The Determination of the Admissible Nilpotent Orbits in Real Classical Groups.

by

James O. Schwartz

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ABSTRACT

This thesis classifies the admissible nilpotent orbits, in the sense of Duflo, in the following groups: $SL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $O(p,q)$, $U(p,q)$, $SO(p,q)$ and $SU(p,q)$. The philosophy of coadjoint orbits suggests that the admissible orbits should be the ones to which one can attach representations.

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Dedication

This thesis is dedicated to my parents
with love and gratitude.

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INTRODUCTION

This thesis is a study of the connection between representations of a semisimple Lie group and the partition of the dual of its Lie algebra g_o^* into orbits under the coadjoint representation. Kirillov first introduced this method in the study of simply connected nilpotent groups and it was generalized by Auslander-Kostant and Duflo to simply connected type I solvable groups. In these cases the orbit method has been successful in setting up a correspondence between equivalence classes of irreducible unitary representations and coadjoint orbits of G on g_o^*. The connection between the algebraic picture of the unitary dual and the more geometric orbit picture has provided useful insight and added coherence to the representation theory.

To implement the correspondence discussed above, Duflo introduced a method for picking out a subset of orbits he calls <u>admissible orbits</u> in g_o^*. He describes a bijection between admissible orbits and unitary representations for nilpotent and Type I solvable groups. In the case of a real semisimple Lie group it is not at present understood which nilpotent coadjoint orbits should correspond to representations of the group.

Moreover, there isn't a general technique for attaching representations to nilpotent coadjoint orbits -- geometric quantization has not been applied successfully to nilpotent orbits except in special cases. The admissible nilpotent orbits are good candidates on which to attempt quantization.

We have classified the nilpotent orbits and determined which of them are admissible in the following real classical groups: $Sp(2n,\mathbb{R})$, $O(p,q)$ and $U(p,q)$ -- the first two are the groups preserving a symplectic and a symmetric form, respectively, on a real vector space, and the latter is the group preserving an Hermitian form on a complex vector space. In addition, we have determined the admissibility of the nilpotent orbits in the following semisimple groups: $SO(p,q)$, $SU(p,q)$ and $SL(n,\mathbb{R})$.

Chapters 1, 2, and 4 describe the method used to determine whether an orbit is admissible. In Chapter 3 we recall how the real nilpotent orbits under each of the real classical groups above are parametrized by equivalence classes of representations of $SL(2,\mathbb{R})$ on a vector space preserving the appropriate non-degenerate sesquilinear symplectic or Hermitian form (Springer, Steinberg). In Chapter 6 we do the same in the case of $SL(n,\mathbb{R})$, except we look at representations of $SL(2,\mathbb{R})$

preserving a multilinear n-form. A nilpotent orbit under a complex group may break up into a disjoint union of several nilpotent orbits under a real form of the complex group, but it turns out, in the cases we have studied, that the issue of admissibility of a nilpotent orbit depends on its orbit under the complexification of G. The final two Theorems in Chapter 5 show which complex nilpotent orbits are admissible in terms of the parametrization discussed above. In Chapter 6, we show all nilpotent orbits in $SL(n,\mathbb{R})$ are admissible.

PRELIMINARIES

When G is semisimple, we can make use of the Killing form, B (or any nondegenerate G-invariant bilinear form on $\mathfrak{g}_o$), to identify $\mathfrak{g}_o$ with $\mathfrak{g}_o^*$, and this identifies adjoint orbits on $\mathfrak{g}_o$ with coadjoint orbits on $\mathfrak{g}_o^*$. We denote the orbit of $X \in \mathfrak{g}_o$ by $\mathcal{O}_X = \{Adg(X) \mid g \in G\}$. We will call the isotropy subgroup of X, the subgroup of G which fixes X, G^X; $G^X = \{g \in G \mid Adg(X) = X\}$. The Lie algebra of G^X will be called $\mathfrak{g}_o^X = \{F \in \mathfrak{g}_o \mid [X,F] = 0\}$. The adjoint representation gives us a smooth, transitive action of G on $\mathcal{O}_X$; thus $\mathcal{O}_X$ is a homogeneous space isomorphic to the coset space G/G^X.

We now introduce an example which shows why integrality of an orbit is not a discriminating enough criterion to use to pick out orbits which should correspond to representations. An orbit $\mathcal{O}_X$ is called <u>integral</u> if there exists a finite dimensional, irreducible unitary representation ν of G^E satisfying the following condition:

$$iB(E,X)\cdot I = d\nu|_e(X) \quad \forall X \in \mathfrak{g}_o^E .$$

If E is nilpotent, then $B(E, \mathfrak{g}_o^E) = 0$ (Chapter 1, Lemma 1). It follows immediately that E is admissible for all nilpotent orbits -- take ν to be the trivial representation. But an orbit and the representation attached to it should match in dimension in the following sense. The representation attached to a nilpotent orbit should have Gelfand-Kirillov dimension equal to one half the dimension of the orbit. Howe and Vogan [9] have shown that there are no such representations associated to the minimal nilpotent orbit in $Sp(2n,\mathbb{R})$. This example points out the need for an extended notion of integrality. Duflo's definition of admissibility is closely related to integrality, but it makes use of the symplectic structure of the orbits. We now describe this structure.

$\mathcal{O}_X$ is a symplectic manifold, i.e. there is a closed 2-form which is non-degenerate on each tangent space.

For a fixed $E \in \mathcal{O}_X$, we construct a skew-symmetric bilinear form on $T_E(\mathcal{O}_X)$ as follows: consider the smooth mapping $\phi : G \longrightarrow \mathcal{O}_X$ given by $g \longrightarrow Adg(E)$. Then the map $d\phi|_e : \mathfrak{g}_o \twoheadrightarrow T_E(\mathcal{O}_X)$ is onto and has kernel $\mathfrak{g}_o^E$. Thus we have a canonical isomorphism: $T_E(\mathcal{O}_X) \cong \mathfrak{g}_o/\mathfrak{g}_o^E$. We define our bilinear form on $\mathfrak{g}_o/\mathfrak{g}_o^E$ and carry it over to $\mathcal{O}_X$ by means of this isomorphism. To begin, define the skew-symmetric mapping $\omega_E : \mathfrak{g}_o \times \mathfrak{g}_o \longrightarrow \mathbb{R}$ as follows: $\omega_E(V,W) = B(E,[V,W])$. The radical of ω_E is $\mathfrak{g}_o^E$, and therefore ω_E makes sense when it is considered as a form on the quotient space $\mathfrak{g}_o/\mathfrak{g}_o^E$. Clearly, ω_E is non-degenerate on $\mathfrak{g}_o/\mathfrak{g}_o^E$. We will omit the verification that ω_E gives us a closed 2-form on $\mathcal{O}_X$. (See Guillemin-Sternberg).

The group G^E acts on the symplectic vector space $\mathfrak{g}_o/\mathfrak{g}_o^E$ via the adjoint representation and preserves the form ω_E . For $g \in G^E$ and $Y + \mathfrak{g}_o^E \in \mathfrak{g}_o/\mathfrak{g}_o^E$, we define $g\cdot(Y+\mathfrak{g}_o^E) = Adg(Y) + \mathfrak{g}_o^E$. It is easy to check that this action preserves ω_E , and thus we get a map from G^E into $Sp(\mathfrak{g}_o/\mathfrak{g}_o^E)$, the group which preserves ω_E on the vector space $\mathfrak{g}_o/\mathfrak{g}_o^E$.

The symplectic group of any vector space V, $Sp(V)$, has a well-known two-fold cover called the metaplectic cover, $Mp(V)$ (Shale [6]). Let $\pi : Mp(V) \longrightarrow Sp(V)$ and let τ

denote our mapping $G^E \longrightarrow Sp(\mathfrak{g}_o/\mathfrak{g}_o^E)$. We define $(G^E)^{mp}$, the "pullback" cover of G^E , as follows: $(G^E)^{mp} = \{(x,y) \in G^E \times Mp(\mathfrak{g}_o/\mathfrak{g}_o^E) \mid \tau(x) = \pi(y)\}$. The mapping $p_1 : (G^E)^{mp} \longrightarrow G^E$ defined by projection on the first factor is a two-fold cover of G^E .

$$\begin{array}{ccc} (G^E)^{mp} & \dashrightarrow & Mp(\mathfrak{g}_o/\mathfrak{g}_o^E) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ G^E & \xrightarrow{\ \tau\ } & Sp(\mathfrak{g}_o/\mathfrak{g}_o^E) \end{array}$$

We say a representation ν^{mp} of $(G^E)^{mp}$ is <u>genuine</u> if it is not trivial on the kernel of the map $p_1 : (G^E)^{mp} \longrightarrow G^E$. Finally, we say a genuine representation ν^{mp} is <u>admissible</u> if $d\nu^{mp}|_e(X) = i\cdot B(E,X)\cdot I$ for all $X \in \mathfrak{g}_o^E$. The group G^E and the orbit are said to be <u>admissible</u> if $(G^E)^{mp}$ admits an admissible representation. It is not difficult to see that $i\cdot B(E,X) = 0$ for every $X \in \mathfrak{g}_o^E$ if E is nilpotent. Thus ν^{mp} must be trivial on $(G^E)_0^{mp}$, and G^E will be admissible if and only if $z \notin (G^E)^{mp}$, where z is the nontrivial element of the kernel of the map $p_1 : (G^E)^{mp} \longrightarrow G^E$.

For $E \in \mathfrak{g}_o$ nilpotent, the algebraic group G^E can be decomposed as follows: $G^E = L \ltimes U$, where L is reductive and U is unipotent, normal in G^E, and simply connected. It follows that $(G^E)^{mp}$ is isomorphic to $L^{mp} \ltimes U$, which reduces the question of admissibility to the subgroup L and its two-fold cover L^{mp}. Possibly after replacing E by another element of $\mathcal{O}_E$, we can embed E in a θ-stable copy of $\mathfrak{sl}(2)$ inside $\mathfrak{g}_o$, where θ is the Cartan involution of $\mathfrak{g}_o$. Then we may take $\mathfrak{l}_o = \mathfrak{z}_{\mathfrak{g}_o}(\mathfrak{sl}(2))$, or $L = Z_G(SL(2))$, as the reductive subgroup in our decomposition of G^E. The polar decomposition enables us to write $G = K\exp(\mathfrak{p}_o)$. Because L is θ-stable and algebraic, we can decompose L as $L = (L\cap K)\exp(\mathfrak{l}_o \cap \mathfrak{p}_o)$, and, as above, the question of admissibility reduces to a question about the two-fold cover of $L\cap K$. We recall that this cover is constructed by mapping $L\cap K$ into $Sp(\mathfrak{g}_o/\mathfrak{g}_o^E)$. Hence the question of admissibility of an orbit finally reduces to analyzing a representation of a compact group on a finite dimensional vector space.

Chapter 1

We begin by showing how the question of admissibility for the nilpotent orbit $\mathcal{O}_E$ is equivalent to a topological question about the two-fold metaplectic cover $(G^E)^{mp}$ of the subgroup $G^E \subset G$. We begin with an easy Lemma.

Lemma 1. Let $E \in \mathfrak{g}_o$ be nilpotent. Then $B(E,X) = 0$ for all $X \in \mathfrak{g}_o^E$, where B is the Killing form on $\mathfrak{g}_o$.

Proof: We must show that $[E,X] = 0$ implies that $B(E,X) = 0$. Use the Jacobson-Morosov Theorem to embed E in a standard $\mathfrak{sl}(2)$-triple $\{E,F,H\}$. We decompose $\mathfrak{g}_o = \bigoplus_i [\mathfrak{g}_o]_i$ under this $\mathfrak{sl}(2)$, where $[\mathfrak{g}_o]_i$ is the H-eigenspace with eigenvalue i. The $\mathfrak{g}_o$-invariance of B implies $B([\mathfrak{g}_o]_i, [\mathfrak{g}_o]_{-j}) = 0$ unless $i = j$. The $\mathfrak{sl}(2)$-theory tells us that $E \in [\mathfrak{g}_o]_2$ and $\mathfrak{g}_o^E \subset \bigoplus_{i \leq 0} [\mathfrak{g}_o]_i$; therefore $B(E, \mathfrak{g}_o^E) = 0$. □

Proposition 1. A nilpotent orbit $\mathcal{O}_E \subset \mathfrak{g}_o$ is admissible if and only if the kernel of the map $\pi : (G^E)^{mp} \longrightarrow G^E$ is not in the identity component $(G^E)_0^{mp}$ of $(G^E)^{mp}$.

Proof. Let $M = \{e,z\}$ denote the kernel of π -- an order two subgroup of $(G^E)^{mp}$. If $z \in (G^E)^{mp}_0$ it is clear that G^E is not admissible, because if there were an admissible representation υ^{mp} of $(G^E)^{mp}$ then, by the Lemma, $d\upsilon^{mp}|_e = 0$. Hence $\upsilon^{mp}|_{(G^E)^{mp}_0} = I$. In particular, $\upsilon^{mp}(z) = I$ and υ^{mp} is not genuine.

Now assume $z \notin (G^E)^{mp}_0$, and we construct a genuine representation υ^{mp} of $(G^E)^{mp}$ such that $\upsilon^{mp}|_{(G^E)^{mp}_0} = I$; that is υ^{mp} is admissible.

Let $\overline{M} = \{\overline{z},\overline{e}\}$ denote the image of M in the component group $\overline{(G^E)^{mp}} = (G^E)^{mp}/(G^E)^{mp}_0$. Then $\overline{M} \subset \overline{(G^E)^{mp}}$, an order 2 normal subgroup, is contained in the center of the group, hence $\overline{M} \subset Z(\overline{(G^E)^{mp}})$. Let σ be the non-trivial one dimensional representation of $\overline{M}$. Let ρ denote the right regular representation on $\mathrm{Ind}_{\overline{M}}^{\overline{(G^E)^{mp}}}(\sigma)$. For $f \in \mathrm{Ind}_{\overline{M}}^{\overline{(G^E)^{mp}}}(\sigma)$ we calculate $\rho(\overline{z}).f$: Let $\overline{g} \in \overline{(G^E)^{mp}}$, then

$$\rho(\overline{z})f(\overline{g}) = f(\overline{g}\ \overline{z}) = f(\overline{z}\ \overline{g}) = \overline{z}.f(\overline{g}) = -f(\overline{g}) .$$

Thus $\rho(\overline{z}) \neq I$. We define a representation υ^{mp} of $(G^E)^{mp}$ by setting $\upsilon^{mp}(g) = \rho(\overline{g})$ where $g \in (G^E)^{mp}$ and

$\bar{g}$ is the image of g in $\overline{(G^E)^{mp}}$. It is clear that ν^{mp} is an admissible representation of $(G^E)^{mp}$. □

We now show G^E is admissible if and only if a reductive subgroup $L \subset G^E$, which we define below, is admissible. Use the Jacobson-Morosov Theorem to embed $E \in \mathfrak{g}_o$ in a standard $\mathfrak{sl}(2)$-triple $\{E,F,H\} \subset \mathfrak{g}_o$.

Proposition 2. G^E is isomorphic to $L \ltimes U$ where $L = Z_G(\mathfrak{sl}(2))$, U is the closed subgroup of G^E corresponding to the Lie subalgebra $\mathfrak{u}_o = \mathfrak{g}_o^E \cap [E,\mathfrak{g}_o]$, U is unipotent, connected and simply connected.

Proof: Assume E is the nilpositive element in the $\mathfrak{sl}(2)$-triple $\{E,F,H\} \subset \mathfrak{g}_0$. First we show that every element $g \in G^E$ can be decomposed $g = \ell \cdot u$ with $\ell \in L$ and $u \in U$. Fix $g \in G^E$. Then $\{E,F,H\}$ and $\{E, Adg^{-1}(F), Adg^{-1}(H)\}$ are both $\mathfrak{sl}(2)$-triples with the same nilpositive element. By Kostant (1959), all triples with the same nilpositive are conjugate by an element of U . Furthermore, the set of neutral elements for all triples with nilpositive E is the linear coset $H+\mathfrak{u}_o$, where $\mathfrak{u}_o = [E,\mathfrak{g}_o] \cap \mathfrak{g}_o^E$ and there is a 1-1 onto mapping: $U \longrightarrow H+\mathfrak{u}_o$ which sends $u \longrightarrow Adu(H)$. Hence there is a unique $u \in U$ such that

$$\{E, Adu(Adg^{-1}(F)), Adu(Adg^{-1}(H)) = \{E,F,H\} .$$

Thus $u \cdot g^{-1} \in Z_G(\mathfrak{sl}(2)) = L$, and we have decomposed

$$g = (g \cdot u^{-1}) \cdot u \in L \cdot U .$$

Now we show that $L \cap U = \{e\}$, U is unipotent connected, simply connected, and U is normal in G^E . We will show below that $\mathfrak{u}_o$ is a nilpotent Lie algebra. The exponential map: $\mathfrak{u}_o \longrightarrow U$ is 1-1 and onto. To see the exponential map, $\exp : \mathfrak{n} \longrightarrow N$, for a connected nilpotent group N , is onto we use induction on the dimension of $\mathfrak{n}$. This statement is true for $\dim \mathfrak{n} = 1$ because the exponential map is onto for a connected abelian group. $\mathfrak{n}$ has a nontrivial center $\mathfrak{z}$, therefore we may assume $\exp : \mathfrak{n}/\mathfrak{z} \longrightarrow N/Z$ is onto. Thus for $n \in N$ there exists $X+\mathfrak{z} \in \mathfrak{n}/\mathfrak{z}$ such that $\exp(X+Z_1) = n \cdot z$ with $z \in Z$, $Z_1 \in \mathfrak{z}$. Also we can find $Z_2 \in \mathfrak{z}$, such that $\exp Z_2 = z$; therefore $\exp(X+Z_1) \cdot \exp(-Z_2) = n$ and this gives $\exp(X+Z_1-Z_2) = n$. To see $\exp$ is one-to-one, we find a basis for the matrices $\mathfrak{n}$ in which they are strictly upper triangular (Engel's Theorem). It is easy to see by direct computation that exp is one-to-one on those matrices. Therefore U is homeomorphic to Euclidean space, i.e. U is connected and simply connected.

Let $[\mathfrak{g}_o]_k$ be the H-eigenspace with eigenvalue k under the adjoint representation of $\mathfrak{sl}(2)$ on $\mathfrak{g}_o$. As an easy consequence of the $\mathfrak{sl}(2)$-theory, we get $\mathfrak{u}_o \subseteq \bigoplus_{k>0} [\mathfrak{g}_o]_k$. Let $u \in U$ and $W \in \mathfrak{u}_o$ such that $\exp W = u$.

If $X \in \bigoplus_{k\geq 0} [\mathfrak{g}_o]_k$, then $\mathrm{Ad}u(X) = X+Y$ with $Y \in \bigoplus_{k>0} [\mathfrak{g}_o]_k$; thus U is unipotent. By the same argument, if $X \in [\mathfrak{g}_o]_0$ then $\mathrm{Ad}u(X) \notin [\mathfrak{g}_o]_0$. On the other hand, for $\ell \in L$, $\mathrm{Ad}\ell(X) \in [\mathfrak{g}_o]_0$, because $\mathrm{Ad}\ell$ commutes with the $\mathfrak{sl}(2)$ action on $\mathfrak{g}_o$. This shows that $L \cap U = \{e\}$. We now show U is normal in G^E and thus $G^E = L \ltimes U$. Since U is connected it is enough to show $\mathrm{Ad}g(X) \in \mathfrak{u}_o$ for all $g \in G^E$ and $X \in \mathfrak{u}_o$. We have $[E,\mathrm{Ad}g(x)] = \mathrm{Ad}g[E,X] = 0$, hence $\mathrm{Ad}g(X) \in \mathfrak{g}^E$. We know there is a $W \in \mathfrak{g}_o$ such that $[E,W] = X$, so:

$$\mathrm{Ad}g(X) = \mathrm{Ad}g([E,W]) = [\mathrm{Ad}g(e),\mathrm{Ad}g(W)] = [E,\mathrm{Ad}g(W)] ,$$

hence $\mathrm{Ad}g(X) \in [E,\mathfrak{g}_o]$ and we get $\mathrm{Ad}g(X) \in \mathfrak{g}^E \cap [E,\mathfrak{g}_o] = \mathfrak{u}_o$ as desired. We note that L is reductive, because the centralizer of a reductive algebraic subgroup of an algebraic group is reductive. □

Recall the covering map $\pi : (G^E)^{mp} \longrightarrow G^E$. We have shown that G^E is isomorphic to $L \ltimes U$. Define $L^{mp} = \pi^{-1}(L)$ and we prove

<u>Proposition 3</u>. The metaplectic cover $(G^E)^{mp}$ is isomorphic to $L^{mp} \ltimes U$.

<u>Proof</u>: Let $\tilde{U}$ be the analytic subgroup of $(G^E)^{mp}$ with Lie algebra $\mathfrak{u}_o$. Since U is simply connected, π is an isomorphism of $\tilde{U}$ onto U . Since $\mathfrak{u}_o$ is normalized by L , $\tilde{U}$ is normalized by L^{mp} .

We have an injective map $i : L^{mp} \ltimes \tilde{U} \longrightarrow (G^E)^{mp}$ and must show this map is onto. Let π_1 and π_2 denote the covering maps as in the diagram:

$$\begin{array}{ccc} L^{mp} \ltimes \tilde{U} & \longrightarrow & (G^E)^{mp} \\ & \searrow^{\pi_2} \quad \swarrow^{\pi_1} & \\ & G^E & \end{array} .$$

Choose $\tilde{x}_1 \in (G^E)^{mp}$ and $\{\tilde{y}_1, \tilde{y}_2\} = \pi_2^{-1} \circ \pi_1(\tilde{x}_1)$. We will show that either $i(\tilde{y}_1) = \tilde{x}_1$ or $i(\tilde{y}_2) = \tilde{x}_1$.
$i(\tilde{y}_1) \neq i(\tilde{y}_2)$ but they both maps to x_1 under π_1 .

Since $\pi_1(\tilde{x}_1) = x_1$ and π_1 is a two-to-one map, we conclude either $i(\tilde{y}_1)$ or $i(\tilde{y}_2)$ equals $\tilde{x}_1$. □

Proposition 4. Let $\{E,F,H\} \subset \mathfrak{g}_o$ be a standard $\mathfrak{sl}(2)$-triple and θ a Cartan involution on $\mathfrak{g}_o$. Then there exists a $g \in G$ such that the $\mathfrak{sl}(2)$-triple $\{E' = Adg(E),\ F' = Adg(F),\ H' = Adg(H)]$ satisfies the relations $\theta E' = -F'$ and $\theta H' = -H'$.

Proof: We define a Cartan involution θ' on the subalgebra spanned by $\{E,F,H\}$ as follows: $\theta' E = -F$ and $\theta' H = -H$. This corresponds to a Cartan decomposition of $\mathfrak{sl}(2) = \mathfrak{k}_o' \underset{\theta'}{\oplus} \mathfrak{p}_o'$ with $\mathfrak{k}_o' = \mathbb{R}(E-F)$ and $\mathfrak{p}_o' = \mathbb{R}H \oplus \mathbb{R}(E+F)$. One can extend θ' to a Cartan involution θ'' on the whole Lie algebra $\mathfrak{g}_o$ so that $\mathfrak{g}_o = \mathfrak{k}_o'' \underset{\theta''}{\oplus} \mathfrak{p}_o''$ (the Cartan decomposition with respect to θ'') and $\mathfrak{k}_o' \subset \mathfrak{k}_o''$ and $\mathfrak{p}_o' \subset \mathfrak{p}_o''$ (Mostow, p. 277 Helgason). Since all Cartan decompositions are conjugate, we can find $g \in G$ such that $Adg(\mathfrak{k}_o'') = \mathfrak{k}_o$ and $Adg(\mathfrak{p}_o'') = \mathfrak{p}_o$ where $\mathfrak{g}_o = \mathfrak{k}_o \underset{\theta}{\oplus} \mathfrak{p}_o$ is the Cartan decomposition on $\mathfrak{g}_o$ with respect to θ . We have $\theta = Adg \circ \theta'' \circ Adg^{-1}$ and the triple $\{Adg(E), Adg(F), Adg(H)\}$ satisfies the desired relations. □

We recall the polar decompositions for a reductive group G. We can write $G = K\exp(\mathfrak{p}_o)$ which comes from the Lie algebra decomposition of $\mathfrak{g}_o = \mathfrak{k}_o \oplus \mathfrak{p}_o$. $\mathfrak{k}_o$ is the 1-eigenspace and $\mathfrak{p}_o$ the(-1)-eigenspace for a Cartan involution θ defined on $\mathfrak{g}_o$, and K the maximal compact subgroup of G corresponding to the subalgebra $\mathfrak{k}_o$. The following two Lemmas give this decomposition for L.

<u>Lemma 2</u>. The centralizer of a θ-stable subgroup $H \subset G$ is θ-stable.

<u>Proof</u>: It suffices to show that $\mathrm{Ad}(\theta g) = \theta \circ \mathrm{Ad}g \circ \theta$. Let $c_g : G \longrightarrow G$ be defined by $c_g(x) = g\cdot x\cdot g^{-1}$, $x \in G$. We observe that $c_{(\theta g)} = \theta \circ c_g \circ \theta$ because $c_{\theta g}(x) = (\theta g)\cdot x\cdot(\theta g)^{-1} = \theta(g)\cdot x\cdot\theta(g^{-1}) = \theta(g(\theta x)g^{-1}) = \theta(cg(\theta x))$ for $x \in G$. Differentiating this equation gives us what we want. □

<u>Lemma 3</u>. Suppose H is a θ-stable subgroup of G and $|H/H_0| = n$. Then $H = (H \cap K)\exp(\mathfrak{p}_o \cap \mathfrak{h}_o)$ where $\mathfrak{h}_o = \mathrm{Lie}(H)$.

<u>Proof</u>: It is clear that $H_1 = (H \cap K)\exp(\mathfrak{p}_o \cap \ell_0) \subseteq H$ and we want to show $H_1 = H$. Assume for a moment we

know $H_0 = (H \cap K)_0 \exp(\mathfrak{p}_o \cap \mathfrak{h}_o)$. Fix $h \in H$, then we can write $h = k \exp X$, with $h \in k$ and $X \in \mathfrak{p}_o$. We calculate $(\theta h)^{-1} \cdot h = \exp 2X \in H$. Since $|H/H_0| = n$, we get $(\exp 2X)^n = \exp(2nX) \in H_0$. Then $\exp(2nX) \in H_0 \cap \exp \mathfrak{p}_o = \exp(\mathfrak{p}_o \cap \mathfrak{h}_o)$ and, therefore, $2nX \in \mathfrak{p}_o \in \mathfrak{h}_o$, and hence $X \in \mathfrak{p}_o \cap \mathfrak{h}_o$. So $\exp X \in \exp(\mathfrak{p}_o \cap \mathfrak{h}_o)$ and $k \in H \cap K$.

So it remains to prove that $H_0 = (H \cap K)_0 \exp(\mathfrak{p}_o \cap \mathfrak{h}_o)$. $H' = (H \cap K)_0 \exp(\mathfrak{p}_o \cap \mathfrak{h}_o)$ is an open set in H_0 because it contains a neighborhood of the identity, therefore, assuming it's a group, it is both open and closed which implies $H' = H_0$. Now we show H' is a group. The following argument shows that it is sufficient to show for $X,Y \in \mathfrak{h}_o \cap \mathfrak{p}_o$, $\exp X \exp Y \in \exp(\mathfrak{p}_o \cap \mathfrak{h}_o)$. Let $k_1 \cdot \exp X_1, k_2 \cdot \exp X_2 \in H'$, then

$$\begin{aligned}(k_1 \exp X_1)\cdot(k_2 \exp X_2) &= k_1(k_2 k_2^{-1})\cdot \exp X_1 \cdot k_2(\exp X_2) \\ &= k_1 \cdot k_2 \exp X_3 \cdot \exp X_2\end{aligned}$$

which is an element of H' if

$$\exp X_3 \cdot \exp X_2 \in \exp(\mathfrak{p}_o \cap \mathfrak{h}_o) .$$

So we set $\phi(t) = \exp tX \cdot \exp tY = k(t)\exp(Z(t))$,

$k(t) \in K$, $Z(t) \in \mathfrak{p}_o$ and $k(t)$, $Z(t)$ are analytic in t. We want to show that $Z(t) \in \mathfrak{h}_o \cap \mathfrak{p}_o$ for all t, but it is enough to show for small t. It is enough to show that the coefficients in the power series expansion of $Z(t)$ belong to $\mathfrak{h}_o$. We have $(\theta\phi(t))^{-1}\,\phi(t) = \exp(2Z(t))$, hence

$$\begin{aligned} Z(t) &= \frac{1}{2}\log((\theta\phi(t))^{-1}\phi(t)) \\ &= \frac{1}{2}\log(\exp tY\cdot\exp tX\cdot\exp tX\cdot\exp tY)\ . \end{aligned}$$

By Campbell-Baker Hansdorf this last expression can be written as a power series in t whose coefficients are bracket in X and Y. Since X and Y belong to $\mathfrak{h}_o$, these coefficients do as well. □

<u>Proposition 5</u>. We have $L = (L \cap K)\exp(\mathfrak{p}_0 \cap \mathfrak{l}_0)$ where $\mathfrak{l}_0 = \text{Lie}\ L$.

<u>Proof</u>: Follows directly from Lemmas 2 and 3. □

<u>Proposition 6</u>. The group $L^{mp} \cong (L \cap K)^{mp}\ \exp(\mathfrak{p}_0 \cap \mathfrak{l}_0)$.

<u>Proof</u>. Apply the same argument used in Proposition 3. □

Recall that an orbit is admissible if and only if the non-trivial element in the kernel of the map $\pi : (G^E)^{mp} \longrightarrow G^E$ is not in the identity component. Let z be the non-trivial element in the kernel of π. By Proposition 3, $(G^E)^{mp} = L^{mp} \cdot U$ and U is connected. Therefore, $z \in (G^E)^{mp}_0$ if and only if $z \in (L^{mp})_0$. By Proposition 6, $L^{mp} = (L \cap K)^{mp} \cdot \exp(\mathfrak{p}_o \cap \mathfrak{l}_o)$ and $\exp(\mathfrak{p}_o \cap \mathfrak{l}_o)$ is connected. Thus, $z \in (L^{mp})_0$ if and only if $z \in (L \cap K)^{mp}_0$. We conclude that the admissibility of an orbit can be decided by considering the compact subgroup $L \cap K \subset G^E$.

We have seen that G^E maps under the adjoint representation into $Sp(2n)$, where $2n = \dim_{\mathbb{R}}(\mathfrak{g}_o/\mathfrak{g}_o^E)$. The image of the compact subgroup $L \cap K \subset G^E$ must be a compact subgroup of $Sp(2n)$. Since all maximal compacts are conjugate in $Sp(2n)$, we can map $L \cap K$ into $Sp(2n) \cap O(2n)$, a maximal compact subgroup of $Sp(2n)$ isomorphic to $U(n)$. The following Proposition, which we will not prove, enables us to compute the metaplectic cover $U(n)^{mp}$ of $U(n)$ explicitly.

<u>Proposition</u>. The group $U(n)^{mp}$ is isomorphic to

$$\{(k, e^{i\theta}) \in U(n) \times S^1 \mid \det(k) = e^{2i\theta}\} .$$

Define a character χ on $L \cap K$ by composing the following maps:

$$L\cap K \xrightarrow{\tau} Sp(2n)\cap O(2n) \xrightarrow{\cong} U(n) \xrightarrow{\det} S^1 .$$

We construct the pullback cover $(L\cap K)^{mp}$ from the metaplectic cover $U(n)^{mp}$ as described in the Introduction. The next Corollary follows directly from the description of $U(n)^{mp}$ given above.

<u>Corollary</u>. The pullback cover $(L\cap K)^{mp}$ is equal to

$$\{(g,z) \in (L\cap K)\times S^1 \mid \chi(g) = z^2\} .$$

The next Theorem shows that an orbit is admissible if and only if the character χ is a square, that is, there is another character ψ of $L\cap K$ such that $\chi = \psi^2$.

<u>Theorem</u>. Let z be the non-trivial element in the kernel of the covering map $\pi : (L\cap K)^{mp} \longrightarrow L\cap K$. Then $z \notin (L\cap K)_0^{mp}$ <=> the character χ of $L\cap K$ is a square.

<u>Proof</u>: First, assume $\chi = \psi^2$ and we show there cannot be a path from the identity to z in $(L\cap K)_0^{mp}$, that is $z \notin (L\cap K)_0^{mp}$. Let $\{(g(\theta),e^{i\theta}) \in (L\cap K)\times S^1 \mid 0 \leq \theta \leq \pi\}$

be a path from $(e,1)$ to $(e,-1)$ in $(L\cap K)_0^{mp}$. We have $g(0) = e$, $g(\pi) = e$, and $\chi(g(\theta)) = \psi(g(\theta))^2 = e^{2i\theta}$. Therefore $\psi(g(\theta)) = e^{i\theta}$ or $\psi(g(\theta)) = -e^{i\theta}$. But $\psi(g(0)) = \psi(e) = 1$, hence $\psi(g(\theta)) = e^{i\theta}$. Thus $\psi(g(\pi)) = e^{i\pi} = -1$ which is impossible because $g(\pi) = e$.

Conversely, assume $z \notin (L\cap K)_0^{mp}$ and we show χ is a square. Define a character ψ by setting, for $g \in L\cap K$, $\psi(g) = \{w \mid (g,w) \in (L\cap K)_0^{mp}\}$. It is easy to check this is a well defined character of $L\cap K$. For $(g,w) \in (L\cap K)^{mp}$ we have $\chi(g) = w^2$; therefore $\chi(g) = \psi(g)^2$ as desired. □

In the next Chapter we develop the machinery necessary to compute χ .

Chapter 2

In order to construct a map from $L\cap K$ into $U(n)$ we will find an $(L\cap K)$-invariant hermitian form on $\mathfrak{g}_o/\mathfrak{g}_o^E$ considered as a complex vector space with respect to a complex structure J. The vector space $\mathfrak{g}_o/\mathfrak{g}_o^E$ is endowed with the natural G^E-invariant symplectic form ω_E. We seek an $(L\cap K)$-invariant inner product b and a complex structure J such that $\tau = b + i\omega_E$ is a Hermitian form on $\mathfrak{g}_o/\mathfrak{g}_o^E$, that is, the following relation must be satisfied by J and τ:

$$i\tau(x,y) = \tau(Jx,y) \quad \text{all} \quad x,y \in \mathfrak{g}_o/\mathfrak{g}_o^E .$$

Equivalently, the compatibility condition can be expressed in terms of J, b, and ω_E as:

$$b(x,y) = \omega_E(Jx,y) \quad \text{all} \quad x,y \in \mathfrak{g}_o/\mathfrak{g}_o^E .$$

If b and ω_E are $(L\cap K)$-invariant then so is τ:

$$\begin{aligned}\tau(x,y) &= b(x,y) + i\omega_E(x,y)\\ &= b(\ell.x,\ell.y) + i\omega_E(\ell.x,\ell.y)\\ &= \tau(\ell.x,\ell.y) .\end{aligned}$$

Assume we have found an endomorphism J on $\mathfrak{g}_o/\mathfrak{g}_o^E$ satisfying the following four properties:

1) $J^2 = -1$
2) $\omega_E(Jx,y) = -\omega_E(x,Jy)$ all $x,y \in \mathfrak{g}_o/\mathfrak{g}_o^E$
3) $\omega_E(Jx,x) \geq 0$ with equality if and only if $x = 0$
4) J commutes with the action of $L \cap K$.

Then we define a symmetric bilinear form b on $\mathfrak{g}_o/\mathfrak{g}_o^E$ by setting $b(x,y) \overset{\text{def}}{=} \omega_E(Jx,y)$ and we make the following

<u>Claim</u>. $\tau = b + i\omega_E$ is a $(L \cap K)$-invariant hermitian form on $\mathfrak{g}_o/\mathfrak{g}_o^E$ viewed as a complex vector space with complex structure given by J.

<u>Proof</u>: The necessary compatibility among J, ω_E, and b is built into the definitions. b is symmetric by property 2.) as follows: $b(x,y) = \omega_E(Jx,y) = -\omega_E(x,Jy) = \omega_E(Jy,x) = b(y,x)$. b is positive definite because $b(x,x) = \omega_E(Jx,x) > 0$ if $x \neq 0$. Finally, τ is $(L \cap K)$-invariant because J commutes with $L \cap K$. For $\ell \in L \cap K$ we have:

$$\begin{aligned}\tau(x,y) &= \omega_E(Jx,y) + i\omega_E(x,y)\\ &= \omega_E(\ell\cdot Jx,\ell y) + i\omega_E(\ell.x,\ell.y)\\ &= \omega_E(J(\ell.x),\ell.y) + i\omega_E(\ell.x,\ell.y)\\ &= \tau(\ell.x,\ell.y)\ . \qquad \square\end{aligned}$$

We now exhibit a complex structure on $\mathfrak{g}_o/\mathfrak{g}_o^E$ which satisfies properties (1)-(4). Decompose $\mathfrak{g}_o$ into isotypic subspaces under $\mathfrak{sl}(2)$; $\mathfrak{g}_o = \bigoplus_{i=1}^{N} W^i$ where $W^i \cong m_i V^i$, V^i an irreducible $\mathfrak{sl}(2)$-module, $\dim V^i = i$. Let $[W^i]_\ell$ be the H-eigenspace with eigenvalue ℓ . Define J on $[W^i]_\ell$ for $-i-1 \leq \ell \leq i-3$ as follows;

<u>Definition</u>. For $x \in [W^i]_\ell$, define $J(x) = (j_{\ell,i})^{-1/2}\cdot\theta(\mathrm{ad}E(x))$ where $j_{\ell,i}$ is a constant depending on i and ℓ , and defined as follows. Since the weight spaces of V^i are one dimensional, there are constants $j_{\ell,i}$ such that for $x \in [V^i]_\ell$ $[F,[E,x]] = j_{\ell,i}\cdot x$.

We compute $j_{\ell,i}$ by looking at the action of E and F in the $\mathfrak{sl}(2)$ representation on homogeneous polynomials of degree i-1. In this representation, the operator $\pi(\begin{pmatrix}0&1\\0&0\end{pmatrix})$ corresponds to the differential operator $x\frac{\partial}{\partial y}$, $\pi(\begin{pmatrix}0&0\\1&0\end{pmatrix})$ to $y\frac{\partial}{\partial x}$, and $\pi(\begin{pmatrix}1&0\\0&-1\end{pmatrix})$ to

$x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. So we have:

$$j_{\ell,i}(x^{\frac{i-1+\ell}{2}} y^{\frac{i-1-\ell}{2}}) \overset{\text{defn.}}{=} F.E.(x^{\frac{i-1+\ell}{2}} y^{\frac{i-1-\ell}{2}})$$
$$= \left[\frac{i-1-\ell}{2}\right]\cdot(F.\ x^{\frac{i+1+\ell}{2}} y^{\frac{i-3-\ell}{2}})$$
$$= \left[\frac{i-1-\ell}{2}\right]\cdot\left[\frac{i+1+\ell}{2}\right] x^{\frac{i-1+\ell}{2}} y^{\frac{i-1-\ell}{2}} .$$

We conclude that $j_{\ell,i} = \left[\frac{i-1-\ell}{2}\right]\cdot\left[\frac{i+1+\ell}{2}\right]$ and note that $j_{\ell,i} = j_{-\ell-2,i}$.

Proposition. J , defined above, satisfies properties (1)-(4).

Proof: 1.) We check $J^2 = -1$. It is enough to check this on a basis, and we choose a special basis made up of eigenvectors spanning each isotypic component $W^i \subset \mathfrak{g}_o$. Pick $x \in [W^i]_\ell$;

$$J^2x = J((j_{\ell,i}^{-1/2})\cdot\theta(\mathrm{ad}E(x)) = (j_{\ell,i}^{-1/2})\cdot J(\theta[E,x])$$
$$= (j_{\ell,i}^{-1/2})\cdot J(\theta[E,x])$$
$$= (j_{\ell,i}^{-1/2})\cdot(j_{-\ell-2,i}^{-1/2})\theta[E,[\theta E,\theta x]] \text{ since } \theta[E,x]\in[W^i]_{-\ell-2}$$
$$= (-j_{\ell,i}^{-1})\cdot[F.[E,x]] \text{ since } j_{\ell,i} = j_{-\ell-2,i} \text{ and } \theta E = -F$$
$$= -x .$$

2.) Check $\omega_E(Jx,y) = -\omega_E(x,J_y)$: We check this equality on a basis of eigenvectors chosen as in 1.). We note that $\omega_E([\mathfrak{g}_o]_i,[\mathfrak{g}_o]_j] = B([E,[\mathfrak{g}_o]_i],[\mathfrak{g}_o]_j)$ which equals 0 unless $j = -i-2$. Since $J[[\mathfrak{g}_o]_i] = [\mathfrak{g}_o]_{-i-2}$ we conclude that for x,y in different eigenspaces $\omega_E(Jx,y) = -\omega_E(x,Jy) = 0$. Now take $x,y \in [W^i]_\ell$. Then $\omega_E(Jx,y) = B([E,Jx],y) = -B(Jx,[E,y]) =$ $-(j_{\ell,i}^{-1/2})\cdot B(\theta[E,x],[E,y]) = -(j_{\ell,i}^{-1/2})\cdot B([E,x],\theta[E,y]) =$ $-\omega_E(x,Jy)$.

3.) Check $\omega_E(Jx,x) > 0$ if $x \neq 0$: $\omega_E(Jx,x) =$ $-\omega_E(x,Jx) = -B([E,x],(j_{\ell,i}^{-1/2})\cdot\theta[E,x]) =$ $(j_{\ell,i}^{-1/2})\cdot B_\theta([E,x],[E,x])$ which is always positive and non-zero on $\mathfrak{g}_o/\mathfrak{g}_o^E$.

4.) Finally we must see that J commutes with $L\cap K \subset G^E$. Certainly $\mathrm{ad}E$ commutes with $L\cap K$, so we want to see why θ commutes with $L\cap K$. Since K is fixed by θ and $\theta\circ\mathrm{Ad}(g) = \mathrm{Ad}(\theta g)\circ\theta$ for $g \in G$ we conclude for $k \in L\cap K$, $\theta\circ\mathrm{Ad}(k) = \mathrm{Ad}(k)\circ\theta$ as desired. □

Now we decompose $\mathfrak{g}_o = \bigoplus_{i=n}^{n} [\mathfrak{g}_o]_i$ into a direct sum of eigenspaces under the H action of the $\mathfrak{sl}(2)$ representation. Because $L = Z_G(\mathfrak{sl}(2))$, L preserves the eigenspaces $[\mathfrak{g}_o]_i$ for all i. Thus for $\ell \in L$ we have:

$$\det_{\mathbb{R}}(\mathrm{Ad}\ell) = \prod_{i=1}^{n} \det\left[\mathrm{Ad}\ell\,|_{[\mathfrak{g}_o]_i}\right].$$

For $g \in G^E$ we let $\overline{\mathrm{Ad}g}$ denote the action induced by $\mathrm{Ad}g$ on the quotient space $\mathfrak{g}_o/\mathfrak{g}_o^E$. We will consider $\mathrm{Ad}(L\cap K)$ as a group of complex linear transformations of the complex space $\mathfrak{g}_o/\mathfrak{g}_o^E$. This is possible because J commutes with $\mathrm{Ad}(L\cap K)$. The complex structure J maps $[\mathfrak{g}_o]_i$ onto $[\mathfrak{g}_o]_{-i-2}$ for all i, hence

$$\det_{\mathbb{C}}(\overline{\mathrm{Ad}\ell}) = \det_{\mathbb{C}}(\overline{\mathrm{Ad}\ell}\,|_{[\mathfrak{g}_o]_{-1}}) \cdot \prod_{i>-1} \det_{\mathbb{C}}\left[\overline{\mathrm{Ad}\ell}\,|_{[\mathfrak{g}_o]_i \oplus J([\mathfrak{g}_o]_i)}\right].$$

<u>Proposition</u>. Let $V = [\mathfrak{g}_o]_i \oplus [\mathfrak{g}_o]_{-i-2}$, $h \in L\cap K$, $i \neq -1$. Then $\det_{\mathbb{C}}(h|_V) = \pm 1$ ($+1$ if $h \in (L\cap K)_0$).

<u>Proof</u>: $V = [\mathfrak{g}_o]_i \oplus [\mathfrak{g}_o]_{-i-2} = [\mathfrak{g}_o]_i \oplus J([\mathfrak{g}_o]_i) = ([\mathfrak{g}_o]_i)_{\mathbb{C}}$. Since h preserves $[\mathfrak{g}_o]_i$ and commutes with J, we have $\det_{\mathbb{C}}(h|_V) = \det_{\mathbb{R}}\left[h|_{[\mathfrak{g}_o]_i}\right] \in \mathbb{R}$. But $L\cap K$ is compact, therefore its image under the determinant map must be ± 1 (and $+1$ on the identity component). □

By the discussion leading up to the <u>Proposition</u> and the <u>Proposition</u> we can conclude:

$$\det_{\mathbb{C}}(\overline{\mathrm{Adh}}) = \pm \det_{\mathbb{C}}\left[\overline{\mathrm{Adh}}\big|_{[\mathfrak{g}_o]_{-1}}\right] .$$

Our goal is to calculate this determinant. We begin this task by finding a parametrization for the nilpotent in $\mathfrak{g}_o$ (under certain groups). The next chapter addresses this problem.

Chapter 3

Classification of Nilpotents

Let G be a real classical Lie group with real Lie algebra $\mathfrak{g}_o$. We describe a correspondence between Lie algebra maps from $\mathfrak{sl}(2)$ into $\mathfrak{g}_o$ and nilpotent orbits for which we need the following

Definition. A Lie algebra map $\phi : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{g}_o$ is equivalent to another such map $\phi' : \mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{g}_o$ if there exists $g \in G$ such that $\phi(X) = g\phi'(X)g^{-1}$ $\forall x \in \mathfrak{sl}(2,\mathbb{R})$.

Proposition 1 (Kostant). There is a bijection between equivalence classes of maps from $\mathfrak{sl}(2)$ into $\mathfrak{g}_o$ and nilpotent orbits under G.

Proof: By the Jacobson-Morozov Theorem we can embed any nilpotent $E \in \mathfrak{g}_o$ in a subalgebra of $\mathfrak{g}_o$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Thus there is at least one equivalence class of maps corresponding to each orbit. To see there is only one, we use the fact that any two copies of $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{g}_o$ with the same nilpositive element are conjugate by an element of the subgroup $U \subset G^E \subset G$

(Kostant, 1959), where U is the closed, connected subgroup of G^E corresponding to the Lie subalgebra $\mathfrak{u}_0 = [E,\mathfrak{g}_o] \cap \mathfrak{g}_o^E$. this establishes the bijection. □

Thus, we conclude that the classification of nilpotent orbits under (for the moment) an arbitrary real group G is equivalent to the classification of equivalence classes of maps from $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{g}_o$. Now we assume G is a real linear Lie group which preserves a non-degenerate sesquilinear form ω_0 on a vector space V_0 defined over a field $\mathbb{F}$. We denote $G = G(\omega_0)$. Then $\mathfrak{g}_o$ is a real matrix algebra contained in $M_n(\mathbb{F})$ (for some n) preserving ω_0. Because Lie algebra maps from $\mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{g}_o$ lift to Lie group maps from $SL(2,\mathbb{R}) \longrightarrow G$, the classification of nilpotent orbits under G is equivalent to the classification of maps from $SL(2,\mathbb{R}) \longrightarrow G(\omega_0)$ up to conjugacy by an element of $G(\omega_0)$.

We introduce a notion of equivalence among pairs (ω,V) where ω is a non-degenerate sequilinear form on a vector space $V/\mathbb{F}$.

<u>Definition</u>. We define $(\omega_1,V_1)\sim(\omega_2,V_2)$ if there exists an $\mathbb{F}$-linear isomorphism $T : V_1 \longrightarrow V_2$ such that $\omega_1(v,w) = \omega_2(Tv,Tw)$ for all $v,w \in V_1$.

This isomorphism is unique up to left multiplication by an element of $G(\omega_2)$ or right multiplication by an element of $G(\omega_1)$. Let $(\omega_1,V_1)\sim(\omega_2,V_2)$ and $T : V_1 \longrightarrow V_2$ implement the equivalence. Then the isomorphism T induces an isomorphism of groups $\phi_T : G(\omega_2) \longrightarrow G(\omega_1)$ defined as follows: for $g \in G(\omega_2)$ let $\phi_T(g) = T^{-1}\circ g\circ T \in G(\omega_1)$.

Fix an equivalence class $\overline{(\omega_0,V_0)}$, and let (ω_0,V_0) be a representative of $\overline{(\omega_0,V_0)}$. We consider the set of all maps $SL(2,\mathbb{R}) \longrightarrow G(\omega_i)$ such that $(\omega_i,V_i) \in \overline{(\omega_0,V_0)}$ and define an equivalence relation on this set as follows.

<u>Definition</u>. Let $\tau : SL(2,\mathbb{R}) \longrightarrow G(\omega_1)$ and $\phi : SL(2,\mathbb{R}) \longrightarrow G(\omega_2)$. We say $\tau \sim \phi$ if and only if there is an isomorphism $T : V_1 \longrightarrow V_2$ preserving the forms ω_1 and ω_2 and $T^{-1}\circ\phi(x)\circ T = \tau(x)$ for all $x \in SL(2)$.

We observe that the set of equivalence classes of mappings of $SL(2) \longrightarrow G(\omega_i)$, $i \in I$, is in bijection with the set of equivalence classes of mappings from $SL(2) \longrightarrow G(\omega_0)$ for the fixed group $G(\omega_0)$ where the equivalence is defined by conjugacy within $G(\omega_0)$.

Now consider the set of all triples $\{(\pi,\omega,V) \mid \pi$ is a representation of $SL(2)$ on V leaving invariant the form ω and $(\omega,V) \in \overline{(\omega_0,V_0)}\}$. Define an equivalence relation on this set as follows:

<u>Definition</u>. $(\pi_1,\omega_1,V_1)\sim(\pi_2,\omega_2,V_2)$ if there is an isomorphism $T : V_1 \longrightarrow V_2$ which intertwines the $SL(2)$ action and preserves the forms ω_1 and ω_2 in the sense already explained.

Clearly, the following two sets are just descriptions of the same thing: $\{(\pi,\omega,V) \mid \pi$ is a representation of $SL(2)$ on V preserving ω and $(\omega,V) \in \overline{(\omega_0,V_0)}\}$ and $\{\phi : SL(2) \longrightarrow G(\omega_i) \mid (\omega_i,V_i) \in \overline{(\omega_0,V_0)}, i \in I\}$. As we've noted, the set of equivalence classes of the latter set is the same as the set of equivalence classes (under conjugation in $G(\omega_0)$), $\{\phi : SL(2) \longrightarrow G(\omega_0)\}$. Therefore, the classification of nilpotent orbits under G reduces to understanding the partition of the set of triples $\{(\pi,\omega,V) \mid \pi$ is a representation of $SL(2)$ on V preserving ω and $(\omega,V) \in \overline{(\omega_0,V_0)}\}$ into equivalence classes.

First we choose a standard model for an irreducible $\mathfrak{sl}(2)$ representation over $\mathbb{R}$, (π^i,V^i) , where V^i is the vector space of homogeneous polynomials of degree

$i-1$ in two variables (irreducible $\mathfrak{sl}(2)$-representations $/\mathbb{R}$ are determined by their dimension). The action of $SL(2)$ on a monomial X^iY^j is given by substitution of variables; that is, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.X^iY^j = (aX+cY)^i \cdot (bX+dY)^j$. On the Lie algebra the elements $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of $\mathfrak{sl}(2)$ correspond to the operators $x\cdot\frac{\partial}{\partial x} - y\cdot\frac{\partial}{\partial y}$, $x\cdot\frac{\partial}{\partial y}$, and $y\cdot\frac{\partial}{\partial x}$, respectively. Also, we fix an $\mathfrak{sl}(2)$-invariant bilinear form on V^i by setting $\omega^i(X^{i-1},Y^{i-1}) = 1$.

We fix $\overline{(\omega_0,V_0)}$ and we will determine all triples (up to equivalence) (π_i,ω_i,V_i) where π_i is a representation of $SL(2)$ on V_i, ω an $SL(2)$-invariant form and $(\omega_i,V_i) \in \overline{(\omega_0,V_0)}$.

Now we use the semisimplicity of $\mathfrak{sl}(2)$ to break up a representation (π_0,V_0) into its irreducible components.

<u>Lemma</u>. $V_0 \cong \bigoplus_i V^i \otimes_{\mathbb{R}} H^i$ as $\mathfrak{sl}(2)$-modules (V^i is as above) and $H^i = \mathrm{Hom}_{\mathfrak{sl}(2)}(V^i,V)$, where H^i is a vector space over $\mathbb{F}$.

<u>Proof</u>: Define a map $\tau : \bigoplus_i V^i \otimes H^i \longrightarrow V$ by

$$\tau(\bigoplus_i (v_i \otimes T^i)) \overset{\text{defn.}}{=} \bigoplus_i T^i(v_i) \quad \text{where } v_i \otimes T^i \in V^i \otimes H^i .$$

It is clear that τ is 1-1 and since $\dim V =$

$\dim(\bigoplus_i V^i \otimes H^i)$, τ is an isomorphism. We consider $\bigoplus_i V^i \otimes H^i$ as an $\mathfrak{sl}(2)$-module by making $\mathfrak{sl}(2)$ act on the first factor alone, and then it is clear that τ is an intertwining map. □

We want to find all $\mathfrak{sl}(2)$-invariant forms on $V_0 \cong \bigoplus_i V^i \otimes H^i$. We fix our canonical $\mathfrak{sl}(2)$-invariant form ω^i on each of the V^i and put an arbitrary nondegenerate sesquilinear form h^i on H^i .

<u>Definition</u>. Let $\bigoplus_i \omega^i \otimes h^i$ be a nondegenerate sesquilinear form on $\bigoplus_i V^i \otimes H^i$ given by

$$\bigoplus_i \omega_i \otimes h_i (\bigoplus_i (v_i \otimes T^i), \bigoplus_i \omega_i \otimes S^i) = \bigoplus_i \omega^i (v_i, w_i) \cdot h^i (T^i, S^i)$$

where $v_i \otimes T^i$, $w_i \otimes S^i \in V^i \otimes H^i$.

The form $\bigoplus_i \omega_i \otimes h_i$ on $\bigoplus V^i \otimes H^i$ gives a form ω_0 on V_0 as follows:

$$\omega(\tau(\bigoplus_i v_i \otimes T^i) , \tau(\bigoplus_i w_i \otimes S^i))$$
$$= \bigoplus_i \omega^i \otimes h^i (\bigoplus_i v_i \otimes T^i , \bigoplus_i w_i \otimes S^i) .$$

(π_0, ω_0, V_0) and $(\pi, \bigoplus_i \omega^i \otimes h^i , \bigoplus_i V^i \otimes H^i)$ are equivalent triples in the sense defined previously. A choice of nondegenerate sequilinear form h^i on H^i, for each i, fixes an $\mathfrak{sl}(2)$-invariant form on $\bigoplus_i V^i \otimes H^i$. The next Proposition shows this gives all $\mathfrak{sl}(2)$-invariant forms on $\bigoplus_i V^i \otimes H^i$.

<u>Proposition</u>. Fix the canonical $\mathfrak{sl}(2)$-invariant form ω^i on V^i. Then the SL(2)-invariant forms on $\bigoplus_i V^i \otimes H^i$ are all of the form $\bigoplus_i \omega^i \otimes h^i$ where h^i is an arbitrary sequilinear form on H^i.

<u>Proof</u>: SL(2) invariant forms on $\bigoplus_i V^i \otimes H^i$ are in 1-1 correspondence with intertwining operators, $\mathrm{Hom}_{\mathfrak{sl}(2)}(\bigoplus_i V^i \otimes H^i, (\bigoplus_j V^j \otimes H^j)^*)$. We have

$$\begin{aligned}
&\mathrm{Hom}_{\mathfrak{sl}(2)}(\bigoplus_i (V^i \otimes H^i), \bigoplus_j (V^j \otimes H^j)^*) \\
&\cong \mathrm{Hom}_{\mathfrak{sl}(2)}(\bigoplus_i (V^i \otimes H^i), \bigoplus_j ((V^j)^* \otimes (H^j)^*)) \\
&\cong \bigoplus_{i,j} \mathrm{Hom}_{\mathfrak{sl}(2)}(V^i \otimes H^i, (V^j)^* \otimes (H^j)^*) \\
&\cong \bigoplus_{i,j} \mathrm{Hom}_{\mathfrak{sl}(2)}(V^i, (V^j)^*) \otimes \mathrm{Hom}_{\mathbb{F}}(H^i, (H^j)^*) \\
&\cong \bigoplus_i \mathrm{Hom}_{\mathfrak{sl}(2)}(V^i, (V^i)^*) \otimes \mathrm{Hom}_{\mathbb{F}}(H^i, (H^i)^*) .
\end{aligned}$$

But we have fixed an $\mathfrak{sl}(2)$ form ω^i on V^i which identifies $\mathrm{Hom}_{\mathfrak{sl}(2)}(V^i,(V^i)^*)$ with $\mathbb{F}$, and we conclude

$$\cong \bigoplus_i \mathrm{Hom}(H^i,(H^i)^*) \ . \quad \square$$

Chapter 4

We have seen that nilpotent orbits of G -- the group which preserves a nondegenerate sesquilinear form on a vector space V -- on its Lie algebra are in one to one correspondence with equivalence classes of representations of $\mathfrak{sl}(2)$ on a vector space with an $\mathfrak{sl}(2)$-invariant sesquilinear form. We fix a representative $(\pi, \bigoplus_i (V^i \otimes H^i), \oplus\, \omega^i \otimes h^i)$ (same notation as in previous section), where we choose h^i to be Hermitian or skew-Hermitian so that the form $\bigoplus_i \omega^i \otimes h^i$ on the vector space $\bigoplus_i V^i \otimes H^i$ is Hermitian or symplectic. This gives us the nilpotent orbits in $\mathfrak{sp}(2n,\mathbb{R})$ and $\mathfrak{o}(p,q)$ or $\mathfrak{u}(p,q)$ if V is real or complex, respectively.

Definition. We say that a basis $\{T^i_j\}_{1 \leq j \leq \dim H^i}$ is a standard basis for H^i with respect to the sesquilinear form h^i on H^i if 1.) for h^i skew-Hermitian and H^i complex, 2.) for h^i symplectic and H^i real, or 3.) for h^i Hermitian, the matrix $(H)_{jk} = (h^i(T^i_j, T^i_k))$ equals

1.) $\begin{bmatrix} iI_p & \\ & -iI_q \end{bmatrix}$, 2.) $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, or 3.) $\begin{bmatrix} I_p & 0 \\ & -I_q \end{bmatrix}$.

In order to find a representative of the orbit $\mathcal{O}_E$ corresponding to the equivalence class $(\pi, \bigoplus_i V^i \otimes H^i, \bigoplus_i \omega^i \otimes h^i)$ we choose a standard basis on $\bigoplus_i V^i \otimes H^i$ and we realize the linear transformation $\pi(E)$ as a matrix $(E)_{ij}$ with respect to this basis. $(E)_{ij} \in \mathfrak{g}_o$ is a representative of $\mathcal{O}_E$. Let $g = (g)_{ij} \in G$ and $g^{-1} = (g^{-1})_{ij} \in G$. Then $\mathrm{Ad}g((E)_{ij})$ is realized by writing $\pi(E)$ with respect to the basis $\{e_i' = \sum_{j=1}^{n} g_{ji} e_j\}_{1 \leq i \leq n}$.

We now show how to construct a Cartan involution θ, and a θ-stable copy of $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{g}_o$ whose nilpositive element is a representative of $\mathcal{O}_E$.

We define an inner product $Q = \bigoplus_i b_1^i \otimes b_2^i$ on $\bigoplus_i V^i \otimes H^i$ which satisfies the following two properties.

1.) There is a standard basis of $\bigoplus_i V^i \otimes H^i$ with respect to the G-invariant form $\bigoplus_i \omega^i \otimes h^i$ which is orthonormal with respect to $Q = \bigoplus_i b_1^i \otimes b_2^i$.

2.) $Q(\pi(E)v, w) = Q(v, \pi(F)w)$ for all $v, w \in V$.

Assume these two conditions are satisfied. Let $(E)_{ij}$ be the matrix corresponding to $\pi(E)$ in this basis; $(E)_{ij} \in \mathfrak{g}_o$. The adjoint with respect to Q of a linear map L is given by conjugate transpose when L is written in an orthonormal basis; hence $(E)^*_{ij} = \overline{(E)_{ji}}$.

But condition 2.) gives us the following equation, $\pi(E)^* = \pi(F)$. Hence we conclude that $\overline{(E)_{ji}} = (F)_{ij}$ or, in other words, $\theta((E)_{ij}) = -(F)_{ij}$ and the $\mathfrak{sl}(2)$ spanned by $\{(E)_{ij},(F)_{ij},(H)_{ij}\} \subset \mathfrak{g}_o$ is θ-stable.

We now define Q . Choose the following basis for $V^i \otimes H^i$: $\{\binom{i-1}{s}^{1/2} X^{i-1-s}Y^s \otimes T^i_k\}_{0\leq s\leq i-1, 1\leq k\leq \dim H^i}$. Define an inner product b^i_1 on V^i by setting

$$b_1(\sqrt{\tbinom{i-1}{s}}\, X^{i-1-s}Y^s, \sqrt{\tbinom{i-1}{t}}\, X^{i-1-t}Y^t) = \delta_{st}$$

and an inner product b^i_2 on H^i by:

$$b^i_2(T^i_m, T^i_n) = \delta_{mn} .$$

We define an inner product $b^i_1 \otimes b^i_2$ on $V^i \otimes H^i$ as follows: for $v\otimes T, w\otimes S \in V^i \otimes H^i$, $b^i_1\otimes b^i_2(v\otimes T, w\otimes S) = b^i_1(v,w)b^i_2(T,S)$.

<u>Claim</u> I.) Let $\omega^i \otimes h^i$ be a symplectic form on $V^i \otimes H^i$. Then the basis $\{\binom{i-1}{s}^{1/2} X^{i-1-s}Y^s \otimes T^i_k\}_{0\leq s\leq i-1\ 1\leq k\leq \dim H^i}$ up to the sign and order of the elements is a Darboux basis with respect to $\omega^i \otimes h^i$ and orthonormal with respect to $b^i_1 \otimes b^i_2$ on $V^i \otimes H^i$.

II.) Let $\omega_i \otimes h_i$ be an Hermitian form on $V^i \otimes H^i$.

a) If i is odd then (up to order)

$$\{(\sqrt{\binom{i-1}{s}/2})(X^{i-1-s}Y^{s}\otimes T^{i}_{k} \pm X^{s}Y^{i-1-s}\otimes T^{i}_{k})\}$$

$0\leq s\leq i-1$, $1\leq k\leq \dim H^{i}$ is a standard basis with regard to $\omega^{i}\otimes h^{i}$ and orthonormal with respect to $b^{i}_{1}\otimes b^{i}_{2}$.

b) If i is even and H^{i} real, then (up to order)

$$\{(\sqrt{\binom{i-1}{s}/2})(X^{i-1-s}Y^{s}\otimes T^{i}_{k} \pm X^{s}Y^{i-1-s}\otimes T^{i}_{k+m_i})\}$$

$0\leq s\leq i-1$, $1\leq k\leq m_{i}$, where $\dim H^{i} = 2m_{i}$, is a standard basis with respect to $\omega^{i}\otimes h^{i}$ and orthonormal with respect to $b^{i}_{1}\otimes b^{i}_{2}$.

c) If i is even and H^{i} complex then (up to order)

$$\{\sqrt{\binom{i-1}{s}/2}\ (X^{i-1-s}Y^{s}\otimes i\cdot T^{i}_{k} \pm X^{s}Y^{i-1-s}\otimes T^{i}_{k})\}$$

$0\leq s\leq i-1$, $1\leq k\leq m_{i}$, where $\dim H^{i} = m_{i}$, is a standard basis with respect to $\omega^{i}\otimes h^{i}$ and orthonormal with respect to $b^{i}_{1}\otimes b^{i}_{2}$.

<u>Proof</u>: Obvious. □

Hence we have shown that Q satisfies condition (1), and we now show that it satisfies property (2). It suffices to check this on a basis for $V^{i}\otimes H^{i}$. For

$k \neq \ell$ and $m \neq n$ both
$Q(E.\binom{i-1}{k+1}^{1/2}(X^{i-1(k+1)}Y^{k+1}\otimes T^i_m),\binom{i-1}{\ell}^{1/2}(X^{i-1-\ell}Y^{\ell}\otimes T^i_n))$ and
$Q(\binom{i-1}{k+1}^{1/2}(X^{i-1-(k+1)}Y^{k+1}\otimes T^i_m),F.\binom{i-1}{\ell}^{1/2}(X^{i-1-\ell}Y^{\ell}\otimes T^i_n))$
equal zero. Thus, it suffices to check:

$$Q(E.\binom{i-1}{k+1}^{1/2}(X^{i-1(k+1)}Y^{k+1}\otimes T^i_m),\binom{i-1}{k}^{1/2}(X^{i-1-k}Y^{k}\otimes T^i_m)) =$$
$$Q(\binom{i-1}{k+1}^{1/2}(X^{i-1(k+1)}Y^{k+1}\otimes T^i_m),F.\binom{i-1}{k}^{1/2}(X^{i-1-k}Y^{k}\otimes T^i_m)) .$$

The left hand side equals $\dfrac{k+1\ \binom{i-1}{k+1}^{1/2}}{\binom{i-1}{k}^{1/2}}$ and the right hand side is $\dfrac{(i-1-k)\binom{i-1}{k}^{1/2}}{\binom{i-1}{k+1}^{1/2}}$. It is easy to check these are both equal to $(k+1)^{1/2}(i-1-k)^{1/2}$. □

We now describe an explicit realization of L as matrices. Let $V = \bigoplus_i V^i\otimes H^i$ be the decomposition of V under a θ-stable copy of $\mathfrak{sl}(2)$ which has E as its nilpositive element. Any $g \in L = Z_G(\mathfrak{sl}(2))$ gives a linear isomorphism $V \longrightarrow V$. Furthermore, since the linear transformation g commutes with $\mathfrak{sl}(2)$, g intertwines the action of $\mathfrak{sl}(2)$ on V . By Schur's lemma this implies that g preserves isotypic components of V . Hence, L can be characterized abstractly as the group of intertwining operators $T : V \longrightarrow V$ where

T preserves the bilinear form on V . We realize these intertwining operators as matrices with respect to the basis chosen on V to determine the θ-stable copy of $\mathfrak{sl}(2)$ in $\mathfrak{g}_o$. This gives L as a θ-stable matrix group.

<u>Proposition</u>.

a.) $L \cong Sp(2m_1)\times O(p_2,q_2)\times Sp(2m_3)\times\cdots$ if G preserves a symplectic form on a real vector space V , where $2m_i = \dim H^i$ and $p_j+q_j = \dim H^j$.

b.) $L \cong O(p_1,q_1)\times Sp(2m_2)\times O(p_3,q_3)\times\cdots$ if G preserves a Hermitian form on a real vector space V , where $2m_i = \dim H^i$ and $p_j+q_j = \dim H^j$.

c) $L \cong U(p_1,q_1)\times U(p_2,q_2)\times\cdots$ if G preserves a Hermitian form on a complex vector space V , where $p_j+q_j = \dim H^j$.

<u>Proof</u>: We know $V \cong \bigoplus_i V^i\otimes H^i$ with G-invariant bilinear form $\bigoplus_i \omega^i\otimes h^i$ on $\bigoplus_i V^i\otimes H^i$. As we have observed $g \in L$ preserves isotypic components, therefore we can write $g = \bigoplus_i g^i$ where $g^i : V^i\otimes H^i \longrightarrow V^i\otimes H^i$. Recall that H^i can be thought of as $\mathrm{Hom}_{\mathfrak{sl}(2)}(V^i,V)$; g defines a linear transformation g^i of H^i by acting on the range of each map. But $g \in L$ preserves the sesquilinear form

on V and therefore g^i must preserve the sesquilinear form on H^i. On the other hand, an element $(g^1, \cdots, g^N) \in Sp(2m_1)\times\cdots$ specifies a linear transformation $g = \Sigma\, (Id)\otimes g^i$ of $\oplus_i V^i\otimes H^i$ [as follows: let $(g^1, \cdots, g^N)(\oplus_i v_i\otimes T^i) = \oplus_i(v_i\otimes g^i(T^i))$ with $v_i\otimes T^i \in V^i\otimes H^i$ where $g^i = (g^i)_{k\ell}$ and $g^i(T^i_j) = \sum_{k=1}^{\dim H^i} g^i_{kj}T^i_k$]. Clearly, g intertwines the $SL(2)$ action on V, and g preserves the form on H^i, therefore g preserves the invariant form on V. These two observations show that $g \in Z_G(\mathfrak{sl}(2))$. □

We've seen that the question of admissibility comes down to the subgroup $L\cap K$. We now prove the following

<u>Lemma</u>.

a.) $L\cap K \cong (Sp(2m_1)\cap O(2m_1))\times(O(p_2,q_2)\cap O(p_2+q_2)\times\cdots$ (V symplectic)

b.) $L\cap K \cong (O(p_1,q_1)\cap O(p_1+q_1))\times(Sp(2m_2)\cap O(2m_2))\times\cdots$ (V real hermitian)

c.) $L\cap K \cong$
$(U(p_1,q_1)\cap U(p_1+q_1))\times(U(p_2,q_2)\cap U(p_2+q_2))\times\cdots$
(V complex Hermitian)

Proof: $G\cap K$ is the linear group which preserves the sesquilinear form $\omega = \bigoplus_i \omega^i \otimes h^i$ on V as well as the inner product $Q = \oplus\, b_1^i \otimes b_2^i$. The proofs of all three statements are the same, therefore we only do a.). So we must show that $(g_1,\cdots,g_N) \in (Sp(2m_1)\times O(2m_1))\times\cdots$ determines an element of L which preserves the inner product Q . But this is clear because $\{T_j^i\}_{1\leq j\leq \dim H^i}$ is an orthonormal basis of H^i for b_2^i .

On the other hand an element $\ell \in L\cap K$ determines an element $(g_1,\cdots,g_N) \in Sp(2m_1)\times O(p_1,q_1)\times\cdots$ which must preserve the form b_2^i on H^i for each i . Hence $(g_1,\cdots,g_N)\in(Sp(2m_1)\cap O(2m_1))\times(O(p_2,q_2)\cap O(p_2+q_2))\times\cdots$. □

Chapter 5

We now consider the case where $\mathfrak{g}_o$ is a matrix algebra $\subset M_n(\mathbb{F})$ which preserves the standard symplectic or Hermitian (with signature (p,q)) form on $\mathbb{F}^n$. We will show $\mathfrak{g}_o$ is isomorphic in these two cases to the space of symmetric or antisymmetric homogeneous polynomials in V of degree two. We begin with the following:

Definition. Let V be a vector space over $\mathbb{F}$. Then we define $\overline{V}$ to be equal to V as sets, but when $\mathbb{F} = \mathbb{C}$ we let $\mathbb{C}$ act on $\overline{V}$ as follows: for $v \in \overline{V}$ and $c \in \mathbb{C}$, set $c.v = \overline{c} \cdot v$, where $\overline{c} \cdot v$ is the multiplication in V.

Lemma. $\mathrm{Hom}_{\mathbb{F}}(V,V) \cong V \otimes_{\mathbb{F}} V^* \cong V \otimes_{\mathbb{F}} \overline{V}$ as G modules and $\mathfrak{g}_o$ modules where V is a vector space over $\mathbb{F}$ with a nondegenerate G-invariant sesquilinear form ω, and $\overline{V}$ inherits the G-module structure and invariant form ω from V.

Proof: First, observe that the G-invariant form ω on V gives a G-equivariant, $\mathbb{F}$-linear, mapping $\overline{V} \longrightarrow V^*$

defined as follows; for $v, w \in V$, $v \longrightarrow v^*$ where $v^*(w) \stackrel{\text{def.}}{=} \overline{\omega(v,w)}$.

Now define a map $\phi : V\otimes\overline{V} \longrightarrow V\otimes V^*$ by setting, for $v, w \in V$, $\phi(v\otimes w) = v\otimes w^*$. The adjoint actions on V give rise to the contragredient actions on V^* . If we take the representation on the tensor products arising from these actions on V and V^* it is easy to check that ϕ is a G (or g_0) equivariant isomorphism.

Finally, define a map $\tau : V\otimes V^* \longrightarrow \mathrm{Hom}(V,V)$ as follows for $v, w \in V$, $f \in V^*$ let $[\tau(v\otimes f)](w) = f(w)\cdot v$. It is easy to check that this map defines an isomorphism which intertwines the adjoint action of the group G as well as the adjoint action of the whole algebra End V . □

We now want to consider the symmetric or antisymmetric elements in $V\otimes\overline{V}$, by which we mean the real span of elements of the form $\langle v\otimes w+w\otimes v\rangle$ or $\langle v\otimes w-w\otimes v\rangle$, respectively, in $V\otimes\overline{V}$. This makes sense because V and $\overline{V}$ are identical as sets. We note that such elements are not closed under scalar multiplication by elements of the field $\mathbb{F} = \mathbb{C}$. We introduce the following notation.

<u>Definition</u>. Let $\Lambda^2(V)$ and $S^2(V)$ be the set of antisymmetric and symmetric elements in $V \otimes \overline{V}$.

<u>Proposition 1</u>. Let $g_o \subset \mathrm{Hom}(V,V)$ be a Lie algebra preserving ω . Under the identification $\mathrm{Hom}(V,V) \cong V \otimes \overline{V}$ described above we have:

a.) if ω is skew-Hermitian, then g_o corresponds to $S^2(V) \subset V \otimes \overline{V}$.

b.) if ω is Hermitian, then g_o corresponds to $\Lambda^2(V) \subset V \otimes \overline{V}$.

<u>Proof</u>: Let $T \in V \otimes V$ and we define T^a to be the unique element of $V \otimes \overline{V}$ such that $\omega(Tx,y) = -\omega(x,T^a y)$ $\forall x,y \in V$. We want to find $\{T \in V \otimes \overline{V} | T = T^a\}$.

Write $T = \sum_i v_i \otimes w_i$ and we compute T^a .

$$\begin{aligned}
\omega(\sum_i v_i \otimes w_i(x),\ y) &= \omega(\sum_i \overline{\omega(w_i,x)} v_i, y) \\
&= \sum_i \overline{\omega(w_i,x)} \cdot \omega(v_i,y) \\
&= \sum_i \overline{\omega(\overline{\omega(v_i,y)} w_i, x)} \\
&= \overline{\omega(\sum_i w_i \otimes v_i(y), x)} \\
&= \epsilon \cdot \omega(x, \sum_i w_i \otimes v_i(y))
\end{aligned}$$

where $\epsilon = \begin{cases} 1 & \text{if } \omega \text{ is Hermitian} \\ -1 & \text{if } \omega \text{ is skew-Hermitian} \end{cases}$. Thus $T^a = (-\epsilon) \sum_i w_i \otimes v_i$ and the Proposition follows. □

We need the following additional notation:

<u>Definition</u>. Let $S(V,W)$ be the symmetric elements in $V\otimes\overline{W} \oplus W\otimes\overline{V}$.

Let $\Lambda(V,W)$ be the antisymmetric elements in $V\otimes\overline{W} \oplus W\otimes\overline{V}$.

We note that $S(V,W)$ and $\Lambda(V,W)$ are both isomorphic to $V\otimes W$, but it is the natural way each sits inside $(V\oplus W) \otimes (\overline{V\oplus W})$ that is of interest to us.

<u>Proposiition 2</u>. Suppose (π,V) is a representation of $\mathfrak{sl}(2)$ on V and $V = V_1\oplus V_2$ the direct sum of $\mathfrak{sl}(2)$ submodules. Then

a) $S^2(V) = S^2(V_1) \oplus S(V_1,V_2) \oplus S^2(V_2)$

b) $\Lambda^2(V) = \Lambda^2(V_1) \oplus \Lambda(V_1,V_2) \oplus \Lambda^2(V_2)$.

<u>Proof</u>: Obvious. □

<u>Corollary</u>. Let $V = \bigoplus_i V^i\otimes H^i$, V^i the standard irreducible $\mathfrak{sl}(2)$-module over $\mathbb{R}$ with canonical $\mathfrak{sl}(2)$-invariant form ω^i , H^i a vector space with the standard Hermitian or skew-Hermitian form h^i defined with respect to a

basis $\{T^i_j\}_{1\leq i\leq \dim H^i}$. Then

a) $$S^2(\oplus V^i\otimes H^i) = S^2(\bigoplus_i \bigoplus_{j=1}^{\dim H^i} (V^i\otimes\mathbb{F}\cdot T^i_j))$$
$$= \bigoplus_{i,k} S^2(V^i\otimes\mathbb{F}\cdot T^i_k)\oplus$$
$$\bigoplus_{\substack{i>j\\k,\ell}} S(V^i\otimes\mathbb{F}\cdot T^i_k\ ,\ V^j\otimes\mathbb{F}\cdot T^j_\ell)\ \oplus$$
$$\bigoplus_{\substack{i\\k>\ell}} (S(V^i\otimes\mathbb{F}\cdot T^i_k\ ,\ V^i\otimes\mathbb{F}\cdot T^i_\ell))$$

b) $\Lambda^2(\oplus V^i\otimes H^i) = \cdots$ replace S^2 by Λ^2 and S by Λ .

Now we use the Clebsch-Gordon formula to decompose $V^i\otimes V^j$, $S^2(V^i)$, and $\Lambda^2(V^i)$.

<u>Proposition 3</u>.

a) $V^i\otimes\overline{V^j} = M(i+j-1)\oplus M(i+j-3)\oplus\cdots\oplus M(i-j+1)$

b) $S^2(V^i) = P(2i-1)\oplus P(2i-5)\oplus\cdots\oplus P(q)$

c) $\Lambda^2(V^i) = N(2i-3)\oplus N(2i-7)\oplus\cdots\oplus N(p)$

where $\begin{cases} q = 1 \text{ and } p = 3 \text{ if } i \text{ odd} \\ q = 3 \text{ and } p = 1 \text{ if } i \text{ even}\end{cases}$ and $M(r)$ is an irreducible $\mathfrak{sl}(2)$-module $\subset V^i\otimes\overline{V^j}$ with dimension r , $P(r)$ an irreducible $\mathfrak{sl}(2)$-module $\subset S^2(V^i)$ of dimension r , and $N(r)$ an irreducible $\mathfrak{sl}(2)$-module $\subset \Lambda^2(V^i)$ of dimension r.

We let $\{T_k^i\}$ be the basis for H^i described in Chapter 4 and we introduce the following additional notation:

$$\Lambda^2(V^i \otimes \mathbb{R}\cdot T_k^i) = N_{kk}^{ii}(2i-3) \oplus N_{kk}^{ii}(2i-7) \oplus \cdots \oplus N_{kk}^{ii}(p)$$
$$S^2(V^i \otimes \mathbb{R}\cdot T_k^i) = P_{kk}^{ii}(2i-1) \oplus P_{kk}^{ii}(2i-5) \oplus \cdots \oplus P_{kk}^{ii}(q)$$
$$\Lambda(V^i \otimes \mathbb{R}\cdot T_k^i \ , \ V^j \otimes \mathbb{R}\cdot T_\ell^j) = N_{k\ell}^{ij}(i+j-1) \oplus N_{k\ell}^{ij}(i+j-3) \oplus \cdots \oplus N_{k\ell}^{ij}(i-j+1)$$
$$S(V^i \otimes \mathbb{R}\cdot T_k^i, \ V^j \otimes \mathbb{R}\cdot T_\ell^j) = P_{k\ell}^{ij}(i+j-1) \oplus P_{k\ell}^{ij}(i+j-3) \oplus \cdots \oplus P_{k\ell}^{ij}(i-j+1)$$

(for $i \neq j$ or $k \neq \ell$)

$$(V^i \otimes \mathbb{R}\cdot T_k^i) \otimes (V^j \otimes \mathbb{R}\cdot T_\ell^j) = M_{k\ell}^{ij}(1+j-1) \oplus \cdots \oplus M_{k\ell}^{ij}(i-j+1) \ .$$

First, assume $V = \oplus V^i \otimes H^i$ is a vector space over $\mathbb{R}$. Then, using the Corollary to Proposition 2 above, we get:

$$S^2(V) = \bigoplus_{i,k} [P_{kk}^{ii}(2i-1) \oplus \cdots \oplus P_{kk}^{ii}(q)] \ \oplus$$
$$\bigoplus_{\substack{i>j \\ k,\ell}} [P_{k\ell}^{ij}(i+j-1) \oplus \cdots \oplus P_{k\ell}^{ij}(i-j+1)] \ \oplus$$
$$\bigoplus_{\substack{i \\ k>\ell}} [P_{k\ell}^{ii}(2i-1) \oplus \cdots \oplus P_{k\ell}^{ii}(1)]$$

and

$$\Lambda^2(V) = \bigoplus_{i,k} [N^{ii}_{kk}(2i-3)\oplus\cdots\oplus N^{ii}_{kk}(p)] \oplus \bigoplus_{\substack{i>j\\k,\ell}} [N^{ij}_{k\ell}(i+j-1)\oplus\cdots\oplus N^{ij}_{k\ell}(i-j+1)] \oplus \bigoplus_{\substack{i\\k>\ell}} [N^{ii}_{k\ell}(2i-1)\oplus\cdots\oplus N^{ii}_{k\ell}(1)]$$

with q and p as in Proposition 3.

Now let $V = \bigoplus_{i=1}^{N} V^i\otimes H^i$ be a complex vector space, i.e. H^i is complex. We want to decompose $\Lambda^2(V) \cong$ $u(p,q)$ as an SL(2)-module. We define a real form $V_{\mathbb{R}}$ of V as the real span of a subset of V :

<u>Definition</u>. $V_{\mathbb{R}} = \langle X^{i-1-s}Y^s \otimes T^i_k\rangle_{\substack{1\leq i\leq N\\1\leq k\leq \dim H^i}}$.

<u>Proposition 4</u>. $\Lambda^2(V) = \Lambda^2(V_{\mathbb{R}}) + i\cdot\Lambda^2(V_{\mathbb{R}})$.

<u>Proof</u>: Let $v\otimes w-w\otimes v \in \Lambda^2(V)$. We can write $v = v_1+iv_2$ and $w = w_1+iw_2$ with $v_1,v_2,w_1,w_2 \in V_{\mathbb{R}}$. Then

$$\begin{aligned} v\otimes w - w\otimes v &= (v_1+iv_2)\otimes(w_1+iw_2) - (w_1+iw_2)\otimes(v_1+iv_2) \\ &= v_1\otimes w_1 - w_1\otimes v_1 + v_2\otimes w_2 - w_2\otimes v_2 \\ &\quad + i[v_2\otimes w_1 - v_1\otimes w_2 - w_2\otimes v_1 + w_1\otimes v_2] . \\ &\in \Lambda^2(V_{\mathbb{R}}) + i(S^2(V_{\mathbb{R}})) . \quad \square \end{aligned}$$

Thus for V complex, we have

$$\begin{aligned}\Lambda^2(V) = &\bigoplus_{i,k} [N^{ii}_{kk}(2i-3)\oplus\cdots\oplus N^{ii}_{kk}(p)] \oplus \\ &\bigoplus_{\substack{i>j\\ k,\ell}} [N^{ij}_{k\ell}(i+j-1)\oplus\cdots\oplus N^{ij}_{k\ell}(i-j+1)] \oplus \\ &\bigoplus_{\substack{i\\ k>\ell}} [N^{ii}_{k\ell}(2i-1)\oplus\cdots\oplus N^{ij}_{k\ell}(1)] \oplus \\ &\bigoplus_{i,k} i\cdot[P^{ii}_{kk}(2i-1)\oplus\cdots\oplus P^{ii}_{kk}(q)] \oplus \\ &\bigoplus_{\substack{i>j\\ k,\ell}} i\cdot[P^{ij}_{k\ell}(i+j-1)\oplus\cdots\oplus P^{ij}_{k\ell}(i-j+1) \oplus \\ &\bigoplus_{\substack{i\\ k>\ell}} i\cdot[P^{ii}_{k\ell}(2i-1)\oplus\cdots\oplus P^{ii}_{k\ell}(1)]\end{aligned}$$

with p and q as in Proposition 3.

<u>Proposition 5</u>. Let (π, V, ω) be a representation of $\mathfrak{sl}(2)$ be with an $\mathfrak{sl}(2)$-invariant sesquilinear form ω. Then $V = \oplus W^i$ where W^j is isomorphic to m_j copies of V^j and $\omega(W^i, W^j) = 0$ for $i \neq j$.

<u>Proof</u>: Let $[V]_k$ be the eigenspace with eigenvalue k under the H-action. By the $\mathfrak{sl}(2)$-invariance, $\omega([V]_k, [V]_{-\ell}) = 0$ unless $k = \ell$. Assume $i > j$ and let $v \in [W^i]_{j-1}$ and $w \in [W^j]_{-j-1}$. Then $\omega(v,w) = c\cdot\omega(F.E.v,w) = -c\cdot\omega(E.v,F.w) = 0$ for some $c \in \mathbb{R}$ because w is a lowest weight vector in W^j.

<u>Proposition 6</u>. The Killing form B considered as a map

$$B : S^2(V) \times S^2(V) \longrightarrow \mathbb{R} \quad \text{or} \quad B : \Lambda^2(V) \times \Lambda^2(V) \longrightarrow \mathbb{R}$$

is given in terms of the canonical G-invariant sesquilinear form ω on V as follows: for $v\otimes w, v'\otimes w' \in V\otimes V$

$$B(v\otimes w, v'\otimes w') = c\omega(v,v')\cdot\omega(w,w') \quad (c \in \mathbb{R}) .$$

<u>Proof</u>: Up to a multiple, there is a unique nondegenerate G-invariant sesquilinear form on a simple Lie algebra. □

We recall that we want to find the determinant of the linear map induced by the adjoint representation of $(L\cap K)_0$ on the quotient space $\mathfrak{g}_o/\mathfrak{g}_o^E$. By our earlier results, for $\ell \in (L\cap K)_0$, it suffices to restrict our attention to the (-1)-eigenspace of $\mathfrak{g}_o$:

$$\det_{\mathbb{C}}(\overline{\mathrm{Ad}\ell}\,|\,\mathfrak{g}_o/\mathfrak{g}_o^E) = \det_{\mathbb{C}}(\mathrm{Ad}\ell\,|\,[\mathfrak{g}_o]_{-1}) .$$

We now describe an explicit basis for $[\mathfrak{g}_o]_{-1}$ using the identification of $\mathfrak{g}_o$ with $S^2(V)$ or $\Lambda^2(V)$ explained above.

The only irreducible submodules in $S^2(V)$ or $\Lambda^2(V)$ which contribute to the (-1) eigenspace (under the H-action) are those of even dimension; namely those composing $S(V^i \otimes H^i, V^j \otimes H^j)$ or $\Lambda(V^i \otimes H^i, V^j \otimes H^j)$ with $i > j$ and $i \not\equiv j \pmod 2$.

<u>Definitions</u>. 1.) If V is real, then we set:

$$M^{ij} = (V^i \otimes H^i) \otimes (V^j \otimes H^j) = \bigoplus_{\substack{k,\ell \\ 1 \leq s \leq j}} M^{ij}_{k\ell}(i+j+1-2s)$$

$$M^{ij}(t) = \bigoplus_{k,\ell} M^{ij}_{k\ell}(t)$$

$$M^{ij}_{k\ell} = \bigoplus_{1 \leq s \leq j} M^{ij}_{k\ell}(i+j+1-2s)$$

2.) If V is complex, then we set:

$$M^{ij} = (V^i \otimes H^i) \otimes (V^j \otimes H^j)$$

$$= \bigoplus_{\substack{k,\ell \\ 1 \leq s \leq j}} [M^{ij}_{k\ell}(i+j+1-2s) \oplus i \cdot M^{ij}_{k\ell}(i+j+1-2s)]$$

$$M^{ij}(t) = \bigoplus_{k,\ell} [M^{ij}_{k\ell}(t) \oplus i \cdot M^{ij}_{k\ell}(t)]$$

$$M^{ij}_{k\ell} = \bigoplus_{1 \leq s \leq j} [M^{ij}_{k\ell}(i+j+1-2s) \oplus i \cdot M^{ij}_{k\ell}(i+j+1-2s)] .$$

For the remaining calculations in this Chapter we will make use of the isomorphisms

$$\begin{cases} \alpha : & V \otimes W \longrightarrow S(V, W) \\ \beta : & V \otimes W \longrightarrow \Lambda(V, W) \end{cases}$$

defined by:

$$\begin{cases}\alpha(v\otimes w) = v\otimes w + w\otimes v \\ \beta(v\otimes w) = v\otimes w - w\otimes v\end{cases}.$$

We get:

$$(V^i\otimes H^i)\otimes(V^j\otimes H^j) \cong \begin{cases}S(V^i\otimes H^i,\ V^j\otimes H^j) \\ \Lambda(V^i\otimes H^i,\ V^j\otimes H^j)\end{cases}$$

and

$$M^{ij}_{k\ell}(r) \cong \begin{cases}P^{ij}_{k\ell}(r) \\ N^{ij}_{k\ell}(r)\end{cases}.$$

We will often suppress the distinction between $(V^i\otimes H^i)\otimes(V^j\otimes H^j)$ and $\Lambda(V^i\otimes H^i,\ V^j\otimes H^j)$ or $S(V^i\otimes H^i,\ V^j\otimes H^j)$.

<u>Claim 1</u>.

a.) $\omega_E(M^{ij},M^{i'j'}) = 0$ unless $i=i'$ and $j=j'$.

b.) $\omega_E(M^{ij}(t),M^{ij}(t')) = 0$ unless $t=t'$.

<u>Proof</u>: We have $V = \bigoplus_i V^i\otimes H^i$ with invariant sesquilinear form ω . Proposition 5 implies that $\omega(V^i\otimes H^i,V^j\otimes H^j) = 0$ unless $i = j$. We observe that $\omega_E(M^{ij},M^{i'j'}) = B([E,M^{ij}],M^{i'j'})$. Since $[E,M^{ij}] \subset M^{ij}$ we conclude $\omega_E(M^{ij},M^{i'j'}) = 0$ if and only if $B(M^{ij},M^{i'j'}) = 0$.
But,

$$\begin{aligned}B(M^{ij},M^{i'j'}) &= B((V^i\otimes H^i)\otimes(V^j\otimes H^j),(V^{i'}\otimes H^{i'})\otimes(V^{j'}\otimes H^{j'})) \\ &= c\omega(V^i\otimes H^i,V^{i'}\otimes H^{i'})\cdot\omega(V^j\otimes H^j,V^{j'}\otimes H^{j'})\end{aligned}$$

and this last expression is 0 unless $i = i'$ and $j = j'$ by Proposition 6. □

Claim 2. Fix $V = \bigoplus_i V^i \otimes H^i$ and $i > j$, and let $1 \leq k, k' \leq m_i$, and $1 \leq \ell, \ell' \leq m_j$.

If $M^{ij}_{k\ell} \subset S^2(V)$, then

a) for i even, j odd, $m_i = \dim H^i$ and $2m_j = \dim H^j$

$B(M^{ij}_{k'\ell'}, M^{ij}_{k,\ell+m_j}) \neq 0 \iff k'=k$ and $\ell'=\ell$

b) for i odd, j even, $2m_i = \dim H^i$ and $m_j = \dim H^j$

$B(M^{ij}_{k'\ell'}, M^{ij}_{k+m_i,\ell}) \neq 0 \iff k'=k$ and $\ell'=\ell$

If $M^{ij}_{k\ell} \subset \Lambda^2(V)$ and V real, then

c) for i odd, j even, $m_i = \dim H^i$ and $2m_j = \dim H^j$

$B(M^{ij}_{k'\ell'}, M^{ij}_{k,\ell+m_j}) \neq 0 \iff k'=k$ and $\ell'=\ell$

d) for i even, j odd, $2m_i = \dim H^i$ and $m_j = \dim H^j$

$B(M^{ij}_{k'\ell'}, M^{ij}_{k+m_i,\ell}) \neq 0 \iff k'=k$ and $\ell'=\ell$.

If $M^{ij}_{k\ell} \subset \Lambda^2(V)$, V complex, $\dim H^i = m_i$ and $\dim H^j = m_j$, then

e) $B(M^{ij}_{k'\ell'}, M^{ij}_{k\ell}) \neq 0 \iff k' = k$ and $\ell' = \ell$.

Proof: Use Proposition 6. □

In order to compute J on $[\mathfrak{g}_o]_{-1}$ we will need to know how θ looks on $S^2(V)$ and $\Lambda^2(V)$.

Proposition 7. The Cartan involution $\theta : X \longrightarrow -\overline{X}^t$ for $X \in \mathfrak{g}_o$ is given on $S^2(V)$ and $\Lambda^2(V)$ by: for $(X^sY^{i-1-s} \otimes T_k^i) \otimes (X^{j-1-t}Y^t \otimes T_\ell^j) \in M^{ij}$

$$\theta((X^sY^{i-1-s} \otimes T_k^i) \otimes (X^{j-1-t}Y^t \otimes T_\ell^j)) =$$
$$(-1)^{s+t+1}(X^tY^{j-1-t} \otimes \check{T}_\ell^j) \otimes (X^{i-1-s}Y^s \otimes \hat{T}_k^i)$$

where $h^j(T_m^j, \check{T}_\ell^j) = \delta_{\ell m}$, $h^i(\hat{T}_k^i, T_m^i) = \delta_{km}$ and h^i and h^j are the appropriate forms on H^i and H^j, respectively.

Proof: The Cartan involution θ is given by negative conjugate transpose on $\mathfrak{g}_o = \mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{so}(p,q)$ or $\mathfrak{su}(p,q)$. We use the inner product $Q = \bigoplus_i b_1^i \otimes b_2^i$ (defined earlier) on $V = \oplus V^i \otimes H^i$ to compute θ on $S^2(V)$ and $\Lambda^2(V)$. If we write a linear transformation $X \in S^2(V)$ (or $\Lambda^2(V)$) with respect to an orthonormal basis on V (under Q) then $X^* = \overline{X}^t$, hence $\theta X = -X^*$. So we now compute X^*. We take the usual orthonormal basis $\{\binom{i-1}{s}^{1/2} \cdot X^{i-1-s}Y^s \otimes T_k^i\}_{\substack{1 \leq k \leq \dim H^i \\ 0 \leq s \leq i-1}}$ for W^i under Q and set:

$$v_1 = \binom{i-1}{s}^{1/2} X^s Y^{i-1-s} \otimes T^i_k$$
$$v_2 = \binom{j-1}{t}^{1/2} X^{j-1-t} Y^t \otimes T^j_\ell$$
$$v_3 = \binom{i-1}{s}^{1/2} X^{i-1-s} Y^s \otimes \hat{T}^i_k$$
$$v_4 = \binom{j-1}{t}^{1/2} X^t Y^{j-1-t} \otimes \check{T}^j_\ell$$

Let $\{v_i\}$ be the complete orthonormal basis (renamed), and $L = v_1 \otimes v_2$. We must demonstrate that $L^* = (-1)^{s+t}$ $v_4 \otimes v_3$, that is, we want $Q(Lv_i, v_j) = Q(v_i, L^* v_j)$ for all i and j . To begin, observe that $L^* v_i = 0$ if $i \neq 1,2$ and $Lv_j = 0$ if $j \neq 3,4$. Therefore, $Q(Lv_i, v_j) = 0$ for $i \neq 3,4$, and since the image of $L^* = \langle v_3, v_4 \rangle$, $Q(v_i, L^* v_j) = 0$ for $i \neq 3,4$. Also, $Q(Lv_3, v_j) = Q(v_3, L^* v_j) = 0$ if and only if $j \neq 2$. Likewise, $Q(Lv_4, v_k) = Q(v_4, L^* v_k) = 0$ if and only if $k \neq 1$. Thus it remains to show that $Q(Lv_3, v_2) = Q(v_3, L^* v_2)$ and $Q(Lv_4, v_1) = Q(v_4, L^* v_1)$. We have $Q(Lv_4, v_1) = (-1)^t Q(v_1, v_1)$ and $Q(v_4, L^* v_1) = (-1)^t Q(v_4, v_4)$, hence $Q(Lv_4, v_1) = Q(v_4, L^* v_1)$. Finally, we break the verification that $Q(Lv_3, v_2) = Q(v_3, L^* v_2)$ into two cases:

I.) $\mathfrak{g}_o$ preserves a symplectic form:
$Q(Lv_3, v_2) = (-1)^s (-1)^{i+1} h^i (T^i_k, \hat{T}^i_k) Q(v_2, v_2)$ and
$Q(v_3, L^* v_2) = (-1)^{t+s} (-1)^t (-1)^{j+1} h^j (\check{T}^j_\ell, T^j_\ell) Q(v_3, v_3)$.

Since $h^i(T^i_k, \hat{T}^i_k) = (-1)^i$ and $h^j(\check{T}^j_k, T^j_k) = (-1)^j$, we conclude $Q(Lv_3, v_2) = Q(v_3, L^* v_2)$.

II.) $\mathfrak{g}_o$ preserves an Hermitian form:

$Q(Lv_3, v_2) = -(-1)^s(-1)^{i+1}h^i(T^i_k, \hat{T}^i_k)Q(v_2, v_2)$ and
$Q(v_3, L^* v_2) = -(-1)^{t+s}(-1)^t(-1)^{j+1}h^j(\check{T}^j_\ell, T^j_\ell)$. Now
$h^i(T^i_k, \hat{T}^i_k) = (-1)^{i+1}$ and $h^j(\check{T}^j_\ell, T^j_\ell) = (-1)^{j+1}$ and thus
$Q(Lv_3, v_2) = Q(v_3, L^* v_2)$ and we're done. □

We know that θ preserves the isotypic parts of $\mathfrak{g}_o$ under the action of a θ-stable $\mathfrak{sl}(2) \subset \mathfrak{g}_o$; i.e. $\theta : W^k \longrightarrow W^k$. Our formula for θ on $S^2(V)$ and $\Lambda^2(V)$ tells us, in addition, that θ preserves $M^{ij}(k) \subset W^k$. We have already observed that $J = c\theta \circ \mathrm{ad}E : [\mathfrak{g}_o]_{-1} \longrightarrow [\mathfrak{g}_o]_{-1}$, and thus we get $J : [M^{ij}(k)]_{-1} \longrightarrow [M^{ij}(k)]_{-1}$ (of course k must be even or $[M^{ij}(k)]_{-1} = \{0\}$).

Our description of the action of L on $S^2(V)$ and $\Lambda^2(V)$ shows that L preserves M^{ij}. Moreover, since L commutes with $\mathfrak{sl}(2)$, L preserves $M^{ij}(k)$ and also the (-1)-eigenspace $[M^{ij}(k)]_{-1}$. Putting all of this together we get the following conclusion: for $\ell \in (L \cap K)_0$.

$$\det_{\mathbb{C}}(\overline{Ad\ell}) = \prod_{\substack{i>j \\ i\not\equiv j(2)}} \det_{\mathbb{C}}(\overline{Ad\ell}|_{[M^{ij}]_{-1}}) = \prod_{\substack{i>j \\ i\not\equiv j(2) \\ t}} \det_{\mathbb{C}}(\overline{Ad\ell}|_{[M^{ij}(t)]_{-1}})$$

We want an explicit basis for $[M^{ij}(t)]_{-1}$. We begin by finding a basis for the irreducible submodule $M^{ij}_{k\ell}(i+j+1-2s) \subset M^{ij}$.

We define a basis on $M^{ij}_{k\ell}(i+j+1-2s)$ which is isomorphic to the standard basis for an $\mathfrak{sl}(2)$ representation on homogeneous polynomials of degree $i+j-2s$ ($= S^{i+j-2s}(X,Y)$). Let ϕ be an intertwining operator, $\phi : S^{i+j-2s}(X,Y) \longrightarrow M^{ij}(i+j+1-2s)$. We set $X^{ij}_{k\ell}(i+j-2s) = \phi(X^{i+j-2s})$ and this fixes a canonical basis on $M^{ij}_{k\ell}(i+j+1-2s)$.

<u>Definition</u>. $X^{ij}_{k\ell}(i+j-2s-2m) = \phi(X^{i+j-2s-m}Y^m)$, $0 \leq m \leq i+j-2s$.

<u>Lemma 1</u>. The highest weight vector $X^{ij}_{k\ell}(i+j-2s)$ in $M^{ij}_{k\ell}(i+j+1-2s)$ is a multiple of

$$\sum_{p=1}^{s} (-1)^{p-1}\binom{s-1}{p-1}(X^{i-p}Y^{p-1}\otimes T^i_k)\otimes(X^{j-1-s+p}Y^{s-p}\otimes T^j_\ell) .$$

<u>Proof</u>: We know $X^{ij}_{k\ell}(i+j-2s)$ must be a linear combination of eigenvectors under the H-action with eigenvalue $i+j-2s$; that is, a sum of the following vectors in $M^{ij}_{\ell k}(i+j+1-2s)$:

$$\{(X^{i-p}Y^{p-1}\otimes T^{i}_{k})\otimes(X^{j-1-s+p}Y^{s-p}\otimes T^{j}_{\ell})\}_{1\leq p\leq s} .$$

Set $X^{ij}_{k\ell}(i+j-2s) = \sum_{p=1}^{s} c_p(X^{i-p}Y^{p-1}\otimes T^{i}_{k})\otimes(X^{j-1-s+p}Y^{s-p}\otimes T^{j}_{\ell})$ where c_p are some constants. Use the fact that $adE(X^{ij}_{k\ell}(i+j-2s)) = 0$ to solve for c_p in terms of c_1 .

$$\begin{aligned} 0 &= adE(X^{ij}_{k\ell}(i+j-2s)) \\ &= \sum_{p=1}^{s-1}[(s-p)c_p(X^{i-p}Y^{p-1}\otimes T^{i}_{\ell})\otimes(X^{j-1-s+p+1}Y^{s-p-1}\otimes T^{j}_{\ell}) \\ &\quad + pc_{p+1}(X^{i-p}Y^{p-1}\otimes T^{i}_{\ell})\otimes(X^{j-1-s+p+1}Y^{s-p-1}\otimes T^{j}_{\ell})] . \end{aligned}$$

Therefore $(s-p)\cdot c_p + p\cdot c_{p+1} = 0$, $1 \leq p \leq s-1$, which implies that $c_p = (-1)^{p-1}\binom{s-1}{p-1}$ as desired. □

We normalize so that $X^{ij}_{k\ell}(i+j-2s) =$
$\sum_{p=1}^{s} (-1)^{p-1}\binom{s-1}{p-1}(X^{i-p}Y^{p-1}\otimes T^{i}_{k})\otimes(X^{j-1-s+p}Y^{s-p}\otimes T^{j}_{\ell})$.

Lemma 2. $X^{ij}_{k\ell}(-i-j+2s)$, the lowest weight vector in our canonical basis on $M^{ij}_{k\ell}(i+j-1-2s)$ equals

$$(-1)^s \cdot \sum_{p=1}^{s} (-1)^{p-1}\binom{s-1}{p-1}[(X^{p-1}Y^{i-p}\otimes T^i_k)\otimes(X^{s-p}Y^{j-1-s+p}\otimes T^j_\ell)].$$

Proof: Consider the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $SL(2)$. It sends X^iY^j to $(-1)^i \cdot X^jY^i$. So $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot (X^{i-p}Y^{p-1}\otimes T^i_k)\otimes(X^{j-1-s+p}Y^{s-p}\otimes T^j_\ell)$ goes to $(-1)^{i+j-1-s} \cdot (X^{p-1}Y^{i-p}\otimes T^i_k)\otimes(X^{s-p}Y^{j-1-s+p}\otimes T^j_\ell)$, and $(-1)^{i+j-1-s} = (-1)^s$ because $i \not\equiv j \pmod 2$. □

We will also need a basis for the irreducible submodule $i \cdot M^{ij}_{k\ell}(i+j+1-2s) \subset i \cdot S(V^i\otimes\mathbb{R}\cdot T^i_k , V^j\otimes\mathbb{R}\cdot T^j_\ell)$. Set $\tilde{X}^{ij}_{k\ell}(i+j-2s) =$

$$\sum_{p=1}^{s} (-1)^{p-1}\binom{s-1}{p-1}[(X^{i-p}Y^{p-1}\otimes i \cdot T^i_k)\otimes(X^{j-1-s+p}Y^{s-p}\otimes T^j_\ell)] .$$

The same reasoning used in Lemmas 1 and 2 shows that $\tilde{X}^{ij}_{k\ell}(i+j-2s)$ is a highest weight vector in $i \cdot M^{ij}_{k\ell}(i+j+1-2s)$ and that $\tilde{X}^{ij}_{k\ell}(-i-j+2s) =$

$$(-1)^s \cdot \sum_{p=1}^{s} (-1)^{p-1}\binom{s-1}{p-1}[(X^{p-1}Y^{i-p}\otimes i \cdot T^i_k)\otimes(X^{s-p}Y^{j-1-s+p}\otimes T^j_\ell)]$$

is a lowest weight, where we define a canonical basis $\{\tilde{X}^{ij}_{k\ell}(i+j-2s-2m)\}_{0\le m\le i+j-2s}$ as in the previous case.

Lemma 3. Fix $V = \bigoplus_p V^p\otimes H^p$, $i > j$, $\oplus\omega^i\otimes h^i$ canonical form on V , and let $1\le k\le m_i$ and $1\le \ell\le m_j$.

If $M^{ij} \subset S^2(V)$, then:

a) for i even, j odd, $m_i = \dim H^i$, $2m_j = \dim H^j$

$$\theta(X^{ij}_{k\ell}(i+j-2s)) = -h^i(T^i_k, T^i_k)(X^{ij}_{k,\ell+m_j}(-1-j+2s))$$

b) for i odd, j even, $2m_i = \dim H^i$, $m_j = \dim H^j$

$$\theta(X^{ij}_{k\ell}(i+j-2s)) = h^j(T^j_\ell, T^j_\ell)(X^{ij}_{k+m_j,\ell}(-i-j+2s))$$

If $M^{ij} \subset \Lambda^2(V)$ and V is real, then:

c) for i odd, j even, $m_i = \dim H^i$, $2m_j = \dim H^j$

$$\theta(X^{ij}_{k\ell}((i+j-2s)) = -h^i(T^i_k, T^i_k)(X^{ij}_{k,\ell+m_j}(-i-j+2s))$$

d) for i even, j odd, $2m_i = \dim H^i$, $m_j = \dim H^j$

$$\theta(X^{ij}_{k\ell}((i+j-2s)) = h^j(T^j_\ell, T^j_\ell)(X^{ij}_{k+m_i,\ell}(-i-j+2s))$$

If $M^{ij} \subset \Lambda^2(V)$, V is complex, $m_i = \dim H^i$ and $m_j = \dim H^j$ then:

(e)
$$\theta(X^{ij}_{k\ell}(i+j-2s)) = -i \cdot h^j(T^j_\ell, T^j_\ell) h^i(T^i_k, T^i_k)(\tilde{X}^{ij}_{k\ell}(-i-j+2s))$$

$$\theta(\tilde{X}^{ij}_{k\ell}(i+j-2s)) = i \cdot h^j(T^j_\ell, T^j_\ell) h^i(T^i_k, T^i_k)(X^{ij}_{k\ell}(-i-j+2s))$$

<u>Proof</u>: By Lemma 2 we have

$$X^{ij}_{k\ell}(-i-j+2s) = (-1)^s \sum_{p=1}^{s} (-1)^{p-1} \binom{s-1}{p-1} [(X^{p-1}Y^{i-p} \otimes T^i_k) \otimes (X^{s-p}Y^{j-1-s+p} \otimes T^j_\ell)].$$

Compare this expression with:

$$\theta(X^{ij}_{k\ell}(i+j-2s)) =$$

$$(-1)^{i+s+1}\sum_{p=1}^{s}(-1)^{p-1}\binom{s-1}{p-1}[(X^{s-p}Y^{j-1-s+p}\otimes\check{T}^{j}_{\ell})\otimes(X^{p-1}Y^{i-p}\otimes\hat{T}^{i}_{k})]$$

□

Lemma 4. (Same setup as Lemma 3)

a) $J(X^{ij}_{k\ell}(-1)) = (-1)^{\frac{i+j+1-2s}{2}} h^{i}(T^{i}_{k},T^{i}_{k})(X^{ij}_{k,\ell+m_j}(-1))$

b) $J(X^{ij}_{k\ell}(-1)) = (-1)^{\frac{i+j-1-2s}{2}} h^{j}(T^{j}_{\ell},T^{j}_{\ell})(X^{ij}_{k+m_i,\ell}(-1))$

c) $J(X^{ij}_{k\ell}(-1)) = (-1)^{\frac{i+j+1-2s}{2}} h^{i}(T^{i}_{k},T^{i}_{k})(X^{ij}_{k,\ell+m_j}(-1))$

d) $J(X^{ij}_{k\ell}(-1)) = (-1)^{\frac{i+j-1-2s}{2}} h^{j}(T^{j}_{\ell},T^{j}_{\ell})(X^{ij}_{k+m_i,\ell}(-1))$

e) $J(X^{ij}_{k\ell}(-1)) =$

$(-1)^{\frac{i+j+1-2s}{2}} i\cdot h^{j}(T^{j}_{\ell},T^{j}_{\ell})h^{i}(T^{i}_{k},T^{i}_{k})(\tilde{X}^{ij}_{k\ell}(-1))$

$J(\tilde{X}^{ij}_{k\ell}(-1)) =$

$(-1)^{\frac{i+j-1-2s}{2}} i\cdot h^{j}(T^{j}_{\ell},T^{j}_{\ell})h^{i}(T^{i}_{k},T^{i}_{k})(X^{ij}_{k\ell}(-1))$

<u>Proof</u>: (for Case a) only)

$$J(X^{ij}_{k\ell}(-1)) = (j_{-1})^{-1/2}\cdot\theta\circ\mathrm{ad}E(X^{ij}_{k\ell}(-1))$$
$$= \theta(X^{ij}_{k\ell}(1)) ,$$
because $[E, X^{ij}_{k\ell}(-1)] = j_{-1}^{1/2}\cdot X^{ij}_{k\ell}(1)$.
$$= \theta(c_0(\mathrm{ad}F)^{\frac{i+j-1-2s}{2}}(X^{ij}_{k\ell}(i+j-2s))$$
where $c_0 = \dfrac{(\frac{i+j+1-2s}{2})!}{(i+j-2s)!}$:
$$= (-1)^{\frac{i+j-1-2s}{2}} c_0(\mathrm{ad}E)^{\frac{i+j-1-2s}{2}} \theta(X^{ij}_{k\ell}(i+j-2s)) .$$
Apply Lemma 3-a.):
$$= (-1)^{\frac{i+j-1-2s}{2}} c_0(\mathrm{ad}E)^{\frac{i+j-1-2s}{2}}$$
$$-h^i(T^i_k, T^i_k)(X^{ij}_{k,\ell+m_j}(-i-j+2s))$$
$$= (-1)^{\frac{i+j+1-2s}{2}} h^i(T^i_k, T^i_k)(X^{ij}_{k,\ell+m_j}(-1)) . \quad \square$$

<u>Lemma 5</u>. In each of the cases in Lemma 3, $\{X^{ij}_{k\ell}(-1)\}_{k,\ell}$ is a complex orthogonal basis for $[M^{ij}(i+j+1-2s)]_{-1}$ and all the vectors in this basis have the same length with respect to the inner product b on $\mathfrak{g}_o/\mathfrak{g}_o^E$.

<u>Proof</u>: The orthogonality follows from Claim 2. The basis elements all have the same length because we constructed them the same way on each submodule $M^{ij}_{k\ell}(i+j+1-2s)$. □

We recall the following identification described in a previous section:

If $L \subset Sp(V)$,

then $L \cap K \cong (Sp(2m_1) \cap O(2m_1)) \times ((O(p_2) \times O(q_2)) \times \cdots$.

If $L \subset O(p,q)$,

then $L \cap K \cong (O(p_1) \times O(q_1)) \times (Sp((2m_2) \cap O(2m_2)) \times \cdots$.

If $L \subset U(p,q)$,

then $L \cap K \cong (U(p_1) \times U(q_1)) \times (U(p_2) \times U(q_2)) \times \cdots$.

The product of two elements on the left side of the isomorphisms above is the usual product on a Cartesian product of groups. Therefore, if $\ell \in L \cap K$ corresponds to

$$(g_1, \cdots, g_N) \in (Sp(2m_1) \cap O(2m_1)) \times (O(p_2) \times (O(q_2)) \times \cdots$$

we conclude

$$\det_{\mathbb{C}}(\overline{Ad\ell}) = \prod_j \det_{\mathbb{C}}(\overline{Adg_j}) ,$$

and we can compute the determinant of any element in $L \cap K$ if we know $\det_{\mathbb{C}}(\overline{Adg_j})$ for all j . Since M^{ij} is $\overline{Adg_j}$ invariant we must compute $\det_{\mathbb{C}}(\overline{Adg_j}|_{M^{ij}})$ for all $i \not\equiv j \mod (2)$.

<u>Proposition 8</u>. Assume $i > j$.

I. <u>Symplectic Group</u>

a) For i even, j odd, let $g_j \in Sp(2m_j) \cap O(2m_j)$ be given by the matrix $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ (where A and B are $m_j \times m_j$ matrices) with respect to the standard symplectic basis on the $2m_j$ dimensional space H^j . Then

$$\det_{\mathbb{C}}(\overline{Adg_j}|_{M^{ij}}) = \begin{cases} \det(A+iB)^{\text{odd power}} & \text{if } m_i \text{ odd} \\ \det(A+iB)^{\text{even power}} & \text{if } m_i \text{ even} \end{cases} .$$

b) For i odd, j even, $g_i = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in Sp(2m_i) \cap O(2m_i)$ with the same conventions as above, then

$$\det_{\mathbb{C}}(\overline{Adg_i}|_{M^{ij}}) = 1 .$$

c) For i even, j odd, let $g_i \in [O(p_i) \times O(q_i)]_0$, then

$$\det_{\mathbb{C}}(\overline{Adg_i}|_{M^{ij}}) = 1 .$$

d) For i odd, j even, let $g_j \in [O(p_j) \times O(q_i)]_0$, then

$$\det_{\mathbb{C}}(\overline{\mathrm{Adg}_i}|_{M^{ij}}) = 1 \quad .$$

II. <u>Group preserving a symmetric form on a real vector space</u>.

a) For i even, j odd,
$g_i = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in Sp(2m_i) \cap O(2m_i)$, $\dim H^i = 2m_i$,
$\dim H^j = m_j$, then

$$\det_{\mathbb{C}}(\overline{\mathrm{Adg}_i}|_{M^{ij}}) = \begin{cases} \det(A+iB)^{\text{odd power}} & \text{if } m_j \text{ is odd} \\ \det(A+iB)^{\text{even power}} & \text{if } m_j \text{ is even} \end{cases} .$$

b) For i odd, j even, $g_j = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in Sp(2m_j) \cap O(2m_j)$

$$\det_{\mathbb{C}}(\overline{\mathrm{Adg}_j}|_{M^{ij}}) = 1 \ .$$

c) For i odd, j even, $g_i \in [O(p_i) \times O(q_i)]_0$,

$$\det_{\mathbb{C}}(\overline{\mathrm{Adg}_i}|_{M^{ij}}) = 1 \ .$$

d) For i even, j odd, $g_j \in [O(p_j) \times O(q_j)]_0$,

$$\det_{\mathbb{C}}(\overline{\mathrm{Adg}_j}|_{M^{ij}}) = 1 \ .$$

III. <u>Group preserving an Hermitian form on a complex vector space</u>.

a) For i even, j odd

$g_i = A \in U(p_i)$(or $U(q_i)$) , $\dim H^i = p_i+q_i$,

$\dim H^j = m_j$,

$$\det_{\mathbb{C}}(\overline{Adg_i}|_{M^{ij}}) = \begin{cases} (\det A)^{\text{odd power}} & \text{if } m_j \text{ is odd} \\ (\det A)^{\text{even power}} & \text{if } m_j \text{ is even} \end{cases}$$

b) For i even, j odd

$g_j = A \in U(p_j)$ (or $U(q_j)$), $\dim H^j = p_j$

$\dim H^i = m_i$,

$$\det_{\mathbb{C}}(\overline{Adg_j}|_{M^{ij}}) = \begin{cases} (\det A)^{\text{odd power}} & \text{if } m_i \text{ is odd} \\ (\det A)^{\text{even power}} & \text{if } m_i \text{ is even} \end{cases}$$

c) For i odd, j even, $g_j \in [U(p_j)\times U(q_j)]_0$

$$\det_{\mathbb{C}}(\overline{Adg_j}|_{M^{ij}}) = 1 .$$

d) For i odd, j even, $g_i \in [U(p_i)\times U(q_i)]_0$

$$\det_{\mathbb{C}}(\overline{Adg_i}|_{M^{ij}}) = 1 .$$

Proof: We will only prove the statements for the Symplectic group because the proofs for the other groups are very similar. L preserves isotypic submodules of g_o, hence g_j preserves $M^{ij}(i+j+1-2s)$. We compute $\det_{\mathbb{C}}(\overline{Adg}_j|_{[M^{ij}(i+j+1-2s)]_{-1}})$ which equals $\det_{\mathbb{C}}(\overline{Adg}_j|_{M^{ij}(i+j+1-2s)})$.

Case (a): We have $g_j(T^j_\ell) = \sum_{m=1}^{m_j} a_{m\ell} T^j_m + \sum_{m=1}^{m_j} (-b_{m\ell}) T^j_{m+m_j}$, $1 \leq \ell \leq m_j$. Therefore

$$\begin{aligned}\overline{Adg}_j(X^{ij}_{k\ell}(-1)) &= \sum_{m=1}^{m_j} a_{m\ell} X^{ij}_{km}(-1) + \sum_{m=1}^{m_j} (-b_{m\ell}) X^{ij}_{k,m+m_j}(-1) \\ &= \sum_{m=1}^{m_j} a_{m\ell} X^{ij}_{km}(-1) + \sum_{m=1}^{m_j} (-b_{m\ell}) \epsilon_{k,s} J(X^{ij}_{km}(-1))\end{aligned}$$

where $\epsilon_{k,s} = (-1)^{\frac{i+j+1-2s}{2}} h_i(T^i_k, T^i_k)$

$$= \sum_{m=1}^{m_j} (a_{m\ell} - i\epsilon_{k,s} b_{m\ell}) X^{ij}_{km}(-1) .$$

Thus

$$\det_{\mathbb{C}} \overline{Adg}_j|_{\overset{2m_j}{\underset{\ell=1}{\oplus}} [M^{ij}_{k\ell}(i+j+1-2s)]_{-1}} = \prod_{s=1}^{j} \det(A - i\epsilon_{k,s} B)$$

and

$$\det_{\mathbb{C}} \overline{Adg}_j |_{[M^{ij}]_{-1}} = \prod_{k=1}^{m_i} \prod_{s=1}^{j} \det(A - i\epsilon_{k,s}B) .$$

Since j is odd and $\epsilon_{k,s} = -\epsilon_{k,s+1}$ we get:

$$\det_{\mathbb{C}}(\overline{Adg}_j |_{M^{ij}}) = \prod_{k=1}^{m_i} \det(A - i\epsilon_{k,1}B) .$$

Case (b): Same reasoning as part (a) leads to the equation,

$$\det_{\mathbb{C}}(\overline{Adg}_i |_{[M^{ij}]_{-1}}) = \prod_{\ell=1}^{m_j} \prod_{s=1}^{j} \det(A - i\epsilon_{\ell,s}B) .$$

In this case j is even $(\epsilon_{\ell,s} = -\epsilon_{\ell,s+1})$, therefore

$$\prod_{\ell=1}^{m_j} \det(\overline{Adg}_i |_{M^{ij}}) = 1 .$$

Case (c): Set $U = \bigoplus_{s=1}^{j} \bigoplus_{k=1}^{m_i} \bigoplus_{\ell=1}^{m_j} [M_{k\ell}^{ij}(i+j+1-2s)]_{-1}$. By Lemma 2-c.) we get $[M^{ij}]_{-1} = U \oplus J(U)$. Since $\overline{Adg}_i(U) \subset U$, and $\overline{Adg}_i(JU) \subset JU$ we conclude

$$\det_{\mathbb{C}}(\overline{Adg_i}|_{[M^{ij}]_{-1}}) = \det_{\mathbb{R}}(\overline{Adg_i}|_U) \in \mathbb{R} .$$

In fact $\det_{\mathbb{R}}(\overline{Adg_i}|_U)$ equals 1 because we are looking at the image in $\mathbb{R}$ of a connected, compact set containing the identity and $\det(\overline{Ad(e)}) = 1$.

<u>Case</u> (d): Use the same reasoning as in part (c). □

<u>Theorem</u>.

(I) Let G be the symplectic group with same notation as above, then

$$\det_{\mathbb{C}}(\overline{Adg_j}) = \begin{cases} 1 & \text{if } j \text{ is even} \\ \det(A+iB)^{\text{odd power}} & \text{if } j \text{ is odd and there are an odd number of even dimensional irreducibles of dimension} > j \text{ in } \bigoplus_i V^i \otimes H^i \\ \det(A+iB)^{\text{even power}} & \text{otherwise} \end{cases}$$

(II) Let G be a group preserving an Hermitian form on a real vector space, then

$$\det_{\mathbb{C}}(\overline{\mathrm{Adg}_j}) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ \det(A+iB)^{\text{odd power}} & \text{if } j \text{ is even and there are an odd number of odd dimensional irreducibles of dimension} < j \text{ in } \bigoplus_i V^i \otimes H^i \\ \det(A+iB)^{\text{even power}} & \text{otherwise} \end{cases}$$

III.) Let G be a group preserving an Hermitian form on a complex vector space, then

$$\det_{\mathbb{C}}(\overline{\mathrm{Adg}_j}) = \begin{cases} (\det A)^{\text{odd power}} & \text{if } j \text{ is odd and there are an odd number of even dimensional irreducibles of dimension} > j \text{ in } \bigoplus_i V^i \otimes H^i \\ (\det A)^{\text{odd power}} & \text{if } j \text{ is even and there are an odd number of odd dimensional irreducibles of dimension} < j \text{ in } \bigoplus_i V^i \otimes H^i \\ (\det A)^{\text{even power}} & \text{otherwise} \end{cases}$$

<u>Corollary</u>. Fix $E \in \mathfrak{g}_o$ and $\mathcal{O}$ the orbit of E under G. Let $g = (g_1, \cdots, g_N) \in L\cap K$. Then $\det_{\mathbb{C}}(\overline{\mathrm{Ad}g}) = \prod_{i=1}^{N} (\det_{\mathbb{C}} g_i)^{n_i}$ and E is admissible $\Leftrightarrow$ n_i is even for all i.

<u>Proof</u>: The proof follows directly from the Theorem above and the last Theorem in Chapter 1. □

<u>Theorem</u>. Fix a nilpotent $E \in \mathfrak{su}(p,q)$. Then the orbit of E under $SU(p,q)$ is admissible $\Leftrightarrow$ 1.) the orbit of E under $U(p,q)$ is admissible, or 2.) If for all odd j with $H^j \neq \{0\}$ the number of even dimensional irreducibles of dim $>$ j in $\bigoplus_i V^i \otimes H^i$ is odd and for all even j with $H^j \neq \{0\}$ the number of odd dimensional irreducibles of dimension $<$ j in $\bigoplus_i V^i \otimes H^i$ is even.

<u>Proof</u>: Fix a nilpotent element $E \in \mathfrak{su}(p,q)$. The orbit of this element under $U(p,q)$ is the same as the orbit under $SU(p,q)$, and, as we've seen, this orbit corresponds to a decomposition of $\mathbb{C}^{p,q} \cong \bigoplus_{i=1}^{N} V^i \otimes H^i$ under the action of $\mathfrak{sl}(2)$. Let the set $\{(g_1, \cdots, g_N) \in \prod_{i=1}^{N} (U(p_i) \times U(q_i))\}$ be the subgroup $L\cap K \subset U(p,q)$

determined by E (see Chapter 4). Then the subgroup $L \cap K \subset SU(p,q)$ determined by E is

$$\{(g_1,\cdots,g_N) \in \prod_{i=1}^{N} (U(p_i)\times U(q_i)) \mid \prod_{i=1}^{N} (\det(g_i))^i = 1\} .$$

By the preceding Corollary we know that there is a sequence of integers $(n_1,\cdots,n_N)$ which determine the determinant character:

$$\det_{\mathbb{C}}(\overline{Adg}) = \prod_{i=1}^{N} (\det g_i)^{n_i} \text{ where } g = (g_1,\cdots,g_N) .$$

For $g = (g_1,\cdots,g_N) \in L \cap K \subset SU(p,q)$, $\prod_i (\det g_i)^i = 1$, therefore on this subgroup the character has the following form:

$$\det_{\mathbb{C}}(\overline{Adg}) = \prod_i (\det g_i)^{n_i+ki}$$

where k is an arbitrary integer.

Thus $E \in \mathfrak{su}(p,q)$ is admissible if and only if this character is a square, which is true if and only if n_{2j} is even for all j and n_{2j+1} is either odd for all j or even for all j . □

<u>Theorem</u>. Fix a nilpotent $E \in \mathfrak{so}(p,q)$. Then the orbit of E under $SO(p,q)$ is admissible <=> the orbit of E under $O(p,q)$ is admissible.

<u>Proof</u>: The $SO(p,q)$ case follows directly from the next lemma because $SO(p,q)_0 = O(p,q)_0$.

Lemma. The orbit of a nilpotent $E \in \mathfrak{g}_o$ is admissible for G if and only if it is admissible for G_0.

Proof of Lemma: E is admissible for G if and only if $z \notin (G^E)_0^{mp}$. We have the following coverings of groups:

$$\begin{array}{ccc} (G^E)^{mp} & \supseteq & (G_0^E)^{mp} \\ \downarrow & & \downarrow \\ G^E & \supseteq & G_0^E \end{array}$$

Since $\mathrm{Lie}(G^E) = \mathrm{Lie}(G_0^E)$, it follows from the inclusions above that $(G^E)_0^{mp} = (G_0^E)_0^{mp}$. Therefore $z \notin (G_0^E)_0^{mp}$ if and only if $z \notin (G^E)_0^{mp}$. □

Chapter 6

$SL(n,\mathbb{R})$

In Chapter III we analyzed nilpotent orbits under a group which preserves a nondegenerate symplectic or orthogonal bilinear form on a vector space V . We now consider nilpotent orbits under a group which preserves a nondegenerate, multilinear, alternating top-dimensional form on a vector space V_0 ; that is, we are considered nilpotent orbits in $\mathfrak{sl}(n,\mathbb{R})$ under $SL(n,\mathbb{R})$ where n is the dimension of V_0 .

As before, Kostant's 1959 results give us a bijection between nilpotent orbits in $\mathfrak{sl}(n,\mathbb{R})$ under $SL(n,\mathbb{R})$ and equivalence classes of mappings from $SL(2,\mathbb{R})$ into $SL(n,\mathbb{R})$ (where two maps are equivalent if they are conjugate by an element of $SL(n,\mathbb{R})$).

We have a notion of equivalence among vector spaces with non-zero top dimensional multi-linear alternating forms. Let the pair (ω,V) be a vector space with such a form.

<u>Definition</u>. $(\omega_1,V_1) \sim (\omega_2,V_2)$ if there exists a linear isomorphism $T : V_1 \longrightarrow V_2$ such that $\omega_1(v_1,\cdots,v_n) = \omega_2(Tv_1,\cdots,Tv_n)$ where $\{v_1,\cdots,v_n\}$ is a basis for V_1 .

The isomorphism T is unique up to right or left multiplication by an element of $SL(n,\mathbb{R})$, and the map T induces an isomorphism $\phi_T : G(\omega_2) \longrightarrow G(\omega_1)$ where $G(\omega_i)$ is the group preserving the form ω_i, $i = 1,2$. (We note that $G(\omega_1) = G(\omega_2) = SL(n,\mathbb{R})$).

Fix an equivalence class $\overline{(\omega_0,V_0)}$ and let (ω_0,V_0) be a representative of $\overline{(\omega_0,V_0)}$ with dimension of V_0 equal to n. We consider the set of all maps $SL(2,\mathbb{R}) \longrightarrow G(\omega_i)$ where $(\omega_i,V_i) \in \overline{(\omega_0,V_0)}$. Define an equivalence relation on this set in the same way we did for the orthogonal and symplectic groups (see Chapter III). The same reasoning used in the earlier cases shows that the classification of nilpotent orbits in $\mathfrak{sl}(n,\mathbb{R})$ under $SL(n,\mathbb{R})$ reduces to understanding the partition of the set of triples

$$\{(\pi,\omega,V) \mid \pi \text{ is a representation of } SL(2,\mathbb{R}) \text{ on } V \text{ which preserves a (dimension of V)-nondegenerate, multilinear alternating form, and } (\omega,V) \in \overline{(\omega_0,V_0)}\}$$

into equivalence classes (where the notion of equivalence is analogous to the equivalence in Chapter III).

Let (π,V) be a representation of $\mathfrak{sl}(2,\mathbb{R})$ on V. We recall that $V \cong \bigoplus_i V^i \otimes H^i$ (see Chapter III for notation).

Proposition.

a.) If $H^i \neq \{0\}$ for some odd i, then there is one (up to equivalence) non-zero $SL(2,\mathbb{R})$-invariant multilinear alternating form on $\bigoplus_i V^i \otimes H^i$.

b.) If $H^i = \{0\}$ for all odd i, then there are two equivalence classes of such forms.

Proof: Let $V = \bigoplus_i V^i \otimes H^i$ and $W^i = V^i \otimes H^i$. Any intertwining operator $T : V \longrightarrow V$ preserves isotypic components and eigenspaces. Therefore $T(W^i) \subset W^i$ and $T([W^i]_p) \subset [W^i]_p$.

First, assume $H^j \neq \{0\}$. We take $\{v_1 = X^{j-1} \otimes T_1^j,\ v_2 = X^{j-2} Y \otimes T_1^j, \cdots, v_j = Y^{j-1} \otimes T_1^j,\ v_{j+1}, \cdots, v_n\}$ to be a basis for V . If ω is an n-form on V then $\omega(Tv_1, \cdots, Tv_n) = (\det T)\omega(v_1, \cdots, v_n)$. Also by the observations above

$$\det T = \prod_i \det(T|_{W^i})$$
$$= \prod_i \prod_{i-(\frac{i-1}{2}) \leq p \leq (\frac{i-1}{2})} \det(T|_{[W^i]_p}) .$$

Let ω_0, ω_1 be two n-forms on V defined by putting $\omega_0(v_1, \cdots, v_n) = 1$ and $\omega(v_1, \cdots, v_n) = c$, for an arbitrary constant c . We want to show that $(\pi, V, \omega_0) \sim (\pi, V, \omega_1)$. To do this we define a linear isomorphism $T : V \longrightarrow V$ as follows:

$$T(v_\ell) = \epsilon|c^{-\frac{1}{n}}|v_\ell \quad \text{for} \quad 1 \le \ell \le j \quad \text{and} \quad \epsilon = \text{sign of } c$$
$$T(v_k) = v_k \qquad \text{for} \quad j+1 \le k \le n .$$

It is clear that T is an intertwining operator, and we calculate: $\omega_1(Tv_1,\cdots,Tv_n) = (\det T)\omega_1(v_1,\cdots,v_n) = \epsilon|c^{-1}|\cdot c = 1 = \omega_0(v_1,\cdots,v_n)$. Thus we've shown $(\pi,V,\omega_0)\sim(\pi,V,\omega_1)$.

Now assume $H^i = \{0\}$ for all odd i. An argument similar to the one above shows that $(\pi,V,\omega_0)\sim(\pi,V,\omega_1)$ where $\omega_0(v_1,\cdots,v_n) = 1$ and $\omega_0(v_1,\cdots,v_n) = c > 0$. So we assume $(\pi,V,\omega_0)\sim(\pi,V,\omega_1)$, $\omega_0(v_1,\cdots,v_n) = 1$. $\omega_1(v_1,\cdots,v_n) = c < 0$, and we derive a contradiction. Assume that $T : V_1 \longrightarrow V_2$ intertwines the $\mathfrak{sl}(2)$-representations and preserves the forms. Because T is an intertwining operator, $\det(T|_{[W^i]_p})$ does not depend on the eigenvalue p . Thus,

$$\det(T|_{W^{2\ell}}) = \prod_p \det(T|_{[W^{2\ell}]_p})$$
$$= (\det T|_{[W^{2\ell}]_{-1}})^{2\ell} > 0 \text{ , for all } \ell .$$

Since $(\pi,V,\omega_0)\sim(\pi,V,\omega_1)$ we must have
$1 = \omega_0(v_1,\cdots,v_n) = \omega_1(Tv_1,\cdots,Tv_n) =$
$(\det T)\cdot\omega_1(v_1,\cdots,v_n) = (\det T)\cdot c = [\prod_\ell (\det T|_{W^{2\ell}})]\cdot c$.

But this is impossible because $\prod_{\ell}(\det T|_{W^{2\ell}})$ is positive and c is negative. We conclude there are two equivalence classes of n-forms on V if $H^i = \{0\}$ for all odd i. □

Fix $(\pi, V = \bigoplus_{i=1}^{N} V^i \otimes H^i, \omega)$ (notation as above) and, as we've shown, this triple corresponds to a nilpotent orbit $\mathcal{O}_E$ in $\mathfrak{sl}(V)$.

We construct a nondegenerate $\mathfrak{sl}(2)$-invariant bilinear form $\tau = \bigoplus_i \omega^i \otimes h^i$ on V, where ω^i is our canonical $\mathfrak{sl}(2)$-invariant form on V^i and h^i is an inner product on H^i defined as follows: let $\{T_1^i, \cdots, T_{\dim H^i}^i\}$ be a basis for H^i and set $h^i(T_j^i, T_k^i) = \delta_{jk}$.

The form τ gives us an SL(2)-equivariant mapping $V \longrightarrow V^*$. This map gives us an SL(2)-equivariant isomorphism $V \otimes V \longrightarrow V \otimes V^*$.

Now we decompose $\mathfrak{gl}(V) \cong V \otimes V$ under the action of $SL(2, \mathbb{R})$. Using the notation introduced in Chapter 4, we have:

$$\begin{aligned} \mathfrak{gl}(V) &\cong \bigoplus_{i,j} (V^i \otimes H^i) \otimes (V^j \otimes H^j) \\ &= \bigoplus_{i,j} M^{ij} \\ &= \bigoplus_{i,j} \bigoplus_{\ell,k} \bigoplus_{s=1}^{j} M^{ij}_{k\ell}(i+j+1-2s) \quad . \end{aligned}$$

We want to compute the $\det_{\mathbb{C}}((L \cap K)_0)_{[\mathfrak{sl}(V)]_{-1}})$. In order to do this we must see how $\mathfrak{sl}(V)$ sits inside the decomposition of $\mathfrak{gl}(V)$ given above.

<u>Proposition</u>. $[\mathfrak{gl}(V)]_p = [\mathfrak{sl}(V)]_p$ for $p \neq 0$.

<u>Proof</u>: We write $\mathfrak{gl}(V) = \mathfrak{z} \oplus [\mathfrak{gl}(V), \mathfrak{gl}(V)]$, where $\mathfrak{z}$ denotes the center of $\mathfrak{gl}(V)$. Clearly both summands are $\mathrm{ad}(\mathfrak{gl}(V))$-invariant. Therefore we may decompose the p^{th} eigenspace under $\mathrm{ad}H$ as follows:

$$[\mathfrak{gl}(V)]_p = [\mathfrak{z}]_p \oplus ([\mathfrak{gl}(V), \mathfrak{gl}(V)]]_p \ .$$

Since $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$ and $[\mathfrak{z}]_p = 0$ when $p \neq 0$, we conclude $[\mathfrak{gl}(V)]_p = [\mathfrak{sl}(V)]_p$ for $p \neq 0$. □

So we may now restrict our attention to $[\mathfrak{gl}(V)]_{-1} = [\mathfrak{sl}(V)]_{-1}$. In order to compute $\det_{\mathbb{C}}(\overline{\mathrm{Adh}}|_{[\mathfrak{sl}(V)]_{-1}})$ for $h \in L \cap K$, we construct a basis for $([\mathfrak{sl}(V)]_{-1})_{\mathbb{C}}$.

To do this we want to know how $J = j_{-1}^{-1/2} \cdot \theta \circ adE$ looks on our canonical basis for $[\mathfrak{g}_o]_{-1}$.

The calculation of θ made in Chapter 4 goes through here, and we have:

$$\theta((X^sY^{i-1-s}\otimes T_k^i)\otimes(X^{j-1-t}Y^t\otimes T_\ell^j) = (-1)^{s+t+1}(X^tY^{j-1-t}\otimes T_\ell^j)\otimes(X^{i-1-s}Y^s\otimes T_k^i) .$$

We will need the following:

<u>Lemma</u>. $\theta(X_{k\ell}^{ij}(i+j-2s) = (-1)^i \; X_{\ell k}^{ji}(-i-j+2s)$.

<u>Proof</u>: $\theta(X_{k\ell}^{ij}(i+j-2s)$

$$= \theta(\sum_{p=1}^{s}(-1)^{p-1}\binom{s-1}{p-1}(X^{i-p}Y^{p-1}\otimes T_k^i)\otimes(X^{j-1-s+p}Y^{s-p}\otimes T_\ell^j))$$

$$= (-1)^{i+s+1}\sum_{p=1}^{s}(-1)^{p-1}\binom{s-1}{p-1}(X^{s-p}Y^{j-1-s+p}\otimes T_\ell^j)\otimes(X^{p-1}Y^{i-p}\otimes T_k^i)$$

$$= (-1)^{s-1}(-1)^{i+s+1}\sum_{p=1}^{s}(-1)^{p-1}(X^{p-1}Y^{j-p}\otimes T_\ell^j)\otimes(X^{s-p}Y^{i-s+p}\otimes T_k^i)$$

$$= (-1)^i \; X_{\ell k}^{ji}(-i-j+2s) . \quad \square$$

<u>Proposition</u>. $J(X_{k\ell}^{ij}(-1)) = (-1)^{\frac{3i+j-1-2s}{2}} \; X_{\ell k}^{ji}(-1)$.

Proof:

$$J(X^{ij}_{k\ell}(-1)) = (j_{-1})^{-1/2}\cdot\theta\circ\mathrm{ad}E(X^{ij}_{k\ell}(-1))$$

$$= \theta(X^{ij}_{k\ell}(-1))$$

because $[E, X^{ij}_{k\ell}(-1)] = j_{-1}^{1/2}\ X^{ij}_{k\ell}(1)$.

$$= \theta\ c_0(\mathrm{ad}F)^{\frac{i+j-1-2s}{2}}\ X^{ij}_{k\ell}(i+j-2s)$$

where $c_0 = \dfrac{(\frac{i+j+1-2s}{2})!}{i+j-2s}$;

$$= (-1)^{\frac{i+j-1-2s}{2}}\ c_0(\mathrm{ad}E)^{\frac{i+j-1-2s}{2}}\ \theta(X^{ij}_{k\ell}(i+j-2s))$$

$$= (-1)^{\frac{i+j-1-2s}{2}}\ c_0(\mathrm{ad}E)^{\frac{i+j-1-2s}{2}}\ (-1)^i X^{ji}_{\ell k}(-i-j+2s)$$

$$= (-1)^{\frac{2i+j-1-2s}{2}}\cdot X^{ji}_{\ell k}(-1)\ .\ \square$$

Corollary. $J([M^{ij}]_{-1}) = [M^{ji}]_{-1}$.

Proof: Clear. □

An argument analogous to the one given in Proposition 4.1 shows L is isomorphic to a subgroup of $GL(1)\times\cdots\times GL(N)$ and that $L\cap K$ is isomorphic to a subgroup of $O(1)\times\cdots\times O(N)$.

Theorem. The determinant character is trivial on $(L\cap K)_0$, therefore all nilpotent orbits are admissible.

<u>Proof</u>: We know $([M^{ij}]_{-1})_{\mathbb{C}} = [M^{ij}]_{-1} \oplus J([M^{ij}]_{-1}) = [M^{ij}]_{-1} \oplus [M^{ji}]_{-1}$. Fix $g_j \in O(j)$. Because $\overline{Adg}_j$ preserves $[M^{ij}]_{-1}$ and $[M^{ji}]_{-1}$ and $L \cap K$ commutes with J we conclude:

$$\det_{\mathbb{C}}(\overline{Adg}_j|_{([M^{ij}]_{-1})_{\mathbb{C}}}) = \det_{\mathbb{R}}(\overline{Adg}_j|_{[M^{ij}]_{-1}}) \in \mathbb{R} .$$

If $g_j \in (L \cap K)_0$, then $\det_{\mathbb{C}}(Adg_j|_{([M^{ij}]_{-1})_{\mathbb{C}}}) = 1$. □

References

1. M. Duflo, "Construction de representations unitairies d'un groupe de Lie," in Harmonic Analysis and Group Representations, C.I.M.E., 1982.

2. V. Guillemin and S. Sternberg, Geometric Asymptotics, Math Surveys 14. American Mathematical Society, Providence, Rhode Island (1978).

3. S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.

4. B. Kostant, "The Principal Three-Dimensional Subgroup and the Betti Numbers of a Complex Simple Lie Group," The American Journal of Mathematics 81 (1959), 973-1032.

5. A.A. Kirillov, "Unitary Representations of Nilpotent Lie Groups," Uspehi Mat. Nauk 17 (1962), 57-110.

6. D. Shale, "Linear Symmetrics of Free Bozon Fields", American Mathematical Society Transactions 103 (1962), 149-167.

7. T.A. Springer and R. Steinberg, "Conjugacy Classes", in Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Mathematics 131, Springer-Verlag, Berlin/Heidelberg/New York, 1981.

8. D. Vogan, "Representations of Reductive Lie Groups", to appear in Proceedings of International Conference of Mathematicians (1986).

9. D. Vogan, "Singular Unitary Representations", Non-Commutative Harmonic Analysis and Lie Groups, Lecture Notes in Mathematics 880, 506-536.