricing kernel of the government is not trivial – i.e. has a positive variance – despite the face that agents are risk neutral. The variation in the pricing kernel comes only from the imperfect ability to smooth taxes across time under incomplete markets. Indeed, if markets were complete, the pricing kernel of the government and the pricing kernel of the agents would be colinear – in this instance, constant.

These equations can be compared to the results in Bohn (1990). Following Barro (1979), Bohn considers an environment with incomplete markets, no capital and risk neutral consumers, where the government must finance an exogenous stream of expenditures using distortionary taxes. Taxes τ are assumed to impose an ad hoc increasing convex deadweight cost $h(\tau)$. Bohn derives the following formula for the return of any traded asset R

$$\mathbb{E}\left\{[R_t - R] \, h'(\tau_t)\right\} = 0 \qquad (1.45)$$

Hence (1.44) can be seen as a microfounded version of (1.45). Some important differences are worth noting. In particular it is not generally true in my model that ν_t is a function of τ_t^l and τ_t^k or even of tax revenues, as a perfect analogy with (1.45) would require. In fact, ν_t is a function of the $\#\mathbb{S}^2 + 1$ variables s_t, s_{t-1} and $\widetilde{b}_{t-1}$. Generically, ν_t is therefore not a function of the $\#\mathbb{S} + 1$ functions τ_s^l and τ^k. In the specification with a utility $v(g_t, s_t)$ from government expenditures, $\nu_t = v_g(g_t, s_t) - 1$, so that ν_t is a function of g_t and s_t – which can be expressed as a function of s_{t-1}, s_t and $\widetilde{b}_{t-1}$. Importantly, the state s_t appears in this formula along with government expenditures. Even if I were to assume that $v(g_t, s_t)$ is independent of s_t, this discussion would suggest that a non linear function of government expenditures $v_g(g_t) - 1$ is better suited for approximating the marginal cost of public funds that an increasing function of taxes or tax revenues $h'(\tau_t)$ as in (1.45). This discussion is important, as tests and implications of this theory rely crucially on sorting out correctly the time series properties of the pricing kernel of the government.

I have focused so far on traded assets. Uncovering the pricing kernel of the government allows me to determine the price the government would be willing to pay for non traded assets. In particular, consider a marginal public investment project requiring an outlay I, and whose payoff X_s is not spanned by traded assets. Then the government should follow the following capital budgeting rule: invest in the project if and only if

$$\frac{\mathbb{E}\{X_s | s_-\}}{R - \frac{\mathrm{Cov}\{X_s / I, 1 + \nu_s | s_-\}}{1 + \mathbb{E}\{\nu_s | s_-\}}} \geq I \qquad (1.46)$$

where $1 + r = \frac{1}{\beta}$. This capital budgeting rule (1.46) shows that the government should discount cash flows using the *beta* associated with its own pricing kernel, and not with that of

consumers

$$-\frac{\mathrm{Cov}\left\{X_s/I, 1 + \nu_s | s_-\right\}}{1 + \mathbb{E}\{\nu_s | s_-\}}$$

Hence the government should attach a public risk premium to returns that covary negatively with shocks affecting adversely its budget.

It is interesting to note that the asset holdings in the optimal portfolio of government liabilities $\{x_t^i, k_{g,t}\}$ has a unit root component. Hence not only do the government's total liabilities $\tilde{b}_t$ have a random walk component, but the composition of its liabilities display a similar kind of persistence.

1.7 Numerical Simulations

1.7.1 Numerical method and parameter values

Numerical method. I approach the problem by solving the dynamic programming problems (1.10), (1.29), (1.15) and (1.38), and then back out the optimal policies. In my calculations, I restrict the state space to be rectangular and bounded. I check numerically that enlarging the rectangle doesn't alter the results. The dynamic programming problem is then solved by a value iteration algorithm with Howard acceleration. I approximate the value function with cubic splines.

Calibration of the risk averse case. To permit comparability of my results to those in Chari, Christiano and Kehoe (1994), I consider the same parameters and functional forms. I assume that preferences are of the form

$$u(c, l) = (1 - \gamma) \log(c) + \gamma \log(1 - l)$$

Technology is described by a production function

$$F(k, l, z, t) = k^\alpha (\exp(\rho t + \tilde{z})l)^{1-\alpha} - \delta k$$

This incorporates two kinds of labor augmenting technological change in the production function. The variable ρ captures deterministic growth in this technical change. The variable $\tilde{z}$ is a

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zero mean technological shock that follows a two-state Markov chain with mean $\bar{z}$ and autocorrelation ρ_z. Let government expenditure be given by $g_t = G \exp(\rho t + \tilde{g})$ where G is a constant and $\tilde{g}$ follows a two-state Markov chain with mean $\bar{g}$ and autocorrelation ρ_g. I take $\gamma = 0.75$, $\beta = 0.98$, $\alpha = 0.34$, $\rho = 0.016$, $G = 0.07$, $\rho_g = 0.89$, $\sigma_g = 0.07$, $\rho_z = 0.81$ and $\sigma_z = 0.04$.

I impose fixed debt limits $\underline{M} = -0.2\overline{GDP}^{fb}$ and $\bar{M} = \overline{GDP}^{fb}$, where $\overline{GDP}^{fb}$ is the mean across states of first best level of GDP that would occur if the state were absorbing.

Notice that without technological shocks, the economy has a balanced growth path along which consumption, capital and government spending grow at rate ρ and labor is constant. As in Chari, Christiano and Kehoe (1994), it is straightforward to modify the model to allow for exogenous growth.

The length of the accounting period where capital are held fixed by assumption is an important parameter. I analyze two series of simulations, one where the period length is one year, and one where the period length is five years.

Calibration of the quasi-linear case. I also calibrate a model with quasi-linear preferences given by

$$u(c, l) = c + \gamma \log(1 - l)$$

To ensure that labor is stationary, I impose that growth is zero in this case. I also adjust γ to 0.5.

1.7.2 Results

Results in the quasi-linear case. Figure 1 displays several variables along a typical path, in an economy with only government expenditure shocks and no capital ownership. The top left panel displays the path of shocks, which take only two possible values, one and two, corresponding to low and high expenditures respectively. Note that debt, labor taxes, labor and capital .all appear to be more persistent than the shock process, reflecting the fact that they incorporate a random walk component. The bottom left panel represents taxes on capital. Taxes on capital are negative – as predicted by Proposition 2– but are never larger than 10^{-5}. I experimented with other specifications of the production function (CES) and never got a result larger than 10^{-4}. Note that capital taxes also seem to incorporate some persistence, although less than other variables in the economy. Capital taxes are relatively larger in absolute value

when debt approaches its upper limit.

Figures 2 (respectively 3) displays the policy functions when the previous government expenditure shock was low (respectively high). On the horizontal axis of every graph is inherited debt. On the vertical axis is the variable whose name indexes the graph. The solid blue (respectively dashed green) lines correspond to the policy functions when the contemporaneous shock is low (respectively high). The second graph on the top row is a zoom on the top left graph representing the policy functions for debt. The thick red line is the 45 degree line. As is apparent from this graph, a high shock is partially absorbed through an increased debt, and a low shock through a decrease in debt. The bottom left graph displays the policy functions for the multipliers. Both are increasing in the level of inherited debt. The multiplier associated with the high shock is always larger than the multiplier associated with the low shock. Moreover, the bottom right panel shows that the variance of ν_s across states for a given s_- is increasing in the level of inherited debt: as inherited debt gets close to the upper debt limit, the government is more limited in its ability to use debt as a shock absorber if the high shock hits. Note that this variance is always minuscule here, reflecting the fact that with reasonable debt limits, risk free debt is a very good shock absorber. Capital taxes are small for two reasons: first because of the Cobb Douglas benchmark, second because the variance of ν_s is small. As government debt approaches the upper limit, the variance of ν_s increases and with it the absolute value of the tax rate.

Table 1 displays the optimal capital ownership level in the invariant distribution, depending on the environment. I first discuss the results when the period length is set to one year. With only government expenditure shocks, Proposition 5 shows that the government takes an infinite long or short position in capital and replicates the complete markets allocation. With only two possible productivity shocks, the government can also replicate the complete markets allocation, with a finite – but large – short position of -295% of $\overline{k}^{fb}$, .where is the mean across states of first best level of capital that would prevail if the state were absorbing That a short position is required is easily understood: the marginal product of capital correlates positively with productivity shocks, and hence with government revenues. The magnitude of the position results from the fact that capital is very colinear with risk free debt. Hence a big leveraged position, short in capital, long in the risk free bond, is required to provide the government with

a state contingent source of revenues that matches the desired variations in the net present value of government surpluses. With government expenditure shocks and productivity shocks, the government cannot replicate the complete markets allocation anymore. The optimal government capital ownership level is almost identical to the one that prevails with only technological shocks. This reflects the fact that in this business cycle calibration, technological shocks are a bigger source of variation in the government needs for funds than government expenditure shocks.

Increasing the period length to five years does not alter the result of Proposition 5. By contrast, optimal government capital ownership in the case of productivity shocks, or both productivity shocks and government expenditure shocks drops from -295% to -59%. The reason is that as the period length is increased, shocks per period become less persistent - getting closer and closer to i.i.d. Hence the variation of the net present value of government surpluses that the government seeks to hedge becomes smaller – approximately 5 times smaller. Hence a smaller position in capital is required.

In this model, the welfare gains from completing markets are small. In all simulations, I compute them to be less than 0.01% of lifetime consumption. This confirms the finding of AMSS for business cycle type calibrations. The size of welfare gains is well understood from AMSS: they depend on the size and persistense of the shocks, the curvature in the uitlity function, and the debt limits. The welfare gains are much larger in a war and peace exercise I report below. Nevertheless, an interesting question is: How much of the gap between welfare in the incomplete markets, no government ownership allocation and welfare in the complete markets allocation can government ownership cover? In all simulations, I find this number to be above 85%. Moreover, in the simulation with a 5 year period length and both government and productivity shocks, I find that a short position of 15% of $\overline{k}^{fb}$ allows to realize 47% – a large, but more reasonable position– of these welfare gains.

I also calibrate a war and peace example, with a one year period length and only government expenditure shocks. The parameters are the same as in the business cycle simulation, except for σ_g which I take to be equal to 0.7 instead of 0.07. I compute the mean welfare gains to be 0.6% of lifetime consumption. The welfare gains reach 1.5% of lifetime consumption if debt is close to the debt limit and the economy experienced a high government expenditure shock in the previous period. With such large shocks, taxes on capital are larger in absolute value than

in the business-cycle simulations, but small – between 0 and −1% with probability 75%. Taxes on capital become large – up to −20% – when debt is very close to the debt limit.

Results in the risk averse case. I first start with a limited experience: I start the economy in period 0 with a given value of $(b_0, k_0, \theta_0, s_-)$ with $s_- = 1$, corresponding to the low government expenditure shock. In period 1, a permanent government expenditure shock hits the economy. This shock can be high – correpsonding to $s = 2$ – or low – corresponding to $s = 1$ – with probabilities 5% and 95% respectively. Capital ownership by the government is disallowed. All the uncertainty in the economy is resolved in period 1. Figures 4 and 5 display the impulse responses of several variables in the economy corresponding to the high and low shock respectively.

First note that the hedging term is non-zero only in the first period, and is smaller than 10^{-3}. Hence taxes on capital are dominated by the intertemporal term. Following a high shock, taxes on capital spike at 150% and then fall back to 0. The interest rate drops to −2% and then reverts almost immediately to 2%. This effect would be present even in an economy without capital as in AMSS. Werning (2005) coins the term "interest rate manipulation".

Debt initially goes up to absorb the high government expenditure shock in period 1, and then drops permanently to a level lower than b_0. From the government's budget constraint, it is apparent that this is the mechanical result of the spike in capital taxation revenues and the low interest rate that prevails between period 1 and period 2. The lower post period 2 debt level helps reduce the burden of interest payments on the government's budget constraint. Capital taxes therefore play a important role in absorbing the shock: they help reduce the debt burden after a high shocks both by lowering the interest rate and by directly collecting revenues. Note that in the case where the low shock hits, capital is subsidized between period 1 and period 2. Capital taxes are therefore not used to raise revenues on average, but rather to help absorb the variations in the net present value of government expenditures.

Consumption initially increases. This is just an intertemporal substitution effect as consumers face a low interest rate between period 1 and period 2. From period 2 on, consumption is permanently lower. Labor and labor taxes also moves up permanently after period 2, hinting at the random walk properties of these variables. The fact that labor and labor taxes increase is just the result of a wealth effect, as consumption drops after period 2. Capital drops after the

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shock. From the resource constraint, one can see that this drop in investment is the mirror image of the increase in consumption stemming from interest rate manipulation and the increase in government expenditure.

Figures 6 and 7 display impulse responses for a similar exercise with respectively a low and high productivity shock. The technological shock inperiod 1 can be low – correpsonding to $s = 2$ – or high – corresponding to $s = 1$ – with probabilities 5% and 95% respectively.

Figure 8 displays a typical path for several variables in an economy with only government expenditure shock and no capital ownership. After a transition from a low shock to a high shock, the tax on capital spikes at about 200% and then reverts to a level of about 5%. That taxes on capital are positive until a new low shock hits reflects the fact that resources are scarcer today than tomorrow on average, given the possibility that a low shock occurs. Symmetrically, after a transition from a high shock to a low shock, the tax on capital spikes at about -200% and then reverts to a level of about -5%.

The volatility of capital taxes depends crucially on the period length. The reason can be explained as follows. Ultimately, the welfare costs associated with capital taxes come through the distortion of the path of consumption that they impose. The distortion associated with a one time capital tax becomes larger as the period length is increased, because consumption is distorted for a longer period. By contrast, the benefits in terms of reduced debt following a high shock decrease as shocks become less persistent per period. In the continuous time limit, the costs are zero, and the government is able to replicate the complete markets allocation with infinite taxes or subsidizes during an infinitely small period following the shock.

Figure 9 displays a typical path for several variables in the economy when the period length is set to five years. The positive and negative spikes in capital taxes are now 20% and -20%. Table 2 summarizes the statistical properties of capital and labor taxes depending on the period length. Standard deviations and autocorrelations are reported per period and not per year. The standard deviation of labor taxes – 5.1% – is close to the number reported by Chari, Christiano and Kehoe (1994) – 6%–for this case. Labor taxes are quite persistent, and more persistent when the period length is 5 years: this is the consequence of less abrupt movements in capital taxes and the capital stock when the period length is longer. Importantly, note that capital taxes hardly display any persistence. This is due to the fact that the magnitudes of capital taxes

is essentially by a difference term $\frac{\psi_s}{u_{c,s}} - \frac{\psi_{s-}}{\theta}$, which tends to remove the unit root component in $\frac{\psi_s}{u_{c,s}}$.

For an economy with only two government expenditure shocks and a one year period length, the government can replicate the complete markets allocation with a capital ownership of about 2300% of $\overline{k}^{fb}$. This position drops to 680% of $\overline{k}^{fb}$ when the period length is extended to five years. For an economy with only two productivity shocks and a one year period length, the government can replicate the complete markets allocation with a capital ownership of about -400% of $\overline{k}^{fb}$. This position drops to -157% of $\overline{k}^{fb}$ when the period length is extended to five years. The welfare gains from completing markets are small – 0.09% of lifetime consumption – albeit larger than in the quasi-linear case.

1.8 Conclusion

I have characterized optimal capital taxation and government ownership when markets are incomplete. In this context, capital taxation and ownership are two ways for the government to collect state dependent revenues and hedge its burden from distortionary taxation across states. Although I have focused on capital taxation, the insight that with incomplete markets and a policy lag, taxes acquire a direct hedging role, and that the indirect hedging consequences of the distortions they generate should be taken into account, is more general.

I have found that this hedging motive for capital taxation is always negligible: Capital taxes come with distortions through the adjustment of capital. This affects the capital tax base and labor tax revenues across states in a way that almost perfectly undoes the direct hedging benefits. In a baseline case, capital taxes are exactly zero. Away from this benchmark, the hedging component of capital taxes is always computed to be minuscule.

By contrast, capital ownership provides the government with a powerful hedging instrument. The reason is that trading, unlike taxing, does not involve distorsions. In a baseline case, I show that the government can perfectly approximate the complete markets allocation by taking an infinitely long or short position in capital. Away from this benchmark, optimal positions are large and decrease sharply when the period length is increased. Substantial benefits can be reaped from smaller positions. Government expenditure shocks call for a long position while

productivity shocks push in the direction of a short position. In a business cycle calibration, I show that productivity shocks is the leading force – resulting in an optimal long position.

Refining these propositions would require developing a more realistic model for investment – incorporating adjustment costs and time to build – and asset valuation. It would also be interesting to move away from the representative agent framework I have analyzed. Unobservable agent heterogeneity together with the government's concern for redistribution would provide an endogenous reason for the use distorsionary taxes. Finally, the large capital positions called for by the model put strain on the assumption of a benevolent government with full commitment. In this light, incorporating relevant political economy constraints into Ramsey type models of optimal taxation appears as a promising research avenue. I leave these issues for future work.

1.9   Appendix

The three terms in (1.28)

The three terms in (1.28) are

$$\tau^k = T^h(k, \tilde{b}, \theta, s_-) + T^i(k, \tilde{b}, \theta, s_-) + T^b(k, \tilde{b}, \theta, s_-)$$

where

$$T^h(k, \tilde{b}, \theta, s_-) = \frac{\frac{\mathbb{E}\{-k(1-\tau^k)F_{kk,s}u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}}{\frac{\mathbb{E}\{F_{k,s}u_{c,s}\frac{\psi_s}{u_{c,s}}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}} + \frac{\mathbb{E}\{F_{k,s}\nu_s u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}} \left[\frac{\text{Cov}\{kF_{k,s}u_{c,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{k,s}u_{c,s}|s_-\}} - \frac{\text{Cov}\{kF_{kk,s}u_{c,s}, \nu_s|s_-\}}{\mathbb{E}\{kF_{kk,s}u_{c,s}|s_-\}} \right]$$

$$T^i(k, \tilde{b}, \theta, s_-) = - \frac{\frac{\mathbb{E}\left\{\left[1+(1-\tau^k)F_{k,s}\right]u_{c,s}\left(\frac{\psi_s}{u_{c,s}} - \frac{\psi_{s-}}{\theta}\right)|s_-\right\}}{\beta\mathbb{E}\{u_{c,s}F_{k,s}|s_-\}}}{\frac{\mathbb{E}\{F_{k,s}u_{c,s}\frac{\psi_s}{u_{c,s}}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}} + \frac{\mathbb{E}\{F_{k,s}\nu_s u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}}$$

$$T^b(k, \tilde{b}, \theta, s_-) = \frac{\frac{\frac{\nu_{2,s-}}{\beta}M_{k,s_-} - \frac{\nu_{1,s-}}{\beta}\overline{M}_{k,s_-}}{\beta\mathbb{E}\{u_{c,s}F_{k,s}|s_-\}}}{\frac{\mathbb{E}\{F_{k,s}u_{c,s}\frac{\psi_s}{u_{c,s}}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}} + \frac{\mathbb{E}\{F_{k,s}\nu_s u_{c,s}|s_-\}}{\mathbb{E}\{F_{k,s}u_{c,s}|s_-\}}}$$

where ψ_{s-} is the multiplier on the resource constraint in the previous period.

Proof of Lemma 1

The consumer's problem is a convex program, and u and F are not satiated. Then the first order conditions (1.6), (1.7) and (1.8) together with (1.5) holding with equality are necessary and sufficient for an optimum in the consumer's problem. It is straightforward to see that (1.1) and (1.3) imply that (1.5) holds with equality – a version of Walras law.

A derivation of (1.26) and (1.28) using a Lagrangian approach

The approach in the text relies on the assumption that the value function is differentiable. An alternative to making that assumption is to approach the task of characterizing the Ramsey allocation by composing a Lagrangian for the Ramsey problem. I keep notations as close as possible to those in the text.

I attach stochastic processes $\{\beta^t \Pr(s^t)\mu_t, \beta^t \Pr(s^t)\nu_t, \beta^t \Pr(s^t)\nu_{1,t}, \beta^t \Pr(s^t)\nu_{2,t}, \beta^t \Pr(s^t)\psi_t\}_{t\geq 0}$ to the constraints

$$u_{c,t} = \mathbb{E}_t \left\{ \beta u_{c,t+1} \left[1 + \left(1 - \tau^k_{t+1}\right) F_{k,t+1} \right] \right\}, \qquad \forall t \geq 0 \text{ and } s^t \in \mathbb{S}^t,$$

$$\tilde{b}_{t-1} \frac{u_{c,t}}{\beta \mathbb{E}_{t-1}\{u_{c,t}\}} + g_t u_{c,t} \leq l_t F_{l,t} u_{c,t} + l_t u_{l,t} + \tau^k_t k_{t-1} F_{k,t} u_{c,t} + \tilde{b}_t, \qquad \forall t \geq 0 \text{ and } s^t \in \mathbb{S}^t,$$

$$\underline{M}(k_t, u_{c,t}, s_t) \leq \tilde{b}_t \leq \overline{M}(k_t, u_{c,t}, s_t), \qquad \forall t \geq 0 \text{ and } s^t \in \mathbb{S}^t \text{ and}$$

$$c_t + g_t + k_t \leq F_t + k_{t-1}, \qquad \forall t \geq 0 \text{ and } s^t \in \mathbb{S}^t.$$

Then the Lagrangian for the Ramsey problem can be represented as

$$\begin{aligned}
L\left(b^g_{-1}, r_0, k_0, \tau^k_0\right) &= \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t u_t \\
&\quad \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t \left[-\mu_t u_{c,t} + \mu_{t-1} u_{c,t} \left(1 + \left(1 - \tau^k_t\right) F_{k,t} \right) \right] \\
&\quad + \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t \nu_t \left(-\tilde{b}_{t-1} \frac{u_{c,t}}{\beta \mathbb{E}_{t-1}\{u_{c,t}\}} - g_t u_{c,t} + l_t F_{l,t} u_{c,t} + l_t u_{l,t} + \tau^k_t k_{t-1} F_{k,t} u_{c,t} + \tilde{b}_t \right) \\
&\quad + \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t \left[\nu_{2,t} \left(\tilde{b}_t - \underline{M}(k_t, u_{c,t}, s_t) \right) + \nu_{1,s} \left(\overline{M}(k_t, u_{c,t}, s_t) - \tilde{b}_t \right) \right] \\
&\quad + \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t \psi_t \left(F_t + k_{t-1} - c_s - g_t - k_t \right)
\end{aligned}$$

where I have introduced for notational convenience the multiplier μ_{-1}, which I impose to

be zero: $\mu_{-1} = 0$.

I am now in position to derive the first order conditions associated with this Lagrangian.
The first order condition for $\tilde{b}_t$ is

$$\nu_t - \frac{\mathbb{E}_t \left\{ u_{c,t} \nu_{t+1} \right\}}{\mathbb{E}_t \left\{ u_{c,t} \right\}} + \nu_{2,t} - \nu_{1,t} = 0$$

which proves equation (1.26).

The formula for the optimal tax on capital (1.28) can be derived by combining the first
order conditions for τ_t^k and k_{t-1}. The first order condition for τ_t^k can be written as

$$-\mu_{t-1} \mathbb{E}_{t-1} \left\{ u_{c,t} F_{k,t} \right\} + \mathbb{E}_{t-1} \left\{ u_{c,t} k_{t-1} F_{k,t} \nu_t \right\} = 0 \qquad (1.47)$$

For $t \geq 1$, the first order condition for k_{t-1} is

$$
\begin{aligned}
0 = & \ \mu_{t-1} \mathbb{E}_{t-1} \left\{ \beta u_{c,t} \left(1 - \tau_t^k \right) F_{kk,t} \right\} + \mathbb{E}_{t-1} \left\{ \beta \nu_t \left(l_t F_{kl,t} u_{c,t} + \tau_t^k F_{k,t} u_{c,t} + \tau_t^k k_{t-1} F_{kk,t} u_{c,t} \right) \right\} \\
& - \nu_{2,t-1} \underline{M}_{k,t-1} + \nu_{1,t-1} \overline{M}_{k,t-1} + \mathbb{E}_{t-1} \left\{ \beta \psi_t (1 + F_{k,t}) \right\} - \psi_{t-1}
\end{aligned}
$$

I use (1.47) to replace μ_{t-1}. I then use the constant return to scale assumption to replace
$l_t F_{kl,t}$ by $-k_{t-1} F_{kk,t}$, to rearrange (1.47) as follows

$$
\begin{aligned}
0 = & \ \left(1 - \tau_t^k \right) \frac{\mathbb{E}_{t-1} \left\{ u_{c,t} k_{t-1} F_{k,t} \nu_t \right\}}{\mathbb{E}_{t-1} \left\{ u_{c,t} F_{k,t} \right\}} \mathbb{E}_{t-1} \left\{ \beta u_{c,t} F_{kk,t} \right\} + \tau_t^k \mathbb{E}_{t-1} \left\{ \beta \nu_t F_{k,t} u_{c,t} \right\} - (1 - \tau_t^k) \mathbb{E}_{t-1} \left\{ \beta \nu_t k_{t-1} F_{kk,t} u_{c,t} \right\} \\
& + \tau_t^k \mathbb{E}_{t-1} \left\{ \beta \psi_t F_{k,t} \right\} + \mathbb{E}_{t-1} \left\{ \frac{\psi_t}{u_{c,t}} \beta (1 + (1 - \tau_t^k) F_{k,t}) u_{c,t} \right\} - \frac{\psi_{t-1}}{u_{c,t-1}} \mathbb{E}_{t-1} \left\{ \beta (1 + (1 - \tau_t^k) F_{k,t}) u_{c,t} \right\} \\
& - \nu_{2,t-1} \underline{M}_{k,t-1} + \nu_{1,t-1} \overline{M}_{k,t-1}
\end{aligned}
$$

which in turn can be rewritten as

$$\tau_t^k = T_t^h + T_t^i + T_t^b$$

where

$$T_t^h = \frac{\frac{\mathbb{E}_{t-1}\{-k_{t-1}(1-\tau_t^k)F_{kk,t}u_{c,t}\}}{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\}}}{\frac{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\frac{\psi_t}{u_{c,t}}\}}{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\}} + \frac{\mathbb{E}_{t-1}\{F_{k,t}\nu_t u_{c,t}\}}{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\}}} \left[\frac{\mathbb{E}_{t-1}\{k_{t-1}F_{k,t}u_{c,t},\nu_t\}}{\mathbb{E}_{t-1}\{k_{t-1}F_{k,t}u_{c,t}\}} - \frac{\mathbb{E}_{t-1}\{k_{t-1}F_{kk,t}u_{c,t},\nu_t\}}{\mathbb{E}_{t-1}\{k_{t-1}F_{kk,t}u_{c,t}\}} \right]$$

$$T_t^i = -\frac{\frac{\mathbb{E}_{t-1}\left\{[1+(1-\tau^k)F_{k,t}]u_{c,t}\left(\frac{\psi_t}{u_{c,t}}-\frac{\psi_{t-1}}{u_{c,t-1}}\right)\right\}}{\beta\mathbb{E}_{t-1}\{u_{c,t}F_{k,t}\}}}{\frac{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\frac{\psi_t}{u_{c,t}}\}}{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\}} + \frac{\mathbb{E}_{t-1}\{F_{k,t}\nu_t u_{c,t}\}}{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\}}}$$

and

$$T_t^b = \frac{\frac{\frac{\nu_{2,t-1}}{\beta}M_{k,t-1}-\frac{\nu_{1,t-1}}{\beta}\overline{M}_{k,t-1}}{\beta\mathbb{E}_{t-1}\{u_{c,t}\}}}{\frac{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\frac{\psi_t}{u_{c,t}}\}}{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\}} + \frac{\mathbb{E}_{t-1}\{F_{k,t}\nu_t u_{c,t}\}}{\mathbb{E}_{t-1}\{F_{k,t}u_{c,t}\}}}$$

This in turn shows that (1.28) holds.

Proof of Proposition 1

If $F(k,l,s) = A(s)k^\alpha l^{1-\alpha}$, then $kF_{kk,s} = (\alpha-1)F_{k,s}$ so that

$$\frac{\text{Cov}\{kF_{kk,s},\nu_s|s_-\}}{\mathbb{E}\{kF_{kk,s}|s_-\}} = \frac{(\alpha-1)\text{Cov}\{F_{k,s},\nu_s|s_-\}}{(\alpha-1)\mathbb{E}\{F_{k,s}|s_-\}} = \frac{\text{Cov}\{kF_{k,s},\nu_s|s_-\}}{\mathbb{E}\{kF_{k,s}|s_-\}}$$

By (1.28), this implies that $\tau^k = 0$.

Since (1.28) applies from $t = 1$ on, this shows that $\tau_t^k = 0$ for all $t \geq 1$.

Proof of Proposition 2

I first prove the following lemma

Lemma 16 *Consider $\{x_n, y_n, z_n\}_{1\leq n\leq N} \in \mathbb{R}_{++}^N$, and probability distribution $\{p_n\}_{1\leq n\leq N}$ with $p_n > 0$ for all n. Assume that there exists n and n' such that $z_n \neq z_{n'}$. The following holds: (i) assume that $x_n < x_{n'}$ if and only if $z_n > z_{n'}$, then $\frac{\mathbb{E}\{x_n y_n z_n\}}{\mathbb{E}\{x_n y_n\}} < \frac{\mathbb{E}\{y_n z_n\}}{\mathbb{E}\{y_n\}}$; (ii) assume that $x_n < x_{n'}$ if and only if $z_n < z_{n'}$, then $\frac{\mathbb{E}\{x_n y_n z_n\}}{\mathbb{E}\{x_n y_n\}} > \frac{\mathbb{E}\{y_n z_n\}}{\mathbb{E}\{y_n\}}$.*

Proof.

$$\frac{\mathbb{E}\{x_n y_n z_n\}}{\mathbb{E}\{x_n y_n\}} < \frac{\mathbb{E}\{y_n z_n\}}{\mathbb{E}\{y_n\}}$$

if and only if

$$\mathbb{E}\{x_n y_n z_n\}\mathbb{E}\{y_n\} < \mathbb{E}\{y_n z_n\}\mathbb{E}\{x_n y_n\}$$

if and only if

$$\sum_{n,n'} p_n p_{n'} x_n y_n z_n y_{n'} < \sum_{n,n'} p_n p_{n'} y_n z_n x_{n'} y_{n'}$$

if and only if

$$0 < \sum_{n,n'} p_n p_{n'} y_n z_n y_{n'} [x_{n'} - x_n]$$

if and only if

$$0 < \frac{1}{2} \sum_{n \neq n'} p_n p_{n'} y_n y_{n'} \left[z_n - z_{n'} \right] \left[x_{n'} - x_n \right]$$

which trivially proves claims (i) and (ii) in the lemma. ■

Assume F is CES with elasticity of substitution σ with Hicks neutral technology shocks:

$$F(k,l) = A(s) \left[\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha) l^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

Then

$$F_k(k,l) = \alpha A(s) k^{\frac{-1}{\sigma}} \left[\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha) l^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}}$$

and

$$-k F_{kk}(k,l) = \frac{\frac{1}{\sigma}}{\frac{\alpha}{1-\alpha} \left(\frac{k}{l}\right)^{\frac{\sigma-1}{\sigma}} + 1} \alpha A(s) k^{\frac{-1}{\sigma}} \left[\alpha k^{\frac{\sigma-1}{\sigma}} + (1-\alpha) l^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}}$$

so that

$$-k F_{kk}(k,l) = \frac{\frac{1}{\sigma}}{\frac{\alpha}{1-\alpha} \left(\frac{k}{l}\right)^{\frac{\sigma-1}{\sigma}} + 1} F_k(k,l)$$

Proposition 2 now follows. Consider for example case (i). Then $\dfrac{\frac{1}{\sigma}}{\frac{\alpha}{1-\alpha} \left(\frac{k}{l_s}\right)^{\frac{\sigma-1}{\sigma}} + 1}$ is a decreasing function of ν_s. Lemma 3 shows that

$$\frac{\mathbb{E}\{-k F_{kk,s}, \nu_s | s_-\}}{\mathbb{E}\{-k F_{kk,s} | s_-\}} < \frac{\mathbb{E}\{k F_{k,s}, \nu_s | s_-\}}{\mathbb{E}\{k F_{k,s} | s_-\}}$$

which proves that $\tau^k > 0$. Cases (ii), (iii) and (iv) in Proposition 2 can be proved along the same lines.

I then prove the second part of the proposition. If $F(k, l, s) = A(s)k^\alpha l^{1-\alpha} - \delta k$, then $kF_{kk,s} = (\alpha - 1)(F_{k,s} + \delta)$ so that

$$\frac{\text{Cov}\{kF_{kk,s}, \nu_s | s_-\}}{\mathbb{E}\{kF_{kk,s} | s_-\}} = \frac{(\alpha - 1)\text{Cov}\{F_{k,s}, \nu_s | s_-\}}{(\alpha - 1)\delta + (\alpha - 1)\mathbb{E}\{F_{k,s} | s_-\}} = \frac{\text{Cov}\{kF_{k,s}, \nu_s | s_-\}}{\mathbb{E}\{kF_{k,s} | s_-\} + \delta}$$

Hence

$$\frac{\text{Cov}\{kF_{kk,s}, \nu_s | s_-\}}{\mathbb{E}\{kF_{kk,s} | s_-\}} < \frac{\text{Cov}\{kF_{k,s}, \nu_s | s_-\}}{\mathbb{E}\{kF_{k,s} | s_-\}}$$

if and only if $\text{Cov}\{kF_{k,s}, \nu_s | s_-\} > 0$. This proves the second part of the proposition.

Proof of Lemma 2

It is clear that $\widetilde{V}$ is decreasing in $\widetilde{b}$. Since $\widetilde{V}$ is differentiable, this is equivalent to $\widetilde{V}_b \leq 0$.

Since, $\beta \widetilde{V}_{b,t} = -\nu_t + \nu_{1,t}$, this proves that $\nu_t - \nu_{1,t} \geq 0$.

Under natural debt limits, (1.21) becomes

$$\nu_{s-} = \mathbb{E}\{\nu_s | s_-\} + \nu_{1,s-}$$

which I can rewrite as

$$\nu_{s-} - \nu_{1,s-} = \mathbb{E}\{\nu_s - \nu_{1,s} | s_-\} + \mathbb{E}\{\nu_{1,s} | s_-\}$$

This proves that $\left\{\widetilde{V}_{b,t}\right\}_{t \geq 1}$ is a nonnegative supermartingale. Therefore, the supermartingale convergence theorem (see Lòeve (1977)) asserts that $\widetilde{V}_{b,t}$ converges almost surely to a finite nonnegative random variable $\widetilde{V}_{b,\infty}$. Since $\widetilde{V}$ is continuously differentiable and concave, this implies in turn that $\widetilde{b}_t$ converges to a finite random variable $\widetilde{b}_\infty$. Since policy functions in (1.15) are continuous, this implies that every point $\widetilde{b}$ in the support of $\widetilde{b}_\infty$ is such that for every states s and s_- in the unique ergodic set of $\{s_t\}_{t \geq 0}$, $\widetilde{b}'_s(\widetilde{b}, s_-) = \widetilde{b}$. Clearly, this is only possible if $\widetilde{b} = -\underline{M}_n$. This proves Lemma 2.

Proof of Proposition 3

If $F(k, l, s) = A(s)k^\alpha l^{1-\alpha}$, then $kF_{kk,s} = (\alpha - 1)F_{k,s}$ so that

$$\frac{\text{Cov}\{kF_{kk,s}u_{c,s}, \nu_s | s_-\}}{\mathbb{E}\{kF_{kk,s}u_{c,s} | s_-\}} = \frac{(\alpha - 1)\text{Cov}\{F_{k,s}u_{c,s}, \nu_s | s_-\}}{(\alpha - 1)\mathbb{E}\{F_{k,s}u_{c,s} | s_-\}} = \frac{\text{Cov}\{kF_{k,s}u_{c,s}, \nu_s | s_-\}}{\mathbb{E}\{kF_{k,s}u_{c,s} | s_-\}}$$

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Proof of Proposition 4

Note that $T_{2,t} > 0$ if and only if

$$\frac{\mathbb{E}_{t-1}\left\{\left[1 + \left(1 - \tau^k\right) F_{k,t}\right] u_{c,t}\frac{\psi_t}{u_{c,t}}\right\}}{\mathbb{E}_{t-1}\left\{\left[1 + \left(1 - \tau^k\right) F_{k,t}\right] u_{c,t}\right\}} > \frac{\psi_{t-1}}{u_{c,t-1}}$$

$T_{2,t} < 0$ if and only if

$$\frac{\mathbb{E}_{t-1}\left\{\left[1 + \left(1 - \tau^k\right) F_{k,t}\right] u_{c,t}\frac{\psi_t}{u_{c,t}}\right\}}{\mathbb{E}_{t-1}\left\{\left[1 + \left(1 - \tau^k\right) F_{k,t}\right] u_{c,t}\right\}} < \frac{\psi_{t-1}}{u_{c,t-1}}$$

and $T_{2,t} = 0$ if and only if

$$\frac{\mathbb{E}_{t-1}\left\{\left[1 + \left(1 - \tau^k\right) F_{k,t}\right] u_{c,t}\frac{\psi_t}{u_{c,t}}\right\}}{\mathbb{E}_{t-1}\left\{\left[1 + \left(1 - \tau^k\right) F_{k,t}\right] u_{c,t}\right\}} = \frac{\psi_{t-1}}{u_{c,t-1}}$$

Let me denote $\left[1 + \left(1 - \tau^k\right) F_{k,t}\right] u_{c,t}$ by K_t and $\frac{\psi_t}{u_{c,t}}$ by ξ_t. Since policy functions in (1.10) are continuous, K_t and ξ_t are continuous functions $K(x_t)$ and $\xi(x_t)$ of $x_t = \{k_t, \tilde{b}_t, u_{c,t}, s_t\}$. Call $\pi(x', x)$ the transition function.

Let Υ be the operator mapping the space F of continuous functions of x into itself, defined by

$$\Upsilon(f)(x) \equiv \frac{\int f(x')K(x')\pi(x', x)}{\int K(x')\pi(x', x)}$$

Therefore, $T_2(x) > 0$ if and only if $\Upsilon(\xi)(x) > \xi(x)$, $T_2(x) < 0$ if and only if $\Upsilon(\xi)(x) < \xi(x)$ and $T_2(x) = 0$ if and only if $\Upsilon(\xi)(x) = \xi(x)$. Thus the sign of T_2 is entirely determined by the sign of $\Upsilon(\xi) - \xi$.

Suppose $P^\infty\{\Upsilon(\xi)(x) \leq \xi(x)\} = 1$. Let $\underline{\xi} \equiv \sup\{\xi, P^\infty\{\xi(x) \geq \xi\} = 1\}$. Define

$$\Gamma = \left\{x, \lim_{j \to \infty} \Pr\{x_t \in A | x_0 = x\} = P^\infty\{A\}\right\}$$

$A_t = \{x, \xi(x) \geq \xi^t(x)\}$, $B_t = \{x, \Pr\{\xi^t(x) \geq \underline{\xi}\} = 1\}$, $A = \cap_{t=0}^\infty A_t$ and $B = \cap_{t=0}^\infty B_t$. Then $P^\infty\{A_t\} = P^\infty\{B_t\} = P^\infty\{\Gamma\} = 1$. Hence $P^\infty\{A \cap B \cap \Gamma\} = 1$. For every $\varepsilon > 0$, there exists $x^\varepsilon \in A \cap B \cap \Gamma$ such that $\xi(x^\varepsilon) < \underline{\xi} + \varepsilon$. This implies that $\Pr\{\underline{\xi} \leq \xi^t(x^\varepsilon) < \underline{\xi} + \varepsilon\} = 1$. Using

the ergodicity of x_t, this implies that

$$P^\infty\{\underline{\xi} \leq \xi(x) < \underline{\xi} + \varepsilon\} = \lim_{t \to \infty} \Pr\{\underline{\xi} \leq \xi^t(x^\varepsilon) < \underline{\xi} + \varepsilon\} = 1$$

Since this is true for all $\varepsilon > 0$, this in turn implies that $P^\infty\{\xi(x) = \underline{\xi}\} = 1$.

Similarly $P^\infty\{\Upsilon(\xi)(x) \geq \xi(x)\} = 1$ implies $P^\infty\{\xi(x) = \overline{\xi}\} = 1$ where $\overline{\xi} \equiv \inf\{\xi, P^\infty\{\xi(x) \leq \xi\} = 1\}$. This proves Proposition 4.

Proof of Proposition 5

Consider the complete markets allocation $\{k_c, l_c, \widetilde{b}'_c\}$ in state $(\widetilde{b}, s_-)$. Labor l_c and net government liabilities in the end of the period $\widetilde{b}'_c$ are constant across states of the world s. Define $X_s \equiv \frac{1}{\beta}\widetilde{b} + g_s - \widetilde{b}'_c$.

Consider next the incomplete markets environment. Denote by $R(k, l_s) \equiv l_s F_{l,s} + l_s H_{l,s}$ the revenues from labor taxation in state s. Set $\tau^k = 0$ and solve, for a given k_g, the following system in $\{k, l_s\}$.

$$\mathbb{E}\{R(k, l_s)|s_-\} = \mathbb{E}\{X_s|s_-\} \tag{1.49}$$

$$(F_k(k, l_s) - r) + \frac{R(k, l_s)}{k_g} = \frac{X_s}{k_g}, \qquad \forall s \in \mathbb{S} \tag{1.50}$$

Note that if (1.50) holds and $|k_g| < \infty$, then (1.49) is equivalent to $\mathbb{E}\{F_k(k, l_s)|s_-\} = r$. If (1.50) holds and $|k_g| = \infty$, then $\mathbb{E}\{F_k(k, l_s)|s_-\} = r$ holds automatically.

Denote by $\{k(k_g), l_s(k_g)\}$ the solution: together with $\widetilde{b}'_{c,s}(k_g) \equiv \widetilde{b}'_c$ for all s, the variables $\{k(k_g), l_s(k_g), \widetilde{b}'_{c,s}\}$ satisfy the constraints in (1.15). Then

$$\lim_{k_g \to \infty} \{k(k_g), l_s(k_g)\} = \lim_{k_g \to -\infty} \{k(k_g), l_s(k_g)\} = \{k_c, l_c\}$$

Therefore, by choosing k_g large and picking $\{k(k_g), l_s(k_g), \widetilde{b}'_{c,s}\}$, the government can approximate as well as desired the complete markets allocation in state $(\widetilde{b}, s_-)$. By doing this in every state and date, the government can therefore perfectly approximate the complete markets allocation. This proves proposition 5.

1.10 Figures and Tables

Table 1: Optimal capital ownership (as a fraction of k) in the quasi-linear model

	Government shocks	Productivity shocks	Government and productivity shocks
Period length = 1 year	infinity	-295%	-295%
Period length = 5 years	infinity	-59%	-59%

Table 2: Summary statistics in the general model

	Period length = 1 year	Period length = 5 years
mean of labor taxes	26.2%	26.9%
per period std of labor taxes (% of mean)	5.1%	5.2%
per period autocorrelation of labor taxes	0.62	0.94
mean of capital taxes	-4.8%	1.2%
per period std of capital taxes	0.72	0.10
per year autocorrelation of capital taxes	-0.06	0.03
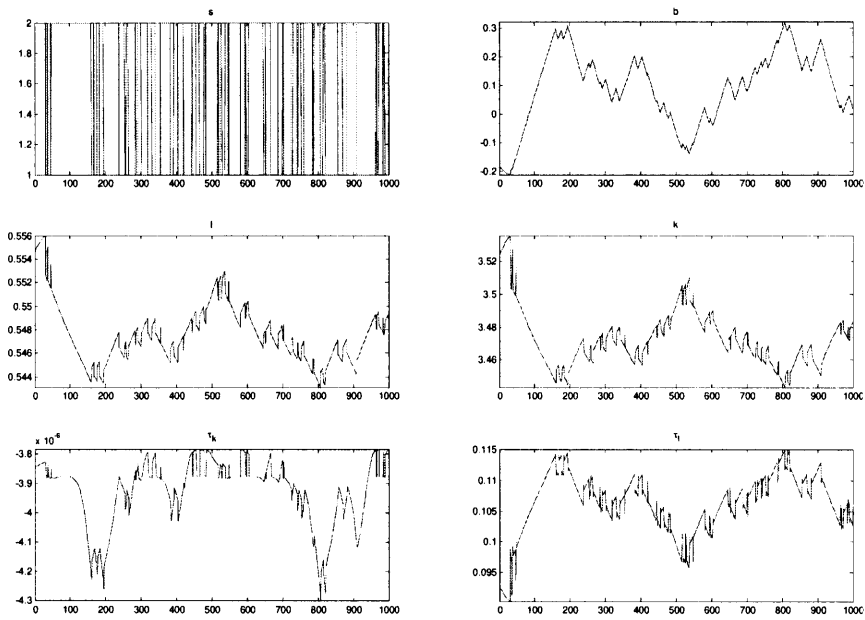
Figure 1: A typical path, quasi-linear preferences, government expenditure shocks.
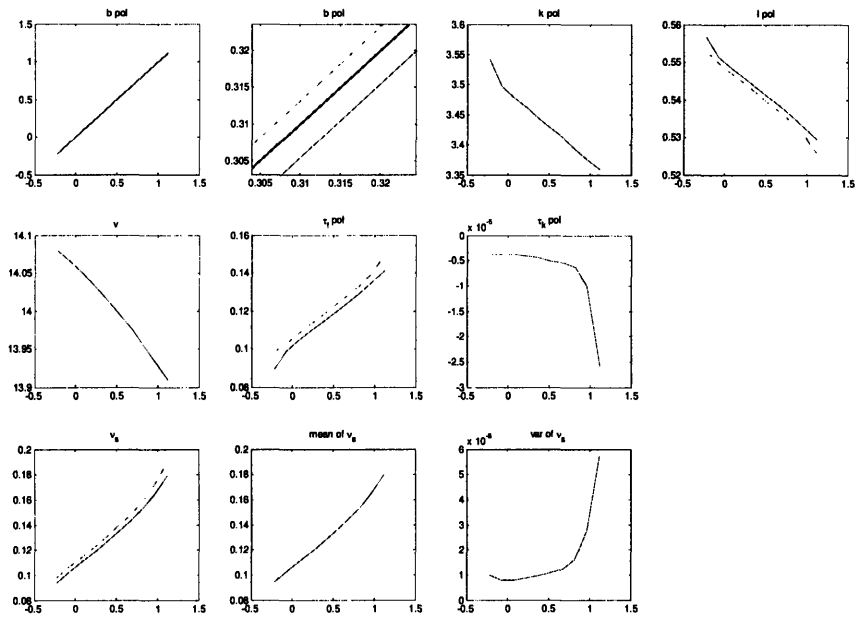
Figure 2: Policy functions, quasi-linear preferences, government expenditure shocks, s_- low.
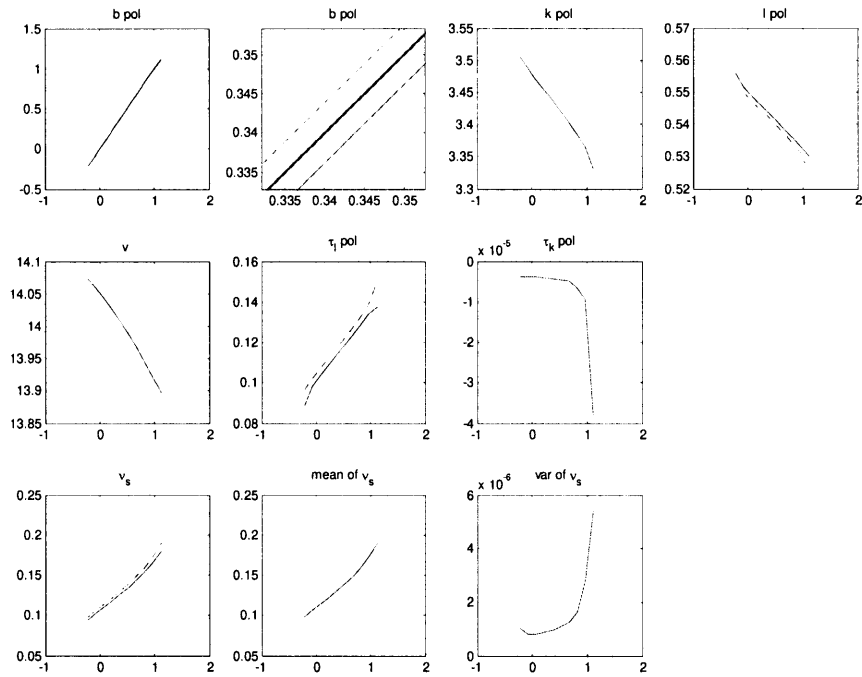
Figure 3: Policy functions, quasi-linear preferences, goverment expenditure shocks, s_- high.
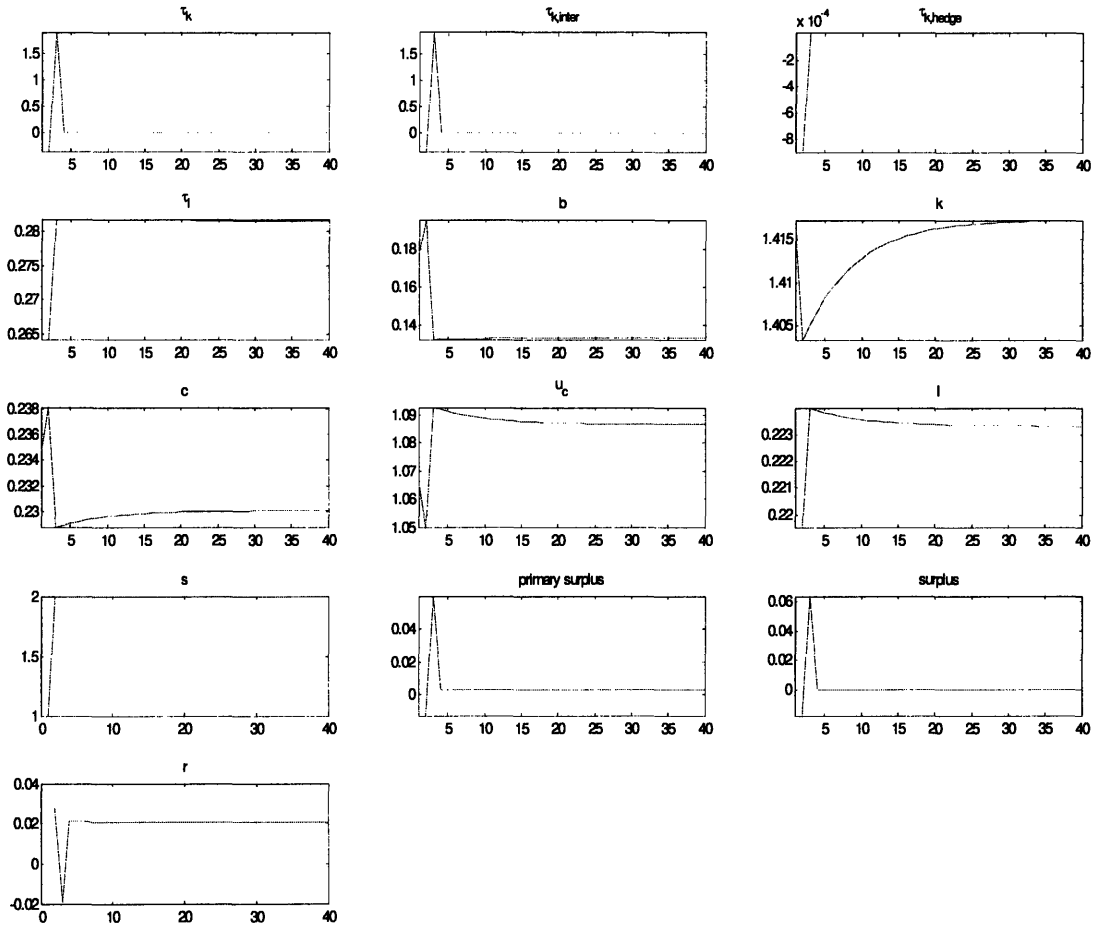
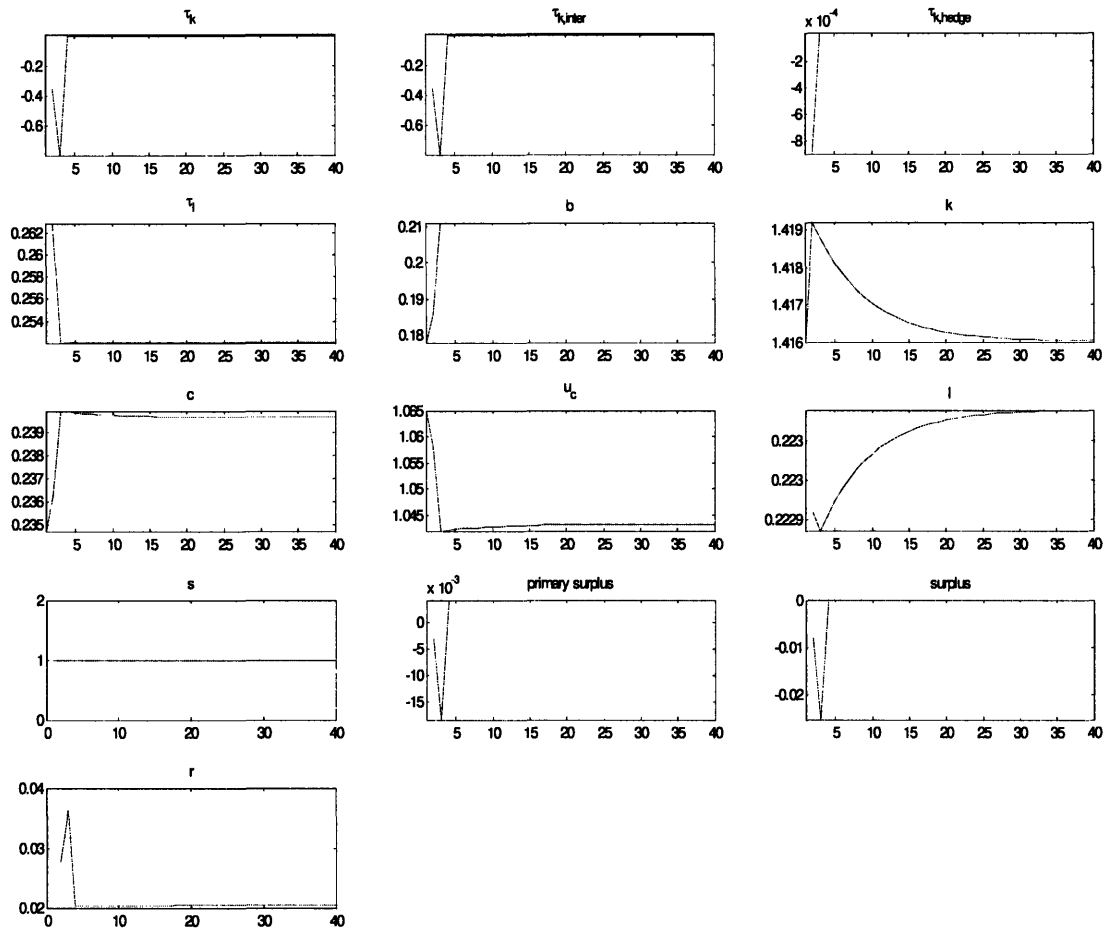Figure 4: Impulse responses, high permanent government expenditure shock

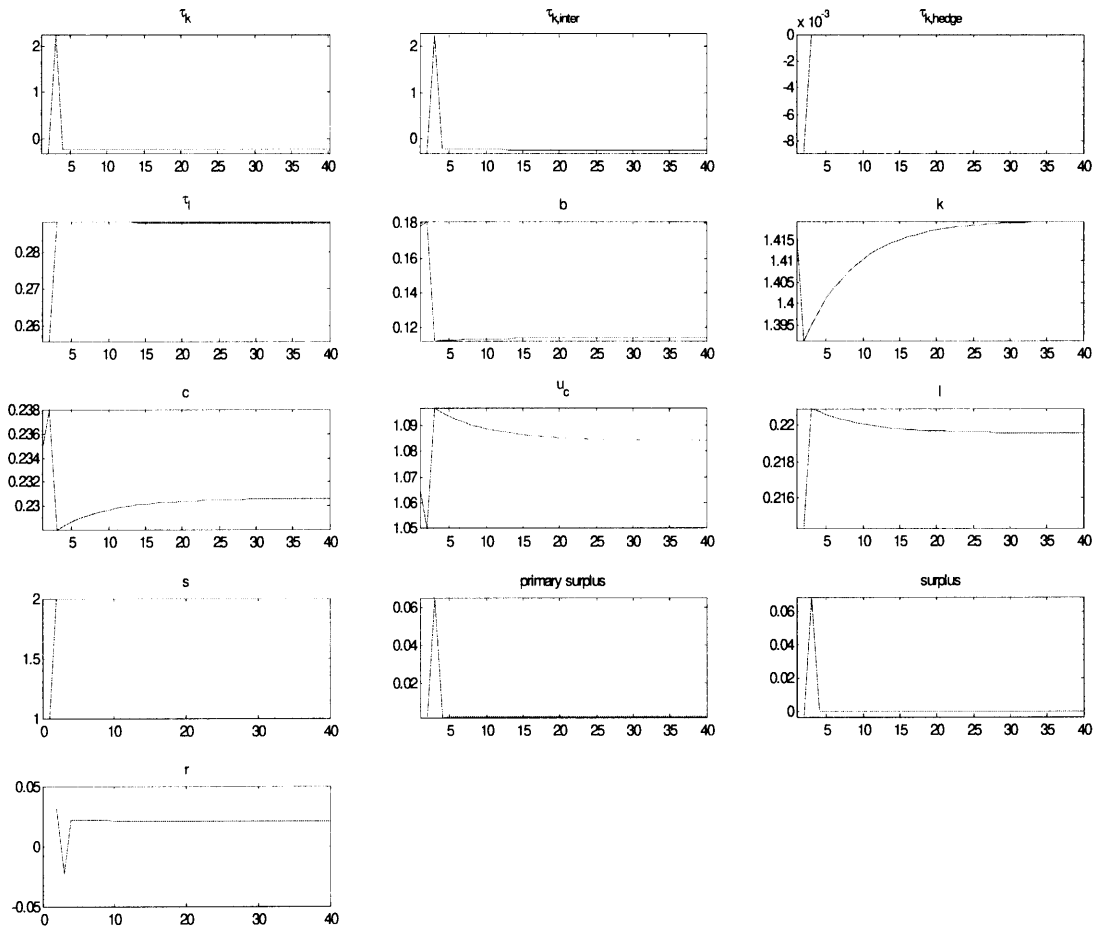Figure 5: Impulse responses, low permanent government expenditure shock

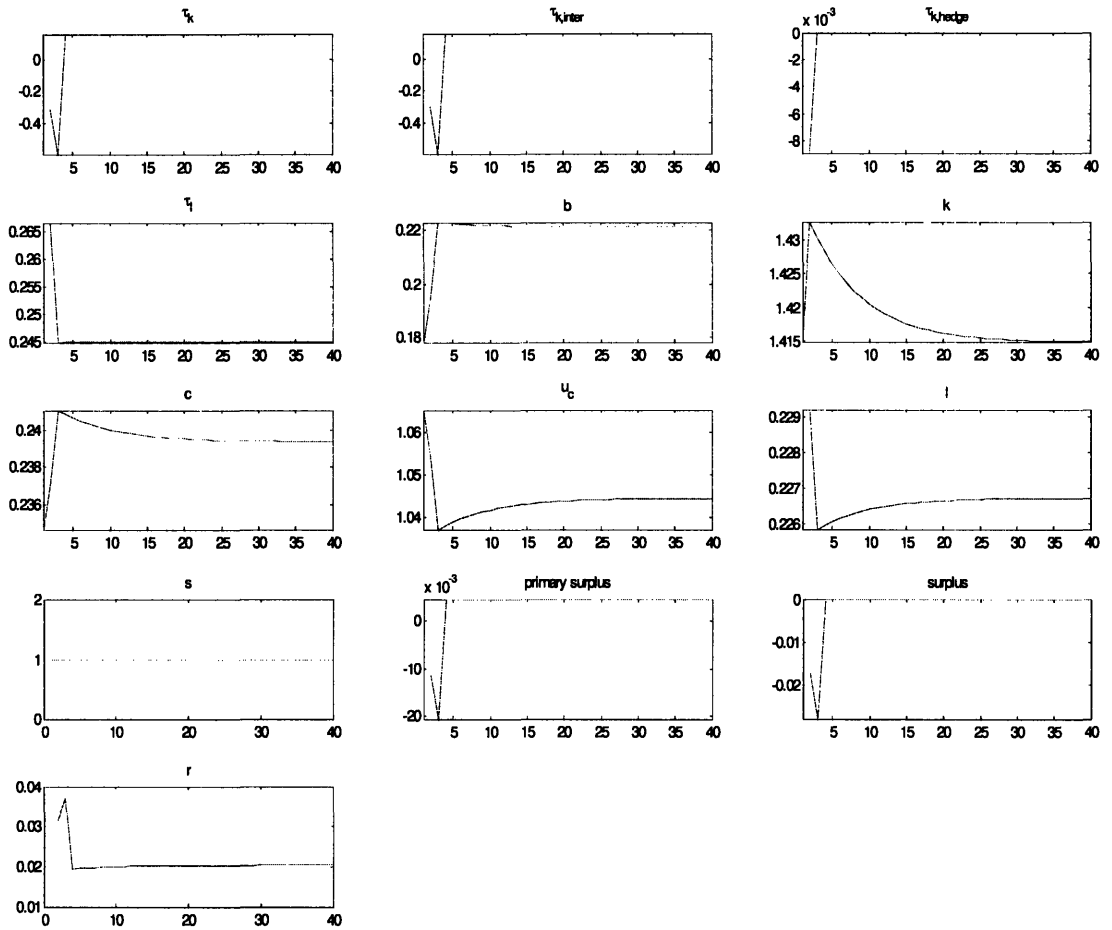Figure 6: Impulse responses, low permanent technology shock

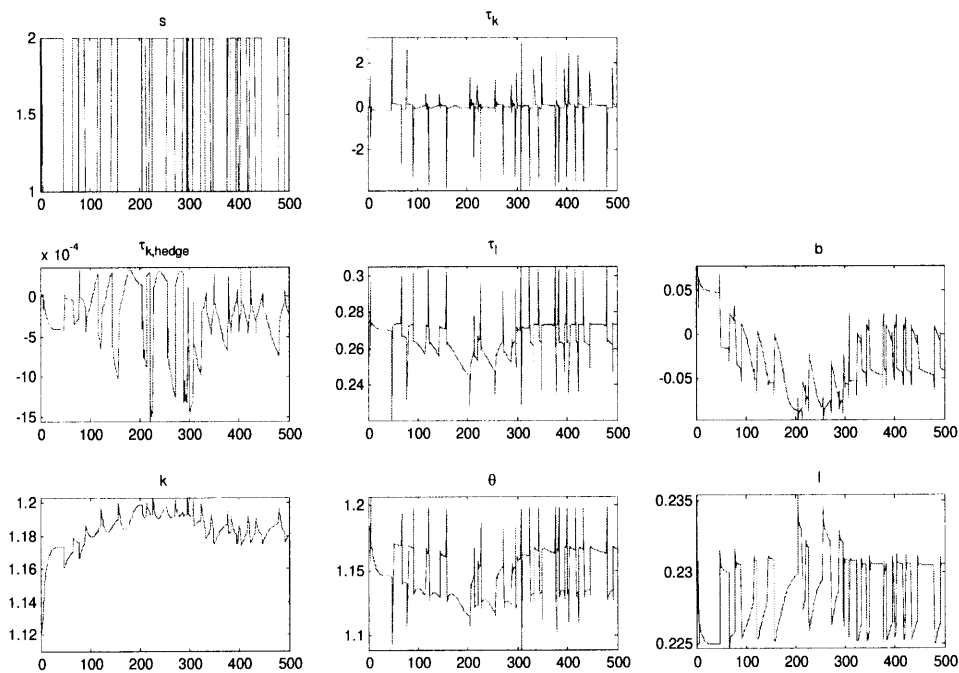Figure 7: Impulse responses, high permanent technology shock

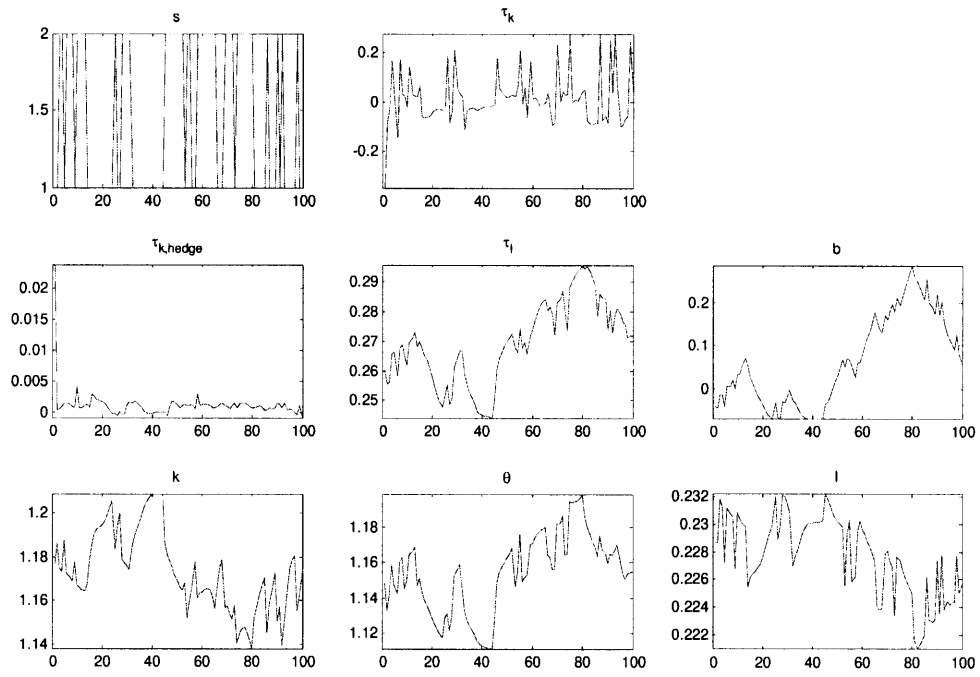Figure 8: A typical path: government expenditure shocks, one year period length

Figure 9: A typical path: government expenditure shocks, five years period length

Bibliography

[1] Aiyagari, S.R, Marcet, A., Sargent, T.J, and Juha Seppälä (2002): "Optimal Taxation without State Contingent Debt", *Journal of Political Economy*, 110, 1220-1254.

[2] Angeletos, George-Marios (2002): "Fiscal Policy with Non-Contingent Debt and the Optimal Maturity Structure", *Quarterly Journal of Economics,* 117, 1105-1131.

[3] Atkeson, A., Chari, V.V., and Patrick J. Kehoe (1999): "Taxing Capital Income: A Bad Idea", *Federal Reserve Bank of Minneapolis Quarterly Review,* 23, 3-17..

[4] Atkinson, A.B and Joseph E. Stiglitz (1972): "The Structure of Indirect Taxation and Economic Efficiency", *Journal of Public Economics,* 1, 97-119.

[5] Barro, Robert J. (1979): "On the Determination of Public Debt", *Journal of Political Economy,* 87, 940-971.

[6] Bohn, Henning (1990): "Tax Smoothing with Financial Instruments", *American Economic Review,* 80, 1217-1230.

[7] Buera, F. and Juan P. Nicolini (2004): "Optimal Maturity Structure of Government Debt without State Contingent Bonds", *Journal of Monetary Economics,* 51, 531-554.

[8] Chamley, Christophe (1986): "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives", *Econometrica,* 54, 607-622.

[9] Chari, V.V., Christiano, L.J. and Patrick J. Kehoe (1994): "Optimal Fiscal Policy in a Business Cycle Model", *Journal of Political Economy,* 102, 617-652.

[10] Farhi, E. and Ivan Werning (2005): "The Optimal Maturity Structure of Government Debt when Markets are Incomplete", *mimeo MIT.*

[11] Judd, Kenneth L. (1985): "Redistributive Taxation in a Simple Perfect Foresight Model", *Journal of Public Economics, 28, 59-83.*

[12] Loève, M. (1977): "Probability Theory II", *Springer Verlag, 4th edition.*

[13] Lucas, R. and Nancy Stokey (1983): "Optimal Fiscal and Monetary Policy in an Economy without Capital", *Journal of Monetary Economics, 12, 55-93.*

[14] Ramsey, Frank P. (1927): "A Contribution to the Theory of Taxation", *Economic Journal,* XLVIII, 47-61.

[15] Stiglitz, Joseph E. (1969): "The Effects of Income, Wealth, and Capital Gains Taxation on Risk-Taking", *Quarterly Journal of Economics, 83, 263-283.*

[16] Werning, Ivan (2005): "Tax Smoothing with Incomplete Markets", *MIT mimeo.*

[17] Zhu, Xiaodong (1992): "Optimal Fiscal Policy in a Stochastic Growth Model", *Journal of Economic Theory, 58, 250-289.*

Chapter 2

Progressive Estate Taxation[1]

Introduction

Arguably, the biggest risk in life is the family one is born into. In particular, newborns partly inherit the luck, good or bad, of their parents and ancestors, passed on by the wealth accumulated within their dynasty. This makes them concerned not only with their own uncertain skills and earning potential, but also with that of their progenitors. They value insurance, from behind the veil of ignorance, against these risks. On the other hand, altruistic parents are partly motivated to work because of the impact their effort can have, through bequests, on their children's wellbeing. The intergenerational transmission of welfare determines the balance between insuring newborns and parental incentives.

One instrument societies use to regulate the degree of this intergenerational transmission is estate taxation. This paper examines the optimal design of the estate tax by characterizing

Pareto efficient allocations in an economy featuring the tradeoff between incentives of parents and insurance of newborns. Our main result is that estate taxation should be progressive: fortunate parents should face a higher marginal tax rate on their bequests.

We begin with a two-period Mirrleesian economy with non-overlapping generations linked by parental altruism; we then extend our analysis to an infinite horizon economy like Atkeson and Lucas (1995) and Albanesi and Sleet (2004). In our simplest economy, a continuum of parents live during the first period. In the second period each is replaced by a single descendent and parents are altruistic towards this child. Parents work, consume and bequeath; children simply consume.[2] Following Mirrlees (1971), parents first observe a random productivity draw and then exert work effort. Both productivity and work effort are private information; only output, the product of the two, is publicly observable. We study the entire set of constrained Pareto efficient allocations and derive their implications for marginal tax rates.

For this economy, if the social welfare criterion is assumed to coincide with the parent's expected utility, then Atkinson and Stiglitz (1976) celebrated uniform-taxation result applies, and the optimal estate tax is zero. That is, when no direct weight is placed on the welfare of children, income should be taxed nonlinearly Mirrlees (1971), but bequests should go untaxed. This arrangement ensures that the intertemporal consumption choice made by parents—trading off their own consumption against their child's consumption—is undistorted. As a result, the inheritability of welfare across generations is perfect: the child consumption rises one-for-one with parental consumption. In effect, efficiency dictates that altruism be exploited to provide higher incentives for parents, by manipulating their children's consumption. Inequality for the children's generation is created as a byproduct, since their expected welfare is of no direct concern.

While this describes one efficient allocation, the picture is incomplete. In this economy the parent and child are distinct individuals, albeit linked through parental altruism, a form of externality. A complete welfare analysis then requires examining the ex-ante utility of both parents and children. Our economy's Pareto frontier is peaked because the parent is altruistic towards the child, so parental welfare decreases if the child is made too miserable. The allocation

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[2] Although some readers have remarked that they find this assumption realistic, it will be relaxed when we extend the time horizon.

discussed in the previous paragraph is a particular point lying on the Pareto frontier: the peak which maximizes the welfare of parents. In this paper we explore other efficient arrangements representing points which lie on on the downward sloping section of the the Pareto frontier, to the right of its peak.

A role for estate taxation emerges: efficient allocations which lie to the right of the peak can be implemented by confronting parents with a simple tax system with separate nonlinear schedules for income and estate taxes. Our main result is that optimal estate taxation is progressive: fortunate parents face a higher marginal tax rate on their bequests. The estate-tax schedule is convex.

Progressive estate taxation emerges to insure children against their parent's luck, it lowers consumption inequality within the children's generation, while still providing some incentives to parents. Child consumption still varies with their parent's, but less than one for one. Consumption mean reverts across generations, the inheritability of welfare is imperfect. The optimal progressivity in taxes reflects this mean reversion: fortunate dynasties must face a lower net return on bequests so that they choose a consumption path declining towards the mean.

We extend the two-period model to an infinite horizon economy, where everyone lives for a single period, during which they observe a productivity draw and work, to be replaced by a single descendant in the next period. With perfect altruism, dynasties behave as if they were an infinite-lived individual.

This extension is important for at least two reasons. First, it provides a motivation for focusing on efficient allocations which do not maximize the expected utility of the very first generation—the analogues of the downward sloping section of the Pareto frontier. Indeed, for the infinite horizon, the allocation that maximizes the welfare of the first generation features everyone in distant generations converging to misery, with zero consumption Atkeson and Lucas (1992). As we show here, by extending the analysis in Farhi and Werning (2005), this result is special to placing no weight on future generations: when some weight is placed on future generations a steady state exists. Second, an infinite horizon allows us to make contact with a growing literature on dynamic Mirrleesian models, such as Golosov, Kocherlakota and Tsyvinski (2003) and Golosov, Tsyvinsky and Werning (2006). In particular, our model environment is identical to that of Albanesi and Sleet (2004).

The main difference between the two-period and infinite horizon economies is that tax implementations are more involved in the latter. We adapt Kocherlakota (2004) implementation. The progressivity of estate taxes extends to the infinite horizon setup: fortunate parents face a higher average marginal tax rate on their bequests. Indeed, the average marginal estate tax rate formula is the same as in the two-period economy.

Our stark conclusion on the progressivity of estate taxation strongly contrasts with the theoretical ambiguity in the shape of the optimal income tax schedule.Mirrlees (1971) seminal paper showed that for bounded distributions of skills the optimal marginal income tax rates are regressive at the top (see also Seade 1982, Tuomala 1990 and Ebert 1992). More recently, Diamond (1998) has shown that the opposite—progressivity at the top—is true with certain unbounded skill distributions Saez (2001). In contrast, our results on the progressivity of the estate tax do not depend on any assumptions regarding the distribution of skills.

Throughout this paper, we study an economy without capital, where aggregate consumption equals aggregate produced output plus an endowment. This no-aggregate-savings assumption allows us to focus on redistribution *within* generations and abstract from transfers *across* generations.

Farhi, Kocherlakota and Werning (2005) extend this model among several dimensions— including capital accumulation, life-cycle elements and general skill processes—and show that the main results are insensitive to this assumption.

The rest of this paper is organized in the following way.Section 1 describes the two period model environment and Section 2 introduces the associated planning problem. Our main results for this two-period economy are in Section 3. In Section 4 we describe the extension to an infinite horizon. The main results for that economy are contained in Section 5. We summarize our conclusions in Section 6.

2.1 Parent and Child: A Two Period Economy

There are two periods labelled $t = 0, 1$. The parent lives during $t = 0$ and is replaced by a single child at $t = 1$. The parent works and consumes, while the child only consumes. Thus, an allocation is a triplet of functions $(c_0(w_0), c_1(w_0), y_0(w_0))$, where c_0 and y_0 represents the

parent's consumption and output, and c_1 represents the child's consumption.

Preferences. The parent is altruistic towards the child

$$v_0 = \mathbb{E}\left[u(c_0) - h\left(\frac{y_0}{w_0}\right) + \beta v_1\right], \tag{2.1}$$

where the expectations is over w_0 and $\beta < 1$. The child's utility is simply

$$v_1 = u(c_1) \tag{2.2}$$

The utility function $u(c)$ is increasing, concave and differentiable; the disutility function $h(n)$ is assumed increasing, convex and differentiable.

Substituting (2.2) into (2.1) yields the alternative expression for the parent's utility:

$$v_0 = \mathbb{E}\left[u(c_0) + \beta u(c_1) - h\left(\frac{y_0}{w_0}\right)\right] \tag{2.3}$$

As usual, the parent's expected utility can be reinterpreted as that of a fictitious dynasty that lives for two periods and discounts at rate β.

Technology. An allocation is *resource feasible* if aggregate consumption in both periods is not greater than the sum of endowments and production.:

$$\int_0^\infty c_0(w_0)dF(w_0) \leq e_0 + \int_0^\infty y_0(w_0)dF(w_0) \tag{2.4}$$

$$\int_0^\infty c_1(w_0)dF(w_0) \leq e_1 \tag{2.5}$$

Incentives. Productivity is private information so incentives need to be provided for truthful revelation. We say that an allocation is *incentive compatible* if the parent finds it optimal to reveal her shock truthfully:

$$u(c_0(w_0)) + \beta u(c_1(w_0)) - h\left(\frac{y_0(w_0)}{w_0}\right) \geq u(c_0(w)) + \beta u(c_1(w)) - h\left(\frac{y_0(w)}{w_0}\right) \tag{2.6}$$

for all productivity realizations w_0.

2.2 Social Welfare and Efficient Allocations

We now study all constrained efficient allocations for the two-period economy introduced in the previous section. We begin with by introducing and discussing our welfare criterion. Consider the general welfare criterion

$$W \equiv v_0 + \alpha \mathbb{E} v_1, \tag{2.7}$$

which places some weight $\alpha \geq 0$ on the expected utility of children. As we vary α we can trace out the entire Pareto frontier, since the latter is convex.

Substituting equations (2.2) and (2.3) into (2.7) implies the alternative expression

$$W = \mathbb{E}[u(c_0) + (\beta + \alpha)u(c_1) - h(y_0/w_0)].$$

Thus, the social welfare function is equivalent to the parent's preferences but with a social discount factor $\hat{\beta} = \beta + \alpha$ that is higher than the private one as long as $\alpha > 0$.

The planning problem maximizes the welfare criterion W over allocations that are resource feasible and incentive compatible. Formally, the problem is

$$\max_{c_0, c_1, y_0} \int_0^\infty [u(c_0(w_0)) + \hat{\beta}u(c_1(w_1)) - h(y_0(w_0)/w_0)]dF(w_0)$$

subject to the resource constraints in equations (2.4)-(2.5) and the incentive compatibility constraints in (2.6).

It is useful to divide the planning problem into two stages. In the first stage the planner chooses the profile of output $y_0(w_0)$ and a schedule of incentives $\Delta(w_0)$, which is equal to utility from consumption $u(c_0(w_0)) + \beta u(c_1(w_0))$ up to a constant. In the second stage, the planner solves the subproblem of how best to provide the incentives $\Delta(w_0)$, using $c_0(w_0)$ and $c_1(w_0)$. The key feature is that the second stage involves no incentive constraints, these are imposed in the first stage. Formally, by introducing Δ and $\underline{U}$ the full problem can be written as

$$\max_{c_0, c_1, y_0, \Delta, \underline{U}} \int_0^\infty [u(c_0(w_0)) + \hat{\beta}u(c_1(w_1)) - h(y_0(w_0)/w_0)]dF(w_0)$$

subject to $\Delta(w) + \underline{U} = u(c_0(w_0)) + \beta u(c_1(w_0))$, the resource constraints in equations (2.4)-(2.5)

and the incentive compatibility constraints $\Delta(w_0) - h(y(w_0)/w_0) \geq \Delta(w) - h(y(w)/w_0)$ for all w_0. Note that the incentive constraint does not involve c_0, c_1 or $\underline{U}$; only Δ and y_0.

For our purposes, it suffices to focus on the second stage that takes Δ and y_0 as given, which allows us to drop the incentive constraint:

$$\max_{c_0, c_1, \underline{U}} \int_0^\infty [u(c_0(w_0)) + \hat{\beta} u(c_1(w_1))] dF(w_0)$$

subject to $\Delta(w_0) + \underline{U} = u(c_0(w_0)) + \beta u(c_1(w_1))$ and the resource constraints in (2.4)-(2.5).

It is convenient to rewrite this problem by changing variables, from consumption to utility assignments $U_0(w) = u(c_0(w))$ and $U_1(w) = u(c_1(w))$, since then the objective is then linear and the constraints strictly convex. After substituting $U_0(w_0) = \Delta(w_0) + \underline{U} - \beta U_1(w_1)$ out the problem becomes

$$\max_{U_1, \underline{U}} \int_0^\infty [\underline{U} + (\hat{\beta} - \beta) U_1(w_1)] dF(w_0)$$

subject to

$$\int_0^\infty C(\Delta(w_0) + \underline{U} - \beta U_1(w_1)) dF(w_0) \leq e_0 + \int_0^\infty y_0(w_0) dF(w_0)$$
$$\int_0^\infty C(U_1(w_0)) dF(w_0) \leq e_1$$

It is easy to see that both resource constraints must bind at an optimum.

2.3 The Main Result: Progressive Estate Taxation

In this section we derive two main results for the two-period economy laid out in the previous section. We first show that implicit marginal tax rates on bequests must be progressive. We then provide a simple tax implementation that relies on two separate schedules for labor income and estates.

2.3.1 Implicit Marginal Taxes

For any allocation and constant $R > 0$ we can define the associated marginal tax rates $\tau(w_0)$ solving the Euler equation

$$1 = \beta R(1 - \tau(w_0))\frac{u'(c_1(w_0))}{u'(c_0(w_0))}. \qquad (2.8)$$

Below, the constant R plays the role of the pre-tax gross interest rate. Since our economy has no savings technology, this value is not uniquely pinned down in equilibrium—it is completely unimportant for anything that follows. Different values of R are associated with different levels for the tax, but they do not affect its shape.

The first-order condition for $U_1(w_0)$, which is necessary and sufficient for optimality, is

$$\hat{\beta} - \beta + \beta\lambda_0 C'(U_0(w_0)) = \lambda_1 C'(U_1(w_0)).$$

where λ_t is strictly positive lagrange multiplier on the resource constraint for period t. From this equation it follows that $U_0(w_0)$ and $U_1(w_0)$ move in the same direction with w_0. Since $U_0(w_0) + \beta U_1(w_0)$ must be increasing, in order to provide incentives, it follows that both $U_0(w_0)$ and $U_1(w_0)$ are increasing; hence, both consumptions $c_0(w_0)$ and $c_1(w_0)$ are increasing in w_0.

Using the fact that $C(u)$ is the inverse of $u(c)$, so that $C'(U_t(w_0)) = 1/u'(c_t(w_0))$, and rearranging we obtain

$$1 = \beta\frac{\lambda_0}{\lambda_1}\left(1 + \left(\frac{\hat{\beta}}{\beta} - 1\right)\frac{u'(c_0(w_0))}{\lambda_0}\right)\frac{u'(c_1(w_0))}{u'(c_0(w_0))}. \qquad (2.9)$$

From the first order condition for $\underline{U}$ it follows that $1/\lambda_0 = \int_0^\infty (1/u'(c_0(w)))dF(w)$. For what follows we normalize so that $R = \lambda_0/\lambda_1$.

Our first result, derived from (2.9) when $\hat{\beta} = \beta$, can be viewed as simply restating the celebrated Atkinson-Stiglitz uniform taxation result for our economy.

Proposition 17 *When $\hat{\beta} = \beta$ the optimal allocation implies a zero marginal estate tax rate: $\tau(w_0) = 0$ in (2.8) and the marginal rate of substitution $u'(c_1(w_0))/u'(c_0(w_0))$ is equated across all dynasties, i.e. for all w_0.*

Atkinson and Stiglitz (1976) showed that, provided preferences over a group of goods is

separable from work effort, then consumption within this group should not be distorted. In other words, the implied marginal taxes for these goods should be equalized to avoid distorting their relative consumption—uniform taxation is optimal. In our context, this result applies to consumption at both dates, c_0 and c_1, and implies that the ratio of marginal utilities is equalized across agents—the estate tax can be normalized to zero.[3]

In contrast, whenever $\hat{\beta} > \beta$ (2.9) implies that the ratio of marginal utilities is not equalized across agents: there must be some distortion, so the marginal estate tax cannot be zero. Indeed, since consumption increases with productivity estate taxation must be progressive.

Proposition 18 *When $\hat{\beta} > \beta$ the optimal allocation implies a nonzero and progressive marginal estate tax: $\tau(w_0) \neq 0$ for all w_0 and $\tau(w_0)$ is increasing in w_0. For $R = \hat{\beta}$ the marginal tax rate is*

$$\tau(w_0) = -(\hat{\beta}/\beta - 1)u'(c_0(w_0)) \left(\int_0^\infty u'(c_0(w))^{-1} dF(w) \right) \tag{2.10}$$

and $c_0(w_0)$, $c_1(w_0)$ and $y_0(w_0)$ are increasing in w_0.

We emphasize that the interesting implication for the tax rate here is that it increases with productivity: taxation is progressive. Without an aggregate savings technology the overall level of estate tax cannot be uniquely pinned down, it is completely irrelevant. Farhi, Kocherlakota and Werning (2005) extends the analysis to an economy with capital, which pins down the level of estate taxation.

2.3.2 A Simple Tax Implementation

We next show that we can implement efficient allocations, and the progressive implicit marginal tax rates that go with them, with a simple tax system. In our implementation, the government confronts parents with two separate schedules: an income tax and an estate tax. We say that an allocation is *implementable by non-linear income and estate taxation $T_1^y(y_0)$, T_2^y and $T^b(b)$*

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[3] One difference is that AtkSti76 assume a linear technological transformation between goods, whereas we assume no possible transformation. Their result on uniform taxation implies that marginal rates of substitution are equalized across agents *and* that they are all equal to the marginal rate of transformation. Our result only emphasizes the former.

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if, for all w_0, the allocation $(c_0(w_0), c_1(w_0), y_0(w_0))$ solves

$$\max_{c_0, c_1, y_0} \{u(c_0) + \beta u(c_1) - h(y_0/w_0)\}$$

subject to

$$c_0 + b_1 = y_0 - T^b(b_1) - T_1^y(y_0),$$
$$c_1 = Rb_1 + y_2 - T_2^y.$$

It is trivial to change things so that it is the child that pays the estate tax at $t = 1$. Furthermore, without loss of generality we can assume that $y_2 - T_2^y = 0$. To see this, define $\hat{b}_1 \equiv b_1 + (y_2 - T_2)/R$ then

$$c_0 + \hat{b}_1 = y_0 - T^b(\hat{b}_1 - (y_2 - T_2)/R) - T_1^y(y_0) - T_2^y(y_0)$$
$$= y_0 - \hat{T}^b(\hat{b}_1) - \hat{T}^y(y_0)$$

where $\hat{T}^y(y_0) \equiv T_1^y(y_0) + T_2^y(y_0)$ and $\hat{T}^b(\hat{b}_1) = T^b(\hat{b}_1 - (y_2 - T_2)/R)$.

Our next result establishes formally that efficient allocations can be implemented with separate nonlinear income and estate taxation. The idea is to define $T^b(b)$ so that

$$\frac{1}{1 + T^{b\prime}(c_1(w))} = 1 - \tau(w)$$

The proof then exploits the fact that marginal tax rates are progressive to ensure that the bequest problem faced by the parent is convex.

Proposition 19 *Suppose $c_0(w_0)$, $c_1(w_0)$, $y_0(w_0)$ and $\tau(w_0)$ are increasing functions. Then there exists tax functions $T^y(y)$ and $T^b(b)$ that implements this allocation, with $T^b(b)$ convex.*

Proof. Use the generalized inverse of $c_1(w)$, where possible flat portions of $c_1(w)$ define discontinuous jumps, to define

$$T^{b\prime}(c) = \frac{1}{1 - \tau((c_1)^{-1}(c))} - 1 \tag{2.11}$$

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and normalize so that $T^b(0) = 0$. Note that by the monotonicity of $\tau(w)$ and $c_0(w)$, the function $T^b(b)$ is convex. Next define net income

$$I(w_0) \equiv c_0(w_0) + R^{-1}c_1(w_0) + T^b(c_1(w_0))$$

We can express this in terms of output y by using the inverse of $y_0(w_0)$: $I^y(y) \equiv I(y_0^{-1}(y))$. Then we let $T^y(y_0) \equiv y_0 - I^y(y_0)$. Finally, let the consumption allocation as a function of net income I be: $(\hat{c}_0(I), \hat{c}_1(I)) \equiv (c_0(I^{-1}(I)), c_1(I^{-1}(I)))$.

We now show that the constructed tax functions, $T^y(y)$ and $T^b(b)$, implement the allocation. For any given net income I the consumer solves the subproblem:

$$V(I) \equiv \max\{u(c_0) + \beta u(c_1)\}$$

subject to $c_0 + R^{-1}c_1 + T^b(c_1) \leq I$. This problem is convex, the objective is concave and the constraint set is convex, since T^b is convex. It follows that the first-order condition

$$1 = \frac{\beta R}{1 + T^{b\prime}(b)} \frac{u'(c_1)}{u'(c_0)}$$

sufficient for optimality. Combining (2.8) and (2.11) it follows that these conditions for optimality are satisfied by $\hat{c}_0(I), \hat{c}_0(I)$ for all I. Hence $V(I) = u(\hat{c}_0(I)) + \beta u(\hat{c}_0(I))$.

Next, consider the worker's maximization over y_0 given by

$$\max_y \{V(I(y)) - h(y/w_0)\}.$$

We need to show that $y_0(w_0)$ solves this problem, which implies that the allocation is implemented since consumption would be given by $\hat{c}_0(I(y_0(w_0))) = c_0(w_0)$ and $\hat{c}_1(I(y_0(w_0))) = c_1(w_0)$. Now, from the previous paragraph and our definitions it follows that

$$y_0(w_0) \in \arg\max_y \{V(I(y)) - h(y; w_0)\}$$

$$\Leftrightarrow y_0(w_0) \in \arg\max_y \{u(\hat{c}_0(I(y))) + \beta u(\hat{c}_1(I(y))) - h(y/w_0)\}$$

$$\Leftrightarrow w_0 \in \arg\max_w \{u(c_0(w)) + \beta u(c_1(w)) - h(y_0(w)/w_0)\}$$

Thus, the first line follows from the last, which is guaranteed by the assumed incentive compatibility of the allocation, (2.6). Hence, $y_0(w_0)$ is optimal and it follows that $(c_0(w_0), c_1(w_0), y_0(w_0))$ is implemented by the constructed tax functions. ∎

2.3.3 Discussion

Without estate taxation there is perfect inheritability of welfare. In particular, consumption of parents and child move in tandem, one for one. This situation is only optimal when the children are not considered independently in the welfare criterion, so that insuring them against the risk of their parent's fortune is not valued.

In contrast, when insurance is provided to the children's generation their consumption still varies with their parent's, but less than one-for-one. The intergenerational transmission of welfare is imperfect: consumption mean reverts across generations. The progressivity of the estate tax schedule reflects this mean reversion. Fortunate parents must face a lower net returns on bequests in order to give them incentives to tilt their consumption towards the present, that is, towards themselves. Likewise poorer parents need to face higher net returns so that their consumption slopes upward. This explains the progressivity of estate taxes.

Another intuition is based on the interpretation of altruism as a form of externality. In the presence of externalities, some form of corrective Pigouvian taxes are generally desirable. Think of a parental bequest as a consumption good with a positive externality to the child; then the Pigouvian logic implies that we should subsidize bequests. Since expected utility is our concern, and utility is concave, this externality is greatest for children with low consumption. Thus, the subsidy rate should be highest—or equivalently, the negative tax should be lowest— for poor parents. Optimal estate taxation is thus progressive. Since our economy has no capital, the Pigouvian level of taxation turns out to be irrelevant—we may tax or subsidize estates. However, the relative tax conclusion in this argument remains robust.

None of these arguments require the private-information structure. However, if productivity or effort were observable, then the first-best allocation would be achievable. Consumption and wealth would then be equated across parents. Although one can still think of a progressive estate tax in this situation for out-of-equilibrium levels of parental wealth, it becomes irrelevant given the lack of parental inequality. In this sense, our results rely on an interaction between

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redistributive and corrective motives for taxation (see also Amador Angeletos and Werning 2005).

2.4 A Dynamic Mirrleesian Economy with Infinite Horizon

We now turn to a repeated version of this economy with an infinite horizon, as in Albanesi and Sleet (2004). All generations work and receive a random productivity draw. An individual born into generation t has ex-ante welfare v_t solving

$$v_t = \mathbb{E}_{t-1}[u(c_t) - h(n_t) + \beta v_{t+1}],$$

where $\beta < 1$ is the coefficient of altruism. We assume that the utility function over consumption satisfies the Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$. We adopt a power disutility function $h(n) = n^\gamma / \gamma$ with $\gamma > 1$ to ensure that the planning problem is convex.

An individual with productivity w, exerting work effort n, produces output $y = w \cdot n$. Utility can then be written as

$$V_t = \sum_{s=0}^{\infty} \beta^s \, \mathbb{E}_{t-1}\big[u(c_{t+s}) - \theta_{t+s} h(y_{t+s})\big] \tag{2.12}$$

where $\theta_t \equiv w_t^{-\gamma}$ can be interpreted as a taste shock to producing output. Productivity w_t, and hence θ_t, is independently and identically distributed across dynasties and generations $t = 0, 1 \ldots$ With innate talents assumed nonheritable, intergenerational transmission of welfare is not mechanical linked through the environment but may arise to provide incentives for altruistic parents.

Since productivity shocks are assumed to be privately observed by individuals and their descendants each dynasty faces a sequence of consumption functions $\{c_t\}$, where $c_t(\theta^t)$ represents an individual's consumption after reporting the history $\theta^t \equiv (\theta_0, \theta_1, \ldots, \theta_t)$. A dynasty's reporting strategy $\sigma \equiv \{\sigma_t\}$ is a sequence of functions $\sigma_t : \Theta^{t+1} \to \Theta$ that maps histories of shocks θ^t into a current report $\hat{\theta}_t$. Any strategy σ induces a history of reports $\sigma^t : \Theta^{t+1} \to \Theta^{t+1}$. We use σ^* to denote the truth-telling strategy with $\sigma_t^*(\theta^t) = \theta_t$ for all $\theta^t \in \Theta^{t+1}$.

Given an allocation $\{c_t\}$, the utility obtained from any reporting strategy σ is

$$U(\{c_t\}, \sigma; \beta) \equiv \sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \beta^t [u(c_t(\sigma^t(\theta^t))) - \theta_t h(y(\sigma^t(\theta^t)))] \Pr(\theta^t).$$

An allocation $\{c_t\}$ is *incentive compatible* if truth-telling is optimal, so that

$$U(\{c_t\}, \sigma^*; \beta) \geq U(\{c_t\}, \sigma; \beta) \tag{2.13}$$

for all strategies σ.

We identify dynasties by their initial utility entitlement v_0 with distribution ψ in the population. An allocation is a sequence of functions $\{c_t^v, y_t^v\}$ for each v, where $c_t^v(\theta^t)$ and $y_t^v(\theta^t)$ represents the consumption and income that a dynasty with initial entitlement v gets at date t after reporting the sequence of shocks θ^t. For any given initial distribution of entitlements ψ and resources e, we say that an allocation $\{c_t^v\}$ is *feasible* if: (i) it is incentive compatible for all dynasties; (ii) it delivers expected utility of v to all initial dynasties entitled to v; and (iii) average consumption in the population does not exceed the fixed endowment e plus income generated in all periods:

$$\int \sum_{\theta^t} c_t^v(\theta^t) \Pr(\theta^t) \, d\psi(v) \leq e + \int \sum_{\theta^t} y_t^v(\theta^t) \Pr(\theta^t) \, d\psi(v) \qquad t = 0, 1, \dots \tag{2.14}$$

Consider the sum of expected utilities weighted by geometric Pareto weights $\alpha_t = \hat{\beta}^t$

$$\sum_{t=0}^{\infty} \alpha_t \, \mathbb{E}_{-1} v_t = \left(1 - \frac{1}{\hat{\beta} - \beta}\right) v_0 + \frac{1}{\hat{\beta} - \beta} \sum_{t=0}^{\infty} \hat{\beta}^t \, \mathbb{E}_{-1} [u(c_t) - \theta_t h(y_t)]. \tag{2.15}$$

with $\hat{\beta} > \beta$. The first term is exogenously given, since we take as given a distribution for the initial utility entitlements v_0. Thus, the welfare criterion is given by

$$\sum_{t=0}^{\infty} \hat{\beta}^t \, \mathbb{E}_{-1} [u(c_t) - \theta_t h(y_t)] \tag{2.16}$$

Future generations are already indirectly valued through the altruism of the current generation. If, in addition, they are also *directly* included in the welfare function the social discount factor

must be higher than the private one (for more details, see Farhi and Werning 2005).

When $\hat{\beta} = \beta$, the planning problem seeks the lowest constant resource level e to ensure that there exists a feasible allocation that delivers the distribution of utility entitlements ψ. This is precisely the efficiency problem studied in Albanesi and Sleet (2004). When $\hat{\beta} > \beta$ we define the social optimum as maximizing the average social welfare function (2.16), weighed by ψ, over all feasible allocations. That is, the *social planning problem* given an initial distribution of entitlements ψ and an endowment level e is to maximize

$$\int U\big(\{c_t^v\}, \sigma^*, \hat{\beta}\big) \, d\psi(v) \tag{2.17}$$

subject to the the resource constraints (2.14), as well as the promise keeping and incentive constraints: $v = U(\{c_t^v\}, \sigma^*; \beta)$ and $U(\{c_t^v\}, \sigma^*; \beta) \geq U(\{c_t^v\}, \sigma; \beta)$ for all initial entitlements v and strategies σ.

We are interested in distributions of utility entitlements ψ such that the solution to the planning problem features, in each period, a cross-sectional distribution of continuation utilities v_t that is also distributed according to ψ. We also require the cross-sectional distribution of consumption and income to replicate itself over time. We term any initial distribution of entitlements with these properties a *steady state* and denote them by ψ^*. Following Farhi and Werning (2005), we approach the planning problem by studying a relaxed version of it. The solutions to both problems coincide for steady state distributions ψ^*, which is all we seek to characterize. The relaxed problem has continuation utility as a state variable that follows a Markov process. Steady states are then invariant distributions of this Markov process.

Define the *relaxed planning problem* to be equivalent to the *social planning problem* except that the sequence of resource constraints (2.14) is replaced by the single intertemporal condition

$$\int \sum_{t=0}^{\infty} \hat{\beta}^t \sum_{\theta^t} \big(c_t^v(\theta^t) - y_t^v(\theta^t) \Pr(\theta^t)\big) \, d\psi(v) \leq \frac{1}{1 - \hat{\beta}} e. \tag{2.18}$$

Letting $\hat{\lambda}$ be the multiplier for this intertemporal resource constraint we form the Lagrangian

$L \equiv \int L^v \, d\psi(v)$ where

$$L^v \equiv \sum_{t=0}^{\infty} \sum_{\theta^t} \hat{\beta}^t \left(u(c_t^v(\theta^t)) - \hat{\lambda} c_t^v(\theta^t) - \theta_t h\big(y_t^v(\theta^t)\big) + \hat{\lambda} y_t^v(\theta^t) \right) \Pr(\theta^t) \qquad (2.19)$$

and study the maximization of L subject to $v = U(\{c_t^v\}, \sigma^*; \beta) \geq U(\{c_t^v\}, \sigma; \beta)$ for all v and σ. For any endowment level e, there exists a unique positive multiplier $\hat{\lambda}(e)$ so that the maximizing this Lagrangian is equivalent to solving the relaxed problem. Maximizing L is equivalent to the pointwise optimization, for each v, of the subproblem:

$$k(v) \equiv \sup L^v \qquad (2.20)$$

subject to $v = U(\{c(u_t^v)\}, \sigma^*; \beta) \geq U(\{c(u_t^v)\}, \sigma; \beta)$ for all σ.

The value function of the component planning problem $k(v)$ defined by (2.20) is continuous, concave, and satisfies the Bellman equation

$$k(v) = \max_{u,h,w} \mathbb{E}\big[u(\theta) - \hat{\lambda} c(u(\theta)) - \theta h(\theta) + \hat{\lambda} y\big(h(\theta)\big) + \hat{\beta} k\big(w(\theta)\big) \big] \qquad (2.21)$$

subject to

$$v = \mathbb{E}\big[u(\theta) - \theta h(\theta) + \beta w(\theta) \big] \qquad (2.22)$$

$$u(\theta) - \theta h(\theta) + \beta w(\theta) \geq u(\theta') - \theta h(\theta') + \beta w(\theta') \qquad \text{for all} \quad \theta, \theta' \in \Theta. \quad (2.23)$$

Denote by $g^w(v, \theta)$ and $g^u(v, \theta)$ the optimal policy function for w and u. The next lemma characterizes some key properties of the value function $k(v)$.

Lemma 20 *The value function $k(v)$ is strictly concave and continuously differentiable on $(\underline{v}, \overline{v})$ where $\underline{v} = -\infty$; it is unbounded below on both sides $\lim_{v \to \underline{v}} k(v) = \lim_{v \to \overline{v}} k(v) = -\infty$; and the derivative has $\lim_{v \to \underline{v}} k'(v) = 1$ and $\lim_{v \to \overline{v}} k'(v) = -\infty$.*

2.5 Steady States and Progressive Taxation

We are interested in steady state distributions ψ^* that have no mass at misery $u(0)/(1 - \beta)$. Our first result is that this is not possible when future generations are not weighed directly,

so that $\hat{\beta} = \beta$. We then show that, in contrast, whenever $\hat{\beta} > \beta$ a steady state distribution exists with no mass at misery. The efficient allocation displays a form of mean reversion across generations that keeps inequality bounded. The mean reversion is characterized by a modified inverse Euler equation which implies that estate taxation is progressive.

2.5.1 An Immiseration Result

For $\beta = \hat{\beta}$, we have to modify our definition for the Social Planning problem. For any distribution ψ of initial welfare entitlements, the planning problem is to minimize the net resources required to deliver the utility entitlements in an incentive compatible way:

$$\inf e \tag{2.24}$$

subject to,

$$\int \sum_{\theta^t} (c_t^v(\theta^t) - y_t^v(\theta^t)) d\psi(v) \leq e \tag{2.25}$$

$$U(\{c_t^v\}, \sigma; \beta) = v \text{ for all } v \tag{2.26}$$

$$U(\{c_t^v\}, \sigma^*; \beta) \geq U(\{c_t^v\}, \sigma; \beta) \text{ for all } v \text{ and } \sigma \tag{2.27}$$

From this program, we can define an invariant distribution exactly as in Section 4 of the paper. We are interested in steady state distributions ψ^* without full mass at misery. Our first result is that this is basically not possible when $\beta = \hat{\beta}$.

Proposition 21 *Suppose that* $\lim_{u \to \infty} \sup c''(u)/c'(u) < \infty$. *Then if* $\beta = \hat{\beta}$, *there exists no invariant distribution* ψ^* *without full mass at misery.*

This result extends the immiseration result in Atkeson and Lucas (1992), who study an endowment economy with privately observed taste shocks, instead of the Mirrleesian production economy with privately observed productivity shocks studied here. They show that the cross-sectional distribution of consumption disperses steadily over time, with inequality growing without bound. As a result, almost everyone converges to the misery, consuming nothing, while a vanishing fraction tend towards bliss, consuming the entire aggregate endowment. Thus, no steady state distribution with positive consumption exists. To the best of our knowledge

Proposition 21 is the first formal statement of an analogous result in the context of a Mirrleesian economy, where private information is regarding productivity shocks. Researchers that assume $\hat{\beta} = \beta$ have been typically forced to impose an ad hoc lower bound on continuation utility to avoid misery and ensure that an steady-state distribution exists (Atkeson and Lucas 1995, Albanesi and Sleet 2004).

2.5.2 Steady States and a Modified Inverse Euler Equation

We now return to efficient allocations where future generations are given positive weight. We first derive an important intertemporal condition that must be satisfied by the optimal allocation. This condition has interesting implications for the optimal estate tax, computed later.

Let λ be the multiplier on the promise-keeping constraint and let $\mu(\theta, \theta')$ represent the multipliers on the incentive constraints. Then the first-order conditions for interior solutions for $u(\theta)$ and $w(\theta)$ are

$$p(\theta) - \hat{\lambda}c'(u(\theta))p(\theta) - \lambda p(\theta) - \sum_{\theta'} \mu(\theta, \theta') + \sum_{\theta'} \mu(\theta', \theta) = 0 \tag{2.28}$$

$$\hat{\beta}k'(w(\theta))p(\theta) - \beta\lambda p(\theta) - \beta\sum_{\theta'} \mu(\theta, \theta') + \beta\sum_{\theta'} \mu(\theta', \theta) = 0 \tag{2.29}$$

The envelope condition is $k'(v) = \lambda$. From the first-order condition for $w(\theta)$ we obtain the CLAR equation

$$\frac{\beta}{\hat{\beta}} k'(v) = \sum_{\theta} k'(g^w(v, \theta))p(\theta). \tag{2.30}$$

This equation encapsulates the mean-reversion force in the model. In sequential notation

$$\frac{\beta}{\hat{\beta}}k'(v_t) = \mathbb{E}_t[k'(v_{t+1})], \tag{2.31}$$

so that $\beta/\hat{\beta} < 1$ acts as an autoregressive coefficient ensuring that over time the derivative $k'(v_t)$ mean reverts back to zero, where the function $k(v)$ finds its interior maximum. The mean-reverting force provided by $\hat{\beta} > \beta$ is crucial for the existence of steady state distributions with bounded inequality, which we prove below. In contrast, when $\hat{\beta} = \beta$ no such central tendency exists, increasing inequality and immiseration ensues and no steady state exists (Proposition 21).

The optimal resolution of the tradeoff between incentives for altruistic parents and insurance for newborns gives rise to a less than one-for-one intergenerational transmission of welfare—in contrast to the case where $\hat{\beta} = \beta$. The descendants of a rich parent are more fortunate than those of a poor parent, but less and less so the more distant is the descendant: the impact of the initial fortune of dynasties dies out over generations.

The more weight is put on future generations, the higher is $\hat{\beta}$ compared to β, and the less intense is the link between the welfare of parents and child. But as we will now show, even the smallest amount of mean-reversion in the form of $\hat{\beta} > \beta$ puts enough limits on the transmission of shocks across generations to prevent the distribution of consumption and welfare from exploding.

The first-order conditions (2.28)–(2.29) imply that

$$\frac{\hat{\beta}}{\beta}k'(w(\theta)) = 1 - \hat{\lambda}c'(u(\theta)) \quad \text{and} \quad \frac{\hat{\beta}}{\beta}k'(v) = 1 - \hat{\lambda}c'(u_-),$$

where u_- should be interpreted as the previous period's assignment of utility from consumption. Substituting these relations into the CLAR (2.30) we arrive at a *Modified Inverse Euler equation*

$$\frac{1}{u'(c_-)} = \frac{\hat{\beta}}{\beta}\sum_\theta \frac{1}{u'(c(\theta))}\, p(\theta) - \hat{\lambda}^{-1}\left(\frac{\hat{\beta}}{\beta} - 1\right). \tag{2.32}$$

The left-hand side together with the first term on the right-hand side is the standard inverse Euler equation. The second term on the right-hand side is novel, since it is zero when $\beta = \hat{\beta}$ and is strictly negative when $\hat{\beta} > \beta$.[4]

We now show that a steady state exists whenever the welfare criterion places direct weight on children so that $\hat{\beta} > \beta$. The proof follows Farhi and Werning (2005) quite closely, which proves such a result for an economy with taste shocks.

Proposition 22 *There exists an invariant distribution ψ^* for the Markov process $\{v_t\}$ implied by g^w.*

Moreover any invariant distribution ψ^* has a support bounded away from misery $\underline{v}$.

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[4] *FarKocWer05 show that this equation, and its implications for estate taxation, generalize to an economy with capital and an arbitrary process for skills.

Suppose in addition that $\lim_{u\to\infty} \sup c''(u)/c'(u) < \infty$. *Then any invariant distribution* $\hat{\psi}$ *necessarily has a support bounded away from* $\bar{v}$.

Note in particular that if $\lim_{u\to\infty} \sup c''(u)/c'(u) < \infty$, then there exists an invariant distribution, and any invariant distribution has a compact support.

The result relies heavily on the force for mean reversion that is behind (2.31) and (2.32). To see this mean-reversion force most clearly consider, as an example, the logarithmic utility case, $u(c) = \log(c)$. Then $1/u'(c) = c$ and (2.32) can be written with sequential time notation as

$$\mathbb{E}_t[c_{t+1}] = \frac{\beta}{\hat{\beta}} c_t + \left(1 - \frac{\beta}{\hat{\beta}}\right) \bar{c},$$

or simply

$$c_{t+1} = \frac{\beta}{\hat{\beta}} c_t + \left(1 - \frac{\beta}{\hat{\beta}}\right) \bar{c} + \varepsilon_{t+1} \qquad \text{with } \mathbb{E}_t[\varepsilon_{t+1}] = 0$$

where $\bar{c} \equiv \hat{\lambda}^{-1}$ is average consumption at the steady-state cross-sectional distribution. As the last expression indicates, with logarithmic utility, consumption itself is autoregressive with an autoregressive coefficient equal to $\beta/\hat{\beta} < 1$.

2.5.3 Tax Implementation

Any allocation that is incentive compatible and feasible, and has strictly positive consumption, can be implemented by a combination of taxes on labor income and estates. Here we first describe this implementation, and explore some features of the optimal estate tax in the next subsection.

For any incentive-compatible and feasible allocation $\{c_t^v(\theta^t), y_t^v(\theta^t)\}$ we propose an implementation along the lines of Kocherlakota (2005). In each period, conditional on the history of their dynasty's reports $\hat{\theta}^{t-1}$ and any inherited wealth, individuals report their current shock $\hat{\theta}_t$, produce, consume, pay taxes and bequeath wealth subject to the following set of budget constraints

$$c_t + b_t \leq y_t\left(\hat{\theta}^t\right) - T_t\left(\hat{\theta}^t\right) + \left(1 - \tau_t\left(\hat{\theta}^t\right)\right) R_{t-1,t} b_{t-1} \qquad t = 0, 1, \ldots \qquad (2.33)$$

where $R_{t-1,t}$ is the before-tax interest rate across generations, and initially $b_{-1} = 0$. Individuals

are subject to two forms of taxation: a labor income tax $T_t(\hat{\theta}^t)$, and a proportional tax on inherited wealth $R_{t-1,t}b_{t-1}$ at rate $\tau_t(\hat{\theta}^t)$.[5]

Given a tax policy $\{T_t^v(\theta^t),\ \tau_t^v(\theta^t),\ y_t^v(\theta^t)\}$, an equilibrium consists of a sequence of interest rates $\{R_{t,t+1}\}$; an allocation for consumption, labor income and bequests $\{c_t^v(\theta^t),\ b_t^v(\theta^t)\}$; and a reporting strategy $\{\sigma_t^v(\theta^t)\}$ such that: (i) $\{c_t,\ b_t,\ \sigma_t\}$ maximize dynastic utility subject to (2.33), taking the sequence of interest rates $\{R_{t,t+1}\}$ and the tax policy $\{T_t,\ \tau_t,\ y_t\}$ as given; and (ii) the asset market clears so that $\int \mathbb{E}_{-1}[b_t^v(\theta^t)]\,d\phi(v) = 0$ for all $t = 0, 1, \ldots$ We say that a competitive equilibrium is incentive compatible if, in addition, it induces truth telling.

For any feasible, incentive-compatible allocation $\{c_t^v,\ y_t^v\}$, with strictly positive consumption we construct an incentive-compatible competitive equilibrium with no bequests by setting $T_t^v(\theta^t) = y_t(\theta^t) - c_t(\theta^t)$ and

$$\tau_t^v(\theta^t) = 1 - \frac{1}{\beta R_{t-1,t}} \frac{u'(c_{t-1}^v(\theta^{t-1}))}{u'(c_t^v(\theta^t))} \tag{2.34}$$

for any sequence of interest rates $\{R_{t-1,t}\}$. These choices work because the estate tax ensures that for any reporting strategy σ, the resulting consumption allocation $\{c_t^v(\sigma^t(\theta^t))\}$ with no bequests $b_t^v(\theta^t) = 0$ satisfies the consumption Euler equation

$$u'\big(c_t^v(\sigma^t(\theta^t))\big) = \beta R_{t,t+1} \sum_{\theta_{t+1}} u'\big(c_{t+1}^v(\sigma^{t+1}(\theta^t,\theta_{t+1}))\big)\big(1 - \tau_{t+1}^v(\sigma^{t+1}(\theta^t,\theta_{t+1}))\big)\Pr(\theta_{t+1}).$$

The labor income tax is such that the budget constraints are satisfied with this consumption allocation and no bequests. Thus, this no-bequest choice is optimal for the individual regardless of the reporting strategy followed. Since the resulting allocation is incentive compatible, by hypothesis, it follows that truth telling is optimal. The resource constraints together with the budget constraints then ensure that the asset market clears.[6]

As noted above, in our economy without capital only the after-tax interest rate matters

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[5] In this formulation, taxes are a function of the entire history of reports, and labor income y_t is mandated given this history. However, if the labor income histories $y^t \colon \Theta^t \to \mathbb{R}^t$ being implemented are invertible, then by the taxation principle we can rewrite T and τ as functions of this history of labor income and avoid having to mandate labor income. Under this arrangement, individuals do not make reports on their shocks, but instead simply choose a budget-feasible allocation of consumption and labor income, taking as given prices and the tax system.

[6] Since the consumption Euler equation holds with equality, the same estate tax can be used to implement allocations with any other bequest plan with income taxes that are consistent with the budget constraints.

so the implementation allows any equilibrium before-tax interest rate $\{R_{t-1,t}\}$. In the next subsection, we set the interest rate to the reciprocal of the social discount factor, $R_{t-1,t} = \hat{\beta}^{-1}$. This choice is natural because it represents the interest rate that would prevail at the steady state in a version of our economy with capital.

2.5.4 Optimal Progressive Estate Taxation

In our environment, the relevant past history is encoded in the continuation utility so the estate tax $\tau(\theta^{t-1}, \theta_t)$ can actually be reexpressed as a function of $v_t(\theta^{t-1})$ and θ_t. Abusing notation we then denote the estate tax by $\tau_t(v, \theta_t)$. Since we focus on the steady-state, invariant distribution, we also drop the time subscripts and write $\tau(v, \theta)$.

The average estate tax rate $\bar{\tau}(v)$ is then defined by

$$1 - \bar{\tau}(v) \equiv \sum_{\theta} \big(1 - \tau(v, \theta)\big) p(\theta) \tag{2.35}$$

Using the modified inverse Euler equation (2.32) we obtain

$$\bar{\tau}(v) = -\hat{\lambda}^{-1} u'\big(c_-(v)\big) \left(\frac{\hat{\beta}}{\beta} - 1\right)$$

In particular, this implies that the average estate tax rate is negative, $\bar{\tau}(v) < 0$, so that bequests are subsidized. However, recall that before-tax interest rates are not uniquely determined in our implementation. As a consequence, neither are the estate taxes computed by (2.34). With our particular choice for the before-tax interest rate, however, the tax rates are pinned down and acquires a corrective, Pigouvian role. Differences in discounting can be interpreted as a form of externalities from future consumption, and the negative average tax can then be seen as a way of countering these externalities as prescribed by Pigou. In our setup without capital, this result depends on the choice of the before-tax interest rate. However, the negative tax on estates would be a robust steady-state outcome in a version of our economy with capital.

In our model it is more interesting to understand how the average tax varies with the history of past shocks encoded in the promised continuation utility v. The average tax is an increasing function of consumption, which, in turn, is an increasing function of v. Thus, estate taxation is progressive: the average tax on transfers for more fortunate parents is higher.

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Proposition 23 *In the repeated Mirrlees economy, an optimal allocation with strictly positive consumption can be implemented by a combination of income and estate taxes. At a steady-state, invariant distribution ψ^*, the optimal average estate tax $\bar{\tau}(v)$ defined by (2.34) and (2.35) is increasing in promised continuation utility v.*

The progressivity of the estate tax reflects the mean-reversion in consumption. The fortunate must face lower net rates of return so that their consumption path decreases towards the mean.[7]

2.6 Conclusions

Societies that do not value future generations directly should help their citizens lead their descendants into misery. But when a society cares about future generations then it should be concerned with insuring children against the greatest risk of all: the family they are born into. A natural tax instrument to for this goals is estate taxation. We find that estate taxation should be progressive.

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[7]Farhi, Kocherlakota and Werning (2005) explore more general versions of this result and discuss other intuitions.

Appendix

2.7 Proof of Lemma 20

Strict concavity and differentiability follow from standard arguments. In order to derive the limits of k and k' at the bounds of the domain, we derive a lower bound $k^{\min}$ and an upper bound $k^{\max}$, for which we can easily compute the corresponding limits.

Consider the solution $\{u^{v_0}(\theta^t), y^{v_0}(\theta^t)\}$ to the relaxed planning problem for a given v_0. For all $v \leq v_0$, define $\{u_v^{v_0}(\theta^t), y_v^{v_0}(\theta^t)\}$ by

$$u_v^{v_0}(\theta^t) = u^{v_0}(\theta^t) \text{ for all } t \geq 0$$

$$h(y_v^{v_0}(\theta^0)) = h(y^{v_0}(\theta^0)) + v_0 - v$$

$$y_v^{v_0}(\theta^t) = y^{v_0}(\theta^t) \text{ for all } t \geq 1$$

Let

$$k^{\min}(v) = \sum_{t=0}^{\infty} \hat{\beta}^t E_{-1}[u_v^{v_0}(\theta^t) - \hat{\lambda}c(u_v^{v_0}(\theta^t)) + \hat{\lambda}y_v^{v_0}(\theta^t) - \theta_t h(y_v^{v_0}(\theta^t))]$$

Since $\{u_v^{v_0}(\theta^t), y_v^{v_0}(\theta^t)\}$ is incentive compatible and delivers welfare level v, we have $k(v) \geq k^{\min}(v)$, for all $v \leq v_0$. We have

$$k^{\min\prime}(v) = 1 - \hat{\lambda}E\left[\frac{1}{h'(h(y^{v_0}(\theta^0)) + v_0 - v)}\right]$$

Hence

$$\lim_{v \to -\infty} k^{\min\prime}(v) = 1$$

Since $k(v) \geq k^{\min}(v)$, for all $v \leq v_0$ and both k and $k^{\min}$ are concave, this implies that

$$\lim_{v \to -\infty} k'(v) \leq 1$$

Next define

$$\bar{k}(v) = \sup \sum_{t=0}^{\infty} \hat{\beta}^t E_{-1}[u(\theta^t) - \hat{\lambda}c(u(\theta^t)) + \hat{\lambda}y(\theta^t) - \theta_t h(y(\theta^t))]$$

s.t.

$$v = \sum_{t=0}^{\infty} \beta^t E_{-1}[u(\theta^t) - \theta^t h(y(\theta^t))]$$

This corresponds to the relaxed planning problem, but without the incentive constraints. Hence we have $k(v) \leq \bar{k}(v)$.

Let

$$m = \max_{u,y,\theta} u - \hat{\lambda}c(u) + \hat{\lambda}y - \theta h(y)$$

Then

$$\bar{k}(v) \leq \sup \sum_{t=0}^{\infty} \beta^t E_{-1}[u(\theta^t) - \hat{\lambda}c(u(\theta^t)) + \hat{\lambda}y(\theta^t) - \theta_t h(y(\theta^t))] + m\left[\frac{1}{1-\hat{\beta}} - \frac{1}{1-\beta}\right]$$

$$\leq v + \sup\left\{\sum_{t=0}^{\infty} \beta^t E_{-1}[-\hat{\lambda}c(u(\theta^t)) + \hat{\lambda}y(\theta^t)]\right\} + m\left[\frac{1}{1-\hat{\beta}} - \frac{1}{1-\beta}\right]$$

Hence if we define

$$C(v) = \inf \sum_{t=0}^{\infty} \beta^t E_{-1}[c(u(\theta^t)) - y(\theta^t)]$$

s.t

$$v = \sum_{t=0}^{\infty} \beta^t E_{-1}[u(\theta^t) - \theta_t h(y(\theta^t))]$$

and

$$k^{\max}(v) = v - C(v) + m\left[\frac{1}{1-\hat{\beta}} - \frac{1}{1-\beta}\right]$$

we have

$$\bar{k}(v) \leq k^{\max}(v)$$

Denote by $\{u^C(\theta^t, v), y^C(\theta^t, v)\}$ the solution of the program defining C. Combining the first order conditions for $u(\theta^t)$ and the envelope theorem, we get

$$c'(u^C(\theta^t, v)) = C'(v) \text{ for all } t \geq 0$$

$$\frac{1}{\theta_t h'(y^C(\theta^t, v))} = C'(v) \text{ for all } t \geq 0$$

This implies that

$$\lim_{v \to -\infty} C'(v) = 0$$

$$\lim_{v \to -\infty} u^C(\theta^t, v) = \underline{u}$$

$$\lim_{v \to -\infty} y^C(\theta^t, v) = \infty$$

Hence

$$\lim_{v \to -\infty} k^{\mathrm{max}\,\prime}(v) = 1$$

Since $k \leq k^{\mathrm{max}}$ and both k and k^{max} are concave, this implies that

$$\lim_{v \to -\infty} k'(v) \geq 1$$

Since we already have

$$k'(v) \leq 1$$

this implies that

$$\lim_{v \to -\infty} k'(v) = 1$$

Note that we always have

$$\lim_{v \to \bar{v}} C'(v) = +\infty$$

$$\lim_{v \to \bar{v}} k^{\mathrm{max}\,\prime}(v) = -\infty$$

Since $k(v) \leq k^{\mathrm{max}}(v)$, and both k and k^{max} are concave, this implies that

$$\lim_{v \to \bar{v}} k'(v) = -\infty$$

Finally, note that

$$\lim_{v \to \bar{v}} k^{\mathrm{max}}(v) = \lim_{v \to \underline{v}} k^{\mathrm{max}}(v) = -\infty$$

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Hence

$$\lim_{v \to \bar{v}} k(v) = \lim_{v \to \underline{v}} k(v) = -\infty$$

2.8 Proof of Proposition 21

In order to characterize the optimal allocation it is convenient to study a relaxed problem. The Lagrangian theorem guarantees that there exists a unique sequence of multipliers $\{q_t\}$ with $q_0 = 1$ on (2.25) such that solving (2.24) is equivalent to solving the following program:

$$\inf \sum_{t \geq 0} q_t \int \sum_{\theta^t} (c_t^v(\theta^t) - y_t^v(\theta^t)) d\psi(v)$$

subject to (2.26) and (2.27). Note that this problem is equivalent to the minimization v by v of

$$C(v; \{q_t\}) = \sum_{t \geq 0} q_t \int \sum_{\theta^t} (c_t^v(\theta^t) - y_t^v(\theta^t))$$

subject to

$$U(\{c_t^v\}, \sigma; \beta) = v$$

$$U(\{c_t^v\}, \sigma^*; \beta) \geq U(\{c_t^v\}, \sigma; \beta) \text{ for all } \sigma$$

Hence $C(v; \{q_t\})$ is the least possible cost of an incentive compatible allocation delivering welfare v to the first generation. It is trivial to see that $C(v; \{q_t\})$ is the solution of the following Bellman equation

$$C(v; \{q_{t+s}\}_{s \geq 1}) = \inf E[c(u_\theta) - y(h_\theta) + q_{t+1} C(w_\theta, \{\frac{q_{t+s}}{q_{t+1}}\}_{s \geq 2})] \qquad (2.36)$$

subject to

$$v = E[u_\theta + \beta w_\theta - \theta h_\theta]$$

$$u_\theta + \beta w_\theta - \theta h_\theta \geq u_{\theta'} + \beta w_{\theta'} - \theta h_{\theta'}$$

For future use, let us denote by $g^w(v, \theta^t)$ the continuation utility after a history of shock θ^t when the initial welfare entitlement is v.

Suppose there exists an invariant distribution ψ^*, and let $\{q_t\}$ be the associated sequence of multipliers. Since ψ is a state variable for (2.24), this shows that q_{t+1}/q_t is independent of t. Hence there exists $0 < q < 1$ such that $q_{t+1}/q_t = q$ for all t. We can therefore drop the time dependence on the sequence $\{q_t\}$ in $C_t(v; \{q_t\})$, and simply write $C(v)$ as a shortcut for $C(v, \{q^t\}_{t \geq 0})$.

Lemma 24 *Suppose there exists an invariant distribution ψ^* without full mass at misery. Then $q \geq \beta$.*

Proof. We will make use of two possible state variables. The first state variable is the natural one: v, promised future utility. The other one is utility attained by the previous generation u_-. Indeed, from the first order conditions, it is easy to see that these two state variables are related by

$$c'(u_-) = \frac{q}{\beta} C'(v)$$

The existence of an invariant distribution $\psi^*(v)$ with not mass at misery is equivalent to the existence of an invariant distribution $\hat{\psi}^*(u_-)$ with no mass at misery.

Let $x_\theta = u_\theta + \beta w_\theta$. Then we can rewrite the Bellman equation (2.36) as

$$C(v) = \inf E[c(u_\theta) - y(h_\theta) + qC(w_\theta)]$$

subject to

$$v = [x_\theta - \theta h_\theta]$$

$$x_\theta - \theta h_\theta \geq x_{\theta'} - \theta h_{\theta'}$$

$$u_\theta + \beta w_\theta = x_\theta$$

Hence, given a value x for x_θ, u_θ and w_θ are given by the sub-program

$$\min c(u) + qC(w)$$

subject to

$$u + \beta w = x$$

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The solution is given by the first order condition

$$c'(x - \beta w) + \frac{q}{\beta} C'(w) = 0$$

Using the implicit function theorem, we can then compute

$$\frac{du}{dx} = \frac{\frac{q}{\beta} C''(w)}{\frac{q}{\beta} C''(w) + \beta c''(x - \beta w)}$$

Hence

$$0 \leq \frac{du}{dx} \leq 1$$

This in turn implies that there exists $M > 0$ such that

$$\max_{\theta, \theta'} |u_{\theta'} - u_\theta| < M \max_\theta h_\theta$$

The first order conditions for u_θ in in (2.36) imply that

$$\frac{\beta}{q} c'(u_-) = E[c'(u_\theta)]$$

Hence

$$\frac{\beta}{q} c'(u_-) = E[c'(u_\theta)] \leq c'(u_{\underline{\theta}})$$

Therefore,

$$\log(\frac{\beta}{q}) + \log(c'(u_-)) \leq \log(c'(u_{\underline{\theta}}))$$

and hence

$$\log(\frac{\beta}{q}) + \log(c'(u_-)) \leq \log(c'(u_-)) + \left(\max_{u \in [u_-, u_{\underline{\theta}}]} \frac{c''(u)}{c'(u)} \right) (u_{\underline{\theta}} - u_-)$$

which we can rewrite as

$$\frac{\log(\frac{\beta}{q})}{\left(\max_{u \in [u_-, u_{\underline{\theta}}]} \frac{c''(u)}{c'(u)} \right)} \leq u_{\underline{\theta}} - u_-$$

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Hence for all $\theta \in \Theta$,

$$\frac{\log(\frac{\beta}{q})}{\left(\max_{u \in [u_-, u_\theta]} \frac{c''(u)}{c'(u)}\right)} - M \max_{\theta' \in \Theta} h_{\theta'} \leq u_\theta - u_-$$

In order to allow for bunching in (2.36), it is convenient to consider the following program

$$\inf_{u,w} \sum_n \bar{p}_n \{c(u_n) - y(h_n) + qC(w_n)\}$$

$$v = \sum_n \bar{p}_n (u_n + \beta w_n - \bar{\theta}_n h_n)$$

$$-\theta_n h_n + u_n + \beta w_n \geq -\theta_n h_{n+1} + u_{n+1} + \beta w_{n+1} \text{ for } n = 1, 2, \ldots, K-1,$$

This problem and its notation require some discussion. We do not incorporate the monotonicity constraint on h. But this notation allows us to consider bunching in the following way. If any set of neighboring agents is bunched, then we group these agents under a single index and let $\bar{p}_n$ be the total probability of this group. Likewise let $\bar{\theta}_n$ represent the conditional average of θ within this group, which is what is relevant for the promise-keeping constraint and the objective function. Let θ_n be the shock of the highest agent in the group. The incentive constraint must rule the highest agent in each group from deviating and taking the allocation of the group above him.

Of course, every combination of bunched agents leads to a different program. The optimal allocation of our problem must solve one of these programs with a strictly monotone allocation—since bunching can be characterized by regrouping agents. Thus, below we characterize solutions to these programs with strict monotonicity of the solution.

The first order conditions for h_n is

$$y'(h_n) = C'(v)\bar{\theta}_n + \theta_n \mu_{n,n+1} - \theta_{n-1} \mu_{n-1,n}.$$

This implies in particular that at the optimum, for any of these programs (and hence for the program solved by the true optimal allocation),

$$y'(h_{\underline{\theta}}) \geq C'(v)\underline{\theta}.$$

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It is easy to verify that $C \geq \hat{C}$, where $\hat{C}$ is the solution to (2.36) without the incentive compatibility constraints. Let $\bar{v}$ be the upper bound of the domain for v. Since both C and $\hat{C}$ are increasing and convex, and since

$$\lim_{v \to \bar{v}} \hat{C}(v) = \infty \quad \text{and} \quad \lim_{\bar{v} \to \infty} \hat{C}'(v) = \infty$$

we have

$$\lim_{v \to \bar{v}} C(v) = \infty \quad \text{and} \quad \lim_{v \to \bar{v}} C'(v) = \infty$$

Therefore,

$$\lim_{v \to \bar{v}} y'(h_{\underline{\theta}}(v)) = \infty \quad \Rightarrow \quad \lim_{v \to \bar{v}} h_{\underline{\theta}}(v) = 0$$

and since h_θ has is decreasing in θ,

$$\lim_{v \to \bar{v}} h_\theta(v) = 0 \text{ for all } \theta \in \Theta$$

But this in turn implies that

$$\frac{\log(\frac{\beta}{q})}{\left(\max_{u \in [u_-, u_{\underline{\theta}}]} \frac{c''(u)}{c'(u)} \right)} \leq \lim_{v \to \bar{v}} \inf(u_\theta - u_-)$$

Suppose that $q < \beta$. This implies that for v or equivalently u_- high enough, the policy functions u_θ are all such that $u_\theta > u_-$. This in turn implies that $\hat{\psi}^*$ necessarily has a support bounded away from $\bar{u}$. This in turn implies that

$$\int C'(v) d\psi^*(v) = \int c'(u_-) d\hat{\psi}^*(u_-) < \infty$$

Integrating

$$\frac{\beta}{q} C'(v) = E[C'(w_\theta)]$$

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over v, we get

$$\int C'(v) d\psi^*(v) = \frac{\beta}{q} \int C'(v) d\psi^*(v)$$

Since ψ^* doesn't have full mass at misery, we have $\int C'(v) d\psi^*(v) > 0$. This in turn implies that $\beta = q$, a contradiction. ∎

We have therefore proved that $q \geq \beta$ at ψ^*. But then from the equation

$$\frac{\beta}{q} C'(v) = \mathbb{E}[C'(w_\theta)]$$

we see that $C'(v_t)$ is a positive supermartingale. By the martingale convergence theorem, for any initial value v_0 for v, the sequence of random variables $\{v_t\}$ converges almost surely to a random variable $C_v'^\infty$ with

$$\mathbb{E}[C_v'^\infty] \leq C'(v).$$

Suppose that there must exists a v^* such that $\Pr(C_{v^*}'^\infty > 0)$. We will show that this is not possible.

For any realization θ^∞ define the set of periods where θ_t takes on some particular value $\theta \in \Theta$ as

$$O_\theta(\theta^\infty) \equiv \{t, \theta_t(\theta^\infty) = \theta\}.$$

Then since Θ is finite, we have that with probability one all values of θ occur infinitely often

$$\Pr(\#O_\theta(\theta^\infty) = \infty \text{ for all } \theta \in \Theta) = 1.$$

Hence there exists an event θ^∞ such that $C'(g^w(v^*, \theta^t(\theta^\infty)))$ converges to a positive and finite value, and $\#O_\theta(\theta^\infty) = \infty$ for all $\theta \in \Theta$. Hence $g^w(v^*, \theta^t(\theta^\infty))$ converges to a finite value w^*. Since $g^w(v, \theta)$ is continuous in v, and $\#O_\theta(\theta^\infty) = \infty$ this implies that $g^w(w^*, \theta) = w^*$ for all $\theta \in \Theta$. This implies that the incentive constraints are not binding at w^*, a contradiction.

Hence $\Pr(C_v'^\infty > 0) = 0$ for all v. Therefore for all v, $C'(g^w(v, \theta^t))$ converges almost surely to 0. This in turn implies that the stochastic process $C'(v_t)$ converges almost surely to 0. This implies that $C'(v_t)$ converges in distribution to 0. Since ψ^* is an invariant distribution, $C'(v_t)$ is distributed as $C'(v_0)$. This implies that the distribution of $C'(v_0)$ has full mass at zero, i.e.

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that ψ^* has full mass at misery.

2.9 Proof of Proposition 22

We start with two lemmas, and then proceed to prove the proposition.

Lemma 25 *The following inequalities hold*

$$\underline{\gamma}(1 - k'(v)) + \left(1 - \frac{\beta}{\hat{\beta}}\right) \leq 1 - k'(g^w(\theta, v)) \leq \bar{\gamma}(1 - k'(v)) + \left(1 - \frac{\beta}{\hat{\beta}}\right)$$

for all $\theta \in \Theta$, where the constants are given by $\bar{\gamma} \equiv (\beta/\hat{\beta}) \max\limits_{1 \leq n \leq N} \{(1 + \theta_n - \mathbb{E}[\theta \mid \theta \leq \theta_n])/\theta_n\}$ and $\underline{\gamma} \equiv (\beta/\hat{\beta}) \min\limits_{2 \leq n \leq N} \{1 + \theta_{n-1} - \mathbb{E}[\theta \mid \theta \geq \theta_n]/\theta_{n-1}\}$.

Proof. Consider the program

$$\max_{u,w} \sum_n \bar{p}_n \{u_n - \hat{\lambda}c(u_n) + \hat{\lambda}y(h_n) - \bar{\theta}_n h_n + \hat{\beta}k(w_n)\}$$

$$v = \sum_n \bar{p}_n(u_n + \beta w_n - \bar{\theta}_n h_n)$$

$$-\theta_n h_n + u_n + \beta w_n \geq -\theta_n h_{n+1} + u_{n+1} + \beta w_{n+1} \text{ for } n = 1, 2, \ldots, K - 1,$$

This problem and its notation require some discussion. We do not incorporate the monotonicity constraint on h. But this notation allows us to consider bunching in the following way. If any set of neighboring agents is bunched, then we group these agents under a single index and let $\bar{p}_n$ be the total probability of this group. Likewise let $\bar{\theta}_n$ represent the conditional average of θ within this group, which is what is relevant for the promise-keeping constraint and the objective function. Let θ_n be the shock of the highest agent in the group. The incentive constraint must rule the highest agent in each group from deviating and taking the allocation of the group above him.

Of course, every combination of bunched agents leads to a different program. The optimal allocation of our problem must solve one of these programs with a strictly monotone allocation— since bunching can be characterized by regrouping agents. Thus, below we characterize solutions to these programs with strict monotonicity of the solution.

The first-order conditions are

$$\bar{p}_n\{\hat{\lambda}y'(h_n) - \bar{\theta}_n + \lambda\bar{\theta}_n\} - \theta_n\mu_n + \theta_{n-1}\mu_{n-1} \leq 0$$

$$\bar{p}_n\{\hat{\beta}k'(w_n) - \beta\lambda\} + \beta(\mu_n - \mu_{n-1}) = 0$$

where, by the envelope condition $\lambda = k'(v)$.

Summing the first-order conditions for h_n, we get

$$\hat{\lambda}[y'(h(\theta))] = 1 - k'(v)$$

Summing up the first-order conditions for w_n, we get

$$[k'(g^w(v,\theta))] = \frac{\beta}{\hat{\beta}}k'(v)$$

The first-order conditions for $n = 1$ imply

$$(1 - \lambda) + \frac{\theta_1}{\bar{\theta}_1}\frac{\mu_1}{\bar{p}_1} = \frac{\hat{\lambda}y'(h_1)}{\bar{\theta}_1} \leq \frac{\hat{\lambda}\,\mathbb{E}[y'(h_\theta)]}{\bar{\theta}_1} = \frac{1 - \lambda}{\bar{\theta}_1}.$$

This implies

$$\frac{\mu_1}{\bar{p}_1} \leq \frac{1 - \lambda}{\theta_1} - (1 - \lambda)\frac{\bar{\theta}_1}{\theta_1}.$$

Using

$$k'(w_1) = \frac{\beta}{\hat{\beta}}\lambda - \frac{\beta}{\hat{\beta}}\frac{\mu_1}{\bar{p}_1},$$

we get

$$k'(w_1) \geq \frac{\beta}{\hat{\beta}}\left[\lambda - \frac{1 - \lambda}{\theta_1} + (1 - \lambda)\frac{\bar{\theta}_1}{\theta_1}\right] = \frac{\beta}{\hat{\beta}}\left[1 + \frac{1}{\theta_1} - \frac{\bar{\theta}_1}{\theta_1}\right]k'(v) + \frac{\beta}{\hat{\beta}}\left[\frac{\bar{\theta}_1}{\theta_1} - \frac{1}{\theta_1}\right].$$

Similarly, writing the first-order conditions for $n = K$, we get

$$(1 - \lambda) - \frac{\theta_{K-1}}{\bar{\theta}_K}\frac{\mu_{K-1}}{\bar{p}_K} = \frac{\hat{\lambda}y'(h_K)}{\bar{\theta}_K} \geq \frac{\hat{\lambda}\,\mathbb{E}[c'(h_\theta)]}{\bar{\theta}_K} = \frac{1 - \lambda}{\bar{\theta}_K}.$$

This implies

$$-\frac{\mu_{K-1}}{\bar{p}_K} \geq \frac{1-\lambda}{\theta_{K-1}} - (1-\lambda)\frac{\bar{\theta}_K}{\theta_{K-1}}.$$

Using

$$k'(w_K) = \frac{\beta}{\hat{\beta}}\lambda + \frac{\beta}{\hat{\beta}}\frac{\mu_{K-1}}{\bar{p}_K},$$

we get

$$k'(w_K) \leq \frac{\beta}{\hat{\beta}}\left[\lambda - \frac{1-\lambda}{\theta_{K-1}} + (1-\lambda)\frac{\bar{\theta}_K}{\theta_{K-1}}\right] = \frac{\beta}{\hat{\beta}}\left[1 + \frac{1}{\theta_{K-1}} - \frac{\bar{\theta}_K}{\theta_{K-1}}\right]k'(v) + \frac{\beta}{\hat{\beta}}\left[\frac{\bar{\theta}_K}{\theta_{K-1}} - \frac{1}{\theta_{K-1}}\right].$$

For any n, $w_K \leq w_n \leq w_1$,

$$\frac{\beta}{\hat{\beta}}\left[1 + \frac{1}{\theta_1} - \frac{\bar{\theta}_1}{\theta_1}\right]k'(v) + \frac{\beta}{\hat{\beta}}\left[\frac{\bar{\theta}_1}{\theta_1} - \frac{1}{\theta_1}\right] \leq k'(w_n)$$

$$\leq \frac{\beta}{\hat{\beta}}\left[1 + \frac{1}{\theta_{K-1}} - \frac{\bar{\theta}_K}{\theta_{K-1}}\right]k'(v) + \frac{\beta}{\hat{\beta}}\left[\frac{\bar{\theta}_K}{\theta_{K-1}} - \frac{1}{\theta_{K-1}}\right].$$

After rearranging, we obtain

$$\frac{\beta}{\hat{\beta}}\left[1 + \frac{1}{\theta_1} - \frac{\bar{\theta}_1}{\theta_1}\right]\left(1 - k'(v)\right) + 1 - \frac{\beta}{\hat{\beta}} \geq 1 - k'(g^w(\theta, v))$$

$$\geq \frac{\beta}{\hat{\beta}}\left[1 + \frac{1}{\theta_{K-1}} - \frac{\bar{\theta}_K}{\theta_{K-1}}\right]\left(1 - k'(v)\right) + 1 - \frac{\beta}{\hat{\beta}}.$$

∎

By Lemma 20 we have that $\lim_{v \to \underline{v}} k'(v) = 1$, then using the bounds we obtain that

$$\lim_{v \to \underline{v}} k'(g^w(v, \theta)) = \frac{\beta}{\hat{\beta}} < 1 = \lim_{v \to \underline{v}} k'(v),$$

for all $\theta \in \Theta$.

The following lemma describes the behavior of the optimal allocation when v goes to $\underline{v}$.

Lemma 26 *We have $g^u(v, \theta) > \underline{u}$ and $\lim_{v \to \underline{v}} g^u(v, \theta) = \underline{u}$, $\lim_{v \to \underline{v}} g^h(v, \theta) = \infty$*

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Proof. Consider the program

$$\max_{u,w} \sum_n \bar{p}_n \{u_n - \hat{\lambda}c(u_n) + \hat{\lambda}y(h_n) - \bar{\theta}_n h_n + \hat{\beta}k(w_n)\}$$

$$v = \sum_n \bar{p}_n(u_n + \beta w_n - \bar{\theta}_n h_n)$$

$$-\theta_n h_n + u_n + \beta w_n \geq -\theta_n h_{n+1} + u_{n+1} + \beta w_{n+1} \text{ for } n = 1, 2, \ldots, K-1,$$

The first order condition for u_n is $1 - \hat{\lambda}c'(u_n) = \frac{\hat{\beta}}{\beta}k'(w_n)$. Hence $1 - \hat{\lambda}c'(g^u(v,\theta)) = \frac{\hat{\beta}}{\beta}k'(g^w(v,\theta))$. Since $k'(g^w(v,\theta)) < \frac{\beta}{\hat{\beta}}$, we have $u_n > \underline{u}$. Moreover, since $\lim_{v \to \underline{v}} k'(v,\theta) = \frac{\beta}{\hat{\beta}}$, we have

$$\lim_{v \to \underline{v}} c'(g^u(v,\theta)) = 0 \qquad \text{or equivalently} \qquad \lim_{v \to \underline{v}} g^u(v,\theta) = \underline{u}$$

That $\lim_{v \to \underline{v}} g^h(v,\theta) = \infty$ follows from

$$\hat{\lambda}[y'(g^h(v,\theta))] = 1 - k'(v)$$

and $\lim_{v \to \underline{v}} k'(v) = 1$. ∎

Since the derivative $k'(v)$ is continuous and strictly decreasing, we can define the transition function

$$Q(x,\theta) = k'(g^w((k')^{-1}(x),\theta))$$

for all $x < l$ if utility is unbounded below. For any probability distribution μ, let $T_Q(\mu)$ be the probability distribution defined by

$$T_Q(\mu)(A) = \int 1_{\{Q(x,\theta) \in A\}} \, d\mu(x) \, dp(\theta)$$

for any Borel set A. Define

$$T_{Q,n} \equiv \frac{T_Q + T_Q^2 + \cdots + T_Q^n}{n}$$

For example, $T_{Q,n}(\delta_x)$ is the empirical average of $\{k'(v_t)\}_{t=1}^n$ over all histories of length n starting with $k'(v_0) = x$. The following lemma establishes the existence of an invariant distribution by considering the limits of $\{T_{Q,n}\}$.

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We are now able to prove a proposition that implies the first part Proposition ??existence, and describes an algorithm to construct an invariant distribution.

Proposition 27 *For each $x < l$ there exists a subsequence $\{T_{Q,\phi(n)}(\delta_x)\}$ that converges weakly, i.e. in distribution, to an invariant distribution on $(-\infty, 1)$ under Q.*

Proof. For all $\theta \in \Theta$

$$\lim_{x \uparrow 1} Q(x, \theta) = \lim_{v \to -\infty} k'(g^w(\theta, v)) = \frac{\beta}{\hat{\beta}} < 1.$$

Note that we have a continuous transition function $Q(x, \theta) \colon (-\infty, 1) \times \Theta \to (-\infty, 1)$.

We next show that the sequence $\{T_Q^n(\delta_x)\}$ is tight, in that for any $\varepsilon > 0$ there exists a compact set K_ε such that $T_Q^n(\delta_x)(K_\varepsilon) \geq 1 - \varepsilon$, for all n. The expected value of the distribution $T_Q^n(\delta_x)$ is simply $\mathbb{E}_{-1}[k'(v_t(\theta^{t-1}))]$ with $x = k'(v_0) < 1$. Recall that $\mathbb{E}_{-1}[k'(v_t(\theta^{t-1}))] = (\beta/\hat{\beta})^t k'(v_0) \to 0$. This implies that

$$\min\{0, k'(v_0)\} \leq \mathbb{E}_{-1}[k'(v_t(\theta^{t-1}))]$$

$$\leq T_Q^n(\delta_x)(-\infty, -A)(-A) + (1 - T_Q^n(\delta_x)(-\infty, -A))1$$

for all $A > 0$. Rearranging,

$$T_Q^n(\delta_x)(-\infty, -A) \leq \frac{1 - \min\{0, x\}}{A + 1}$$

Hence we can find $A_\varepsilon > 0$ such that

$$T_Q^n(\delta_x)(-\infty, -A_\varepsilon) \leq \frac{\varepsilon}{2}$$

Define a_ε by

$$1 - a_\varepsilon = \sup_{\substack{x \in [A_\varepsilon, 1) \\ \theta \in \Theta}} Q(x, \theta)$$

Since for all $\theta \in \Theta$, $\lim_{v \to -\infty} k'(v, \theta) < \frac{\beta}{\hat{\beta}} < 1$, we have $a_\varepsilon > 0$. In addition, for all $n \geq 1$, $T_Q^n(\delta_x) = T_Q(T_Q^{n-1}(\delta_x))$, so that

$$T_Q^n(\delta_x)(1 - a_\varepsilon, 1) \leq T_Q^{n-1}(\delta_x)(-\infty, A_\varepsilon) \leq \frac{\varepsilon}{2}$$

Since we also have

$$T_Q^n(\delta_x)(-\infty, -A_\varepsilon) \leq \frac{\varepsilon}{2}$$

this implies

$$T_Q^n(\delta_x)[A_\varepsilon, 1 - a_\varepsilon] \geq \varepsilon$$

Taking $K_\varepsilon = [A_\varepsilon, 1 - a_\varepsilon]$, this implies that $\{T_Q^n(\delta_x)\}_{n \geq 1}$ is tight, and therefore $\{T_Q^n(\delta_x)\}_{n \geq 0}$, is tight.

Tightness implies that there exists a subsequence $T_Q^{\phi(n)}(\delta_x)$ that converges weakly, i.e. in distribution, to some probability distribution π on $(-\infty, 1)$. Since $Q(x, \theta)$ is continuous in x, then $T_Q(T_Q^{\phi(n)}(\delta_x))$ converges weakly to $T_Q(\pi)$. But the linearity of T_Q implies that

$$T_Q(T_Q^{\phi(n)}(\delta_x)) = \frac{T_Q^{\phi(n)+1}(\delta_x) - T_Q(\delta_x)}{\phi(n)} + T_Q^{\phi(n)}(\delta_x)$$

and since $\phi(n) \to \infty$ we must have $T_Q(\pi) = \pi$. ∎

Note that for any invariant distribution π, $T_Q(\pi) = \pi$ implies that the support of π is contained in $(-\infty, \frac{\beta}{\hat{\beta}}]$. This proves the second part of Proposition 22. We finally prove a lemma that implies the last part Proposition ??existence.

Lemma 28 *Suppose that* $\lim_{u \to \infty} \sup c''(u)/c'(u) < \infty$. *Then any invariant distribution* $\hat{\psi}$ *necessarily has a support bounded away from* $\bar{v}$.

Proof. We will make use of two possible state variables. The first state variable is the natural one: v, promised future utility. The other one is utility attained by the previous generation u_-. Indeed, from the first order conditions, it is easy to see that these two state variables are related by

$$1 - \hat{\lambda} c'(u_-) = \frac{\hat{\beta}}{\beta} k'(v)$$

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The existence of an invariant distribution $\psi^*(v)$ with not mass at misery is equivalent to the existence of an invariant distribution $\hat{\psi}^*(u_-)$ with no mass at misery.

Let $x_\theta = u_\theta + \beta w_\theta$. Then we can rewrite the Bellman equation (2.21) as

$$k(v) = \sup E[u_\theta - \hat{\lambda}c(u_\theta) - \theta h_\theta + \hat{\lambda}y(h_\theta) + \hat{\beta}k(w_\theta)]$$

subject to

$$v = E[x_\theta - \theta h_\theta]$$

$$x_\theta - \theta h_\theta \geq x_{\theta'} - \theta h_{\theta'}$$

$$u_\theta + \beta w_\theta = x_\theta$$

Hence, given a value x for x_θ, u_θ and w_θ are given by the sub-program

$$\max u - \hat{\lambda}c(u) + \hat{\beta}k(w)$$

subject to

$$u + \beta w = x$$

The solution is given by the first order condition

$$1 - \hat{\lambda}c'(u) = \frac{\hat{\beta}}{\beta}k'(\frac{x-u}{\beta}) = 0$$

Using the implicit function theorem, we can then compute

$$\frac{du}{dx} = \frac{-\frac{\hat{\beta}}{\beta}k''(\frac{x-u}{\beta})}{-\frac{\hat{\beta}}{\beta}k''(\frac{x-u}{\beta}) + \hat{\lambda}c''(u)}$$

Hence

$$0 \leq \frac{du}{dx} \leq 1$$

This in turn implies that there exists $M > 0$ such that

$$\max_{\theta,\theta'} |u_{\theta'} - u_\theta| < M \max_\theta h_\theta$$

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Consider the program (2.9). The first order condition for h_n is

$$\bar{p}_n\{\hat{\lambda}y'(h_n) - \bar{\theta}_n + \lambda\bar{\theta}_n\} - \theta_n\mu_n + \theta_{n-1}\mu_{n-1} \leq 0$$

where $\lambda = k'(v)$. This implies that

$$y'(h_{\underline{\theta}}) \geq \frac{\theta}{\hat{\lambda}}(1 - k'()v)$$

This shows that

$$\lim_{v\to\bar{v}} y'(h_{\underline{\theta}}(v)) = \infty \quad \Rightarrow \quad \lim_{v\to\bar{v}} h_{\underline{\theta}}(v) = 0$$

and since h_θ has is decreasing in θ,

$$\lim_{v\to\bar{v}} h_\theta(v) = 0 \text{ for all } \theta \in \Theta$$

The first order condition(2.32) implies that

$$c'(u_-) \geq \frac{\hat{\beta}}{\beta}c'(u_{\underline{\theta}}) - \hat{\lambda}^{-1}(\frac{\hat{\beta}}{\beta} - 1)$$

which can be rewritten as

$$\frac{\beta}{\hat{\beta}}c'(u_-) + \hat{\lambda}^{-1}(1 - \frac{\beta}{\hat{\beta}}) \geq c'(u_{\underline{\theta}})$$

This in turn implies that for all $\theta \in \Theta$

$$\exp\left(M \max_\theta h_\theta \max_{u\in[u_{\bar{\theta}},u_{\underline{\theta}}]} \frac{c''(u)}{c'(u)}\right)\left(\frac{\beta}{\hat{\beta}}c'(u_-) + \hat{\lambda}^{-1}(1 - \frac{\beta}{\hat{\beta}})\right) \geq c'(u_\theta)$$

Since

$$\lim_{v\to\bar{v}} h_\theta(v) = 0 \text{ for all } \theta \in \Theta$$

we have

$$\lim_{u_-\to\bar{u}} \exp\left(M \max_\theta h_\theta \max_{u\in[u_{\bar{\theta}},u_\theta]} \frac{c''(u)}{c'(u)}\right) = 1 \text{ for all } \theta \in \Theta$$

This in turn proves that for u_- high enough, all the policy functions u_θ are such that $u_\theta < u_-$. Hence any invariant distribution $\hat{\psi}^*$ necessarily has a support bounded away from $\overline{u}$. This is equivalent to saying that any invariant distribution $\hat{\psi}$ necessarily has a support bounded away from $\overline{v}$. $\blacksquare$

Bibliography

[1] Albanesi, Stefania and Christoper Sleet, "Dynamic Optimal Taxation with Private Information," 2004. mimeo. 2, 4, 11, 12, 15

[2] Amador, Manuel, George-Marios Angeletos, and Iv´an Werning, "Corrective vs. Redistributive Taxation," 2005. work in progress. 11

[3] Atkeson, Andrew and Robert E. Lucas Jr., "On Efficient Distribution with Private Information," Review of Economic Studies, 1992, v59(3), 427–453. 3, 15

[4] Atkeson, Andrew and Robert Lucas, "Efficiency and Equality in a Simple Model of Efficiency Unemployment Insurance," Journal of Economic Theory, 1995, 66, 64–88. 2, 15

[5] Atkinson, A.B. and J.E. Stiglitz, "The Design of Tax Structure: Direct vs. Indirect Taxation,"Journal of Public Economics, 1976, 6, 55–75. 2, 8

[6] Diamond, Peter A., "Optimal Income Taxation: An Example with a U-Shaped Pattern of Optimal Marginal Tax Rates," American Economic Review, 1998, 88 (1), 83–95. 4

[7] Ebert, Udo, "A reexamination of the optimal nonlinear income tax," Journal of Public Economics, 1992, 49 (1), 47–73. 4

[8] Farhi, Emmanuel and Ivan Werning, "Inequality, Social Discounting and Progressive Estate Taxation," NBER Working Papers 11408 2005. 4, 12, 13, 16

[9] Farhi, Emmanuel, Narayana Kocherlakota, and Ivan Werning, 2005. work in progress. 4, 8, 16, 18

[10] Golosov, Mikhail, Aleh Tsyvinski, and Iv´an Werning, "New Dynamic Public Finance: A User's Guide," forthcoming in NBER Macroeconomics Annual 2006, 2006. 4

[11] Golosov, Mikhail,Narayana Kocherlakota, and Aleh Tsyvinski, "Optimal Indirect and Capital Taxation,"Review of Economic Studies, 2003, 70 (3), 569–587. 4

[12] Kocherlakota, Narayana, "Zero Expected Wealth Taxes: A Mirrlees Approach to Dynamic Optimal Taxation," 2004. mimeo. 4, 16

[13] Mirrlees, James A., "An Exploration in the Theory of Optimum Income Taxation," Review of Economic Studies, 1971, 38 (2), 175–208. 2, 4

[14] Saez, Emmanuel, "Using Elasticities to Derive Optimal Income Tax Rates," Review of Economic Studies, 2001, 68 (1), 205–29. 4

[15] Seade, Jesus, "On the Sign of the Optimum Marginal Income Tax," Review of Economic Studies, 1982, 49 (4), 637–43. 4

[16] Tuomala, Matti, Optimal Income Taxation and Redistribution, Oxford University Press, Clarendon Press, 1990. 4

[17] Seade, Jesus, "On the Sign of the Optimum Marginal Income Tax," Review of Economic Studies, 1982, 49 (4), 637–43. 4

Chapter 3

Saving and Investing for Early Retirement: A Theoretical Analysis[1]

Introduction

> Two years ago, when the stock market was soaring, 401(k)'s were swelling and
> (...) early retirement seemed an attainable goal. All you had to do was invest
> that big job-hopping pay increase in a market that produced double-digit gains like
> clockwork, and you could start taking leisurely strolls down easy street at the ripe
> old age of, say, 55. (Business Week December 31, 2001)

The dramatic rise of the stock market between 1995 and 2000 significantly increased the
proportion of workers opting for early retirement (Gustman and Steinmeier, 2002). The above

quote from Business Week demonstrates the rationale behind the decision to retire early: A booming stock market raises the amount of funds available for retirement and allows a larger fraction of the population to exit the workforce prematurely.

Indeed, for most individuals, increasing one's retirement savings seems to be one of the primary motivations behind investing in the stock market. Accordingly, there is an increased need to understand the interactions among optimal retirement, portfolio choice, and savings, especially in light of the growing popularity of 401(k) retirement plans. These plans give individuals a great amount of freedom when determining how to save for retirement. However, such increased flexibility also raises concerns about the extent to which agents' portfolio and savings decisions are rational. Having a benchmark against which to determine the rationality of people's choices is crucial for both policy design and in order to form the basis of sound financial advice.

In this paper we develop a theoretical model with which we address some of the interactions among savings, portfolio choice, and retirement in a utility maximizing framework. We assume that agents face a constant investment opportunity set and a constant wage rate while still in the workforce. Their utility exhibits constant relative risk aversion and is nonseparable in leisure and consumption. The major point of departure from preexisting literature is that we model the labor supply choice as an optimal stopping problem: An individual can work for a fixed (nonadjustable) amount of time and earn a constant wage but is free to exit the workforce (forever) at any time she chooses. In other words, we assume that workers can work either full time or retire. As such, individuals face three decision problems: 1) how much to consume, 2) how to invest their savings, and 3) when to retire. The incentive to quit work comes from a discrete jump in their utility due to an increase in leisure once retired. When retired, individuals cannot return to the workforce.[2] We also consider two extensions of the basic framework. In the first extension we disallow the agent from choosing retirement past a pre-specified deadline. In a second extension we disallow her from borrowing against the net present value (NPV) of her human capital (i.e., we require that financial wealth be nonnegative).

The major results that we obtain can be summarized as follows:

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[2]This assumption can actually be easily relaxed. For instance, we could assume that retirees can return to the workforce (at a lower wage rate) without affecting any of the major predictions of the model.

First, we show that the agent will enter retirement when she reaches a certain wealth threshold, which we determine explicitly. In this sense, wealth plays a dual role in our model: Not only does it determine the resources available for future consumption, but it also controls the "distance" to retirement.

Second, the option to retire early strengthens the incentives to save compared to the case in which early retirement is not allowed. The reason is that saving not only increases consumption in the future but also brings retirement "closer." Moreover, this incentive is wealth dependent. As the individual approaches the critical wealth threshold to enter retirement, the "option" value of retiring early becomes progressively more important and the saving motive becomes stronger.

Third, the marginal propensity to consume (MPC) out of wealth declines as wealth increases and early retirement becomes more likely. The intuition is simple: An increase in wealth will bring retirement closer, therefore decreasing the length of time the individual remains in the workforce. Conversely, a decline in wealth will postpone retirement. Thus, variations in wealth are somewhat counterbalanced by the behavior of the remaining NPV of income and in turn the effect of a marginal change in wealth on consumption becomes attenuated. Once again this attenuation is strongest for rich individuals who are closer to their goal of early retirement.

Fourth, the optimal portfolio is tilted more towards stocks compared to the case in which early retirement is not allowed. An adverse shock in the stock market will be absorbed by postponing retirement. Thus, the individual is more inclined to take risks as she can always postpone her retirement instead of cutting back her consumption in the event of a declining stock market. Moreover, in order to bring retirement closer, the most effective way is to invest the extra savings in the stock market instead of the bond market.

Fifth, the choice of portfolio over time exhibits some new and interesting patterns. We show that there exist cases in which an agent might optimally increase the percentage of financial wealth that she invests in the stock market as she ages (in expectation), *even though* her income and the investment opportunity set are constant. This result obtains, because wealth increases over time and hence the option of early retirement becomes more relevant. Accordingly, the tilting of the optimal portfolio towards stocks becomes stronger. Indeed, as we show in a calibration exercise, the model predicts that, prior to retirement, portfolio holdings could increase,

especially when the stock market exhibits extraordinary returns as it did in the late 1990s during which time many workers experienced rapid increases in wealth, and allow the individual to opt for an earlier retirement date. In fact our model suggests a possible partial rationalization of the (apparently irrational) behavior of individuals who increased their portfolios as the stock market was rising and then liquidated stock as the market collapsed.[3]

This paper is related to a number of strands in the literature that are surveyed in Ameriks and Zeldes (2001) and Jagannathan and Kocherlakota (1996). The paper closest to ours is that of Bodie, Merton, and Samuelson (1992) (henceforth BMS). The major difference between BMS and this paper is the different assumption we make about the ability of agents to adjust their labor supply. In BMS, labor can be adjusted in a continuous fashion. However, a significant amount of evidence suggests that labor supply is to a large extent indivisible. For example, in many jobs workers work either full time or they are retired. Moreover, it appears that most people do not return to work after they retire, or if they do, they return to less well-paying jobs or they work only part time. As BMS claim in the conclusion of their paper,

> Obviously, the opportunity to vary continuously one's labor without cost is a far cry from the workings of actual labor markets. A more realistic model would allow limited flexibility in varying labor and leisure. One current research objective is to analyze the retirement problem as an optimal stopping problem and to evaluate the accompanying portfolio effects.

This is precisely the direction we take here. There are at least two major directions in which our results differ from BMS. First, we show that the optimal retirement decision introduces a nonlinear option-type element in the decision of the individual that is entirely absent if labor is adjusted continuously. Second, the horizon and wealth effects on portfolio and consumption choice in our paper are fundamentally different than those in BMS. For instance, stock holdings in BMS are a *constant multiple* of the sum of (financial) wealth and human capital. This multiple is not constant in our setup, but instead depends on wealth.[4] Third, the model we present here allows for a clear way to model retirement, which is difficult in the literature that

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[3]Some (indirect) evidence to this fact is given in the August 2004 Issue Brief of the Employee Benefit Research Institute (Fig. 2 - based on the EBRI/ICI 401(k) Data).

[4]If we impose a retirement deadline, this multiple also depends on the distance to this deadline.

allows for a continuous labor-leisure choice. An important implication is that in our setup, we can calibrate the parameters of the model to observed retirement decisions. In the BMS framework, on the other hand, calibration to microeconomic data is harder because individuals do not seem to adjust their labor supply continuously.[5]

The model is also related to a strand of the literature that studies retirement decisions. A partial listing includes Stock and Wise (1990), Rust (1994), Laezar (1986), Rust and Phelan (1997), and Diamond and Hausman (1984). Most of these models are structural estimations that are solved numerically. Here our goal is different: Rather than include all the realistic ramifications that are present in actual retirement systems, we isolate and very closely analyze the new issues introduced by the indivisibility and irreversibility of the labor supply - retirement decision on savings and portfolio choice. Naturally, there is a trade-off between adding realistic considerations and the level of theoretical analysis that we can accomplish with a more complicated model. Other studies in this literature include Sundaresan and Zapatero (1997), who study optimal retirement, but in a framework without disutility of labor, and Bodie, Detemple, Otruba and Walter (2004), who investigate the effects of habit formation, but without optimal retirement timing.

Some results of this paper share similarities with results that obtain in the literature on consumption and savings in incomplete markets. A highly partial listing includes Viceira (2001), Chan and Viceira (2000), Campbell, Cocco, Gomes and Maenhout (2001), Kogan and Uppal (2001), Duffie, Fleming, Soner and Zariphopoulou (1997), Duffie and Zariphopoulou (1993), Koo (1998), and Carroll and Kimball (1996) on the role of incomplete markets and He and Pages (1993) and El Karoui and Jeanblanc-Pique (1998) on issues related to the inability of individuals to borrow against the NPV of their future income. This literature provides insights on why consumption (as a function of wealth) should be concave, and also offers some implications on portfolio choice. However, while in the incomplete markets literature, the results are driven by the inability of agents to effectively smooth their consumption due to missing markets,[6] in this

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[5]Liu and Neiss (2002) study a framework similar to BMS, but force an important constraint on the maximal amount of leisure. This, however, omits the issues related to indivisibility and irreversibility, which as we show lead to fundamentally different implications for the resulting portfolios. In sum, the fact that labor supply flexibility is modeled in a more realistic way allows a closer mapping of the results to real-world institutions than is allowed for by a model that exhibits continuous choice between labor and leisure.

[6]Chan and Viceira (2000) combines insights of both literatures. However, they assume labor- leisure choices that can be adjusted continuously.

paper the results are driven by an option component in an agent's choices that is related to the ability of agents to adjust their time of retirement.

Throughout the paper we maintain the assumption that agents receive a constant wage. This is done not only for simplicity, but more importantly because it makes the results more surprising. It is well understood in the literature[7] that allowing for a (positive) correlation between wages and the stock market can generate upward-sloping portfolio holdings over time. What we show is that optimal retirement choice can induce observationally similar effects *even when labor income is perfectly riskless.* Since the argument and the intuition for this outcome are orthogonal to those in existing models, we prefer to use the simplest possible setup in every other dimension, thereby isolating the effects of optimal early retirement.

Technically, our model extends methods proposed by Karatzas and Wang (2000) (who do not allow for income) to solve optimal consumption problems with discretionary stopping. The extension that we consider in Section 3.3 uses ideas proposed by Barone-Adesi and Whaley (1987), and in Section 3.5, we extend the framework in He and Pages (1993) to allow for early retirement.

Finally, three papers that present parallel and independent work on similar issues are Lachance (2003), Choi and Shim (2004), and Dybvig and Liu (2005). Lachance (2003) and Choi and Shim (2004) study a model with a utility function that is separable in leisure and consumption, but that abstracts from a deadline for retirement and/or borrowing constraints.[8] The somewhat easier specification of separable utility does not allow consumption to fall upon retirement as we observe in the data. Technically, these papers solve the problem using dynamic programming rather than convex duality methods, which cannot be easily extended to models with deadlines, borrowing constraints, etc. Our approach overcomes these difficulties. Dybvig and Liu (2005) study a very similar model to that in Section 3.5 of this paper, with similar techniques. However, they do not consider retirement prior to a deadline as we do. A deadline makes the problem considerably harder (since the critical wealth thresholds become time dependent). Nonetheless, we are able to provide a fairly accurate approximate closed-form solution for this problem in Section 3.3. One can actually perform simple exercises that demon-

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[7] See, e.g., Jagannathan and Kocherlakota (1996) and BMS.

[8] Another model that makes similar assumptions is that of Kingston (2000).

strate that in the absence of a retirement deadline, the model-implied distribution of retirement times becomes implausible. Most importantly, compared to the papers above, the present paper goes into significantly greater detail in terms of the economic analysis and implications of the results. In particular, we provide applications (like the analysis of portfolios of agents saving for early retirement in the late 1990s) that demonstrate quite clearly the real-world implications of optimal portfolio choice in the presence of early retirement.

The structure of the paper is as follows: Section 3.1 contains the model setup. In Section 3.2 we describe the analytical results if one places no retirement deadline. Section 3.3 contains an extension to the case in which retirement cannot take place past a deadline, Section 3.4 contains some calibration exercises, and Section 3.5 extends the model by imposing borrowing constraints. Section 3.6 concludes. We present technical details and all proofs in the Appendix.

3.1   Model setup

3.1.1   Investment opportunity set

The consumer can invest in the money market, where she receives a fixed, strictly positive interest rate $r > 0$. We place no limits on the positions that can be taken in the money market. In addition, the consumer can invest in a risky security with a price per share that evolves according to

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t,$$

where $\mu > r$ and $\sigma > 0$ are known constants and B_t is a one-dimensional Brownian motion on a complete probability space (Ω, F, P).[9] We define the state-price density process (or stochastic discount factor) as

$$H(t) = \Xi(t)Z^*(t), \quad H(0) = 1,$$

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[9] We shall denote by $F = \{F_t\}$ the P-augmentation of the filtration generated by B_t.

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where $\Xi(t)$ and $Z^*(t)$ are given by

$$\Xi(t) = e^{-rt}$$
$$Z^*(t) = \exp\left\{-\int_0^t \kappa dB_s - \frac{1}{2}\kappa^2 t\right\}, \quad Z^*(0) = 1$$

and κ is the Sharpe ratio

$$\kappa = \frac{\mu - r}{\sigma}.$$

As is standard, these assumptions imply a dynamically complete market (Karatzas and Shreve, 1998, Chapter 1).

3.1.2 Portfolio and wealth processes

An agent chooses a portfolio process π_t and a consumption process $c_t > 0$. These processes are progressively measurable and they satisfy the standard integrability conditions given in Karatzas and Shreve (1998) Chapters 1 and 3. The agent also receives a constant income stream y_0 while she works and no income stream while in retirement. Retirement is an irreversible decision. We assume until Section 3.3 that an agent can retire at any time she chooses.

The agent is endowed with an amount of financial wealth $W_0 \geq -\frac{y_0}{r}$. The process of stock-holdings π_t is the dollar amount invested in the risky asset (the "stock market") at time t. The amount $W_t - \pi_t$ is therefore invested in the money market. Short selling and borrowing are both allowed. We place no extra restrictions on the (financial) wealth process W_t until Section 3.5 of the paper. Additionally, in Section 3.5 we will impose the restriction $W_t \geq 0$. As long as the agent is working, the wealth process evolves according to

$$dW_t = \pi_t \left\{\mu dt + \sigma dB_t\right\} + \left\{W_t - \pi_t\right\} r dt - (c_t - y_0) dt. \tag{3.1}$$

Applying Ito's Lemma to the product of $H(t)$ and $W(t)$, integrating, and taking expectations, we get for any stochastic time τ that is finite almost surely

$$E\left(H(\tau)W(\tau) + \int_0^\tau H(s)\left[c(s) - y_0\right] ds\right) \leq W_0. \tag{3.2}$$

This is the well-known result that in dynamically complete markets one can reduce a dy-

namic budget constraint of the type in Eq. (3.1) to a single intertemporal budget constraint of the type in Eq. (3.2). If the agent is retired, the above two equations continue to hold with $y_0 = 0$.

3.1.3 Leisure, income, and the optimization problem

To obtain closed-form solutions, we assume that the consumer has a utility function of the form

$$U(l_t, c_t) = \frac{1}{\alpha} \frac{\left(l_t^{1-\alpha} c_t^{\alpha}\right)^{1-\gamma^*}}{1 - \gamma^*}, \quad \gamma^* > 0, \tag{3.3}$$

where c_t is per-period consumption, l_t is leisure, and $0 < \alpha < 1$. We assume that the consumer is endowed with $\bar{l}$ units of leisure. Leisure can only take two values, l_1 or $\bar{l}$: If the consumer is working, $l_t = l_1$; if the consumer is retired $l_t = \bar{l}$. We assume that the wage rate w is constant, so that the income stream is $y_0 = w(\bar{l} - l_1) > 0$. We normalize $l_1 = 1$. Note that this utility is general enough so as to allow consumption and leisure to be either complements ($\gamma^* < 1$) or substitutes ($\gamma^* > 1$). The consumer maximizes expected utility

$$\max_{c_t, \pi_t, \tau} E\left[\int_0^\tau e^{-\beta t} U(l_1, c_t) dt + e^{-\beta \tau} \int_\tau^\infty e^{-\beta(t-\tau)} U(\bar{l}, c_t) dt\right], \tag{3.4}$$

where $\beta > 0$ is the agent's discount factor.[10] The easiest way to proceed is to start backwards by solving the problem

$$U_2(W_\tau) = \max_{c_t, \pi_t} E\left[\int_\tau^\infty e^{-\beta(t-\tau)} U(\bar{l}, c_t) dt\right],$$

where $U_2(W_\tau)$ is the value function once the consumer decides to retire and W_τ is the wealth at retirement. By the principle of dynamic programming we can rewrite (3.4) as

$$\max_{c_t, W_\tau, \tau} E\left[\int_0^\tau e^{-\beta t} U(l_1, c_t) dt + e^{-\beta \tau} U_2(W_\tau)\right]. \tag{3.5}$$

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[10] By standard arguments the constant discount factor β could also incorporate a constant hazard rate of death, λ.

It will be convenient to define the parameter γ as

$$\gamma = 1 - \alpha(1 - \gamma^*)$$

so we can then reexpress the per-period utility function as

$$U(l, c) = l^{(1-\alpha)(1-\gamma^*)} \frac{c^{1-\gamma}}{1-\gamma}.$$

Since we have normalized $l = 1$ prior to retirement, the per-period utility prior to retirement is given by

$$U_1(c) = U(1, c) = \frac{c^{1-\gamma}}{1-\gamma}. \tag{3.6}$$

Notice that $\gamma > 1$ if and only if $\gamma^* > 1$, and $\gamma < 1$ if and only if $\gamma^* < 1$. Under these assumptions, it follows from standard results (See, e.g., Karatzas and Shreve, 1998, Chapter 3), that once in retirement, the value function becomes

$$U_2(W_\tau) = \left(\bar{l}^{1-\alpha}\right)^{1-\gamma^*} \left(\frac{1}{\theta}\right)^\gamma \frac{W_\tau^{1-\gamma}}{1-\gamma}, \tag{3.7}$$

where

$$\theta = \frac{\gamma - 1}{\gamma}\left(r + \frac{\kappa^2}{2\gamma}\right) + \frac{\beta}{\gamma}.$$

In order to guarantee that the value function is well defined, we assume throughout that[11] $\theta > 0$ and $\beta - r < \frac{\kappa^2}{2}$.[12] It will be convenient to redefine the continuation value function as

$$U_2(W_\tau) = K \frac{W_\tau^{1-\gamma}}{1-\gamma},$$

where

$$K = \left(\bar{l}^{1-\alpha}\right)^{1-\gamma^*} \left(\frac{1}{\theta}\right)^\gamma. \tag{3.8}$$

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[11] Observe that this is guaranteed if $\gamma > 1$.

[12] As we show in the Appendix, this will guarantee that retirement takes place with probability one in this stochastic setup.

Since $\bar{l} > l_1 = 1$, it follows that

$$K^{\frac{1}{\gamma}} > \frac{1}{\theta} \text{ if } \gamma < 1 \tag{3.9}$$

$$K^{\frac{1}{\gamma}} < \frac{1}{\theta} \text{ if } \gamma > 1. \tag{3.10}$$

3.2 Properties of the solution

Theorem 31 in the Appendix presents a formal solution to the problem. The nature of the solution is intuitive: The agent enters retirement if and only if the level of her assets exceeds a critical level $\overline{W}$, which we analyze more closely in Subsection 3.2.1. As might be expected, another feature of the solution is that the agent's marginal utility of consumption equals the stochastic discount factor, both pre- and post-retirement (up to a constant λ^*, which depends on the wealth of the agent at time 0 and is chosen so that the intertemporal budget constraint is satisfied):

$$e^{-\beta t} U_C(l_t, c_t) = e^{-\beta t} l_t^{(1-\alpha)(1-\gamma^*)} c_t^{-\gamma} = \lambda^* H(t). \tag{3.11}$$

This is just a manifestation of the fact that the market is dynamically complete.[13] Importantly, the marginal utility of consumption is continuous when the agent enters retirement. This is a consequence of a principle in optimal stopping that is known as "smooth pasting," which implies that the derivative of the value function J_W is continuous. When smooth pasting is combined with the standard envelope theorem, that is,

$$U_C(l_t, c_t) = J_W, \tag{3.12}$$

it follows that the marginal utility of consumption (U_C) is continuous. A consequence of the continuity of the marginal utility of consumption is that consumption itself will jump when the agent enters retirement. This is simply because consumption needs to "counteract" the discrete change in leisure, which enters the marginal utility of consumption in a nonseparable way. The jump is given by

$$\frac{c_{\tau+}}{c_{\tau-}} = \bar{l}^{(1-\alpha)\frac{(1-\gamma^*)}{\gamma}} = K^{1/\gamma}\theta. \tag{3.13}$$

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[13]See the monograph of Karatzas and Shreve (1998), Chapters 3 and 4.

Notice that $\gamma^* > 1$ will imply a downward jump and $\gamma^* < 1$ an upward jump (since $\bar{l} > 1$). For the empirically relevant case ($\gamma^* > 1$), the model predicts a downward jump in consumption, consistent with the data.

In the next three subsections we explore some properties of the solution in more detail. The benchmark model against which we compare our results is a model in which there is a constant labor income stream and no retirement (the worker works forever). This is the natural benchmark for this section, since it keeps all else equal except for the the option to retire. The results we obtain in this section allow us to isolate insights related to optimal retirement in a framework in which solutions are not time dependent and therefore are easier to analyze. Fortunately, all of the results continue to hold when we introduce a retirement deadline in Section 3.3, in which case the natural benchmark model will be one in which the agent is forced to work for a fixed amount of time, which is more natural.

3.2.1   Wealth at retirement

For a constant ξ_2, where

$$\xi_2 = \frac{1 - 2\frac{\beta - r}{\kappa^2} - \sqrt{(1 - 2\frac{\beta - r}{\kappa^2})^2 + 8\frac{\beta}{\kappa^2}}}{2},$$

Theorem 31 gives wealth at retirement as

$$\overline{W} = \frac{(\xi_2 - 1)K^{\frac{1}{\gamma}}\theta}{\left(1 + \xi_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta - 1\right)}\frac{y_0}{r}. \tag{3.14}$$

As Theorem 31 asserts, for wealth levels higher than $\overline{W}$, it is optimal to enter retirement, whereas for lower wealth levels, it is optimal to remain in the workforce. In the Appendix we show that $\overline{W}$ is strictly positive, i.e., a consumer will never enter retirement with negative wealth since there is no more income to support post-retirement consumption. It is clear that the critical wealth level $\overline{W}$ does not depend on the initial wealth of the consumer. The discrete decision between work and retirement is a choice variable that by standard dynamic programming depends only on the current state variable of the system, namely, the current level of wealth. If the current level of wealth is above $\overline{W}$, retirement is triggered, otherwise it

is not. However, it is still true that agents who start life with a higher level of wealth are more likely to hit the retirement threshold in a shorter amount of time. Potentially, an agent might be born with a wealth level sufficiently above $\overline{W}$ that she can retire immediately.

To understand the forces behind the determination of $\overline{W}$, we sketch the basic idea behind the derivation of (3.14). The Appendix demonstrates that one can reduce the entire consumption - portfolio - retirement timing problem to a standard optimal stopping problem. After reducing the problem to an optimal stopping problem, one can use well-known intuitions from option valuation. Specifically, retirement can be viewed as an American-type option that allows one to exchange the value of future income with the extra leisure that is brought about by retiring. In this section the option has infinite maturity; in Section 3.3 its maturity is finite. The key idea behind transforming the problem into a standard optimal timing problem is not to use the level of wealth as the state variable, but instead its marginal value, J_W. We can see the advantage of doing so by combining (3.11) and (3.12) to obtain

$$J_W = \lambda^* e^{\beta t} H(t).$$

The right-hand side of this equation is exogenous (up to a constant that is chosen so as to satisfy the intertemporal budget constraint). By contrast, the evolution of wealth itself depends on both optimal consumption and portfolio choice, and thus both the drift and the volatility of the wealth process are endogenous.

The next step is to define $Z_t = J_W = \lambda^* e^{\beta t} H(t)$ as a state variable and pose the entire problem as a standard optimal stopping problem in terms of Z_t. It is straightforward to show that

$$\frac{dZ_t}{Z_t} = (\beta - r)\, dt - \kappa dB_t,$$

and thus that Z_t follows a standard geometric Brownian motion with volatility equal to the Sharpe ratio (κ) and drift equal to the difference between the discount rate and the interest rate ($\beta - r$). To complete the analogy with option pricing, it remains to determine the net payoff from exercising the option of early retirement as a function of Z_t. The key difficulty in achieving this is that the cost of forgone income is expressed in monetary terms, while the benefit of extra leisure is in utility terms. The Appendix shows that the correct notion of net benefit is the

difference in "consumer surplus" enjoyed by an agent who is not working versus someone who is, assuming that the marginal value of wealth is the same for both. In the Appendix we show that this computation leads to the following net payoff upon retirement:

$$Z_\tau \left[\frac{1}{\theta} \frac{\gamma}{1-\gamma} \left(K^{1/\gamma} \theta - 1 \right) Z_\tau^{-\frac{1}{\gamma}} - \frac{y_0}{r} \right].$$

(3.15)

The term Z_τ outside the square brackets is equal to the marginal value of wealth upon retirement and hence transforms monetary units into marginal utility units. The second term inside the square brackets $(-\frac{y_0}{r})$ is negative, capturing the permanent loss of the net present value of income, and can be thought of as the "strike" price of the option.

The first term inside square brackets is always positive and captures the payoff of the option. To analyze this term, it is probably easiest to use the following relation, which we show in the Appendix,

$$\overline{W} = K^{1/\gamma} Z_\tau^{-\frac{1}{\gamma}}$$

so that we can rewrite the payoff once the option is exercised as

$$Z_\tau \left[\frac{\gamma}{1-\gamma} \frac{(K^{1/\gamma}\theta - 1)}{K^{1/\gamma}\theta} \overline{W} - \frac{y_0}{r} \right].$$

The first term inside the square brackets now has an intuitive interpretation: It is the monetary equivalent of obtaining the leisure level $\bar{l}$. Schematically speaking, going into retirement is "as if" the wealth of the agent is increased by $\frac{\gamma}{1-\gamma} \frac{(K^{1/\gamma}\theta-1)}{K^{1/\gamma}\theta} \overline{W}$ at the fixed cost $\frac{y_0}{r}$. Summarizing, in order to obtain the optimal retirement time, it suffices to solve the optimal stopping problem

$$\sup_\tau E \left\{ e^{-\beta\tau} Z_\tau \left[\frac{1}{\theta} \frac{\gamma}{1-\gamma} \left(K^{1/\gamma}\theta - 1 \right) Z_\tau^{-\frac{1}{\gamma}} - \frac{y_0}{r} \right] \right\}.$$

There are many direct consequences of this option interpretation. For instance, a standard intuition in optimal stopping is that an increase in the "payoff" of the option upon exercise will increase the opportunity cost of waiting and will make the agent exercise the option earlier. A consequence of Eq. (3.15) is that an increase in y_0 will reduce the payoff of "immediate exercise" and hence will push the critical retirement wealth upward. Indeed, Eq. (3.14) demonstrates a linear relation between $\overline{W}$ and y_0. This homogeneity of degree one shows that one can express

the target wealth at retirement in terms of multiples of current income, and suggests the normalization $y_0 = 1$, which we adopt in all quantitative exercises.

Furthermore, Eq. (3.15) allows us to perform comparative statics with respect to an agent's disutility of labor. Assume that γ^* is kept fixed and is larger than one for simplicity.[14] Assume now that we increase $\bar{l}$ so that $K^{1/\gamma}$ decreases (by Eq. (3.13) and the fact that $\bar{l} > 1$). The value of immediate exercise in Eq. (3.15) will then increase (since $\gamma > 1$) and the agent will decide to enter retirement sooner. This is intuitive: For agents who work longer hours for the same pay, the relative increase in leisure upon retirement $(\bar{l})$ is larger and hence retirement is more attractive, all else equal. Similar comparative statics follow for variations in the relative importance of leisure in the utility function (α). Another standard intuition from option pricing is that increases in the volatility of the underlying state variable (the Sharpe ratio κ in our case) will lead to postponement of exercise (retirement in our case).

Interestingly, there is a direct link between the change in consumption upon retirement and the critical level of wealth, $\overline{W}$. Note that by combining (3.13) and (3.14), we obtain the following relation:

$$\overline{W} = \frac{(\xi_2 - 1)\frac{c_{\tau+}}{c_{\tau-}}}{\left(1 + \xi_2 \frac{\gamma}{1-\gamma}\right)\left(\frac{c_{\tau+}}{c_{\tau-}} - 1\right)} \frac{y_0}{r}. \tag{3.16}$$

Thus, assuming $\gamma > 1$, lower threshold levels of the critical wealth will be associated with larger decreases in consumption. In the empirical literature this correlation between low levels of wealth at retirement and large decreases in consumption is seen as evidence that workers do not save enough for retirement. Our rational framework suggests the alternative explanation that this correlation is simply the result of preference heterogeneity: Agents who value leisure a lot will be willing to absorb larger decreases in their consumption upon retirement (since leisure and consumption enter nonseparably in the utility function) and will have lower levels of retirement-triggering wealth. In option jargon, the payoff of immediate exercise will be too large, as will the opportunity cost of waiting. Hence, low levels of wealth upon entering retirement and large decreases in consumption are merely two manifestations of the same economic force, namely, a stronger preference for leisure.

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[14]Similar arguments can be given when $\gamma^* < 1$.

3.2.2 Optimal consumption

We concentrate on a consumer with wealth lower than $\overline{W}$, that is, a consumer who has an incentive to continue working. The following proposition characterizes the optimal consumption behavior of the consumer.

Proposition 29 *Assume that $W_t < \overline{W}$, so that the agent has not retired yet. Let c_t be the optimal consumption process, and let c_t^B denote optimal consumption in the benchmark model, in which the consumer has no option of retirement. Then:*

i) Consumption prior to retirement is lower compared to the benchmark case: $c_t < c_t^B$.

ii) The marginal propensity to consume out of wealth, $\frac{\partial c_t}{\partial W_t}$, is a declining function of W_t. By contrast the marginal propensity to consume out of wealth is constant and equal to θ in the benchmark case.

The intuition for the first assertion is straightforward: The desire to attain retirement incentivizes the agent to save and accumulate assets compared to the benchmark case. This explains part i) of the proposition.

Fig. 1 illustrates Part ii) of the proposition. In the standard Merton (1971) framework (with or without an income stream), the marginal propensity to consume out of wealth is fixed at θ. However, in the present model the marginal propensity to consume approaches θ asymptotically as wealth goes to the lowest allowable level, namely, $-\frac{y_0}{r}$. It declines between $-\frac{y_0}{r}$ and $\overline{W}$ and then jumps to θ when wealth exceeds the retirement threshold, $\overline{W}$, so that the agent enters retirement.

The best way to understand why the marginal propensity to consume is not constant, but rather declining, is to consider the following thought experiment. Suppose that we decrease the wealth of an agent by an amount x prior to retirement, due, e.g., to an unexpected stock market crash. In our framework this will have two effects. First, it will reduce the agent's total resources and hence will lead to a consumption cutback, as in the standard Merton framework. Second, it will distantiate the agent from the threshold level of wealth that is required to attain retirement. As a result, the agent can now expect to remain in the workforce longer and thus she will have the opportunity to partially recoup the loss of x units of wealth by the net present
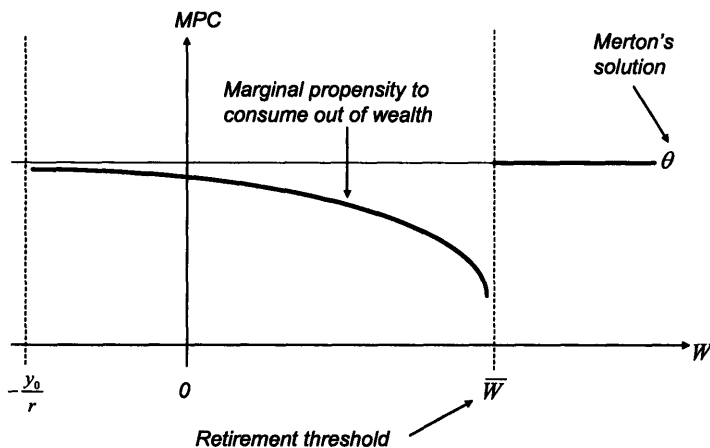
Figure 3-1:

value of the additional income. In short, part of the wealth shock is absorbed by postponing retirement and thus the effect on consumption is moderate.

Naturally, one would expect this effect to be strongest when the distance to the retirement threshold is small (i.e., for wealth levels close to $\overline{W}$). By contrast, when the option of early retirement is completely "out of the money" (for instance, when wealth is close to $-\frac{y_0}{r}$), then marginal changes in wealth will have almost no impact on the net present value of future income and thus the marginal propensity to consume will asymptote to θ as $W_t \rightarrow -\frac{y_0}{r}$ in Fig. 1.

Of course, if $W_t > \overline{W}$, the agent enters retirement and the usual affine relation between consumption and wealth prevails, as is common in Merton-type setups. The marginal propensity to consume is constant at θ since all of the adjustment to wealth shocks goes through consumption.

It is important to note that the key to these results is not the presence of labor supply flexibility per se, but the irreversibility of the retirement decision along with the indivisibility of labor supply. To substantiate this claim, assume that the agent never retires and that her leisure choice is determined optimally on a continuum at each point in time, so that $l_t + h_t = \bar{l}$, where h_t are the hours devoted to work, and the instantaneous income is wh_t, with w defined as in Section 3.1.3.[15] The solution for optimal consumption that one obtains in such a framework

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[15]See BMS for a proof. Liu and Neis (2002) impose the constraint $h_t \geq 0$ and obtain different results. It is interesting to note that in the framework of Liu and Neiss (2002), an individual starts losing labor supply flexibility as she approaches the constraint $h_t = 0$. Hence, she effectively becomes more risk averse. In our

with perfect labor supply flexibility is

$$c_t = \overline{C_1}\left(W_t + \frac{y_0}{r}\overline{C_2}\right)$$

for two appropriate constants $\overline{C_1}$ and $\overline{C_2}$. Notice the simple affine relation between wealth and consumption. These results show an important direction in which the present model sheds some new insights, beyond existing frameworks, into the relations among retirement, consumption, and portfolio choice. In particular, under endogenous retirement, wealth has a dual role. First, as in all consumption and portfolio problems, it controls the amount of resources that are available for future consumption. Second, it controls the distance to the threshold at which retirement is optimal. It is this second channel that is behind the behavior of the marginal propensity to consume that we analyze above.[16]

3.2.3 Optimal portfolio

The following proposition gives an expression for the holdings of stock.

Proposition 30 *Prior to retirement, the holdings of stock are given by*

$$\pi_t = \frac{\kappa}{\sigma}\frac{1}{\gamma}\left(W_t + \frac{y_0}{r}\right) + \left(\frac{J_W(W_t)}{J_W(\overline{W})}\right)^{\xi_2-1}\frac{\kappa}{\sigma}\frac{y_0}{r}\xi_2\left((\xi_2-1)+\frac{1}{\gamma}\right)\left[\frac{\gamma}{1-\gamma}\frac{(\xi_2-1)}{\left(1+\xi_2\frac{\gamma}{1-\gamma}\right)}-1\right].$$

(3.17)

The second term in (3.17) is always

i) positive, and

ii) increasing in W_t.

---

framework this is true only post-retirement. Pre-retirement, the individual exposes herself to more risk because this is the only way in which she can accelerate retirement. This shows that taking indivisibility and irreversibility into account, the properties of the solution are fundamentally different.

[16]The concavity of the consumption function is also a common result in models that combine non-spanned income and/or borrowing constraints of the form $W_t \geq 0$ (e.g., Carroll and Kimball, 1996). A quite important difference between these models and the one we consider here is that in the present model, the effects of concavity are most noticeable for high levels of wealth and not for low wealth levels. In our model the MPC asymptotes to θ as $W_t \to -\frac{y_0}{r}$, and declines from there to the point where $W_t = \overline{W}$. It then jumps back up to θ, reflecting the loss of the real option associated with remaining in the workforce. By contrast, in models such as Koo (1998) or Duffie, Fleming, Soner and Zariphopoulou (1997) the MPC is above θ for low levels of wealth and asymptotes to θ as $W_t \to \infty$. We discuss this issue further in Section 3.5, where we introduce borrowing constraints.
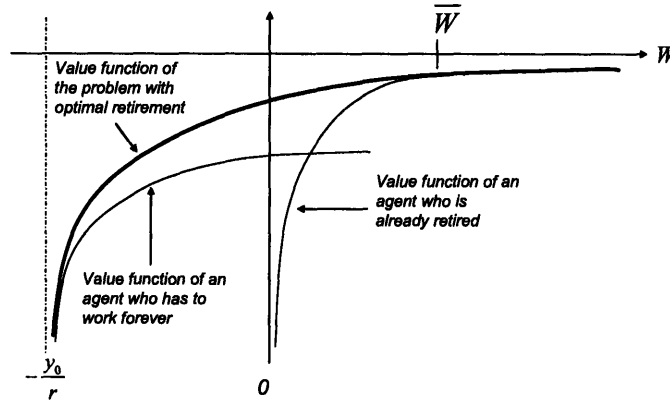
Figure 3-2:

Post-retirement, the optimal holdings of stock are given by the familiar Merton formula:

$$\pi_t = \frac{\kappa}{\sigma}\frac{1}{\gamma}W_t.$$

In simple terms, assertion i) in Proposition 30 implies that the possibility of retirement raises an agent's appetite to take risk (compared to an infinitely lived Merton investor without the option to retire). The intuition is straightforward: First, the ability to adjust the duration of work effectively hedges the agent against stock market variations. Second, the possibility to attain the extra utility associated with more leisure raises the agent's willingness to accept more risk.

Fig. 2 provides a graphical illustration of the first assertion, sketching the following three value functions: a) the value function of a Merton problem wherein the agent has to work forever, b) the value function of an agent who is already retired, and c) the value function of the problem involving optimal retirement choice. As we can see, the third value function looks like an "envelope" of the other two functions which are "more" concave. This implies that relative risk aversion will be lower and hence holdings of stock will be higher in the presence of optimal retirement choice. The value function of the problem involving optimal choice of retirement asymptotes to the first value function as $W_t \to -\frac{y_0}{r}$ (i.e., the option of early retirement becomes completely worthless) and it coincides with the second value function when $W_t \geq \overline{W}$ (i.e., when the agent enters retirement).

Assertion ii) in Proposition 30 is driven by a separate intuition. To see why it holds, it is easiest to think of a fictitious asset, namely, a barrier option that could be used to finance retirement. This option pays off when the agent enters retirement. Its payoff is given by

$$\overline{W} - W^{NR},$$

where $\overline{W}$ is the target wealth given by Eq. (3.14) and W^{NR} is given by the wealth of an agent without the option to retire, keeping the marginal value of a dollar (J_W) the same across the two agents. It can be shown that $\overline{W} - W^{NR}$ is always positive and can be expressed as

$$
\begin{aligned}
\overline{W} - W^{NR} &= E \int_\tau^\infty \frac{H_s}{H_\tau} \left(c_s^R - c_s^{NR} + y_0 \right) ds = \\
&= E \int_\tau^\infty \frac{H_s}{H_\tau} \left(c_s^R - c_s^{NR} \right) ds + \frac{y_0}{r},
\end{aligned}
$$

where c_s^R is the consumption of a retiree and c_s^{NR} is the consumption of a worker. In other words, this expression is equal to the difference in the NPV of the consumption streams of a retired versus a nonretired person plus the net present value of forgone income. Hence, $\overline{W} - W^{NR}$ is the *extra* wealth that is needed to finance retirement.

The second term in (3.17) is just the replicating portfolio of such an option. As for most barrier options, the replicating portfolio becomes largest when the option gets closer to becoming exercised, that is, as $W_t \to \overline{W}$; this is why assertion ii) holds.

It is interesting to relate the above results to BMS. To do so, we start by normalizing the nominal stock holdings by W_t, so as to obtain nominal holdings of stock as a function of financial wealth. This gives the "portfolio" fraction of stocks $\phi_t = \frac{\pi_t}{W_t}$, or using (3.17),

$$
\begin{aligned}
\phi_t &= \frac{\kappa}{\sigma} \frac{1}{\gamma} \left(1 + \frac{y_0}{W_t} \frac{1}{r} \right) + \\
&\quad + \left(\frac{J_W(W_t)}{J_W(\overline{W})} \right)^{\xi_2 - 1} \frac{\kappa}{\sigma} \frac{y_0}{W_t} \frac{1}{r} \xi_2 \left[\left((\xi_2 - 1) + \frac{1}{\gamma} \right) \right] \left[\frac{\gamma}{1 - \gamma} \frac{(\xi_2 - 1)}{\left(1 + \xi_2 \frac{\gamma}{1 - \gamma} \right)} - 1 \right].
\end{aligned}
$$

It is interesting to note the dependence of these terms on W_t. By fixing $\frac{y_0}{r}$ and increasing W_t, one can observe that the first term actually decreases. This is the standard BMS effect: According to BMS, the allocation to stocks depends on the relative ratio of financial wealth

to human capital. If an individual is "endowed" with a lot of human capital compared to her financial wealth, it is as if she is endowed with a bond (since labor income is not risky). Hence, she will invest heavily in stocks in order to make sure that a constant fraction of her total resources (financial wealth + human capital) is invested in risky assets. This key intuition of BMS explains why the first term declines as W_t increases.

However, in the presence of a retirement option this conclusion is not necessarily true, due to the second term. To see why, compute ϕ_W and evaluate it around $\overline{W}$ to obtain, after some simplifications,

$$\phi_W(\overline{W}) = -\frac{1}{\overline{W}}\left(\phi(\overline{W}) - \frac{\kappa}{\sigma}\frac{1}{\gamma}\right) + \frac{1}{\overline{W}}\frac{1}{\phi(\overline{W})}\left(\frac{\kappa}{\sigma}\right)^2 \frac{\left(K^{\frac{1}{\gamma}}\theta - 1\right)}{K^{\frac{1}{\gamma}}\theta}\left((\xi_2 - 1) + \frac{1}{\gamma}\right)\frac{\xi_2}{1 - \gamma}.$$

The first term is clearly negative and captures the increase in the denominator of $\phi = \frac{\pi}{W}$. The second term is positive and potentially larger than the first term, depending on parameters, and captures the increase in the likelihood that the option of retirement will be exercised. Hence, for values of W_t close to $\overline{W}$, it is possible that $\phi_W > 0$. One can easily construct numerical examples whereby this is indeed the case.

It is noteworthy that this result is driven by the option elements introduced by the irreversibility of the retirement decision and not by labor supply flexibility per se. Indeed, one can show (using the methods in BMS) that allowing an agent to choose labor and leisure freely on a continuum would result in

$$\pi_t = \frac{1}{\gamma^*}\frac{\kappa}{\sigma}\left(W_t + \frac{y_0}{r}\frac{\bar{l}}{(\bar{l} - l_1)}\right).$$

This implies that ϕ would have to be decreasing in W_t, despite the presence of labor supply flexibility. The reason for these differences is that in BMS, the amount allocated to stocks as a fraction of total resources (financial wealth + human capital) is a constant. In our framework this fraction depends on wealth. Wealth controls both the resources available for future consumption and the likelihood of "exercising" the real option of retirement.

In summary, not only does the possibility of early retirement increase the incentive to save more, it also increases the incentive of the agent to invest in the stock market because this is the most effective way to attain this goal. Furthermore, this incentive is strengthened as an

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individual's wealth approaches the target wealth level that triggers retirement.

3.3 Retirement before a deadline

None of the claims made so far rely on restricting the time of retirement to lie in a particular interval. The exposition above is facilitated by the infinite horizon setup, which allows for explicit solutions to the associated optimal stopping problem. However, the trade-off is that in the infinite horizon case, there is no notion of aging, since time plays no explicit role in the solution. Moreover, the "natural" theoretical benchmark for the model in the previous section is one without retirement at all. In this section we are able to extend all the insights of the previous section by comparing the early retirement model to a benchmark model with mandatory retirement at time T, which is more natural.

Formally, the only modification that we introduce in this section compared to Section 3.1 is that Eq. (3.5) becomes

$$\max_{c_t, W_\tau, \tau} E\left[\int_t^{\tau \wedge T} e^{-\beta(s-t)} U(l_1, c_s) ds + e^{-\beta(\tau \wedge T - t)} U_2(W_{\tau \wedge T})\right], \tag{3.18}$$

where T is the retirement deadline and $\tau \wedge T$ is shorthand notation for $\min\{\tau, T\}$.

The Appendix presents the solution to the above problem in Theorem 35. As might be expected, one needs to use some approximate method to obtain analytical solutions, because now the optimal stopping problem is on a finite horizon.[17] The extended appendix discusses the nature of the approximation and examines its performance against consistent numerical methods to solve the problem.[18] One can easily verify that the formulas for optimal consumption, portfolio, etc. are identical to the respective formulas of Theorem 31 (the sole exception being that the constants are modified by terms that depend on $T - t$). As a result, all of the analysis in Section 3.2 carries through to this section. This is particularly true for the dependence of consumption, portfolio, etc. on wealth. Here we focus only on the implications of the model

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[17]An important remark on terminology: The term "finite horizon" refers to the fact that the optimal stopping region becomes a function of the deadline to mandatory retirement. The individual continues to be infinitely lived.

[18]The basic idea behind the approximation is to reduce the problem to a standard optimal stopping problem and use the same approximation technique as in Barone-Adesi and Whaley (1987). The most important advantage of this approximation is that it leads to very tractable solutions for all quantities involved.

for portfolio choice as a function of age. The results for consumption are similar.

By Theorem 35 in the Appendix, the optimal holdings of stock as a fraction of financial wealth are given as

$$\phi_t = \frac{\kappa}{\sigma}\frac{1}{\gamma}\left(1+\frac{y_0}{W_t}\frac{1-e^{-r(T-t)}}{r}\right) + \frac{\kappa}{\sigma}\frac{y_0}{W_t}\frac{\left(1-e^{-r(T-t)}\right)}{r}\left(\frac{J_W\left(W_t,T-t\right)}{J_W\left(\overline{W}_{T-t},T-t\right)}\right)^{\xi_{2(T-t)}-1}$$

$$\times\xi_{2(T-t)}\left(\frac{1}{\gamma}+(\xi_{2(T-t)}-1)\right)\left[\frac{\gamma}{1-\gamma}\frac{\left(\xi_{2(T-t)}-1\right)}{\left(1+\xi_{2(T-t)}\frac{\gamma}{1-\gamma}\right)}-1\right], \tag{3.19}$$

where $\overline{W}_{T-t}$ is the critical threshold that leads to retirement when an agent has $T-t$ years to mandatory retirement and the constants $\xi_{2(T-t)}$ are given in the Appendix. As in Section 3.2.3, the first term is the standard BMS term for an investor with $T-t$ years to mandatory retirement. Whether financial wealth increases or the time to mandatory retirement decreases, the first term becomes smaller, which is the standard BMS intuition. Hence, the first term decreases over time (in expectation) because $T-t$ falls, while W_t increases over time (in expectation). The second term captures the replicating portfolio of the early retirement option and is strictly positive. Its relevance is larger a) the closer the option is to being exercised (in/out of the money) and b) the more time is left until its expiration $(T-t)$. Accordingly, the importance of the second term in (3.19) should be expected to decrease when $T-t$ is small. However, it should be expected to increase when W_t increases.

This now opens up the possibility of rich interactions between "pure" horizon effects (variations in $T-t$, keeping W_t constant) and "wealth" effects. As an agent ages but is not yet retired, the "pure" horizon effects will tend to decrease the allocation to stocks. However, in expectation wealth increases as well and thus the option to retire early becomes more and more relevant, counteracting the first effect.

We quantitatively illustrate the interplay of these effects in the next section.

3.4 Quantitative implications

To quantitatively assess the magnitude of the effects described in Section 3.3 we proceed as follows. First, we fix the values of the variables related to the investment opportunity set to $r = 0.03, \mu = 0.1$, and $\sigma = 0.2$. For β we choose 0.07 in order to account for both discounting and a constant probability of death. For γ we consider a range of values (typically $\gamma = 2, 3, 4$). This leaves one more parameter to be determined, namely K. The parameter K controls the shift in the marginal utility of consumption upon entering retirement. It is a well-documented empirical fact that consumption drops considerably upon entering retirement. As such, the most natural way to determine the value of K is to match the agent's declining consumption upon entering retirement. Aguiar and Hurst (2004) report expenditure drops of 17%, whereas Banks, Blundell and Tanner (1998) report changes in log consumption expenditures of almost 0.3 in the five years prior to retirement and thereafter. Since these decreases mainly pertain to food expenditures, which are likely to be inelastic, we also calibrate the model to somewhat larger decreases in consumption.[19]

In light of (3.13), we have

$$\frac{c_{\tau+}}{c_{\tau-}} = K^{1/\gamma}\theta \rightarrow K = \left(\frac{c_{\tau+}}{c_{\tau-}}\right)^\gamma \theta^{-\gamma},$$

where $c_{\tau-}$ is the consumption immediately prior to retirement and $c_{\tau+}$ is the consumption immediately thereafter. By substituting a post- /pre- retirement ratio of $\frac{c_{\tau+}}{c_{\tau-}} = \{0.5, 0.6, 0.7\}$ in the above formula we can determine the respective values of K that will ensure that the retirement decrease is equal to $\{0.5, 0.6, 0.7\}$, respectively. We fix the mandatory retirement age to be $T = 65$ throughout and normalize y_0 to be one. The abbreviation "Ret" indicates the solution implied by a model with optimal early retirement (up to time T) and "BMS" denotes the solution of a model with mandatory retirement at time T, with no option to retire earlier or later.

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[19] Admittedly, not all of these effects are purely due to nonseparability between leisure and consumption. Home production is undoubtedly a key determinant behind these decreases. It is important to note, however, that our model is not incompatible with such an explanation. As long as a) the agent can leverage consumption utility with her increased leisure, and b) time spent on home production is not as painful as work, the present model can be seen as a good reduced-form approximation to a more complicated model that would model home production explicitly.
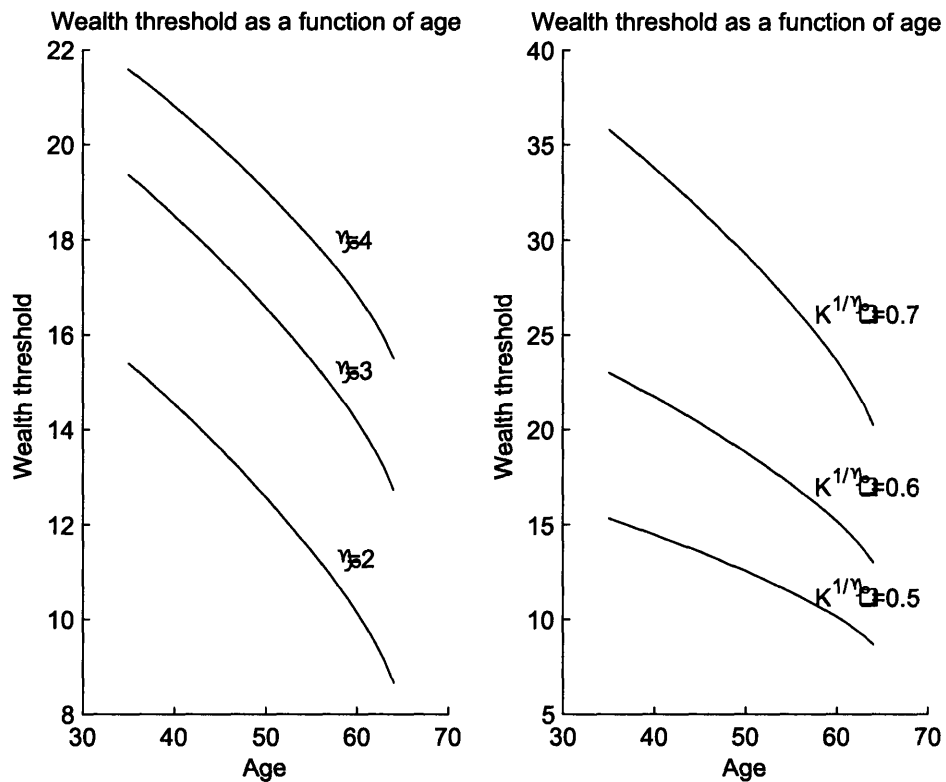
Figure 3-3:

Fig. 3 plots the target wealth that is implied by the model, i.e., the level of wealth required to enter retirement. This figure demonstrates two patterns. First, threshold wealth declines as an agent nears mandatory retirement. This is intuitive, because the option to work is more valuable the longer its "maturity": As a (working) agent ages, the incentive to keep the option "alive" is reduced and hence the wealth threshold declines. Second, the critical wealth implied by this model varies with the assumptions made about risk aversion, and the disutility of work as implied by a lower $K^{1/\gamma}\theta$. Risk aversion tends to shift the threshold upwards, whereas lower levels of $K^{1/\gamma}\theta$ (implying more disutility of labor) bring the threshold down. These are intuitive predictions. An agent who is risk averse wants to avoid the risk of losing the option to work, whereas an agent who cares a lot about leisure will want to enter retirement earlier.

Fig. 4 addresses the importance of the real option to retire for portfolio choice. The figure plots the second term in Eq. (3.19) as a fraction of total stockholdings, π_t. In other words, it
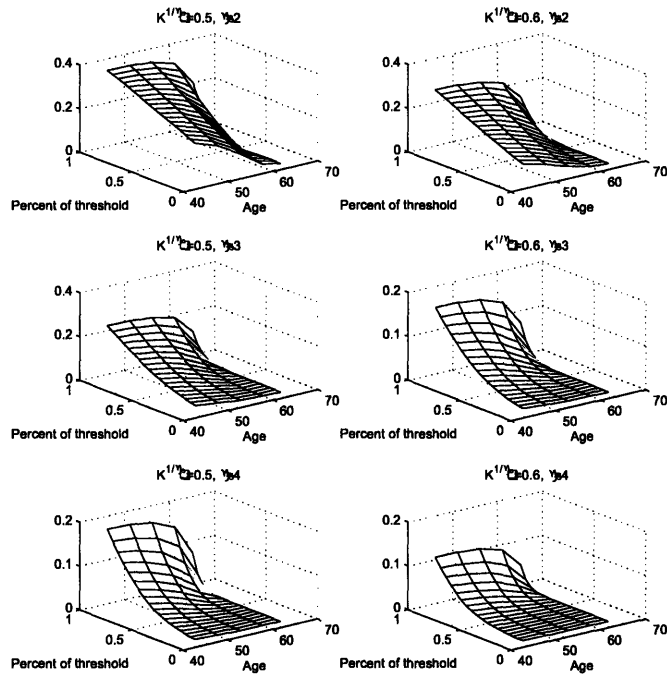
Figure 3-4:

plots the relative importance of stockholdings due to the real option component as a percentage of total stockholdings. This percentage is plotted as a function of two variables, age and wealth. Age varies between 45 and 64 and wealth varies between zero and x, where x corresponds to the level of wealth that would make an agent retire (voluntarily) at 64. We normalize wealth levels by x so that the (normalized) wealth levels vary between zero and one. We then plot a panel of figures for different levels of γ and $K^{1/\gamma}\theta$. Fig. 4 demonstrates the joint presence of "time to maturity" and "moneyness" effects in the real option to retire. Keeping wealth fixed and varying the time to maturity (i.e., increasing age), the relative importance of the real option to retire declines. Similarly, increasing wealth makes the real option component more relevant, because the real option is more in the money. It is interesting to note that the real option component is large, realizing values as large as 40% for some parameter combinations.

In Fig. 5 we consider the implications of the model for portfolio choice as a function of

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age. We fix a path of returns that correspond to the realized returns on the CRSP value-weighted index between 1989 and 1999. We then plot the portfolio holdings (defined as total stockholdings normalized by financial wealth) over time for an individual whose wealth in 1989 was just enough to allow her to retire in the end of 1999 at the age of 58. This is achieved as follows. Assume that in 1989 the investor is 48 years old and has wealth W_0. We treat W_0 as the unknown variable that we need to solve for. For any given W_0, and using both the optimal consumption and portfolio policies and the path of the realized returns between 1989 and 1999, we can determine how much wealth the investor has in 1999, when she is 58 years old. In order to ensure that she retires at that point we know that her wealth must be $\overline{W}$. Hence, we choose W_0 so as to make sure that ten years thereafter (given the optimal policies and the realized path of returns) the wealth has grown to exactly $\overline{W}$. We repeat the same exercise assuming various combinations of $K^{1/\gamma}\theta$ and γ. In order to be able to compare the results, we also plot the portfolio that would be implied if the individual had no option of retiring early and we label this later case as "BMS." Fig. 5 shows that the portfolio of the agent is initially declining and then flat or even increasing over time after 1995. This is in contrast to what would be predicted by ignoring the option to retire early (the "BMS" case). This fundamentally different behavior of the agent's portfolio over time is due to the extraordinary returns during the latter half of the 1990s, which makes wealth grow faster and hence the real option to retire very important towards the end of the sample. By contrast, if one assumed away the possibility of early retirement, the natural conclusion would be that a run-up in prices would change the composition of the agent's total resources (financial wealth + human capital) towards financial wealth. For a constant income stream this would therefore mean a decrease in the portfolio chosen.

Fig. 6 demonstrates the above effect more clearly. In this figure we normalize total stockholdings by total resources (human capital + financial wealth). Consistent with our results above, in the BMS case we get a constant equal to $\frac{\kappa}{\sigma}\frac{1}{\gamma}$. When we allow for an early retirement option, we observe that the fraction of total resources invested in stock exhibits a stark increase towards the latter half of the 1990s, as the option of early retirement becomes more relevant. The increase in this fraction is small for the first half of the 1990s and large for the latter part
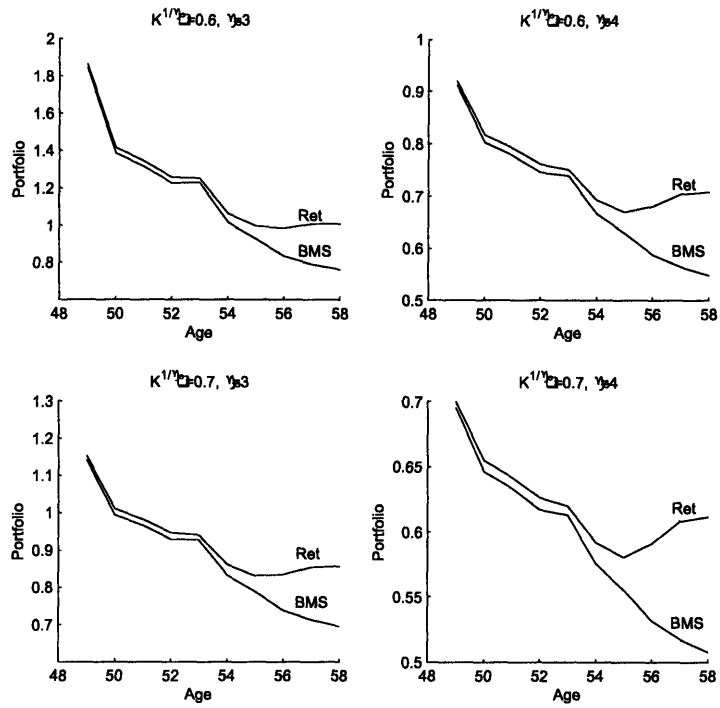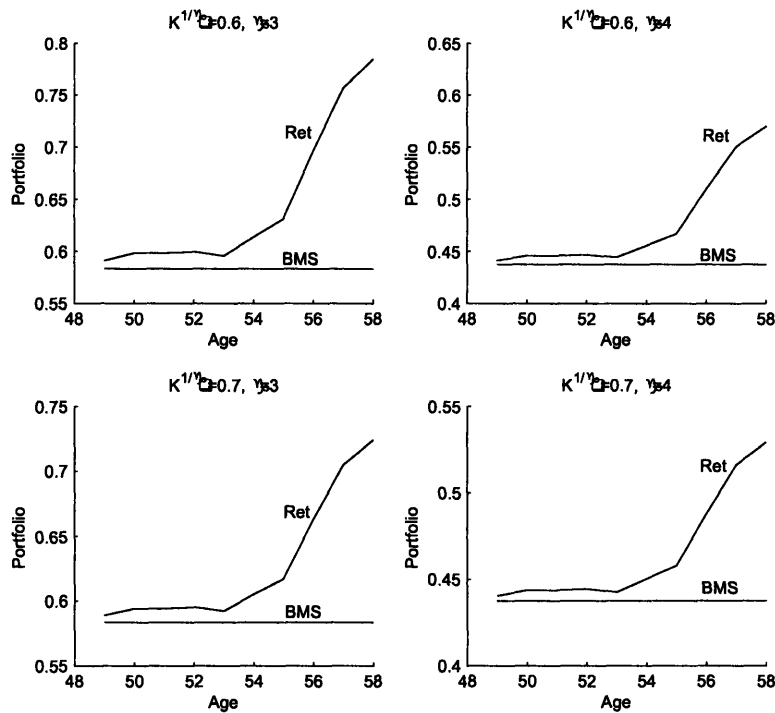
Figure 3-5:

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Figure 3-6:

of the decade.

Fig. 6 is useful in understanding the behavior of the portfolio holdings in Fig. 5. In the first half of the sample the standard BMS intuition applies. The fraction of total resources invested in the stock market is roughly constant even after taking the option of early retirement into account. Hence, by the standard intuition behind the BMS results, the portfolio of the agent (total stockholdings normalized by financial wealth) declines over time. However, in the latter half of the sample, the increase in the real option to retire is strong enough to counteract the decline in the portfolio implied by standard BMS intuitions.

Figs. 7 and 8 repeat the same exercise as in Figs. 5 and 6, only now for an agent who came close to retirement in 1999, that is, we now assume that her wealth in 1999 was slightly less than sufficient for her to actually retire. To achieve this we just assume that in 1989 she started with slightly less initial wealth than necessary to retire by 1999. It is interesting to note what happens after the stock market crash of 2000. Now, the option of early retirement
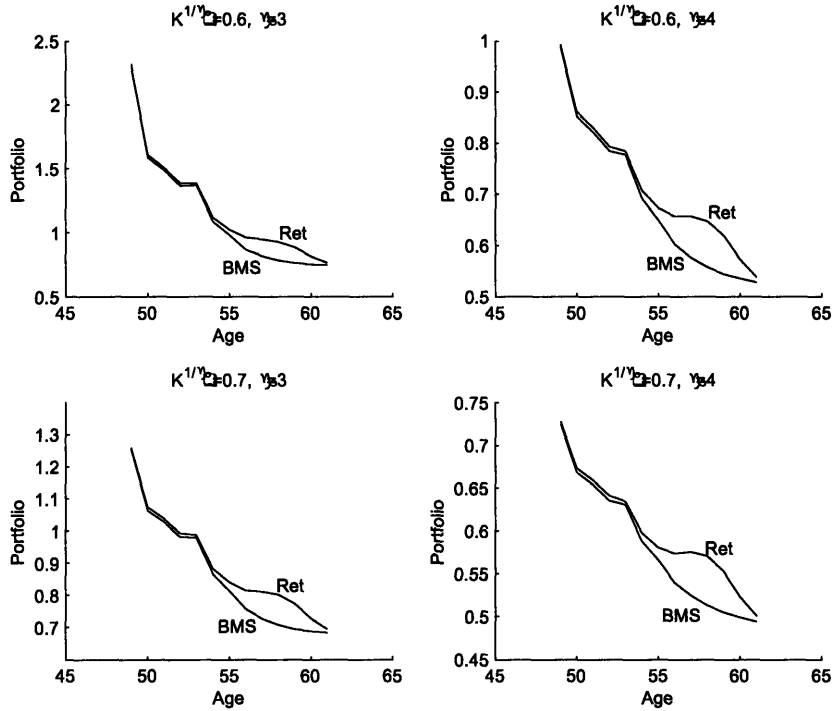
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Figure 3-7:

starts to become irrelevant and the agent's portfolio declines. The effect of a disappearing
option magnifies the decrease in the portfolio. By contrast, in the BMS case the abrupt
decrease in the stock market (and hence wealth) would be counterbalanced by a change in the
composition between financial wealth and human capital towards human capital. This effect
tends to somehow counteract the effects of aging and produces a much more moderate
decrease in portfolio holdings.

These figures are meant to demonstrate the fundamentally different economic implications
that can result once one takes into account the real option to retire. As such they should be
seen as merely an illustrative application. Note, however, that a stronger result can be shown
in the context of this exercise. For wealth levels close to the retirement threshold and for our
preferred base scenario of $K^{1/\gamma}\theta = 0.7$ and $\gamma = 4$, the portfolio would increase with age in
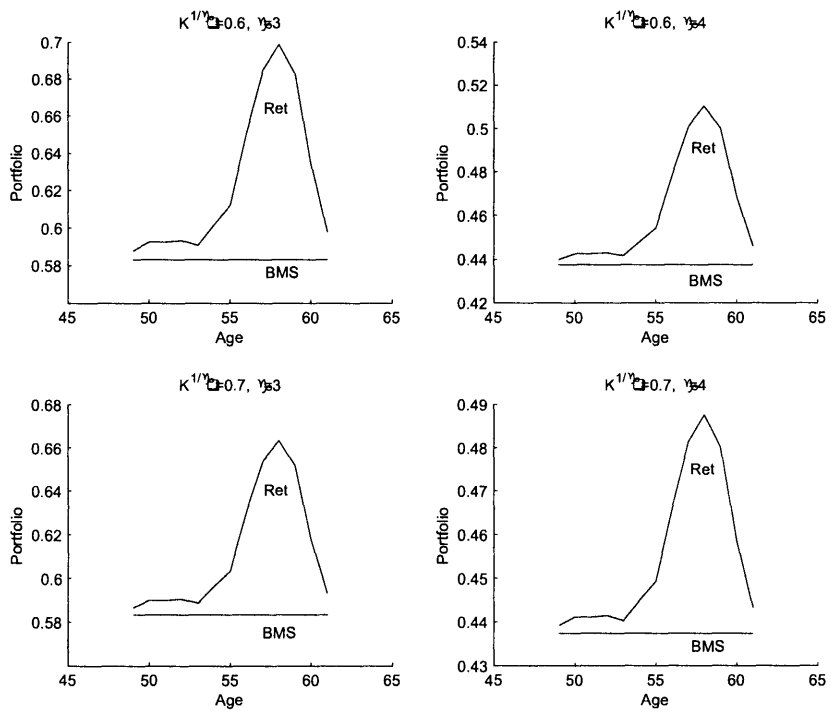expectation as the agent approaches early retirement. Fig. 9 illustrates this effect. The only

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Figure 3-8:

difference from Fig. 5 is that in Fig. 9 we perform the counterfactual exercise of assuming that the stock market moved along an "expected path" between 1989 and 1999. To do so, we assume that the increments of the Brownian motion driving returns are zero, so that the expected returns and the realized returns in the stock market coincide. As can be seen, the qualitative features of Fig. 5 are preserved. For the base scenario $K^{1/\gamma}\theta = 0.7$ and $\gamma = 4$, the fraction invested in stocks is initially decreasing with respect to age, then flat and even slightly increasing (between ages 57 and 58) along the expected path. This increase of the portfolio with age (in expectation) would be impossible in the absence of an early retirement option. Interestingly, the decline in portfolios between ages 48 and 57 is much smaller than what a BMS model would imply. The increase in the importance of the option of early retirement counteracts the pure horizon effect, so that the allocation to stock is almost constant for agents between ages 48 and 57. This may help explain the relatively constant allocations to stock that Ameriks and Zeldes (2001) document empirically.

The present paper is theoretical in nature, and we don't claim to have modeled even a small fraction of all the issues that influence real life retirement, consumption, and portfolio decisions (e.g., shorting and leverage constraints, transaction costs, undiversifiable income and health shocks, etc.). However, note that the model does produce "sensible" portfolios (for the combination $\gamma = 4$ and $K^{1/\gamma}\theta = 0.7$) as well as variations in portfolio shares between 1995 and 2003. In the bottom right plots of Fig. 7, for instance, the portfolio of the agent grows from 0.58 to 0.62 between 1995 and 1999 and then declines to roughly 0.5 by the beginning of 2003. In comparison, the Employee Benefits Research Institute (EBRI)[20] reports that the average equity share in a sample of 401(k)s grew steadily from 0.46 to 0.53 between 1995 and 1999 only to fall to 0.4 by the beginning of 2003. The reason the model performs well is that we are considering an agent close to retirement, that is, at a time when the remaining NPV of her income is not a large component of her total wealth in the first place. For young agents the model has similar problems matching the data as BMS, which is to be expected.

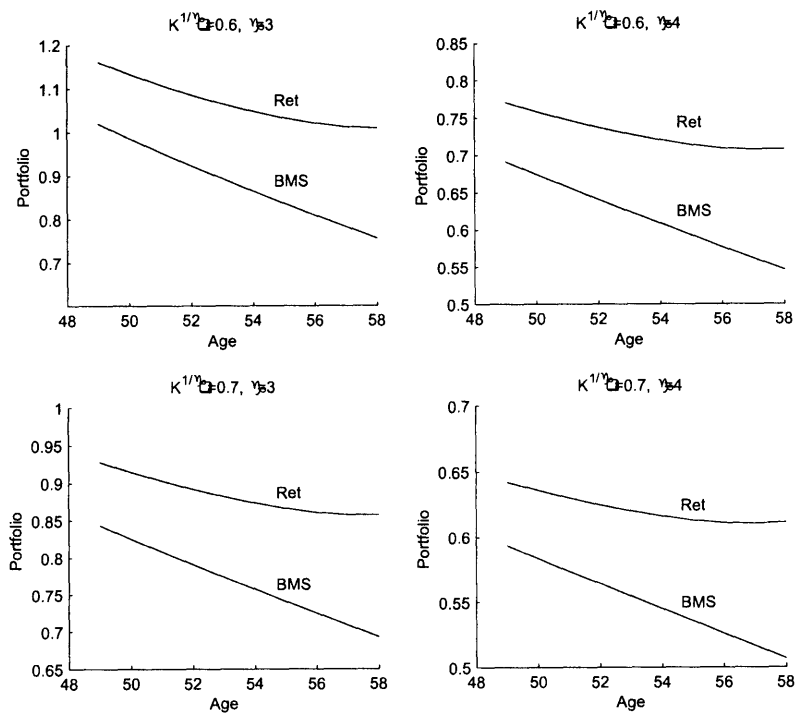[20] Employee Benefits Research Institute, Issue Brief 272 (Aug 2004), especially Figure 2.

Figure 3-9:

3.5 Borrowing constraint

Thus far we assume that the agent is able to borrow against the value of her future labor income. In this section we impose the additional restriction that it is impossible for the agent to borrow against the value of future income. Formally, we add the requirement that $W_t \geq 0$, for all $t > 0$. To preserve tractability, we assume in this section that the agent is able to go into retirement at any time that she chooses without a deadline. This makes the problem stationary and as a result the optimal consumption and portfolio policies will be given by functions of W_t alone.

The borrowing constraint is never binding post-retirement because the agent receives no income and has constant relative risk aversion. This implies that once the agent is retired, her consumption, her portfolio, and her value function are the same with or without borrowing constraints. In particular, if she enters retirement at time τ with wealth W_τ, her expected utility is still $U_2(W_\tau)$.

The problem the agent now faces is

$$\max_{c_t, W_\tau, \tau} E\left[\int_0^\tau e^{-\beta t} U(l_1, c_t) dt + e^{-\beta \tau} U_2(W_\tau)\right] \tag{3.20}$$

subject to the borrowing constraint

$$W_t \geq 0, \forall t \geq 0, \tag{3.21}$$

and the budget constraint

$$dW_t = \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - (c_t - y_0 1\{t < \tau\}) dt. \tag{3.22}$$

We present the solution in Theorem 36 in the Appendix. We devote the remainder of this section to a comparison of results we obtain in Section 3.2 with the resulting optimal policies we obtain in Theorem 36.

The Appendix gives a simple proof as to why wealth at retirement is smaller with borrowing constraints than without (even though quantitatively the effect is negligible). In terms of optimal stockholdings, the presence of borrowing constraints moderates holdings of stock, and
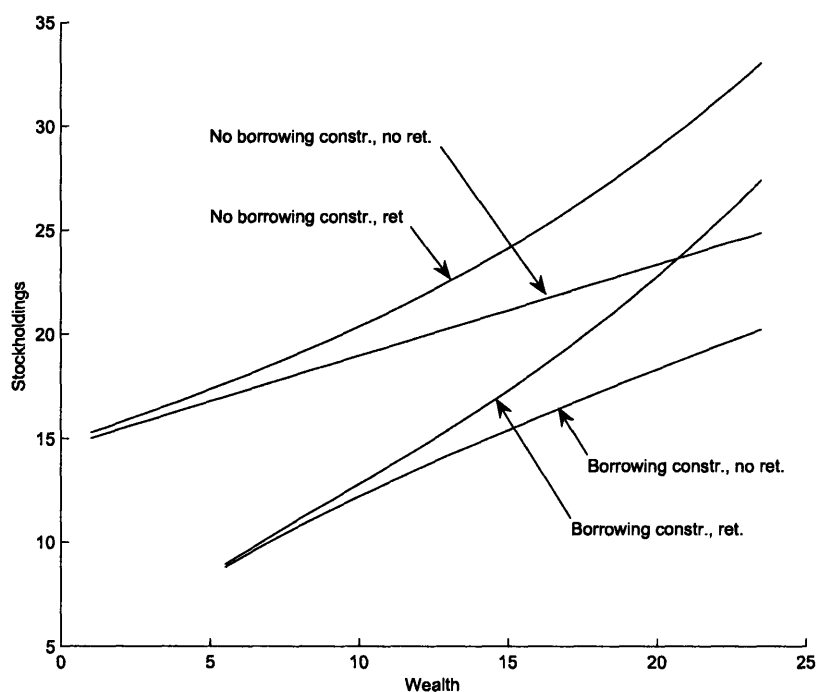
Figure 3-10:

decreasingly so as the wealth of the agent increases.

Fig. 10 compares optimal portfolios for four cases formed by those with and without the early retirement option, and those with and without the imposition of borrowing constraints. For the cases in which we allow retirement, we take wealth levels close to retirement but lower than the threshold that would imply retirement. The figure demonstrates that for levels of wealth close to retirement there are only (minor) quantitative differences between agents with borrowing constrains and agents without. The qualitative properties are the same. Holdings of stock increase with wealth (more than linearly). One can observe that the optimal stockholdings in the presence of early retirement are tilted more towards stocks whether we impose borrowing constraints or not. Similarly, the optimal holdings of stock are smaller when one imposes borrowing constraints (whether one allows for a retirement option or not).

We conclude by summarizing the key insights of this section. Borrowing constraints are relevant for levels of wealth close to zero, where optimal retirement is not an issue. Similarly, the

effects of optimal retirement are relevant for high levels of wealth, where borrowing constraints are unlikely to bind in the future. Hence, as long as one examines the effects of the option to retire close to the threshold levels of wealth, borrowing constraints can be safely ignored. However, it is important to note that borrowing constraints can fundamentally affect quantities related to, e.g., the expected time to retirement for a person who starts with wealth close to zero because they will typically imply lower levels of stockholdings and hence a more prolonged time (in expectation) to reach the retirement threshold.

3.6 Conclusion

In this paper we propose a simple partial equilibrium model of consumer behavior that allows for the joint determination of a consumer's optimal consumption, portfolio, and time to retirement. The Appendix provides essentially closed-form solutions for virtually all quantities of interest. The results can be summarized as follows. The ability to time one's retirement introduces an option-type character to the optimal retirement decision. This option is most relevant for individuals with a high likelihood of early retirement, that is, individuals with high wealth levels. This option in turn affects both an agent's incentive to consume out of current wealth and her investment decisions. In general, the possibility of early retirement will lead to portfolios that are more exposed to stock market risk. The marginal propensity to consume out of wealth will be lower as one approaches early retirement, reflecting the increased incentives to reinvest gains in the stock market in order to bring retirement "closer."

The model makes some intuitive predictions. Here we single out some of the predictions that seem to be particularly interesting. First, the model suggests that during stock market booms, there should be an increase in the number of people that opt for retirement as a larger percentage of the population hits the retirement threshold (some evidence for this may be found in Gustman and Steinmeier (2002) and references therein). Second, the models shows that it is possible that portfolios of aging individuals could exhibit increasing holdings of stock over time, even if there isn't variation in the investment opportunity set and the income stream exhibits no correlation with the stock market (or any risk whatsoever). This is interesting in light of the evidence in Ameriks and Zeldes (2001) that portfolios tend to be increasing or

hump-shaped with age for the data sets that they consider. Third, according to the model, there should be a discontinuity in the holdings of stock and in consumption upon entering retirement. Ample empirical evidence shows that indeed, this is the case. (see, e.g., Aguiar and Hurst (2004) and references therein). The discontinuity in stockholdings seems to have been less tested an hypothesis. Fourth, the model predicts that all else equal, switching to a more flexible retirement system that links portfolio choice with retirement timing should lead to increased stock market allocations. This is consistent with the empirical fact that stock market participation increased in the U.S. as 401(k)s were gaining popularity. Fifth, increasing levels of stockholdings during a stock market run-up and liquidations during a stock market fall might not be due to irrational herding; instead, both effects might be due to the behavior of the real option to retire that emerges during the run-up and becomes irrelevant after the fall.

In this paper we try to outline the basic new insights that obtain by the timing of the retirement decision. By no means do we claim that we address all the issues that are likely to be relevant for actual retirement decisions (e.g., health shocks, unspanned income, etc.). Rather, we view the theory developed in this paper as a complement to our understanding of richer, typically numerically solved, models of retirement. Many interesting extensions to this model should be relatively tractable.

A first important extension would be to include features that are realistically present in actual 401(k)-type plans such as tax deferral, employee matching contributions, and tax provisions related to withdrawals. The solutions developed in such a model could be used to determine the optimal saving, retirement, and portfolio decisions of consumers that are contemplating retirement and taking into account tax considerations.

A second extension would be to allow the agent to reenter the workforce (at a lower income rate) once retired. We doubt this would alter the qualitative features of the model, but it is very likely that it would alter the quantitative predictions. It can be reasonably conjectured that the wealth thresholds would be significantly lower in that case, and the portfolios tilted even more towards stocks because of the added flexibility.

A third extension of the model would be to introduce predictability and more elaborate preferences. If one were to introduce predictability, while keeping the market complete (like Wachter, 2002), the methods of this paper can be easily extended. It is also very likely that

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the model would not loose its tractability if one uses Epstein-Zin utilities in conjunction with the methods recently developed by Schroder and Skiadas (1999).

A fourth extension of the model that we are currently pursuing is to study its general equilibrium implications.[21] This is of particular interest as it would enable one to make some predictions about how the properties of returns are likely to change as worldwide retirement systems begin to offer more freedom to agents in making investment and retirement decisions.

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[21]The role of labor supply flexibility in a general equilibrium model with continuous labor-leisure choice is considered in Basak (1999). It is very likely that the results we present in this paper could form the basis for a general equilibrium extension. It is well known in the macroeconomics literature that allowing for indivisible labor is quite important if one is to explain the volatility of employment relative to wages. See, for example, Hansen (1985) and Rogerson (1988).

.1 Appendix

This version of the Appendix contains the statements of the theorems and a sketch of the proof of the main theorem. An extended appendix containing all the proofs is available online.

.1.1 Theorems and proofs for section 3.2

Theorem 31 *To obtain the solution to the problem we describe in Section 3.1, define the constants*

$$\xi_2 = \frac{1 - 2\frac{\beta-r}{\kappa^2} - \sqrt{(1 - 2\frac{\beta-r}{\kappa^2})^2 + 8\frac{\beta}{\kappa^2}}}{2} \tag{23}$$

$$\lambda = \left(\frac{(\xi_2 - 1)\theta}{\left(1 + \xi_2 \frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta - 1\right)} \frac{y_0}{r} \right)^{-\gamma} \tag{24}$$

and

$$C_2 = \frac{\left[\frac{\gamma}{1-\gamma}\frac{(\xi_2-1)}{\left(1+\xi_2\frac{\gamma}{1-\gamma}\right)} - 1\right]\frac{y_0}{r}}{\lambda^{\xi_2-1}},$$

assume that

$$\frac{r}{\theta}\frac{\left(\frac{1-\gamma}{\gamma} + \xi_2\right)}{\xi_2 - 1} < 1,$$

and let λ^* be the (unique) solution of

$$\xi_2 C_2 (\lambda^*)^{\xi_2-1} - \frac{1}{\theta}(\lambda^*)^{-\frac{1}{\gamma}} + \frac{y_0}{r} + W_t = 0. \tag{25}$$

Then

$$C_2 > 0, \xi_2 < 0,$$

and the optimal policy is given as follows:

a) If $W_t < \overline{W} = \frac{(\xi_2-1)K^{\frac{1}{\gamma}}\theta}{\left(1+\xi_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta-1\right)}\frac{y_0}{r}$,

consumption follows the process

$$c_s = \left(\lambda^* e^{\beta(s-t)} \frac{H(s)}{H(t)}\right)^{-\frac{1}{\gamma}} 1\{t \leq s < \tau^*\} \tag{26}$$

$$c_s = \bar{l}^{(1-\alpha)\frac{(1-\gamma^*)}{\gamma}} \left(\lambda^* e^{\beta(s-t)} \frac{H(s)}{H(t)}\right)^{-\frac{1}{\gamma}} 1\{s \geq \tau^*\}, \tag{27}$$

the optimal retirement time is

$$
\begin{aligned}
\tau^* &= \inf\{s : W_s = \overline{W}\} = \\
&= \inf\left\{s : \lambda^* e^{\beta(s-t)} \frac{H(s)}{H(t)} = \underline{\lambda}\right\},
\end{aligned} \tag{28}
$$

and the optimal consumption and stockholdings as a function of W_t are given by

$$c_t = c(W_t) = (\lambda^*(W_t))^{-\frac{1}{\gamma}} \tag{29}$$

$$\pi_t = \pi(W_t) = \frac{\kappa}{\sigma}\left(\xi_2(\xi_2 - 1)C_2\lambda^*(W_t)^{\xi_2-1} + \frac{1}{\gamma}\frac{1}{\theta}\lambda^*(W_t)^{-\frac{1}{\gamma}}\right). \tag{30}$$

b) If $W_t \geq \overline{W} = \frac{(\xi_2-1)K^{\frac{1}{\gamma}\theta}}{\left(1+\xi_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}\theta}-1\right)}\frac{y_0}{r}$, the optimal solution is to enter retirement immediately ($\tau^* = t$) and the optimal consumption /portfolio policy is given as in Karatzas and Shreve (1998), Chapter 3.

The remainder of this section is to provide a sketch for the proof of Theorem 31. Throughout this section we fix $t = 0$ without loss of generality. For a concave, strictly increasing and continuously differentiable function $U : (0, \infty) \to R$, we can define the inverse $I(\cdot)$ of $U'(\cdot)$. Let $\widetilde{U}$ be given by

$$\widetilde{U}(y) = \max_{x>0}[U(x) - xy] = U(I(y)) - yI(y), \quad 0 < y < \infty.$$

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It is easy to verify that $\widetilde{U}(\cdot)$ is strictly decreasing and convex, and satisfies

$$\widetilde{U}'(y) = -I(y), \quad 0 < y < \infty \tag{31}$$

$$U(x) = \min_{y>0} \left[\widetilde{U}(y) + xy \right] = \widetilde{U}(U'(x)) + xU'(x), \quad 0 < x < \infty.$$

To start, we fix a stopping time τ and define

$$V_\tau(W_0) = \max_{c_t, \pi_t} E \left[\int_0^\tau e^{-\beta t} U_1(c_t) dt + e^{-\beta \tau} U_2(W_\tau) \right], \tag{32}$$

where U_1 and U_2 are as defined in Eqs. (3.6) and (3.7). The following result is a generalization of the equivalent result in Karatzas and Wang (2000) to allow for income.

Lemma 32 *Let*

$$\widetilde{J}(\lambda; \tau) = E \left[\int_0^\tau \left[e^{-\beta t} \widetilde{U_1}(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0 \right] dt + e^{-\beta \tau} \widetilde{U_2}(\lambda e^{\beta \tau} H(\tau)) \right].$$

For any τ that is finite almost surely, there exists λ^* such that

$$V_\tau(W_0) = \inf_{\lambda>0} \left[\widetilde{J}(\lambda; \tau) + \lambda W_0 \right] = \widetilde{J}(\lambda^*; \tau) + \lambda^* W_0$$

and the optimal solution to (32) entails

$$W_\tau = I_2(\lambda^* e^{\beta \tau} H_\tau) \tag{33}$$

$$c_t = I_1(\lambda^* e^{\beta t} H(t)) 1\{t < \tau\} \tag{34}$$

with I_1 and I_2 defined similarly to Eq. (31). Moreover, the value function $V(W_0)$ of the problem outlined in Section 2 satisfies

$$V(W_0) = \sup_\tau V_\tau(W_0) = \sup_\tau \inf_{\lambda>0} \left[\widetilde{J}(\lambda; \tau) + \lambda W_0 \right] = \sup_\tau \left[\widetilde{J}(\lambda^*; \tau) + \lambda^* W_0 \right].$$

Karatzas and Wang (2000) show that one can reduce the entire joint portfolio-consumption-stopping problem into a pure optimal stopping problem by examining whether the inequality

$$V(W_0) = \sup_{\tau} \inf_{\lambda>0} \left[\tilde{J}(\lambda; \tau) + \lambda W_0 \right] \leq \inf_{\lambda>0} \sup_{\tau} \left[\tilde{J}(\lambda; \tau) + \lambda W_0 \right] = \inf_{\lambda} \left[\tilde{V}(\lambda) + \lambda W_0 \right] \quad (35)$$

becomes an equality, with $\tilde{V}(\lambda)$ defined as

$$\tilde{V}(\lambda) = \sup_{\tau} \tilde{J}(\lambda; \tau) = \sup_{\tau} E \left[\int_0^{\tau} \left[e^{-\beta t} \widetilde{U_1}(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0 \right] dt + e^{-\beta \tau} \widetilde{U_2}(\lambda e^{\beta \tau} H(\tau)) \right]. \quad (36)$$

The inequality (35) follows from a standard result in convex duality (see, e.g., Rockafellar, 1997). Reversing the order of maximization and minimization in (35) makes the problem significantly more tractable, since $\tilde{V}(\lambda)$ is the value of a standard optimal stopping problem, for which one can apply well known results. In particular, the parametric assumptions that we make in Section 3.1.3 allow us to solve this optimal stopping problem explicitly. This forms a substantial part of the proof and we present it in the extended appendix.

The cost of reversing the order of the minimization and the maximization, however, is that it will only give us an upper bound to the value function. The rest of the proof here is therefore devoted to showing that the inequality in (35) is actually an equality.

Remark 33 *The option pricing interpretation given in Section 3.2.1 is based on a slight rewriting of equation (36). To see this, note that*

$$\begin{aligned}
\tilde{V}(\lambda) &= \sup_{\tau} \tilde{J}(\lambda; \tau) = \sup_{\tau} E \left[\int_0^{\tau} \left[e^{-\beta t} \widetilde{U_1}(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0 \right] dt + e^{-\beta \tau} \widetilde{U_2}(\lambda e^{\beta \tau} H(\tau)) \right] = \\
&= E \left[\int_0^{\infty} \left[e^{-\beta t} \widetilde{U_1}(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0 \right] dt \right] \\
&\quad + \sup_{\tau} E \left[e^{-\beta \tau} \widetilde{U_2}(\lambda e^{\beta \tau} H(\tau)) - \int_{\tau}^{\infty} \left[e^{-\beta t} \widetilde{U_1}(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0 \right] dt \right].
\end{aligned}$$

The extended appendix shows that the third line can be further rewritten as

$$\sup_{\tau} E \left\{ e^{-\beta \tau} Z_{\tau} \left[\frac{1}{\theta} \frac{\gamma}{1-\gamma} \left(K^{1/\gamma} \theta - 1 \right) Z_{\tau}^{-\frac{1}{\gamma}} - \frac{y_0}{r} \right] \right\},$$

which is precisely the option we analyze in Section 3.2.1.

The following result is a straightforward extension of a result in Karatzas and Wang (2000) and is given without proof.

Lemma 34 *Let $\widehat{\tau_\lambda}$ be the optimal stopping rule associated with λ in Eq. (36). Then*

$$\widetilde{V}'(\lambda) = -E\left[\int_0^{\widehat{\tau_\lambda}} H(t)\left(I_1(\lambda e^{\beta t}H(t)) - y_0\right)dt + H(\widehat{\tau_\lambda})I_2(\lambda e^{\beta\widehat{\tau_\lambda}}H_{\widehat{\tau_\lambda}})\right], \quad \lambda \in (0,\infty).$$

The final steps towards proving Theorem 31 use this observation in order to replace the inequality in (35) with an equality sign. In particular, the extended appendix shows how to use Lemma 34 to demonstrate that the policies we propose in the statement of the theorem are feasible and their associated payoff provides an upper bound to the value function. We then conclude that they are the optimal policies.

.1.2 Proofs for Sections 3.2.2 and 3.2.3

See extended Appendix.

.1.3 Theorems and proofs for Section 3.3

The statement of Theorem 35 is almost identical to the statement of Theorem 31 with the main exception that all the constants now depend on $T - t$. In the following statement of the theorem we isolate the results related to the optimal portfolio and leave the precise statement of the entire theorem along with its proof for the extended Appendix.

Theorem 35 *The optimal portfolio is given as*

$$\pi_t = \pi(W_t) = \frac{\kappa}{\sigma}\left(\xi_{2(T-t)}(\xi_{2(T-t)}-1)C_{2(T-t)}\lambda^*(W_t)^{\xi_{2(T-t)}-1} + \frac{1}{\gamma}\frac{1}{\theta_{(T-t)}}\lambda^*(W_t)^{-\frac{1}{\gamma}}\right),$$

where $\lambda^*(W_t)$ is the unique solution of

$$\xi_{2(T-t)}C_{2(T-t)}(\lambda^*)^{\xi_{2(T-t)}-1} - \frac{1}{\theta_{(T-t)}}(\lambda^*)^{-\frac{1}{\gamma}} + \frac{y_0(1 - e^{-r(T-t)})}{r} + W_t = 0 \qquad (37)$$

and $C_{2(T-t)}, \xi_{2(T-t)}, \theta_{(T-t)}$ are constants that depend on $T - t$, which we give explicitly in the extended appendix.

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Since Theorem 35 involves a finite-horizon optimal stopping problem, we use an approximation (along the lines of Barone-Adesi and Whaley, 1987) in order to solve it. The extended appendix discusses the quality of this approximate solution and finds that it is very accurate.

.1.4 Theorems and proofs for Section 3.5

Theorem 36 *Under technical conditions given in the extended appendix, there exist appropriate constants* $C_1, C_2, Z_L, Z_H, \xi_1$, *and* ξ_2 *and a positive decreasing process* X_s^* *with* $X_t^* = 1$ *so that the optimal policy triple* $< \widehat{c}_s, \widehat{W}_{\widehat{\tau}}, \widehat{\tau} >$ *is*

a) If $W_t < \overline{\overline{W}} = K^{\frac{1}{\gamma}} Z_L^{-\frac{1}{\gamma}}$,

$$
\begin{aligned}
\widehat{c}_s &= \left(\lambda^* e^{\beta(s-t)} X_s^* \frac{H(s)}{H(t)} \right)^{-\frac{1}{\gamma}} 1\{s < \widehat{\tau}\} \\
\widehat{W}_{\widehat{\tau}} &= \overline{\overline{W}} \\
\widehat{\tau} &= \inf\{s : W_s = \overline{\overline{W}}\} = \\
&= \inf\{s : \lambda^* e^{\beta(s-t)} X_s^* \frac{H(s)}{H(t)} = Z_L\}
\end{aligned}
$$

and λ^* *is given by*

$$
\xi_1 C_1 (\lambda^*)^{\xi_1 - 1} + \xi_2 C_2 (\lambda^*)^{\xi_2 - 1} - \frac{1}{\theta} (\lambda^*)^{-\frac{1}{\gamma}} + \frac{y_0}{r} + W_t = 0. \tag{38}
$$

Using the notation $\lambda^*(W_t)$ *to make the dependence of* λ^* *on* W_t *explicit, the optimal consumption and portfolio policy is given by*

$$
\begin{aligned}
c_t &= c(W_t) = (\lambda^*(W_t))^{-\frac{1}{\gamma}} \\
\pi_t &= \pi(W_t) = -\frac{\kappa}{\sigma} \frac{\lambda^*(W_t)}{\lambda_{W_t}^*(W_t)},
\end{aligned}
$$

where $\lambda_{W_t}^*(W_t)$ *denotes the first derivative of* $\lambda^*(W_t)$ *with respect to* W_t.

b) If $W_t \geq \overline{\overline{W}} = K^{\frac{1}{\gamma}} Z_L^{-\frac{1}{\gamma}}$, *the optimal solution is to enter retirement immediately* $(\widehat{\tau} = t)$ *and the optimal consumption policy is given as in the standard Merton (1971) infinite horizon problem.*

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We include the proof and the precise assumptions behind this theorem in the extended appendix to this paper.

Bibliography

[1] Aguiar, M., and E. Hurst (2004): "Consumption vs. Expenditure," NBER Working Paper, w.10307.

[2] Ameriks, J., and S. P. Zeldes (2001): "How Do Household Portfolio Shares Vary With Age?," Working Paper, Columbia GSB.

[3] Banks, J., R. Blundell, and S. Tanner (1998): "Is There a Retirement-Savings Puzzle?," American Economic Review, 88(4), 769—788.

[4] Barone-Adesi, G., and R. E. Whaley (1987): "Efficient Analytic Approximation of American Option Values," Journal of Finance, 42(2), 301—320.

[5] Basak, S. (1999): "On the Fluctuations in Consumption and Market Returns in the Presence of Labor and

[6] Human Capital: An Equilibrium Analysis," Journal of Economic Dynamics and Control, 23(7), 1029—64.

[7] Bodie, Z., J. B. Detemple, S. Otruba, and S. Walter (2004): "Optimal consumption-portfolio choices and retirement planning," Journal of Economic Dynamics and Control, 28(3), 1115—1148.

[8] Bodie, Z., R. C. Merton, and W. F. Samuelson (1992): "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model," Journal of Economic Dynamics and Control, 16(3-4), 427—49.

[9] Campbell, J. Y., J. F. Cocco, F. J. Gomes, and P. J. Maenhout (2004): "Investing Retirement Wealth: A Life-Cycle Model," Review of Financial Studies - forthcoming.

[10] Campbell, J. Y., and L. M. Viceira (2002): Strategic Asset Allocation, Clarendon Lectures in Economics. Oxford University Press.

[11] Carroll, C. D., and M. S. Kimball (1996): "On the Concavity of the Consumption Function," Econometrica, 64(4), 981—92.

[12] Chan, Y. L., and L. Viceira (2000): "Asset Allocation with Endogenous Labor Income: The Case of Incomplete Markets," Working Paper, Harvard University.

[13] Choi, K. J., and G. Shim (2004): "Disutility, Optimal Retirement, and Portfolio Selection," mimeo.

[14] Graduate School of Management, Korea Advanced Institute of Science and Technology.

[15] Cox, J. C., and C.-f. Huang (1989): "Optimal consumption and portfolio policies when asset prices follow a diffusion process," Journal of Economic Theory, 49(1), 33—83.

[16] Cox, J. C., S. A. Ross, and M. Rubinstein (1979): "Option Pricing: A Simplified Approach," Journal of Financial Economics, 7(3), 229—263.

[17] Diamond, P. A., and J. A. Hausman (1984): "Individual Retirement and Savings Behavior," Journal of Public Economics, 23(1-2), 81—114.

[18] Duffie, D. (2001): Dynamic asset pricing theory. Princeton University Press, Princeton and Oxford.

[19] Duffie, D., W. Fleming, M. Soner, and T. Zariphopoulou (1997): "Hedging in Incomplete Markets with HARA Utility," Journal of Economic Dynamics and Control, 21, 753—782.

[20] Duffie, D., and T. Zariphopoulou (1993): "Optimal Investment with Undiversifiable Income Risk," Mathematical Finance, 3, 135—148.

[21] Dybvig, P. H., and H. Liu (2005): "Lifetime Consumption and Investment: Retirement and Constrained Borrowing," mineo. Washington University in St. Louis.

[22] El Karoui, N., and M. Jeanblanc-Picque (1998): "Optimization of Consumption with Labor Income," Finance and Stochastics, 2(4), 409—440.

[23] Gustman, A. L., and T. L. Steinmeier (1986): "A Structural Retirement Model," Econometrica, 54(3), 555—84.

[24] Gustman, A. L., and T. L. Steinmeier (2002): "Retirement and the Stock Market Bubble," NBER Working Paper, 9404.

[25] Hansen, G. (1985): "Indivisible labor and the business cycle," Journal of Monetary Economics, 16(3), 309—327.

[26] Harrison, J. M. (1985): Brownian motion and stochastic flow systems, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York.

[27] He, H., and H. F. Pages (1993): "Labor Income, Borrowing Constraints, and Equilibrium Asset Prices," Economic Theory, 3(4), 663—696.

[28] Jagannathan, R., and N. Kocherlakota (1996): "Why Should Older People Invest Less in Stocks than Younger People?," Federal Reserve Bank of Minneapolis Quarterly Review, pp. 11—23.

[29] Karatzas, I., J. P. Lehoczky, and S. E. Shreve (1987): "Optimal portfolio and consumption decisions for a "small investor" on a finite horizon," SIAM Journal on Control and Optimization, 25(6), 1557—1586.

[30] Karatzas, I., and S. E. Shreve (1991): Brownian motion and stochastic calculus, vol. 113 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edn.

[31] Karatzas, I., and S. E. Shreve (1998): Methods of mathematical finance, Applications of Mathematics. Springer-Verlag, New York.

[32] Karatzas, I., and H. Wang (2000): "Utility maximization with discretionary stopping," SIAM Journal on Control and Optimization, 39(1), 306—329.

[33] Kingston, G. (2000): "Efficient Timing of Retirement," Review of Economic Dynamics, (3), 831—840.

[34] Kogan, L., and R. Uppal (2001): "Risk Aversion and Optimal Portfolio Policies in Partial and General Equilibrium Economies," NBER Working Papers: 8609, 2001.

[35] Koo, H. K. (1998): "Consumption and Portfolio Selection with Labor Income: A Continuous Time Approach," Mathematical Finance, 8(1), 49—65.

[36] Lachance, M.-E. (2003): "Optimal Investment Behavior as Retirement Looms," Working Paper, Wharton.

[37] Lazear, E. P. (1986): Retirement from the Labor Forcepp. 305—55, Handbook of Labor Economics, vol. I eds. O. Ashenfelter and R. Layard. Elsveier.

[38] Liu, J., and E. Neis (2002): "Endogenous Retirement and Portfolio Choice," Working Paper, UCLA.

[39] Merton, R. C. (1971): "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," Journal of Economic Theory, 3(4), 373—413.

[40] Musiela, M., and M. Rutkowski (1998): Martingale Methods in Financial Modelling, vol. 36 of Applications of Mathematics. Springer-Verlag, New York, first edn.

[41] Oksendal, B. (1998): Stochastic differential equations: An Introduction with Applications - Fifth Edition, Universitext. Springer-Verlag, Berlin.

[42] Poterba, J. M., S. F. Venti, and D. A. Wise (1998): "401(k) Plans and Future Patterns of Retirement Saving," American Economic Review, 88(2), 179—184.

[43] Poterba, J. M., S. F. Venti, and D. A. Wise (2000): "Saver Behavior and 401(k) Retirement Wealth," American Economic Review, 90(2), 297—302.

[44] Rockafellar, R. T. (1997): Convex analysis, Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ.

[45] Rogerson, R. (1988): "Indivisible labor, lotteries and equilibrium," Journal of Monetary Economics, 21(1), 3—16.

[46] Rust, J. (1994): Structural Estimation of Markov Decision Processespp. 3081—3143, Handbook of Econometrics, vol. 4 eds. Engle R.F and McFadden D.L,. North-Holland, Amsterdam.

[47] Rust, J., and C. Phelan (1997): "How Social Security and Medicare Affect Retirement Behavior in a World of Incomplete Markets,"Econometrica, 65(4), 781—831.

[48] Schroder, M., and C. Skiadas (1999): "Optimal Consumption and Portfolio Selection with Stochastic Differential Utility," Journal of Economic Theory, 89, 68—126.

[49] Stock, J. H., and D. A. Wise (1990): "Pensions, the Option Value of Work, and Retirement," Econometrica, 58(5), 1151—80.

[50] Sundaresan, S., and F. Zapatero (1997): "Valuation, Asset Allocation and Incentive Retirements of Pension Plans," Review of Financial Studies, 10, 631—660.

[51] Viceira, L. (2001): "Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income," Journal of Finance, 56(2), 433—470.

[52] Wachter, J. (2002): "Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets," Journal of Financial and Quantitative Analysis, 37, 63—91.