Gaussian Distribution Approximation for Localized Effects of Input Parameters

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Abstract—In the application of cycle-to-cycle control to manufacturing processes, the model of the process reduces to a gain matrix and a pure delay. For a general multiple input – multiple output process, this matrix shows the degree of influence each input has on each output. For a system of high order, determining this gain matrix requires excessive numbers of experiments to be performed, and thus a simplified, but non-ideal form for the gain matrix must be developed. In this paper, the model takes the form of a Gaussian distribution with experimentally determined standard deviation and scaling coefficients. Discrete die sheet metal forming, a multiple input-multiple output process with high dimensionality, is chosen as a test application. Results of the prediction capabilities of the Gaussian model, as well as those of two previously existing models, are presented. It is shown that the Gaussian distribution model does the best job of predicting the output for a given input. The model’s invariance over a set of different formed parts is also presented. However, as shown in the companion paper on cycle-to-cycle control, the errors inherent in this model will cause non-ideal performance of the resulting control system. However, this model appears to be the best form for this problem, given the limit of minimal experimentation.

Index Terms—Gaussian distribution model, Cycle-to-Cycle, sheet metal forming.

I. INTRODUCTION

The use of Cycle-to-Cycle (CtC) (or Run-by-Run (RbR)) control has been established as a valid means for improving process quality through the application of discrete control concepts [1, 2]. In a companion paper [3] this method is extended from the more typical single input – single output form to a general multiple input – multiple output (MIMO) solution. However, when the dimension of such problems becomes large, or when the cost of in-process experimentation is high, full order models of the process become prohibitive and approximate, easily parameterized models are needed instead.

This paper presents a general approximation model, consistent with CtC control theory, for processes with spatially distributed input sources. Examples of such processes are the temperature distributions resulting from a distributed heat source or the pressure distribution in a plate that is being acted on by multiple force sources. It could also be applied to spatially distributed surface reactions where a spatially distributed temperature or concentration is imposed.

Cycle-to-Cycle control is a method for applying feedback to processes that are inaccessible (for both measurement and manipulation) during the forming cycle. This limitation stems from the impossibility or the prohibitively high cost of placing sensors and actuators that could result in in-process control. Examples of such processes are sheet-metal forming and chemo-mechanical polishing (CMP) where proper measurements during a cycle are very difficult. This limitation on control results in a simple “gain and delay” model for any process (see Hardt and Siu [1]); because, by definition of process cycle, all process dynamics are over by the time a cycle is complete [1, 4]. This simplicity in model form, coupled with the assumptions of linearity and time invariance, results in an ability to analyze and predict the performance of such systems using common discrete control theory tools.

However, for MIMO problems with \( n \) outputs and \( m \) corresponding inputs, the resulting process gain (see Hardt and Rzepniewski [3]) is a matrix \( G_p \) of dimension \( n \times m \). To fully identify all \( nm \) coefficients, a minimum of \( nm \) experiments must be performed. For any reasonable dimension, this number soon becomes very large. Even if the data can be generated by a suitable simulation, the resulting computational burden is onerous.

However, for problems involving a spatially distributed source acting on a continuum, the influence of each individual input diffuses a limited amount beyond the surface area that is directly acted on by an input. Thus a model that can efficiently capture this influence without having to populate the full process gain matrix would greatly reduce the number of data points required. In this work, a static Gaussian distribution is proposed to describe the influence of each input in the MIMO Cycle-to-Cycle process output.

The Gaussian distribution has been shown to provide a good approximation for distributed effects of sources. Dore and Gaussard [5] and Hebert et al. [6] use the Gaussian distribution shape to approximate the sensed effects of point sources in camera imaging.

Wan et al. [7] show that the thermal distributions in solids can be well approximated by the Gaussian shape. This

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approximation is used in favor of a more complex numerical solution, owing to its good performance and resulting quick computation time for the solution.

II. MODELS FOR CYCLE-TO-CYCLE CONTROL

A. Perturbation Approximation

As was described earlier, Cycle-to-Cycle control uses a linear model to approximate the static behavior of general non-linear processes. As a result, a perturbation approach is used to establish a model that is accurate in the region of operation. This approach is valid given that process control seeks to make small changes to a process that is already close to target, but subject to significant drifts or noise effects. The model takes the form presented in Siu [4]:

$$\Delta_{OUTPUT} = G_p \Delta_{INPUT}$$  \hspace{1cm} (1)

where $G_p$ is a static gain. This perturbation model is used because most manufacturing processes will appear non-linear over their full range of input values. By using a local model, we not only justify the linear assumption but also acknowledge that the purpose of the control is small error reduction for a process near the correct target.

B. Model Dimensionality

Even given the simple structure of the required model, appropriate approximations need to be made to determine models for large numbers of coupled inputs and outputs. Linear models for single input-single output (SISO) systems are simple to obtain with only a few tests (a minimum of 2 to establish a perturbation). Linear models for multiple input-multiple output systems require secondary assumptions and approximations to establish a good model with only a few tests. A system with $n^2$ inputs and $n^2$ outputs requires $n^4$ coefficients to satisfy the matrix form of the CTC model:

$$ \begin{bmatrix} \vdots \\ n^2 \end{bmatrix}_{OUTPUTS} = \begin{bmatrix} n^2 \times n^2 \end{bmatrix}_{GAINS} \begin{bmatrix} \vdots \\ n^2 \end{bmatrix}_{INPUTS}$$  \hspace{1cm} (2)

Even with the realization that each test gives a potential of $n^2$ data points, $n^2$ tests are still required to fill the gain matrix; for a modest size system with 144 inputs, 144 tests would be required to determine the gain matrix. (Note that this does not follow the growth trend for full factorial designs because CTC does not allow multiplicative input interaction.)

Because of the large number of tests required to establish the $G_p$, for systems with high numbers of inputs and outputs some additional approximations need to be made.

III. GAUSSIAN INFLUENCE COEFFICIENTS

To cope with the model dimensionality problem, a general form for the coupling of inputs to outputs is assumed. The Gaussian distribution is chosen because of its generality and physical relevance in addition to its simplicity of description. The Gaussian distribution requires only two parameters to fully describe it, the mean and standard deviation. Thus a whole row of the gain matrix, $n^2$ coefficients, may be filled with just two parameters. This approach is called Gaussian influence coefficients (GIC).

When applied to processes with spatially distributed inputs the mean is placed at the point of application of the input, thus the influence distribution is centered on the input. Although this does eliminate the need to determine the mean, a scaling constant for the whole distribution will also be needed.

With this model, the entire $n^4$ gain matrix $G_p$ has only $2n^2$ unknowns: the magnitude and variance of each Gaussian influence function about each input. This would require a minimum of four perturbation experiments to determine, but the number of required experiments can be further reduced if a fixed variance is used. Under the latter assumption, the problem now defaults to a single perturbation (two successive experiments with slightly different input values) to find the process gain matrix.

Once a scaled Gaussian process model is assumed, the input-output equation takes the form:

$$ \begin{bmatrix} \text{output Vector} \\ \text{Gaussian unscaled Matrix} \end{bmatrix} = \begin{bmatrix} \text{Diagonal Scaling Matrix} \\ \text{Input Vector} \end{bmatrix}$$  \hspace{1cm} (3)

For example, a five input-five output system plant gain matrix, as shown in equation (1), takes the form:

$$ G_p = \begin{bmatrix} .40 & .24 & .05 & 0 & 0 \\ .24 & .40 & .24 & .05 & 0 \\ .05 & .24 & .40 & .24 & 0 \\ 0 & 0 & .05 & .24 & .40 \\ 0 & 0 & 0 & 0 & .24 \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 \\ 0 & 0 & s_3 & 0 & 0 \\ 0 & 0 & 0 & s_4 & 0 \\ 0 & 0 & 0 & 0 & s_5 \end{bmatrix}$$  \hspace{1cm} (4)

where $s_i, i=\{1...5\}$ are the scaling factors. Note that a fixed standard deviation of one was assumed. The result of using the above model matrix to predict the output shape is shown in Figure 1. Note that, although the input is assumed local only to position 3 in the plot, a continuum over all locations is assumed for the outputs.

Figure 1 shows the input-output relationship for a system once a Gaussian distribution of the input influence is assumed.
Note that a scaling and standard deviation of 1 was used to construct the figure. The output shape shown is characteristic of continuum effects such as a temperature and pressure distributions from heat and force inputs, respectively.

Figure 2  Input-Output relation when assuming a Gaussian influence shape. Input (bottom) and output (top). Scaling =1, standard deviation = 1.

IV. EXAMPLE APPLICATION

A test application for the Gaussian influence coefficients model is discrete die sheet metal forming [8]. A traditional monolithic die is divided into many “pins” that are independently actuated. Figure 3 shows such a discrete die. Note that the pins are not actuated in-process and that effective in-process measurement of parts being formed is impractical because of spring back (elastic relaxation of plastically deformed parts).

The capability to freely change the shape of the die allows greater flexibility in manufacturing and offers greater cost effectiveness for parts with low run numbers. Because the reconfigurable die is able to assume many shapes, valuable floor and storage space does not have to be used to house extra machines or spare dies. Such high process flexibility and cost effectiveness is greatly desired in the airplane manufacturing industry, which produces only a few hundred products a year.

Figure 3  Reconfigurable Tool Produced by Northrop Grumman and Cyril Bath. The forming surface is 4 ft x 6 ft in area. (See Papazian [9] for design details).

V. EXISTING CONTROL ALGORITHMS

While offering greater flexibility, the discrete die forming tool requires appropriate control algorithms to function properly. It should be noted that the tool shown in Figure 3 has many thousands of inputs and outputs. Cycle-to-cycle control is a perfect candidate because it is able to cope with this degree of dimensionality and process constraints [3].

As discussed above, Cycle-to-Cycle control needs a plant gain matrix \( G_p \) to function properly. However, the number of experiments needed to identify a process- (and sometimes part-) specific model is limited by both time and cost constraints. Thus, a general model, with very few parameters to identify, is highly desirable. Two widely differing models have been used in previous implementations of CtC control for this process.

A. Spatial Coordinate Algorithm

The spatial coordinate algorithm (SCA) assumes a purely diagonal gain matrix [10]. This algorithm is an attempt to identify a process model with a minimal data set, only one perturbation.

An understanding of the assumptions behind SCA is gained when one observes the plant gain matrix:

\[
G_p = \begin{bmatrix}
    g_{p1} & 0 & 0 \\
    0 & \ddots & 0 \\
    0 & 0 & g_{pn^2}
\end{bmatrix}
\]  \hspace{1cm} (5)

The implication of this gain matrix, when it is substituted into equation (1) is that each output is independent of all but one input. When applied to the example application of discrete die sheet metal forming, this means that the portion of metal (output) above each respective pin (input) is independent of any other. Thus, the model predicts non-continuous parts which are not physically characteristic of sheet metal. This phenomenon is shown in Figure 4.

Figure 4  Input-output relationship for the spatial coordinate algorithm.

B. Deformation Transfer Function

The deformation transfer function (DTF) introduces coupling by using convolution [11]. The multiplication of the plant gain matrix and input vector in equation (1) becomes a convolution. Because convolution in the spatial domain is equivalent to multiplication in the frequency domain, a Fourier transform of both part and die is used in the experimental implementation of this algorithm. Note that this plant model only requires one perturbation (\( n^2 \) data points) to be fully described.

Norfleet [12] shows that convolution of the plant matrix and
input vector in the spatial domain may be carried out as multiplication in the spatial domain once the original plant matrix is replaced by a circulant matrix. The \( n^2 \times n^2 \) matrix is fully populated by repeating one column (\( n^2 \times 1 \)) of information. The plant matrix thus takes the following form:

\[
G_p = \begin{bmatrix}
g_{p1} & g_{pm} & \cdots & g_{p2} 
g_{p2} & g_{p1} & \cdots & \\
\vdots & \vdots & & \\
g_{pm} & g_{p(n^2-1)} & \cdots & g_{p1}
\end{bmatrix}
\]

(6)

Once again the implications of using this gain matrix are analyzed through the CtC viewpoint. This matrix indicates that
1. All the outputs are coupled to all the inputs.
2. The form of coupling is uniform across all the outputs.

These observations yield non-intuitive results. The derived plant matrix is independent of the size of the object to be controlled. Looking at the application of this model to discrete die sheet metal forming, this means that a pin (input) that is meters away from the portion of metal under consideration (output) still has influence on the output. By contrast, with a metal forming process one expects a local rather than a global influence of each input.

It is also reasonable to expect that the magnitude of this influence may be non-uniform across the full die surface. For example, when making a cylinder, one could expect the pins at the center, which are perpendicular to the metal sheet, to have a different influence than pins at the edges, which are not perpendicular to the metal.

VI. SIMULATION EXAMPLE

A. Bump Perturbation Tests

It is now proposed that discrete die sheet metal forming should have a localized influence that follows a Gaussian distribution. This section examines this proposition through simulation examples. An ABAQUS finite element simulation written by Socrate [13] for stretch forming over a reconfigurable die is used for these tests.

The gains for the Gaussian function of a single-pin perturbation are determined by first forming a desired shape, then artificially displacing up a single pin and forming another part. The gains are defined as changes in part shape divided by changes in die shape. Die displacements of 0.02in. and 0.05in are used. A typical result from perturbation tests is shown in Figure 5. Although it is not perfect, the fit between the Gaussian and determined coefficients is deemed satisfactory. Note that these perturbation tests are an extreme case, since they attempt to define the gains from a single-pin displacement. In practice, many pins are changed at the same time to establish the desired output shape.

These tests also revealed that the Gaussian influence function typically has a standard deviation of about 1 across the whole part (ranging from approximately 0.85 to 1.5). With this observation, a uniform unity standard deviation is assumed in all results presented after this section. It is of note that
1. Once a standard Gaussian spread is assumed, there are only \( n^2 \) coefficients to identify. These can be identified with a single perturbation test.
2. An exact system model is not needed. The nature of CtC control is such that it requires a model that is sufficiently representative of reality. Thus reasonable simplifications are tolerated.

Figure 5  Perturbation test gains plotted along with a Gaussian distribution. Standard deviation = 1.5, scale = 0.4.

B. Scaling Distribution

Figure 6 shows the scaling factors, entries in the scaling matrix of equation (4), as a function of their positions on the target part. Note that only a quarter-symmetry matrix is shown; the full matrix is composed of the quarter-symmetry model reflected about the (1,i) and (i,1) lines. Upon visual inspection, it becomes apparent that the scaling factors follow smooth changes from one location in the matrix to the next; note that such observations are not made when observing the SCA and DTF matrices.

A second order model fit is chosen to approximate the general trend of the scaling factors. The solution takes the form:

\[
A\hat{\beta} = \mathbf{y}
\]

where \( \mathbf{y} \) is the scaling matrix rearranged as a vector, \( \hat{\beta} \) is a coefficient vector and

\[
A = \begin{bmatrix}
1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 & x_1x_2^2 & x_1^2x_2 & x_1^2x_2^2
\end{bmatrix}
\]

(8)

where \( x_1 \) and \( x_2 \) are the Cartesian coordinates of the scaling factor \( y_i \). Note that, since there are potentially \( n^2 \) entries in the \( y \) vector and only nine are required to determine the coefficient vector, \( \hat{\beta} \), there is a degree of redundancy. This redundancy increases the robustness to noise.
C. Part Prediction Comparison

A reasonable test of the goodness of a model is its ability to accurately predict parts given a die shape. With two experiments at different shapes a single perturbation gain matrix can be calculated. This can then be used with a new input to predict a third shape. This predicted shape can then be compared to the actual (or simulated as herein) to evaluate the model error.

For this purpose, five cylinders of different radii are used to perform tests on the three different model algorithms. The radii are: 6.65 in., 8.65 in., 10.65 in, 12.65 in, and 14.65 in. The different dies are shown in Figure 7.

All parts are based on the ABAQUS simulation of Socrate [13]. Five dies are sent as inputs into the simulation and the parts that they produce are taken as the “ideal” parts that each model will be asked to predict.

With five different tests, there are 30 possible combinations of models and target parts. Thus, from these five simulations, 30 values of prediction error, spanning a shape and perturbation magnitude space, can be obtained. The test numbering convention is shown in Table 1.

<table>
<thead>
<tr>
<th>Parts used for model</th>
<th>Target part</th>
<th>Test number</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 and 3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2 and 4</td>
<td>1</td>
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<tr>
<td>2 and 5</td>
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<td>1 and 3</td>
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<td>1 and 4</td>
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The root mean square (RMS) shape error from each of the 30 part prediction tests is shown in Figure 8. Using the area under each error curve as a measure of goodness (the net prediction error across all parts) the GIC model has 36% less error than SCA and 64% less error than DTF. This is in line with the physical insights offered into each algorithm; sheet metal is neither fully decoupled nor fully coupled, it is only locally coupled.

D. Model Coefficient Invariance

In addition to part prediction ability, a second criterion is used to establish the GIC model as appropriate for sheet metal forming. A good model is one which remains constant for similar target parts or for the same target part between cycles. This consistency in the model is reflective of its goodness and is the key to predicting and choosing the desired performance of a controller.

As presented in equation (4), a standard Gaussian matrix is used for all GIC models. It is only the scaling matrix that varies from one system to another. Thus, it is sufficient to view only the scaling matrix to judge model consistency.

Once again, data from the five parts in Figure 7 is used for this experiment. Note that each part has 100 inputs and 100 outputs. Because the scaling matrix is diagonal, it is possible to rearrange the $100x100$ scaling matrix into a $10x10$ matrix where the position of each scaling coefficient corresponds to the location of the pin that it is scaling. Note that this is done for visualization purposes only and that this “new” scaling matrix cannot be used in equation (4). In order to view the scaling coefficients more easily, the $10x10$ scaling matrix is also rearranged column-wise as a vector according to:
The scaling vector as a function of test number is shown in Figure 9. Note that these results were obtained by using noise-free simulated experiments. It is readily apparent that the scaling factors remain approximately constant even for different target radii and different starting dies, i.e. different magnitude changes.

A universal scaling matrix is assembled by averaging the available 30 scaling values for each of the 100 pins. This matrix is then used to compute all ten possible combinations of starting and target parts (since there is a total of 5 parts). The error between a “formed” and predicted part is shown in Figure 10. When one compares these results to the GIC algorithm results from Figure 8, one observes that this universal model does the best job thus far of predicting a part.

VII. CONCLUSION

This paper presented a static gain model for spatially distributed, multiple input – output processes, for use in a cycle-to-cycle control scheme. The model takes the form of a Gaussian influence distribution with experimentally determined standard deviation and scaling coefficients. Application of the model to discrete die sheet metal forming, as well as those of two previously existing models, was presented. It was shown that the Gaussian distribution model does the best job of predicting the output when given an input matrix. The model’s invariance for a set of formed parts was also presented. The simple static gain model adequately captured true behavior and is appropriate for application in cycle to cycle control.

However, as shown in [3] when the model is inexact, controller performance cannot be guaranteed. Since the Gaussian model is an approximation, it will always carry with it varying degrees of uncertainty. Research into the robustness of the Cycle-to-Cycle methods with the Gaussian model approximation is underway to assess the limits of performance that can be predicted from this highly practical approximate process model.

REFERENCES