Deformation Quantization of Symplectic Fibrations

by

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Abstract

The deformation quantization of symplectic fibrations is carried out. A symplectic fibration is a fibration of a symplectic manifold (M, ω^M) with a symplectic fibre (F, σ) over a symplectic base (B, ω^B) such that the fibres are symplectic submanifolds of M. Consider the algebra of quantized functions on the manifold M, $\mathbb{A}^\hbar(M)$. Assuming the action of the structure group of the fibration $M \to B$ to be Hamiltonian on fibres one can describe how the quantization of the total space of the fibration arises from the quantization of the base with values in the algebra of quantized functions along the fibres, $\mathbb{A}^\hbar(F)$. An algebra homomorphism $\Gamma(B, ({}^{\mathfrak{A}}W_B)_{\text{flat}}) \to \mathbb{A}^\hbar(M)$, i.e. from flat sections of the bundle ${}^{\mathfrak{A}}W_B$, the Weyl algebra bundle of on the base, with values in the bundle of the quantized functions along the fibres is constructed. The example of a sphere–bundle is considered in detail.

Thesis Supervisor: Ezra Getzler
Title: Professor of Mathematics

To the memory of

Alesha Brodsky (June 11, 1991 - March 13, 1992),
Anya Pogosyants(March 31, 1969 - December 15, 1995),
Igor Slobodkin (October 7, 1967 - December 15, 1995).

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Contents

1. Introduction: statement of the problem and the main theorem.

Quantization is a map from functions on a (phase) space to operators on some Hilbert space. It involves a parameter (usually the Planck's constant h or $\hbar = \frac{h}{2\pi i}$). The product of two operators is given by some series in $\hbar$. The inverse of quantization map allows us to get a noncommutative product on functions, namely, by taking the inverse of the product of corresponding operators. There is also the so called correspondence principle. The zero degree term in the $\hbar$-decomposition of a noncommutative function (series in $\hbar$ with functional coefficients) should give a commutative product of functions. The term of degree one should be expressed through the Poisson bracket. This allows one to think of quantization as deformation of a structure of the algebra (Poisson algebra of functions on a manifold). In the formal deformation quantization one does not consider questions of convergency of series in $\hbar$. However, the formal deformation turns out to be useful tool for describing global properties of a manifold. This concept of deformation quantization was described in [2].

There are other ways to quantize functions on a symplectic manifold, there is already some literature on various connection of deformation quantization with geometric quantization, Berezin quantization [6], Toeplitz quantization [17]. It seems that any possible quantization is isomorphic as an algebra to deformation quantization with a certain characteristic class [8].

From now on quantization means deformation quantization, unless stated otherwise. Quantization of a symplectic (or Poisson) manifold M is a construction of a noncommutative associative product on M. It is called a $*$–product, it is a product on the algebra of series in a formal variable $\hbar$ with functional coefficients. This noncommutative algebra $\mathbb{A}^\hbar$ should be a deformation of the algebra of functions on the manifold. Let (M,ω) be a symplectic manifold. Then the symplectic form ω defines a Lie–algebra structure on $C^\infty(M)$, called the Poisson structure. For $f,g \in C^\infty(M)$ let $\{f,g\} = (df)^\sharp(g)$, where $\sharp : \mathcal{T}^*M \to \mathcal{T}M$, defined by ω (see below section (2.2)).

Definition 1.1. Deformation quantization of a symplectic manifold (M,ω) is an associative algebra $(\mathbb{A}^\hbar, *)$ over $\mathbb{C}[[\hbar]]$ with an isomorphism $\epsilon : \mathbb{A}^\hbar/\hbar\mathbb{A}^\hbar \to C^\infty(M)$ s.t.

1. $\mathbb{A}^\hbar$ is flat as a $\mathbb{C}[[\hbar]]$–module,
2. $\mathbb{A}^\hbar$ is separable and complete in $\hbar$–topology,
3. for any $f,g \in C^\infty(M)$

$$\epsilon(\frac{i}{\hbar}(\check{f} * \check{g} - \check{g} * \check{f})) = \{f,g\}$$

for $\check{f}$ and $\check{g}$ s.t. $\epsilon\{\check{f}\} = f$ and $\epsilon\{\check{g}\} = g$,

4. The structure of the product $*$ on $\mathbb{A}^\hbar/\hbar^n\mathbb{A}^\hbar$ for all $n \geq 0$ is given by bidifferential operators.

DeWilde–Lecomte [9] and also Fedosov [11] proved that on any symplectic manifold there exists a quantization.

The following idea lies behind the Fedosov construction (see [10]): a Koszul–type resolution is considered for $C^\infty(M)[[\hbar]]$. Each term of the resolution has a noncommutative algebraic structure hence providing the algebra of functions with a new noncommutative product. Fedosov constructs such a resolution by using the differential forms on the manifold with values in the Weyl–algebra bundle. The main step then is to find a differential on it which respects the algebra structure. This differential is called Fedosov connection and is obtained by an iteration procedure from a torsion free symplectic connection on the manifold.

Lichnerowitz [20] showed that any connection on a symplectic manifold gives rise to a torsion–free symplectic connection. It is useful to introduce a notion of a quantization triple.

Definition 1.2. A quantization triple Φ is the following three objects (manifold, deformation of a symplectic form, a connection) : $\Phi = (M, \omega, \nabla)$, where $\omega = \omega_0 + \hbar\alpha$ is a characteristic class of deformation, $\alpha \in \Gamma(M, \Lambda^2 T^*M)[[\hbar]]$, a series in $\hbar$ with coefficients being closed 2–forms on the manifold.

Then the deformation quantization theorem ([11]) can be stated as follows

Theorem 1.3. *(Fedosov) Given a quantization triple there is a unique way to construct a $*$-product.*

Deligne [8] and also Nest and Tsygan [24] have shown that the class of isomorphisms of deformed algebras is determined by the class of the form $\omega \in H^2(M)[[\hbar]]$.

I study the deformation of the *twisted products* of two symplectic manifolds B and F. One can consider this product as a fibre bundle of symplectic manifolds over a symplectic base $M \to B$ with a standard fibre F: $M = B \ltimes F$. It turns out that there is a symplectic structure on the total space. This kind of fibre bundle is called a symplectic fibration. They were first described by Sternberg [25] and Weinstein [27]. The structure group of the bundle $M \to B$, G, acts by symplectomorphisms on F. I consider the case when G is a finite dimensional Lie group. The action of the group should be Hamiltonian in order for our construction to work. It is not however a big restriction (see remark (3.6))

The question is then the following: how to define the product of two quantization triples, and what the $*$–product on the total space is. Heuristically, the quantization of M then should provide a sort of a "twisted product of quantizations".

Obviously, the product depends on how twisted the symplectic fibration $M = B \ltimes F$ is. This can be described by a connection on $M \to B$ compatible with the symplectic form on M.

Each fibre is a symplectic manifold, so given a G–invariant symplectic connection along the fibres we can quantize the fibres and consider a new bundle $\mathfrak{A}$ over the base B. This bundle can be described as a bundle of algebras of quantized functions on fibres, i.e. the fibre of $\mathfrak{A}$ over a point b is the algebra of quantized functions on the fibre of $M \to B$ at the point b:

$$\mathfrak{A}_b = \mathbb{A}^{\hbar}(M_b) \tag{1}$$

This leads us to a more general case of quantization with coefficients in some bundle of algebras (some examples of such bundles are considered in [13]). Here I am mainly interested in the particular case of the quantization with coefficients in the auxiliary bundle of the form (1). Quantization with values in an auxiliary bundle of the quantization of fibres is different from the quantization with values in any other auxiliary bundle (see remark (3.11)) and is more difficult to perform. However I believe that it is useful to understand the mechanism in order at least to see that quantization is a fundamental notion like some homology theory and hence it should respect a fibration structure.

Definition 1.4. A twisted quantization triple is a triple

$$\Psi = (\Phi, \mathfrak{A}, \nabla^{\mathfrak{A}})$$

where

- $\Phi = (B, \omega^B, \nabla^B)$ is a quantization triple on B,
- $\mathfrak{A}$ is an auxiliary bundle of algebras over B, as in (1)
- $\nabla^{\mathfrak{A}}$ is a covariant derivative on $\mathfrak{A}$, which respects the algebraic structure.

In our case the connection on the bundle $M \to B$ determines the covariant derivative $\nabla^{\mathfrak{A}}$ on this bundle $\mathfrak{A}$. It respects the algebraic structure, i.e. it satisfies Leibnitz rule with respect to the product on $\mathfrak{A}$. The construction of the covariant derivative is given in section (3.2).

Each twisted quantization triple gives a quantization of the manifold with values in the auxiliary bundle. The twisted quantization triple with an auxiliary bundle with fibres (1) corresponds to a quantization triple modeled on M, the total space of the fibration, and hence provides a quantization of the total space.

Theorem 1.5. *A twisted quantization triple gives a quantization of the total space.*

The main claim of this theorem is that quantization of the base with values in the auxiliary bundle (1) corresponds to a certain quantization triple $\Phi = (M, \omega, \nabla)$, where ω is a polynomial in $\hbar$ starting from a symplectic form on M. By Fedosov theorem (1.3) it leads to quantization of the total space of the fibration $M \to B$. The auxiliary bundle obtained from the symplectic fibration structure as in (1) has a noncommutative algebra structure. Noncommutativity of this auxiliary bundle makes the quantization procedure more complicated than when the structure is commutative.

To carry out the program first of all one has to construct a symplectic form on M. There is a one–parameter family of symplectic forms on the total space. The construction involves the notion of weak coupling limit of Guillemin, Lerman and Sternberg [16]. The behavior of the $*$–product when this parameter tends to zero gives us a way to understand the relation between quantizations of the base and the fibre with the quantization of the total space.

Mazzeo and Melrose ([23]) gave an interpretation of Hodge–Leray spectral sequence from an analytic point of view. In particular they introduced language similar to b–calculus for description of Riemannian fibrations. The idea was to put in a small parameter ϵ, so that all horizontal differential forms had the parameter in some degree. So everything which came from the base was "marked" by this small parameter. This gave the description of terms in the spectral sequence by the coefficients in Taylor decomposition w.r.t this small parameter.

Symplectic fibrations provide somewhat similar picture. One even has the parameter naturally coming in the construction of a symplectic form on the total space. Indeed, when the parameter is zero one gets a fiberwise noncommutative product while along the base it is commutative. The $*$–product then is a bidifferential expression only in vertical coordinates. The term at the first degree of the parameter gives a Poisson bracket in the horizontal direction, i.e. it is a first order bidifferential expression in the horizontal direction. If the fibration is trivial these bidifferential expressions are fiberwise constant.

Our main theorem (1.5) is actually a statement about solutions of two equations, which is discussed in the theorem (3.12). Namely,

1. There exists r, a 1-form on the base with values in the twisted Weyl bundle, such that the initial connection becomes flat when one adds r to it.

2. For each section $\mathbf{a}$ of the auxiliary bundle there exists only one corresponding flat section of the twisted Weyl bundle.

Fedosov's quantization procedure is discussed in the chapter (2), calculations and examples are given in the appendix. Main chapter deals with symplectic fibrations, procedure, interpretation, examples.

Notations. Repeated indices assume summation.

Grading and filtration of the Weyl algebra bundle are $\mathbb{Z}$–grading and $\mathbb{Z}$–filtration, I do not use the natural $\mathbb{Z}_2$–grading on the differential forms.

Let $\mathcal{E}$ be a bundle over some manifold M. Then $\mathcal{A}^n(M,\mathcal{E})$ denotes C^∞–sections of n–form bundle with values in the bundle $\mathcal{E}$,

$$\mathcal{A}^k(M,\mathcal{E}) = \Gamma(B, \Lambda^k \mathcal{T}^* B \otimes \mathcal{E}).$$

$\mathcal{A}^n(M)$ denote the bundle of n–forms on M, and

$$\mathcal{A}(M,\mathcal{E}) = \oplus_{n=0}^{\infty} \mathcal{A}^n(M,\mathcal{E}).$$

I use terms "fibration" and "fibre bundle" interchangeably.

The term "connection" is used in two senses, for a covariant derivative on any vector bundle, usually denoted by ∇ and also for a connection on a fibre bundle i.e. a splitting of the tangent bundle to the total space of a fibration into a sum of a vertical and a horizontal subbundles.

2. Generalities on deformation quantization.

2.1. **Weyl algebra of a vector space.** Let E be a vector space with a non-degenerate skew-symmetric form ω. The algebra of polynomials on E is the algebra of symmetric powers of E^*, $S(E^*)$, and it has a skew-symmetric form on it which is dual to ω. Let e be a point in E and $\{e^i\}$ denote its linear coordinates in E with respect to some fixed basis. Then $\{e^i\}$ define a basis in E^*. Let ω^{ij} be the matrix for the skew-symmetric form on E^*. Let us consider the power series in $\hbar$ with values in $S(E^*)$:

$$W(E^*) = S(E^*)[[\hbar]] : \quad a(e,\hbar) = \sum_{k\geq 0} a_k(e)\hbar^k$$

$W(E^*)$ is called the Weyl algebra of E^*.

Then there is an associative noncommutative product on this algebra called Moyal–Vey product:

$$a \circ b(e,\hbar) = \exp\{-\frac{i\hbar}{2}\omega^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial z^l}\}\ a(x,\hbar)\ b(z,\hbar)\ |_{x=z=e} \tag{2}$$

The Lie bracket is defined with respect to this product. We can look at this algebra as at a completion of the universal enveloping algebra of the Heisenberg algebra on $E^* \oplus \mathbb{C}\hbar$, namely, the algebra with relations

$$e^i \circ e^j - e^j \circ e^j = -i\hbar\omega^{ij}$$

where $\omega^{ij} = \omega(e^i, e^j)$ defines a Poisson bracket on E^*. Let us consider the product of the Weyl algebra and the exterior algebra of the space E^*: $W(E^*) \otimes \Lambda E^*$. Let dx^i be the basis in ΛE^* corresponding to e^i in $W(E^*)$.

There is a decreasing filtration on the Weyl algebra $W(E^*)$ given by the degree of generators. e's have degree 1 and $\hbar$ has degree 2:

$$W_p = \{\text{elements with degree} \geq p\}.$$

$$W_0 \supset W_1 \supset W_2 \supset \dots$$

One can define a grading on W as follows

$$gr_i W = \{\text{elements with degree} = i\}.$$

it is isomorphic to W_i/W_{i+1}. One can see that the product (2) preserves the grading.

Definition 2.1. An operator on $W(E^*) \otimes \Lambda E^*$ is said to be of degree k if it maps $W_i \otimes \Lambda E^*$ to $W_{i+k} \otimes \Lambda E^*$ for all i.

Such an operator defines a sequence of maps $gr_i W \otimes \Lambda E^*$ to $gr_{i+k} W \otimes \Lambda E^*$.

Definition 2.2. Derivation on $W(E^*) \otimes \Lambda E^*$ is a linear operator which satisfies the Leibnitz rule:

$$D(ab) = (Da)b + (-1)^{\tilde{a}\tilde{D}} a(Db)$$

where $\tilde{a}$ and $\tilde{D}$ are corresponding degrees. It turns out that all linear derivations are inner operators.

Lemma 2.3. *Any linear derivation D on $W(E^*) \otimes \Lambda E^*$ can be represented as an action of an adjoint operator, i.e. there exist such $v \in W(E^*)$ so that $Da = \frac{i}{\hbar}[v, a]$ for any $a \in W(E^*)$*

Proof. Indeed, $\frac{\partial}{\partial e^i} a = \frac{i}{2\hbar}[\omega_{ij} e^j, a]$.
So for any derivation one can get a formula: $Da = \frac{i}{\hbar}[\frac{1}{2}\omega_{ij} e^i De^j, a]$. □

One can define two natural operators on the algebra $W(E^*) \otimes \Lambda E^*$: δ and δ^* of degree -1 and 1 correspondingly. δ is the lift of the "identity" operator

$$u : e^i \otimes 1 \to 1 \otimes dx^i$$

and δ^* is the lift of its inverse. On monomials $e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ... \wedge dx^{j_n} \in W^m(E^*) \otimes \Lambda^n E^*$ δ and δ^* can be written as follows:

$$\delta : \; e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ... \wedge dx^{j_n} \mapsto$$
$$\sum_{k=1}^{m} e^{i_1} \otimes ...\widehat{e^{i_k}}... \otimes e^{i_m} \otimes dx^{i_k} \wedge dx^{j_1} \wedge ... \wedge dx^{j_n}$$
$$\delta^* : \; e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ...dx^{j_n} \mapsto$$
$$\sum_{l=1}^{n} (-1)^l e^{j_l} \otimes e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ...\widehat{dx^{j_l}}... \wedge dx^{j_n}.$$

Let a_0 be a projection of $a \in \Lambda^* W(E^*)$ to $gr_0 W(E^*) \otimes \Lambda^0 E^*$, which is the center of the algebra, i.e. the summands in a which do not contain either e-s or dx-s.

Lemma 2.4. *Operators δ and δ^* have the following properties:*

$$\delta a = dx^j \frac{\partial a}{\partial e^j} = [-\frac{i}{\hbar}\omega_{kl}\, e^k dx^l,\; a], \quad \delta^* a = y^j \iota_{\frac{\partial}{\partial x^j}} a, \quad \delta^2 = \delta^{*2} = 0$$

On monomials from $gr_m W(E^*) \otimes \Lambda^n E^*$

$$\delta\delta^* + \delta^*\delta = (m+n)Id,$$

where Id is the identity operator. Any element $a \in gr_m W(E^*) \otimes \Lambda^n E^*$ has a decomposition:

$$a = \frac{1}{m+n}(\delta\delta^* a + \delta^*\delta a) + a_0.$$

2.2. **Symplectic connections.** Let M be a manifold and let us consider connections on M.

Proposition 2.5. *Let ω be a skew-symmetric 2-form on $\mathcal{T}M$. Then ω must be closed in order for torsion-free connection ∇ preserving this form to exist.*

Proof. The skew–symmetry of ω is the following condition: $\omega(X,Y) = -\omega(Y,X)$. The connection ∇ is torsion–free when $\nabla_X Y - \nabla_Y X = [X,Y]$. Suppose such ∇ exists. Then it preserves the form ω when $\nabla\omega = 0$. This means that for all $X, Y, Z \in \mathcal{T}M$:

$$\nabla_X \omega(Y,Z) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) \tag{3}$$

Notice, that $\nabla_X(\omega(Y,Z)) = X\omega(Y,Z)$. Then,

$$\begin{aligned} X\omega(Y,Z) \quad - \quad Y\omega(X,Z) \quad + \quad Z\omega(X,Y) & \\ = \omega(\nabla_X Y, Z) - \omega(\nabla_X Z, Y) - \omega(\nabla_Y X, Z) & \\ + \omega(\nabla_Y Z, X) + \omega(\nabla_Z X, Y) - \omega(\nabla_Z Y, X) & \\ = \omega([X,Y], Z) - \omega([X,Z], Y) + \omega([Y,Z], X) & \end{aligned}$$

which is exactly the condition $d\omega = 0$. □

Notice, that in the Riemannian case when the form is symmetric, there is a unique torsion free connection compatible with the form, the Levi–Civita connection. In the case of a skew–symmetric form there are plenty of connections compatible with the form, provided that the form is closed. So the statement of uniqueness of Levi–Civita connection in the Riemannian case is substituted by the requirement for the form to be closed in the skew–symmetric setting.

We are actually interested mostly in the case when M is a symplectic manifold, i.e. when the symplectic form ω is given. Symplectic form ω is a 2-form on $\mathcal{T}M$ which is closed and nondegenerate.

Definition 2.6. A connection which preserves a symplectic form is called a symplectic connection.

Let us actually find how many symplectic connections exist.

Proposition 2.7. [20] [22]. *Let ω be a closed nondegenerate 2–form. Then for every connection ∇ there exist a three–tensor S, such that*

$$\tilde{\nabla} = \nabla + S$$

is a connection on $\mathcal{T}M$ compatible with ω.
Then for $X, Y \in \mathcal{T}M$

$$\bar{\nabla}_X Y = \tilde{\nabla}_X Y - \frac{1}{2} Tor(X, Y)$$

defines a torsion-free connection compatible with the form ω. Here 2-form Tor is the torsion of $\tilde{\nabla}$

$$Tor(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - \tilde{\nabla}_{[X,Y]}$$

We can describe S once we have introduced the following operators:

$$\flat : \mathcal{T}M \to \mathcal{T}^* M$$
$$u^\flat = \omega(u, .) \ for \ \ u \in \mathcal{T}M$$

Let also $\sharp$ be the inverse to $\flat$, given by :

$$\sharp : \mathcal{T}^* M \to \mathcal{T}M$$

Then S is defined:

$$S_X Y = \frac{1}{2} \{(\nabla_X \omega)(Y, .)\}^\sharp$$

Symplectic connections form an affine space whose associated vector space is

$$\mathcal{A}^1(M, sp(2n)),$$

Lie algebra $sp(2n)$ valued one-forms on M.

2.3. **Deformation quantization of a symplectic manifold.** Let M^{2n} be a symplectic manifold with a symplectic form ω. In local coordinates:

$$\omega = \omega_{ij} dx^i \wedge dx^j$$

The symplectic form on a manifold M defines a Poisson bracket on functions on M. For any two functions $u, v \in C^\infty(M)$:

$$\{u, v\} = \omega^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \tag{4}$$

where $(\omega^{ij}) = (\omega_{ij})^{-1}$.

Then we can define the bundle of Weyl algebras $\mathcal{W}(M)$, with fibre at a point $x \in M$ being the Weyl algebra of $\mathcal{T}^*_x M$. Let $\{e^1, ...e^{2n}\}$ be $2n$ generators in $\mathcal{T}^*_x M$. The form ω^{ij}_x defines pointwise Moyal–Vey product.

The symplectic group $Sp(2n)$ acts on the cotangent bundle to M. Let P be the principal $Sp(2n)$-bundle. $Sp(2n)$ preserves the algebraic structure of W. So we can consider $\mathcal{W}(M)$ – the Weyl algebra bundle over M as an associated bundle to the principal bundle P:

$$\mathcal{W}(M) = P \times_{Sp(2n)} W.$$

The filtration and the grading in $\mathcal{W}(M)$ are inherited from $W(\mathcal{T}_x^*M)$ at each point $x \in M$. Denote by $\mathcal{W}^i$ the i-th graded component in $\mathcal{W}(M)$. $\mathcal{W}(M)$ is an infinite sum of symmetric powers of the cotangent bundle to M with the Moyal–Vey product.

$$\mathcal{W}(M) = \oplus_i \mathcal{W}^i$$

Now let us consider the tangent bundle to M and a symplectic connection, ∇, satisfying (3). This symplectic connection can be naturally lifted to act on any symmetric power of the cotangent bundle (by the Leibnitz rule) and since the cotangent bundle $\mathcal{T}^*M \cong \mathcal{W}^1$ we can lift ∇ to be an operator on sections $\Gamma(M, \mathcal{W}^i)$ with values in $\Gamma(M, \mathcal{T}^*M \otimes \mathcal{W}^i)$. By abuse of notations this operator is also called ∇.

It preserves the grading, i.e. it is an operator of degree zero (see definition (2.1) — an operator is said to be of degree k if it acts $\mathcal{A}(M, \mathcal{W}^i) \to \mathcal{A}(M, \mathcal{W}^{i+k})$).

It is clear that in general this connection is not flat ($\nabla^2 \neq 0$). Fedosov's idea is that for $\mathcal{W}(M)$ bundle one can add to the initial symplectic connection some operators not preserving the grading so that the sum gives flat connection on the Weyl bundle.

Theorem 2.8. *(Fedosov.) One can find uniquely* $r_k : \Gamma(M, \mathcal{W}^i) \to \Gamma(M, \mathcal{T}^*M \otimes \mathcal{W}^{i+k})$ *such that*

$$D = -\delta + \nabla + r_1 + r_2 + \dots \tag{5}$$

is a flat connection and

$$\delta^* r_i = 0.$$

There is a one-to-one correspondence between horizontal sections of this connection and functions on the manifold.

The noncommutative structure on the Weyl bundle determines a $*$–product on functions by this correspondence.

The construction of D, its flat sections and $*$–product in coordinates is given in the Appendix.

Actually, $\delta = dx^i \frac{\partial}{\partial e^i}$ (it is of degree -1). The flatness of the connection is given by some recurrent procedure. It is very similar to Kazhdan connection [14] on the algebra of formal vector fields. While Kazhdan connection does not have a parameter involved it has the same structure – it starts with known -1 and 0 degree terms. Other terms are of higher degree and can be recovered one by one.

In fact, the equation

$$D^2 = 0$$

is just the Maurer–Cartan equation for a flat connection. Locally the connection can be written as $D = d + A$, A being an operator from $\Gamma(M, \mathcal{W})$ to $\Gamma(M, \mathcal{T}^*M \otimes \mathcal{W})$. Then this equation becomes:

$$dA + \frac{1}{2}[A, A] = 0$$

3. Quantization of twisted products.

3.1. Symplectic forms on symplectic fibrations. In this section I give the definition of a symplectic fibration and then construct a one-parameter family of symplectic forms on the total space. One can find a nice exposition in the sixth chapter of the recent book [21], see also [16].

Definition 3.1. A symplectic fibration is a locally trivial fibration $\pi : M \to B$ with a symplectic fibre (F, σ) whose structure group preserves the symplectic form σ on F.

This means that

1. There is an open cover U_α of B and a collection of diffeomorphisms

$$\phi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times F$$

   such that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}U_\alpha & \xrightarrow{\phi_\alpha} & U_\alpha \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle pr} & \\ & U_\alpha & \end{array}$$

2. For the fibre over $b \in B$, $F_b = \pi^{-1}(b)$, let $\phi_\alpha(b)$ denote the restriction of ϕ_α to F_b followed by projection onto F, $\phi_\alpha(b) : F_b \to F$. Then

$$\phi_{\beta\alpha}(b) = \phi_\beta(b) \circ \phi_\alpha(b)^{-1} \quad \in Symp(F, \sigma)$$

   for all α, β and $b \in U_\alpha \bigcap U_\beta$.

If $\pi : M \to B$ is a symplectic fibration then each fibre F_b carries a symplectic structure $\sigma_b \in \Omega^2(F_b)$ defined by

$$\sigma_b = \phi_\alpha(b)^* \sigma$$

for $b \in U_\alpha$. The form is independent of α as follows from the definition. Also, if there is a G–invariant symplectic torsion–free connection ∇^F on F it defines a symplectic torsion–free connection on each fibre:

$$\nabla_b = \phi_\alpha(b)^* \nabla^F$$

Definition 3.2. A symplectic form ω on the total space M of a symplectic fibration is called compatible with the fibration π if each fibre F_b is a symplectic submanifold of (M, ω), with σ_b being the restriction of ω to F_b.

Each symplectic form compatible with a symplectic fibration defines a connection on it. Denote by $\mathcal{V}M$ the bundle of vertical tangent vectors. Then

Definition 3.3. A connection on a fibre bundle $\pi : M \to B$ is a choice of splitting

$$\Gamma : \quad \mathcal{T}M = \mathcal{H}M \oplus \mathcal{V}M \tag{6}$$

of the following short exact sequence of vector bundles:

$$0 \to \mathcal{V}M \to \mathcal{T}M \to \pi^*\mathcal{T}B \to 0$$

The connection (6) is compatible with the symplectic form if at each point $x \in M$:

$$\mathcal{H}_x M := \{X \in \mathcal{T}_x M : \quad \Omega(X, V) = 0 \quad \text{for all } \mathcal{V} \in \mathcal{V}_x M\}$$

So each symplectic form whose restriction on fibres is nondegenerate defines a compatible connection. The horizontal subbundle is defined to have all those vector fields which are perpendicular to the vertical ones with respect to the symplectic form.

Denote by "Pr" the projection operator

$$Pr : \quad \mathcal{T}M \to \mathcal{V}M$$

with kernel equal to the chosen horizontal subbundle $\mathcal{H}M$. For $x \in M, \quad Y \in \mathcal{T}_{\pi(x)}B$, let Y^H denote the lift of Y in $\mathcal{H}_x M$, so that

$$Y^H \in \mathcal{H}_x M, \quad \pi_* Y^H = Y \tag{7}$$

Later I will need a notion of a connection on a principal G–bundle P. The fibres of vertical subbundle $\mathcal{V}P$ are naturally identified with $\mathfrak{g}$ in the case of a principal bundle. Hence the horizontal subbundle $\mathcal{H}P$ can be described not only as a kernel of the projection operator "Pr", but also as a kernel of so called connection 1–form:

$$\lambda : \mathcal{T}P \to \mathfrak{g},$$

a G–invariant form on the principal G–bundle P with values in the Lie algebra $\mathfrak{g}$, such that

$$\imath_{\xi_P}\lambda = \xi, \quad \text{if} \quad \xi \in \mathfrak{g}, \xi \mapsto \xi_P \in Vect(P) \quad \text{under the map} \quad \mathfrak{g} \to Vect(P)$$

Let G also act on a symplectic manifold (F, σ) by symplectomorphisms, i.e. there is a group homomorphism

$$G \to Symp(F, \sigma) : \quad g \mapsto \psi_g.$$

The infinitesimal action determines the Lie algebra homomorphism

$$\mathfrak{g} \to Vect(F, \sigma) : \xi \mapsto X_\xi, \quad \text{defined by} \quad X_\xi = \frac{d}{dt}|_{t=0}\psi_{exp(t\xi)} \quad \text{for} \quad \xi \in \mathfrak{g}.$$

Definition 3.4. The action of G on F is called Hamiltonian if:

1. Each vector field X_ξ is Hamiltonian. This means that there is a Hamiltonian function H_ξ so that the 1-form $\imath_{X_\xi}\sigma = dH_\xi$.

2. The map: $\mathfrak{g} \to C^\infty(F) : \xi \mapsto H_\xi$ can be chosen to be a Lie algebra homomorphism with respect to the Lie algebra structure on $\mathfrak{g}$ and the Poisson structure on $C^\infty(F)$.

(If a group action satisfies only first condition it is called weakly Hamiltonian.) Hamiltonian action determines a moment map:

$$\mu : F \to \mathfrak{g}^*,$$

for each point $x \in F$ defined by $< \mu(x), \xi >= H_\xi(x)$, where $< \cdot, \cdot >$ is the pairing: $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$. The following proposition in its present form is an adaptation of a theorem about weak coupling form from [16] for our purposes.

Proposition 3.5. *Let $G \to Symp(F, \sigma) : g \mapsto \psi_g$ be a Hamiltonian action on (F, σ) with a moment map μ^F. Then every connection on the principal G-bundle $P \to B$ over a symplectic manifold (B, ω^B), gives rise to a one-parameter family of symplectic forms on the associated fibration $M = P \times_G F \to B$, which restricts to the forms σ_b on the fibres:*

$$\Omega_\epsilon = \omega^\Gamma + \frac{1}{\epsilon^2}\pi^*\omega^B \tag{8}$$

where ϵ is a small parameter and ω^Γ is the coupling form, so that at a point $x \in M$, $\pi(x) = b$

$$\omega^\Gamma = \sigma_b + H_T$$

where $T \in \mathcal{A}^2(B)$: $T(X, Y) = -Pr([X^H, Y^H])$, $X, Y \in \mathcal{T}B$ and H_V is a Hamiltonian function of a vector field V with respect to the form σ_b, defined by μ_F.

Notice that σ_b acts on vertical vectors only, but H_T is nonzero only on horizontal vector fields. This extra term H_T is needed for the form to be closed and ϵ makes the form Ω_ϵ to be nondegenerate.

Proof. The main idea is to use the so called Weinstein universal phase space: $W = P \times \mathfrak{g}^*$, which has a G-equivariant symplectic form coming from the canonical symplectic form on the cotangent bundle $\mathcal{T}^*P$, besides the action of the group G on W is Hamiltonian. The moment map:

$$\mu^W : W \to \mathfrak{g}^*$$

is given by the projection: $W = P \times \mathfrak{g}^* \to \mathfrak{g}^*$. Then the symplectic reduction of W at 0 value of the moment map

$$\mu = \mu^W + \mu^F$$

is exactly $M = P \times_G F$, and the symplectic form on M is inherited from W.

Now let us look at the details. Consider the case when F is a cotangent bundle $\mathcal{T}^*G$. The action of G on its cotangent bundle is Hamiltonian with respect to a canonical symplectic form on $\mathcal{T}^*G$. The associated bundle

$$P \times_G \mathcal{T}^*G = \mathcal{V}^*P$$

can be naturally identified with the vertical cotangent bundle of P. The fibre of $\mathcal{V}^*P$ over $b \in B$ is the cotangent bundle $\mathcal{T}^*P_b$ of the fibre.

Let $\mathcal{V}P \subset \mathcal{T}P$ be a bundle of vertical tangent vectors. The fibre at a point p is $\mathcal{V}_pP \subset \mathcal{T}_pP$. Let $\mathcal{V}^*P$ be the dual of this vertical subbundle. A connection 1–form, $\lambda_p : \mathcal{T}_pP \to \mathfrak{g}$, determines a horizontal subbundle in $\mathcal{T}_pP$ by

$$\mathcal{H}_pP = \{v \in \mathcal{T}_pP \mid \lambda_p(v) = 0\}$$

$\mathcal{V}^*P = (\mathcal{H}P)^\perp$ contains those cotangent vectors which are zero on the horizontal tangent vectors. The associated bundle

$$P \times_G \mathcal{T}^*G = \mathcal{V}^*P$$

can be naturally identified with the vertical cotangent bundle of P. The connection $\lambda : \mathcal{T}P \to \mathfrak{g}$ and then $\rho : \mathfrak{g} \to \mathcal{V}P$ define the injection $\imath_\lambda : \mathcal{V}^*P \hookrightarrow \mathcal{T}^*P$. By definition of the connection 1–form, this injection is equivariant under the action of G and hence the 2–form

$$\omega_\lambda = \imath_\lambda^* \omega_{can} \in \mathcal{A}^2(\mathcal{V}^*P)$$

is invariant under the action of G and restricts to the canonical symplectic form on the fibres of $\mathcal{V}^*P$. The pull–back of the canonical symplectic form on $\mathcal{T}^*P$ gives a closed 2–form on $\mathcal{V}^*P$.

For the general case consider any symplectic manifold (F, σ) with a Hamiltonian G–action and the moment map $\mu^F : F \to \mathfrak{g}^*$. Consider the product space

$$W = \mathcal{V}^*P \times F$$

and the closed 2–form on W

$$\check{\omega}_\lambda = \omega_\lambda + \sigma.$$

This form restricts to a standard forms on each fibre $W_b = \mathcal{T}^*P_b \times F$ of the bundle $W \to B$. The action of G on these fibres is Hamiltonian with moment map $\mu_W : W \to \mathfrak{g}$

$$\mu_W = \mu_P \circ \imath_\lambda \oplus \mu_F$$

The group G acts freely on $\mu_W(0)^{-1}$ (0 is a regular value of this map) and the quotient $M = \mu_W(0)^{-1}/G$ can be naturally identified with $P \times_G F$. The 2–form $\tilde{\omega}_\lambda$ descends to a 2–form on M, whose restriction to fibres gives a symplectic form in the fibres. □

Remark 3.6. (From [16]). The symplectic fibrations with connection constructed this way turn out to include all symplectic fibre bundles with connection for which the holonomy group is a finite dimensional Lie group. Let $M \to B$ be a symplectic fibration over a connected base B. Suppose that Γ is a symplectic connection on M. Assume that the holonomy group over $b_0 \in B$ is the finite dimensional Lie group G. If γ is a path from b_0 to some other point $b \in B$ the connection gives a holonomy map on fibres

$$\tau_\gamma : F_{b_0} \to F_b. \tag{9}$$

If γ_1 is another curve joining b_0 and b, then

$$\tau_{\gamma_1} = \tau_\gamma \circ g, \tag{10}$$

where $g \in G$. So $P \to B$ is the fibre bundle with a fibre ober $b \in B$ being the set of all τ_γ of the form (9) which form a principal G–bundle by (10). Furthermore every curve on B starting at B_0 has, by construction, a horizontal lift to P, so P is equipped with a G–invariant connection. The original symplectic bundle with connection, M, then comes from the construction of the proposition (3.5).

Let us take a point $x \in M$. One can introduce a local frame $\{f_\alpha\}$ of vertical tangent bundle $\mathcal{V}M$ and a local frame $\{e_i\}$ in $\mathcal{T}B$ at a point $b = \pi(x)$ of B, with dual frames $\{f^\alpha\}$ and $\{e^i\}$. Using the connection we obtain a local frame on the tangent bundle $\mathcal{T}M = \pi^*\mathcal{T}B \oplus \mathcal{V}M$ at a point x.
Then the form can be written as a block matrix:

$$\Omega_\epsilon = \begin{vmatrix} \frac{\pi^*\omega^B}{\epsilon^2} + H_T & 0 \\ 0 & \sigma_b \end{vmatrix} \tag{11}$$

Hence, the corresponding Poisson bracket is also a block matrix:

$$\begin{vmatrix} (\frac{\pi^*\omega^B}{\epsilon^2} + H_T)^{-1} & 0 \\ 0 & (\sigma_b)^{-1} \end{vmatrix}$$

We see that the Moyal product with respect to this form is a product of those for the base and the fibres.

Given a symplectic connection preserving the form σ_b along the fibre F_b one can write it in these local coordinates as follows:

$$\nabla^F = d^F + \frac{i}{\hbar}[\Gamma_{\alpha\beta\gamma} f^\alpha f^\beta d\xi^\gamma, \cdot]. \tag{12}$$

Here $\Gamma_{\alpha\beta\gamma} = (\sigma_b)_{\alpha\delta}\Gamma^\delta_{\beta\gamma}$, where $\Gamma^\delta_{\beta\gamma}$ are Christoffel symbols of the initial symplectic connection on $\mathcal{T}F_b$.

3.2. **Covariant derivative on the auxiliary bundle.** Again, let $\pi : M \to B$ be a locally trivial fibration with a fibre F. Let F be equipped with a quantization triple (F, σ, ∇). Let $\mathbb{A}^\hbar(F)$ be the quantization of F, i.e. noncommutative algebra of formal series in $\hbar$ with coefficients being C^∞–functions on F.

Now let us define the following bundle $\mathfrak{A}$ over B: a bundle which fibres are algebras of quantized functions on the fibres of the bundle $\pi : M \to B$, i.e. the fibre of $\mathfrak{A}$ at a point $b \in B$ is

$$\mathfrak{A}_b = \mathbb{A}^\hbar(M_b).$$

Since $M = P \times_G F$, an associated bundle to a G–bundle $P \to B$, this auxiliary bundle $\mathfrak{A}$ is also associated to P with the fibre $\mathbb{A}^\hbar(F)$

$$\mathfrak{A} = P \times_G \mathbb{A}^\hbar(F).$$

It follows from the following

Proposition 3.7. *Let G be a group of symplectomorphisms of (F, σ). Let (F, σ, ∇) be a quantization triple on F such that ∇ is a G–invariant connection. Let $\mathbb{A}^\hbar(F)$ be the corresponding algebra of quantized functions. Let ρ be the G–action on $\mathbb{A}^\hbar(F)$. Then infinitesimally it is an adjoint action by Hamiltonians given by the Poisson bracket.*

$$\rho(\xi) f = \frac{i}{\hbar}[H_\xi, f] = \{H_\xi, f\} \tag{13}$$

for $\xi \in \mathfrak{g}$, $f \in \mathbb{A}^\hbar(F)$

Proof. There is a Lie algebra map

$$H : \mathfrak{g} \to C^\infty(F).$$

For any $\xi \in \mathfrak{g}$, let X_ξ be the corresponding vector field under the map $\mathfrak{g} \to Vect(F)$. Then the function H_ξ is defined as follows:

$\{H_\xi, f\} = X_\xi f$, for any $f \in C^\infty(F)$, where $\{f, g\}$ is a Poisson bracket on F.

One can see that the algebra action is given by inner derivations. The quantized version of this action is the map

$$\rho : \mathfrak{g} \to End(\mathbb{A}^{\hbar}(F)).$$

It is also given by inner derivations with respect to the $*$–product on $\mathbb{A}^{\hbar}(F)$:

$$\rho(\xi)f = \frac{i}{\hbar}[H_{\xi}, f], \quad \text{for any } f \in \mathbb{A}^{\hbar}(F) \tag{14}$$

where $[f,g] = f * g - g * f$ is a bracket on $\mathbb{A}^{\hbar}(F)$.

The map $H : \mathfrak{g} \to \mathbb{A}^{\hbar}(F)$ is not however necessary a map of Lie algebras. There might be a cocycle in the center of $\mathbb{A}^{\hbar}(F)$:

$$c(\xi, \eta) = [H_{\xi}, H_{\eta}] - H_{[\xi,\eta]}$$

It starts from the second degree in $\hbar$, since in the first degree $[H_{\xi}, H_{\eta}]$ is nothing but the Poisson bracket and the action is Hamiltonian. For discussion see Astashkevich [1]. However calculations involving those similar to appendix show that the section H_{ξ} lifted to a flat section has only first power in $\hbar$. So the action of the algebra $\mathfrak{g}$ on $\mathbb{A}^{\hbar}(F)$ is given by Hamiltonians and is equal to the Poisson bracket in $C^{\infty}(F)$. S o the central element is always 0:

$$c(\xi, \eta) = \{H_{\xi}, H_{\eta}\} - H_{[\xi,\eta]} = 0.$$

□

Covariant derivative on the bundle $\mathfrak{A} \to B$ respecting the algebra structure can be obtained from a connection 1–form. Understanding of the formula for this covariant derivative is important for the sequel. One finds that the curvature of this covariant derivative differs from the coupling form by at most a central form. The rest of this section is devoted to the proof of the following

Proposition 3.8. *Covariant derivative on $\mathfrak{A} \to B$ corresponding to a connection 1–form λ on $P \to B$ is given by the formula:*

$$^{\mathfrak{A}}\nabla f = df + \frac{i}{\hbar}[H_{\lambda}, f]. \tag{15}$$

Its curvature is a 2–form on B with values in $\mathfrak{A}$:

$$^{\mathfrak{A}}R = \frac{i}{\hbar} ad\{H_T\}, \quad \textit{where } T(X,Y) = -Pr[X^H, Y^H], \quad Pr : TP \to \mathfrak{g}.$$

Proof. The covariant derivative formula follows from (14) and the diagram (16) bellow. The expression for the curvature follows from the definition of the curvature of a covariant derivative $\nabla : \Gamma(B,E) \to \mathcal{A}^1(B,E)$. □

Let me describe how in general situation choice of a connection on the principal G–bundle P determines a covariant derivative on any associated vector bundle following the exposition in [3].

Let $\lambda \in \mathcal{A}^1(P, \mathfrak{g})$ be a connection 1–form on a principal G–bundle P. Let also G act on some vector space E, and the action be given by the map ρ,

$$\rho : G \to End(E).$$

Then the bundle $\mathcal{E} = P \times_G E \to B$ is an associated bundle to the principal bundle P. The space of differential forms on B with values in $\mathcal{E}$, $\mathcal{A}^k(B, P \times_G E)$, can be described as the subspace of the space of differential forms on P with values in E. This subspace is a space of all basic forms with values in E, $\mathcal{A}(P, E)_{bas}$. A basic differential form on a principal bundle P with a structure group G, taking values in the representation (E, ρ) of G, is an invariant and horizontal differential form, that is a form $\alpha \in \mathcal{A}(P, E)$ which satisfies

1. $g \cdot \alpha = \alpha, \quad g \in G$
2. $\imath(X)\alpha = 0$ for any vertical field X on P.

Lemma 3.9. *If $\alpha \in \mathcal{A}^q(P, E)_{bas}$, define $\alpha_B \in \mathcal{A}^q(B, P \times_G E)$ by*

$$\alpha_B(\pi_* X_1, \dots, \pi_* X_q)(b) = [p, \alpha(X_1, \dots, X_q)(p)],$$

where $p \in P$ is any point such that $\pi(p) = b$, and $X_i \in \mathcal{T}_p P$. Then α_B is well defined, and the map $\alpha \to \alpha_B$ is an isomorphism from $\mathcal{A}^q(P, E)_{bas}$ to $\mathcal{A}^q(B, P \times_G E)$.

As a particular case, there is a representation of the sections of $\mathcal{E}$ as G–equivariant functions on P with values in E. Let $C^\infty(P, E)^G$ denote the space of equivariant maps from P to E, that is those maps $s : P \to E$ that satisfy $s(p \cdot g) = \rho(g)s(p)$. There is a natural isomorphism between $\Gamma(B, P \times_G E)$ and $C^\infty(P, E)^G$, given by sending $s \in C^\infty(P, E)^G$ to s_B defined by

$$s_B(b) = [p, s(p)];$$

here p is any element of $\pi^{-1}(b)$ and $[p, s(p)]$ is the element of $\mathcal{E} = P \times_G E$ corresponding to $(p, s(p)) \in C^\infty(P, E)^G$. Infinitesimally, a function s in $C^\infty(P, E)^G$ satisfies the formula:

$$(X_P \cdot s)(b) + \rho(X)s(b) = 0, \quad \text{for } X \in \mathfrak{g}$$

where we also denote by ρ the differential of the representation ρ:

$$\rho : \mathfrak{g} \to Vect(E)$$

Given a connection 1–form λ on P one obtains covariant derivative ∇ on the associated vector bundle $\mathcal{E}$ from the following commutative diagram:

$$\begin{array}{ccc} C^\infty(P,E)^G & \xrightarrow{d+\rho(\lambda)} & \mathcal{A}^1(P,E)_{bas} \\ \big\downarrow & & \big\downarrow \\ \Gamma(B,\mathcal{E}) & \xrightarrow{\quad\nabla\quad} & \mathcal{A}^1(B,\mathcal{E}) \end{array} \tag{16}$$

3.3. **Fedosov connection and flat sections on symplectic fibrations.** Now the bundle we consider for quantization is

$$^{\mathfrak{A}}W_B = \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathfrak{A} \to B$$

where $\mathcal{W}_B$ is the Weyl algebra bundle corresponding to the triple $\{B, \omega, \nabla^B\}$. The covariant derivative on this bundle is given by

$$\nabla = \nabla^B \otimes 1 + 1 \otimes^{\mathfrak{A}} \nabla. \tag{17}$$

From now on the $\otimes$ sign will be omitted for sections of $^{\mathfrak{A}}W_B$ where it does not lead to confusion. One can do it since sections of $\mathcal{W}_B$ and sections of $\mathfrak{A}$ commute with each other.

The complex of differential forms with values in the twisted Weyl bundle should give a resolution of $C^\infty(M)[[\hbar]]$.

The construction of a flat connection with values in an auxiliary bundle follows the ideas of the original Fedosov construction on symplectic manifolds. However, instead of the Weyl algebra bundle we consider $^{\mathfrak{A}}\mathcal{W}_B = \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathfrak{A}$. This bundle is a bundle of graded algebras with degrees assigned as in the original $\mathcal{W}_B$ bundle, namely,

$$deg(\hbar) = 2, \quad deg(e^i) = 1, \quad e^i\text{'s being generators of } \mathcal{W}_B. \tag{18}$$

Let

$$(^{\mathfrak{A}}W_B)_n = \{s \in^{\mathfrak{A}} W_B, \quad \text{such that} \quad deg(s) \geq n\}.$$

Then let us define also the grading

$$gr_n(^{\mathfrak{A}}W_B) = \{s \in^{\mathfrak{A}} W_B, \quad \text{such that} \quad deg(s) = n\}.$$

(It is isomorphic to $(^{\mathfrak{A}}W_B)_n/(^{\mathfrak{A}}W_B)_{n+1}$). The pointwise noncommutative product on $^{\mathfrak{A}}W_B$ is inherited from the Moyal–Vey product $\circ$ for $\mathcal{W}_B$ and the noncommutative product $*$ on the auxiliary bundle $\mathfrak{A}$. Let us denote the product on $^{\mathfrak{A}}W_B$ also by $\circ$.

Remark 3.10. The noncommutative product on $\mathfrak{A}$ contains terms in different degrees in $\hbar$, and does not have any other degree bearing terms. Hence, the product on $^{\mathfrak{A}}W_B$ is not preserving the grading anymore.

Let $\{e^i\}$ be linear coordinates at a point $x \in B$ and $\{dx^i\}$ be corresponding generators in $\mathcal{T}^*B$. Then a symplectic form on the base is $\omega^B = \omega_{kl}dx^k \wedge dx^l$. Let $s(x,e,\hbar)$ denote a section of the quantization bundle ${}^{\mathfrak{A}}\mathcal{W}_B$: $s(x,e,\hbar) \in \Gamma(B, {}^{\mathfrak{A}}\mathcal{W}_B)$. Such section has the following degree decomposition

$$s(x,e,\hbar) = \sum_{2k+L \geq 0} \hbar^k e^L \mathbf{s}_{kL}(x), \tag{19}$$

where $\mathbf{s}_{kL}(x)$ are sections of the auxiliary bundle and L is a multiindex:

$$e^L = (e^1)^{l_1} \dots (e^n)^{l_n}$$

and the degree of L, $|L| = l_1 + \ldots + l_n$. Then $2k + |L|$ is the combined degree, so one can say that $\mathbf{s}_{kL}(x) \in gr_{2k+L}({}^{\mathfrak{A}}\mathcal{W}_B)$. Let me list some useful equalities (I omit the $\otimes$–symbol between section from $\Gamma(B, \mathcal{W}_B)$ and $\Gamma(B, \mathfrak{A})$, i.e. for example $e^k\mathbf{s}$ means $e^k \otimes \mathbf{s}$).

$$\begin{aligned} e^k \circ e^l - e^k \circ e^l &= -i\hbar\omega^{kl}, \\ e^k \circ e^l\mathbf{t} - e^l\mathbf{t} \circ e^k &= -i\hbar\omega^{kl}\mathbf{t}, \\ e^k\mathbf{s} \circ e^l\mathbf{t} - e^l\mathbf{t} \circ e^k\mathbf{s} &= e^k \circ e^l[\mathbf{s},\mathbf{t}] - i\hbar\omega^{kl}(\mathbf{s} * \mathbf{t}) \\ &= -i\hbar\omega^{kl}\mathbf{st} + e^k \circ e^l[\mathbf{s},\mathbf{t}] - i\hbar\omega^{kl}(\mathbf{s} * \mathbf{t} - \mathbf{st}), \\ \text{where} \quad \mathbf{s},\mathbf{t} \in \Gamma(B,\mathfrak{A}), &\quad [\mathbf{s},\mathbf{t}] = \mathbf{s} * \mathbf{t} - \mathbf{t} * \mathbf{s}. \end{aligned} \tag{20}$$

Here $\mathbf{st}$ is the first commutative term in $\mathbf{s} * \mathbf{t}$, it has 0–degree. One can see that in the last equality (20) the degree of the first term $\mathbf{st}$ is not different from the common degree on the left hand side since it involves only the product in $\mathcal{W}_B$ which preserves the degree. However, the two other summands have higher degree because the commutator in the auxiliary bundle does not preserve the degree, it starts from the multiple of $\hbar$ and hence raises the degree at least by 2.

Let for $\mathbf{s},\mathbf{t} \in \mathfrak{A}$, such that $\mathbf{s},\mathbf{t}$ do not have terms with multiples of $\hbar$, the $*$–product be written as

$$\mathbf{s} * \mathbf{t} = \mathbf{st} + \hbar\phi_1(\mathbf{s},\mathbf{t}) + \cdots + \hbar^n\phi_n(\mathbf{s},\mathbf{t}) + \dots \tag{21}$$

also the $*$–commutator in $\mathfrak{A}$ can be represented as:

$$\begin{aligned} [\mathbf{s},\mathbf{t}] &= \mathbf{s} * \mathbf{t} - \mathbf{t} * \mathbf{s} \\ &= \hbar\psi_1(\mathbf{s},\mathbf{t}) + \cdots + \hbar^n\psi_n(\mathbf{s},\mathbf{t}) + \dots, \end{aligned} \tag{22}$$

where ϕ's and ψ's are bidifferential expressions.

The symplectic connection ∇^B on (B, ω^B) can be lifted to an operator ∇^B on $\mathcal{A}(B, {}^{\mathfrak{A}}W_B)$ (denoted by the same symbol). The connection (15), ${}^{\mathfrak{A}}\nabla$ also acts on the bundle ${}^{\mathfrak{A}}W_B$, together they form a connection on ${}^{\mathfrak{A}}W_B$:

$$\nabla = \nabla^B \otimes 1 + 1 \otimes^{\mathfrak{A}} \nabla. \tag{23}$$

The curvature of this connection is the sum of curvatures by proposition (3.8): $R = R^B + {}^{\mathfrak{A}}R$. The curvature of the covariant derivative on the auxiliary bundle is given by the form

$${}^{\mathfrak{A}}R = \frac{i}{\hbar}\mathrm{ad}\{H_{T_{kl}} dx^k \wedge dx^l\}, \tag{24}$$

where $H_{T_{kl}} \in \Gamma(B, \mathfrak{A})$, and "ad" means adjoint action with respect to a $*$–product in $\Gamma(B, \mathfrak{A})$.

One can try to find the procedure which would flatten this connection by adding some terms to ∇ so that

$${}^{\mathfrak{A}}D = \nabla + \delta + r$$

is flat. Again, one would like δ to be a (-1)–degree term and

$$r = r_1 + r_2 + \dots, \quad deg(r_i) = i.$$

Remark 3.11. Let us consider a quantization with coefficients in an auxiliary bundle which does not have degree bearing terms. Let $R^{\mathrm{aux}} = R^{\mathrm{aux}}_{kl} dx^k \wedge dx^l$ be the curvature of that bundle. Then all r_i can be obtained recursively similar to the construction in the Appendix (A.1). For example for r_1 one gets

$$\begin{aligned} r_1 &= \delta^{-1}(R^B + R^{\mathrm{aux}}) \\ &= \frac{i}{\hbar}\mathrm{ad}\{R^B_{ijkl} e^i e^j e^k dx^l\} + R^{\mathrm{aux}}_{kl} e^k dx^l \end{aligned}$$

When one constructs the Fedosov connection and the flat sections of it with values in some auxiliary bundle all what happens is that the initial symplectic connection gets shifted by a connection of auxiliary bundle ∇^{aux}, curvature gets an extra term R^{aux} but the formulas are exactly the same as they are for symplectic manifolds.

In the case of symplectic fibrations the auxiliary bundle has grading by degrees of $\hbar$ which alters the procedure. Namely, in the formula for r_1, which is supposed to be a degree 1 term in the Fedosov connection on gets for

$$R = \frac{i}{\hbar}\mathrm{ad}\{R^B_{ijkl} e^i e^j dx^k dx^l\} + \frac{i}{\hbar}\mathrm{ad}\{H_{T_{kl}} dx^k \wedge dx^l\}$$

which gives in $\delta^{-1}R$ terms of degree -1. Using calculations from (20) one gets two terms in

$$\delta^{-1}(\frac{i}{\hbar}\mathrm{ad}\{H_{T_{kl}}dx^k \wedge dx^l\}) = \frac{i}{\hbar}\mathrm{ad}\{e^k\}H_{T_{kl}}dx^l + \frac{i}{\hbar}\mathrm{ad}\{H_{T_{kl}}\}e^k dx^l \tag{25}$$

where first term starts from (-1) degree i.e. it changes δ as well as r_1.

So the problem is different from the usual quantization of symplectic manifolds because the product on ${}^{\mathfrak{A}}W_B$ does not preserve the degree. However in this case one has the following theorem:

Theorem 3.12. *There is a unique solution $\delta + r$ of the equation for flat connection*

$${}^{\mathfrak{A}}D = \nabla + \delta + r$$

on the bundle ${}^{\mathfrak{A}}W_B$:

$${}^{\mathfrak{A}}D^2 = 0$$

satisfying the normalization condition $\delta^{-1}r = 0$. The flat sections of this connection are in one-to-one correspondence with sections of the auxiliary bundle, $C^\infty(B, \mathfrak{A})$.

Proof. The proof can be given by a construction in the spirit of the one from the Appendix. However, in this case we have some subtleties to iron out. Main equation is

$${}^{\mathfrak{A}}D^2 = \nabla^2 + \frac{1}{2}[\nabla, \delta] + \frac{1}{2}[\nabla, r] + \delta^2 + r^2 = 0. \tag{26}$$

Using (21) and (22) one gets that

$$\frac{i}{\hbar}\mathrm{ad}\{H_\lambda\} = \{\psi_1(H_\lambda, \cdot) + \ldots + \hbar^{n-1}\psi_n(H_\lambda, \cdot) + \ldots\}$$

from the formulas (20) the term (25) has the following decomposition

$$\frac{i}{\hbar}\mathrm{ad}\{H_{T_{kl}}e^k dx^l\}$$

$$= \frac{i}{\hbar}\mathrm{ad}\{e^k\}\{H_{T_{kl}} + \hbar\phi_1(H_{T_{kl}}, \cdot) + \ldots + \hbar^n\phi_n(H_{T_{kl}}, \cdot) + \ldots\}dx^l \tag{27}$$

$$+(e^k \circ \cdot)\{\psi_1(H_{T_{kl}}, \cdot) + \ldots + \hbar^{n-1}\psi_n(H_{T_{kl}}, \cdot) + \ldots\}dx^l \tag{28}$$

From the (13) one can deduce that actually most of the terms are 0, however it is proved by Astashkevich only for semisimple groups, so if the structure group of the fibration is semisimple one has for a section $(\mathbf{s}) \in \mathfrak{A}_B^W$

$$\frac{i}{\hbar}\mathrm{ad}\{H_\lambda\}(\mathbf{s}) = \psi_1(H_\lambda, \mathbf{s})$$

$$\frac{i}{\hbar}\mathrm{ad}\{H_{T_{kl}}e^k dx^l\}(\mathrm{s})$$
$$= \frac{i}{\hbar}\mathrm{ad}\{e^k\}(\mathrm{s})H_{T_{kl}}dx^l + \phi_1(H_{T_{kl}}, \mathrm{ad}\{e^k\}(\mathrm{s}))dx^l + \psi_1(H_{T_{kl}}, (e^k \circ \mathrm{s}))dx^l$$
$$= H_{T_{kl}}dx^l \frac{i}{\hbar}[e^k, \mathrm{s}] + e^k dx^l T_{kl}(\mathrm{s}) + [e^k dx^l, T_{kl}(\mathrm{s})]$$

From (13) follows that

$$[H_{T_{kl}}, \mathrm{s}] = T_{kl}\mathrm{s},$$

where T_{kl} is an element in $\mathfrak{g}$ on M and hence its action on $\mathbb{A}^\hbar(F)$ is defined, so it is also defined on sections of the bundle ${}^{\mathfrak{A}}W_B$. The operator $\frac{i}{\hbar}\mathrm{ad}\{e^k\}H_{T_{kl}}dx^l$ is of degree (-1). This term has to be added to the the initial (-1)–degree operator, δ.

The iteration method still can be applied yielding the unique solution. However, in the (-1)–degree term in Fedosov connection one gets

$$gr_{-1}({}^{\mathfrak{A}}D) = \frac{i}{\hbar}(\omega_{kl} + H_{T_{kl}} + \dots)dx^l \mathrm{ad}\{e^k\}.$$

It produces the term which kills ${}^{\mathfrak{A}}R$, but also by the formula (20) it gives some extra terms in higher degrees, which have to be gotten rid off in the process of flattening the connection. The curvature of this connection starts from the term of (-2)nd degree. So in the lowest degree one has:

$$gr_{(-2)}(\frac{i}{\hbar}\mathrm{ad}(\omega_{kl} + H_{T_{kl}})dx^k \wedge dx^l)$$
$$= \{\frac{i}{\hbar}(\omega_{kl} + H_{T_{kl}} + \frac{1}{2}(H_T\omega^{-1}H_T)_{kl} + \dots)dx^l \mathrm{ad}\{e^k\}\}^2).$$

One can see that the adjoint action of $(\omega_{kl} + H_{T_{kl}})dx^k \wedge dx^l$ can be represented as an action of a second power of some series. This series converges if the ratio of ω and H_T is much bigger than 1 (size of the fibres is very small in comparison to the base). It means that the fiberwise symplectic form should be much smaller than the one on the base.

This forces us to introduce a small parameter ϵ:

$$(\omega_{kl} + H_{T_{kl}})dx^k \wedge dx^l = (\frac{\hat{\omega}_{kl}}{\epsilon^2} + H_{T_{kl}})dx^k \wedge dx^l$$

Let us define an operator in the center $\check{\delta}^2 = \frac{i}{\hbar}\mathrm{ad}(\omega_{kl} + H_{T_{kl}})dx^k \wedge dx^l$. Then the (-1)–degree term in the flat connection should be

$$\check{\delta} = \frac{i}{\hbar}\sqrt{(\frac{\hat{\omega}_k l}{\epsilon^2} + H_{T_{kl}})}dx^l \mathrm{ad}\{e^k\}.$$

Now the equation for the flat connection D should start with $\check{\delta}$ instead of δ. One can define an inverse, $\check{\delta}^{-1}$, of the operator $\check{\delta}$. The flat connection we are looking for is now of the form:

$$D = \check{\delta} + \nabla + r.$$

The condition on flatness and the normalization condition $\delta^{-1}r = 0$ give unique solution for r by iteration procedure.

The entire symmetry of the connection form ∇ gives the equality in the first degree of the equation (26):

$$[\check{\delta}, \nabla] = 0$$

Third equation

$$[\check{\delta}, r_1] + \nabla^2 = 0$$

gives the unique solution for r_1 and the procedure goes like in the usual case, except one has to keep in mind that now δ is different from the usual one. Namely, once found r_1, the rest of

$$r = r_1 + r_2 + \dots$$

can be found from the equation:

$$[\check{\delta}, (r - r_1)] + [\nabla, r] + r^2 = 0.$$

It is solved recursively. Uniqueness follows from the condition

$$\delta^{-1}r = 0,$$

which is the same as

$$\check{\delta}^{-1}r = 0.$$

The flat sections of this new connection corresponding to the sections of the auxiliary bundle can also be obtained by a recursive procedure:

$$S = \mathbf{s} + s_1 + s_2 + \dots, \quad s_i \in gr_i({}^{\mathfrak{A}}W_B).$$

The iteration procedure converges with respect to the total degree in ${}^{\mathfrak{A}}W_B$ given by (18). □

Step leading to the flat connection of D^M on $\mathcal{W}^M$ consists of identifying flat sections of D, with flat sections of the Fedosov connection on B with values in the auxiliary bundle, $\mathfrak{A}$.

3.4. **Weak coupling limit. MM calculus.** Let $\mathcal{W}_M$ be the Weyl algebra bundle on M. Then consider $\mathcal{W}_{M/B}$, the bundle on B with the fibers $\mathcal{W}_z$ being the restriction of the bundle $\mathcal{W}_M$ to M_z, $z \in B$. The set of sections of the product of the bundles $\mathcal{W}_B \otimes_{C[[\hbar]]} \mathcal{W}_{M/B}$ maps into a dense subset in sections of the bundle $\mathcal{W}_M$.

Lemma 3.13. *There is an isomorphism:*

$$\mathcal{A}^n(M, \mathcal{W}_M) = \oplus_{p+q=n} \mathcal{A}^p(B, \mathcal{W}_B \otimes_{C[[\hbar]]} \mathcal{A}^q(M/B, \mathcal{W}_{M/B})) \tag{29}$$

Proof. At each point z of M the Weyl algebra can be defined as the universal enveloping algebra of the Heisenberg algebra of $\mathcal{T}_z^*M$. Hence one can define the Weyl algebra as

$$\mathcal{W}_z = U(\mathcal{T}^*{}_z M \oplus \hbar\mathbb{R}).$$

The PBW–theorem says that $U(\mathcal{T}M) \simeq S(\mathcal{T}M)$ as vector spaces. This gives isomorphisms of algebras:

$$S(\pi^*\mathcal{T}_{\pi(z)}B \oplus \mathcal{V}_z M) = S(\pi^* T_{\pi(z)}B) \otimes S(\mathcal{V}_z M)$$
$$U(\mathcal{T}_z M \oplus \hbar\mathbb{R}) = U(\pi^*\mathcal{T}_{\pi(z)}B \oplus \hbar\mathbb{R}) \otimes_{\mathbb{C}[[\hbar]]} U(\mathcal{V}_z M \oplus \hbar\mathbb{R})$$
$$\mathcal{W}(M)_z = \mathcal{W}(\pi^*\mathcal{T}_{\pi(z)}B \oplus \mathcal{V}_z M) = \mathcal{W}(\pi^*\mathcal{T}_{\pi(z)}B) \otimes \mathcal{W}(\mathcal{V}_z M) \tag{30}$$

Let us define a bundle $\pi_*\mathcal{W}$ on B so that its sections are: $\Gamma(B, \pi_*\mathcal{W}) = \Gamma(M, \mathcal{W})$ with a fibre $(\pi_*\mathcal{W})_z = \Gamma(M_z, \mathcal{W}_z)$ The fibre of the Weyl algebra bundle $\mathcal{W}$ at a point $z \in M$ is

$$\Gamma(M_z, \pi^*(\mathcal{W}_B)_{\pi(z)} \otimes \mathcal{W}_{M_z}) = (\mathcal{W}_B)_{\pi(z)} \otimes \Gamma(M_z, \mathcal{W}_{M_z})$$

$$S(\mathcal{T}^*M) \cong S(\mathcal{T}^*B) \otimes S(\mathcal{T}^*(M/B)).$$

□

Consider a quantization map:

$$Q : \Gamma(B, {}^{\mathfrak{A}}W_B) \to \Gamma(B, \mathcal{W}_B \otimes_{C[[\hbar]]} \mathcal{W}_{M/B}) \tag{31}$$

induced by

$$Q^F : C^\infty(F)[[\hbar]] \to \Gamma(F, \mathcal{W}_F),$$

which sends functions on F to flat sections of the Fedosov connection on $\mathcal{W}^F$. Let $\hat{D}$ be the result of the mapping of ${}^{\mathfrak{A}}D$ under (31) The representation (19) suggests that sections $\Gamma(B, {}^{\mathfrak{A}}W_B)$ may be regarded as sections of some Weyl algebra bundle $\mathcal{W}$ over the total space of the bundle $M \to B$. Identifying the flat sections of ${}^{\mathfrak{A}}D$ (3.12) with the sections $b \in \Gamma(M, \mathcal{W})$, such that

$$D^F b = 0$$
$$\hat{D} b = 0$$

simultaneously leads to the map from twisted quantization to the quantization of the total space. These two equations are equivalent to the equation

$$D^M b = (D^F + \hat{D}) b = 0.$$

The connection $\hat{D}$ together with the Fedosov connection along the fibres, D^F, gives rise to a single connection D^M on $\mathcal{W}$:

$$D^M = d + \frac{i}{\hbar}[\Gamma_{ijk} e^i e^j dx^k + \Gamma_{\alpha\beta\gamma} f^\alpha f^\beta d\xi^\gamma + H_{\lambda_k} dx^k, \cdot] + \dots$$

where d means the exterior differential on the manifold M. Also we use the expression for ∇^F in local coordinates (12).

One can try to represent D^M in the form

$$D^M = \nabla^M + \frac{i}{\hbar}[\gamma, \cdot]$$

where ∇^M is the connection corresponding to a symplectic connection on the bundle $\mathcal{T}^* M$ and γ is a globally defined section of $\mathcal{W}$. We arrive at the following

Proposition 3.14. *Under the map (31) and then (29) the connection D (3.12) on $\Gamma(B, {}^{\mathfrak{A}} W_B)$ goes to a flat connection*

$$D \rightsquigarrow D^M = \hat{D} + D^F$$

on $\Gamma(M, W_M)$. The Weyl algebra structure is defined on $\Gamma(M, W_M)$ from the symplectic form on M, (11). There is an isomorphism of flat sections of these connections which leads to the homomorphism of algebras:

$$\Gamma(B, {}^{\mathfrak{A}} W_B)_{flat}) \cong \mathbb{A}^\hbar(M).$$

A symplectic connection on TM, ∇^M under this homomorphism corresponds to a connection ∇, (23).

In order to see how this homomorphism works I want to introduce calculus similar to the one in [23]. Given a connection (6) on M: $\mathcal{T}M = \mathcal{H}M \oplus \mathcal{V}M$ one implements the splitting into the structure of the product manifold $X = M \times [0, \epsilon_0)$, where $\epsilon_0 \ll 1$ is some fixed small number. (We want it to be small enough so that ϵ involved in the symplectic form (8) is bigger than ϵ_0. The product $X = M \times [0, \epsilon)$ has an induced fibration, with leaves F and the base $B \times [0, \epsilon)$.) Consider the space $\mathcal{L}$ of C^∞ vector fields on X which are tangent to the fibres, M, of the product structure and which are also tangent to the fibres of the fibration $M \to B$, above $M_0 = \{\nu = 0\}$. In local coordinates x^j, ξ^k in M, where the x's give coordinates in B, the elements of $\mathcal{L}$ are the vector fields of the form

$$\sum_{j=1}^{2p} a(x,\xi,\nu)\nu\partial_{x_j} + \sum_{k=1}^{2q} b(x,\xi,\nu)\partial_{\xi_k}.$$

Consider a vector bundle ${}^{\nu}\mathcal{T}M$ for which $\mathcal{L}$ is the set of sections

$$\mathcal{L} = C^{\infty}(X,{}^{\nu}\mathcal{T}M).$$

There is a natural bundle map $\imath_\nu : {}^{\nu}\mathcal{T}M \to \mathcal{T}_X M$, the lift of $\mathcal{T}M$ to X. It is an isomorphism except over M_0, where its range is equal to $\mathcal{V}M$. It is important to define the dual map

$$\imath^{\nu} : \mathcal{T}^*_X M \to {}^{\nu}\mathcal{T}^*M,$$

which range over m_0 is a subbundle which is naturally isomorphic to the bundle of forms on fibres.

Given a connection (6) on M: $\mathcal{T}M = \mathcal{H}M \oplus \mathcal{V}M$, the restriction of ${}^{\nu}\mathcal{T}^*M$ to the boundary, M_0, of X naturally splits

$${}^{\nu}\mathcal{T}^*{}_{M_0}M = \mathcal{V}^*M \oplus \nu^{-1}\mathcal{T}^*B, \quad u_0 = \imath_\nu(\alpha) + \nu^{-1}\pi^*\beta.$$

The exterior powers of ${}^{\nu}\mathcal{T}^*M$ also split at the boundary so one can define a new bundle of rescaled differential forms on X. Namely,

$${}^{\nu}\mathcal{A}^k_x(M) = \sum_{j=0}^{k} \mathcal{A}^j_x(M/B) \oplus \nu^{-(k-j)}\mathcal{A}^{k-j}_{\pi(x)}(B).$$

The symmetric powers of ${}^{\nu}\mathcal{T}^*M$ have the following decomposition

$${}^{\nu}S^k\mathcal{T}^*M = \sum_{j=0}^{k} S^j(M/B) \oplus \nu^{-(k-j)}S^{k-j}\mathcal{T}^*B.$$

This way one can define the rescaled Weyl algebra bundle. At a point $x \in M$ as before one can introduce a local frame $\{f_k\}$ of vertical tangent bundle $\mathcal{V}M$ corresponding to ∂_{ξ_k} and a local frame $\{e_i\}$ in $\mathcal{T}B$, corresponding to ∂_{x_j} at a point $b = \pi(x)$ of B, with dual frames $\{f^k\}$ and $\{e^i\}$. Using the connection we obtain a local frame on the tangent bundle $\mathcal{T}M = \pi^*\mathcal{T}B \oplus \mathcal{V}M$ at a point x. The differential on the bundle of forms ${}^{\nu}\mathcal{A}^k(M)$ is given by

$$d = \nu\frac{dx^j}{\nu}\partial_{x_j} + d\xi^k\partial_{\xi_k},$$

so that $d :^{\nu} \mathcal{A}^k(M) \to^{\nu} \mathcal{A}^{k+1}(M)$. Similarly we can find a symplectic connection on ${}^{\nu}\mathcal{T}M$ from the initial connection on $\mathcal{T}M$.

Then ${}^{\nu}D$, the Fedosov connection on rescaled Weyl algebra bundle ${}^{\nu}\mathcal{W}_M$ will have the Taylor decomposition into degrees of ν. Since Fedosov connection is flat it should give an equation in each degree of ν. The ν–decomposition of $({}^{\nu}D)^2$ gives 0 in each degree of ν. The result for quantization can be stated as follows:

1. The quantization of M for $\nu = 0$ is $C^{\infty}(B, \mathbb{A}^{\hbar}(M/B))$
2. The term at the first power of ν is the product on the base given by the Poisson bracket with values in the quantization of the fiber.
3. n–th power of ν allows one to write a product on the base with values in the product of the fibre upto the n–th power in $\hbar$.

The total space of a symplectic fibration $M \to B$ together with the rescaled symplectic form and a connection preserving it make such a quantization triple which is easily associated with a twisted quantization triple and hence gives a map from quantization of the total space to a quantization of the base with values in the auxiliary bundle.

4. Examples of symplectic fibrations and their quantization.

Fedosov quantization provides a way to construct a $*$–product on a symplectic manifold. In the previous section I showed how Fedosov quantization works for any symplectic fibration. However, explicit formulas become complicated very quickly and one can find explicit formulas only in a few particular cases.

The article [12] treats a symplectic fibration $M \to B$ with a symplectic fibre being a cylinder. The fibre can be represented as $\mathbb{C}^* = \mathcal{T}^*S^1$ – the cotangent bundle to the circle. Locally a point z in M can be described by coordinates (x, r, θ), where $x = (x_1, ..., x_{2n})$ denotes coordinates of the base point $\pi(z)$ while r and θ are coordinates in the fibre. Then $M = P \times_{u(1)} \mathbb{C}^*$, where P is a principal $u(1)$–bundle. The symplectic form is constructed from proposition (3.5). It is an example of a simplest case, with a bundle being a cotangent bundle to a principal bundle. Let λ be a connection one–form on P. Hamiltonian of λ is a function in r only, it does not involve θ. Quantization on the fibres is the Weyl quantization like in $\mathbb{R}^2 n$ (see (34)). So one gets from this fiberwise quantization that fiberwise flat sections depend on $(r + f^1), (\theta + f^2)$, where (f^1, f^2) are coordinates in $T^*(M/B)$ corresponding to $(dr, d\theta)$ in the Weyl algebra bundle. The flat connection in the Weyl algebra bundle on M is a series in $(r + f^1)$ (it does not involve the other coordinate $(\theta + f^2)$.

Other examples with two–dimensional fibres might include any Riemannian surfaces as fibres except a torus (see [21]).

An example of fibration with fibres being $\mathbb{R}^{2n}$ is considered in [13].

I want to present here in details a very natural example of a symplectic fibration, S^2–bundles, which however is already more complicated than the fibration with cylinder–fibres. The symplectic form in this example is considered in [21]. Since I need it , let me give its construction here.

Let P be an S^1-principal bundle. Let $\mathcal{V}P \subset \mathcal{T}P$ be the bundle of vertical tangent vectors. The fibre at a point p being $\mathcal{V}_pP \subset \mathcal{T}_pP$. Let $\mathcal{V}^*P$ be the dual of this vertical subbundle. A connection 1–form, $\lambda : \mathcal{T}_pP \to \mathbb{R}$ determines a horizontal subbundle by

$$\mathcal{H}_p = \{v \in \mathcal{T}_pP \mid \lambda_p(v) = 0\}$$

This horizontal subbundle induces an injection:

$$\imath_\lambda : \mathcal{V}^*P \hookrightarrow \mathcal{T}^*P,$$

namely, a vertical cotangent vector $\xi \in \mathcal{V}^*_pP$ is a linear functional on $\mathcal{T}_pP$ which vanishes on the horizontal subspace $\mathcal{H}_pP$. The manifold

$$W = \mathcal{V}^*P \times S^2$$

carries a natural symplectic form

$$\Omega = pr_B^*\omega_B + \imath_\lambda^*\omega_{can} + pr_S^*\sigma$$

where $pr_B : W \to B$ and $pr_S : W \to S^2$ are the obvious projections and σ is and S^1–invariant volume form on S^2. One can see that

- Ω is invariant under the diagonal action of S^1.
- $\mathcal{V}^*P$ is equivariantly diffeomorphic to $P \times \mathbb{R}$, so the moment map $\mu : W = P \times \mathbb{R} \times S^2 \to \mathbb{R}$ is given by

$$\mu(p, \eta, z) = h(z) - \eta$$

  where $h : S^2 \to \mathbb{R}$ is the height function, i.e. it is a moment map for the action of S^1 on S^2 by rotating about a vertical axis. It's value is X, where X is a vertical coordinate on S^2.
- The level set $\mu^{-1}(0)$ can be identified with the manifold $P \times S^2$ by the map which takes the form Ω to

$$\omega = pr_B^*\omega_B - d(H_\lambda) + pr_S^*\sigma$$

  on $P \times S^2$, where $H(p, z) = h(z)$ is the height function on S^2.

Thus M is the symplectic reduction of (W, Ω) at 0.

Let me consider local coordinates at a point z in M: (x, X, θ), where $x = (x_1, ..., x_{2n})$ denotes coordinates of the base point $\pi(z)$ while X and θ are cylindrical polar coordinates in the fibre, X gives a height function and θ is an angle. Then the symplectic form on the fibre is

$$\sigma = dX \wedge d\theta.$$

The vertical vector field $\frac{\partial}{\partial\theta}$ has a Hamiltonian X:

$$H_{\frac{\partial}{\partial\theta}} = X.$$

The auxiliary bundle $\mathfrak{A}$ is a bundle of quantized functions on fibres, it is associated to S^1–bundle. The connection on $\mathfrak{A}$ is inherited from a connection 1–form by Proposition(3.8). In coordinates it is as follows:

$${}^{\mathfrak{A}}\nabla \mathbf{s} = d\mathbf{s} + \lambda\{X, \mathbf{s}\},$$

where λ is a local 1-form on the base and $\{\cdot,\cdot\}$ is a fibrewise Poisson bracket, $\mathbf{s}$ some section of $\mathfrak{A}$. The curvature of this connection is:

$${}^{\mathfrak{A}}R = \frac{i}{\hbar}\mathrm{ad}H_T = T\{X, \cdot\},$$

where T is a 2–form on the base, the curvature of the connection λ. Since a Hamiltonian of any vector field is a linear function of just one coordinate X. One can see that there is no extra terms in the connection coming from the auxiliary curvature because

$$[H_T, H_T] = 0.$$

It follows from Tamarkin's formula for quantization of a sphere:

$$f(v,\bar{v}) * g(v,\bar{v})|_{v=0} =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{h^n}{(1-h)\cdot\dots\cdot(1-(n-1)h)} \frac{\partial^n}{\partial z^n} \frac{\partial^n}{\partial \bar{w}^n} f(z,\bar{z}) g(w,\bar{w})|_{z=w=0} \tag{32}$$

for z, v, w being holomorphic coordinates at 0. A flat connection is as follows:

$$^{\mathfrak{A}}D = \check{\delta} + \nabla + r,$$

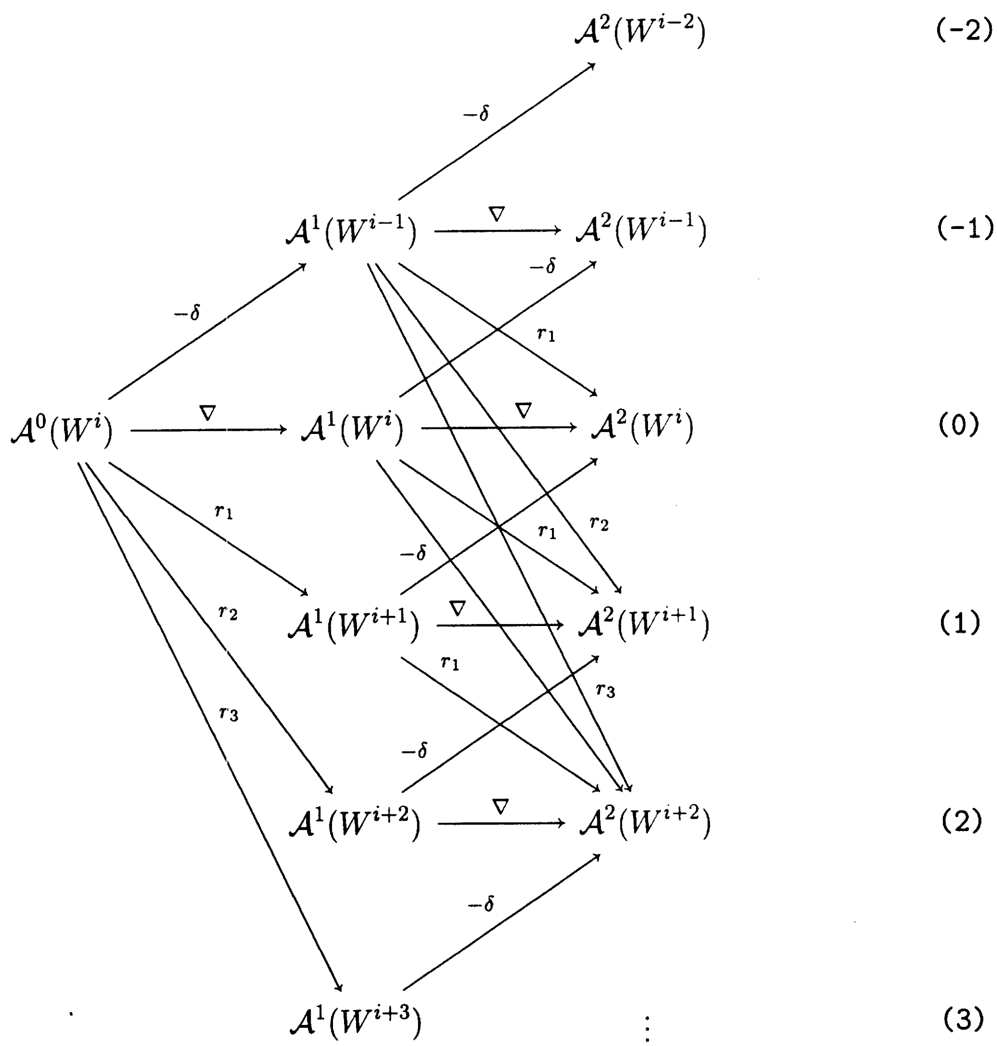
where $\check{\delta} = \mathrm{ad}(\frac{\omega_{kl}}{\epsilon^2}\sqrt{(1+\epsilon^2\omega^{-1}TX)}e^k dx^l)$, ϵ – small number, which makes the $\sqrt{\ }$ to exist. As a result one gets that the flat connection in case of a sphere bundle does not depend on the coordinate θ. The characteristic class of the deformation is $\frac{\omega^B}{\epsilon^2} + TX$.

Appendix A. Constructions.

A.1. **Fedosov connection.** Let $\mathcal{A}^p(W^i) = \Gamma(M, \Lambda^p \mathcal{T}^* M \otimes \mathcal{W}^i)$.
For curvature to be equal 0 we have to get 0 when applying the connection twice to any element $a \in \Gamma(M, \mathcal{W})$: $D^2 a = 0$
Showing that it's true for any element in $\mathcal{A}^0(W^i) = \Gamma(M, \mathcal{W}^i)$ for any i will do.
Let us represent the action of operators constituting the connection

$$D = -\delta + \nabla + r_1 + r_2 + r_3 + r_4 + \dots$$

by arrows pointing in directions corresponding to their degree.



$$\delta \text{ has degree } -1,$$
$$\nabla \text{ has degree } 0,$$
$$\text{each } r_k \text{ has degree } k.$$

The sum of arrows coming to $\mathcal{A}^2(W^i)$ for every i must be 0. For every degree we get an equation on operators. We get an infinite sequence of recursive equations which are uniquely solvable.

- $\delta^2 = 0$
- $-[\delta, \nabla] = 0$
- $-[\delta, r_1] + \nabla^2 = 0$
- $-[\delta, r_2] + [\nabla, r_1] = 0$
- $-[\delta, r_3] + [\nabla, r_2] + \frac{[r_1, r_1]}{2} = 0$
- $-[\delta, r_4] + [\nabla, r_3] + [r_1, r_2] = 0$

All these equations can be written together:

$$D^2 = (-\delta + \nabla + [r, .])^2 = \nabla^2 - [\delta, r] + r^2 = 0$$

where $r = r_1 + r_2 + \ldots$ It is solved recursively: in each degree $k \geq 0$ one gets an equation involving r_i for $i \leq k$. Let us show what happens in the first few equations.

Degree -2. The equation is $\delta^2 = 0$. It is satisfied by the Lemma (2.4).

Degree -1. Next one is $-[\delta, \nabla] = 0$. It is true by a simple calculation.

Degree 0. Here is the first nontrivial calculation. We have to find such r_1 that $-[\delta, r_1] = \nabla^2$.

a) Existence. First of all:

$$[\delta, \nabla^2] = [\delta, \nabla]\nabla - \nabla[\delta, \nabla]$$

which is 0 by the previous equation.

There is an operator δ^* which is a homotopy for δ.

$$\delta^*\delta^* = 0, \quad \delta\delta^* + \delta^*\delta = id\ c \tag{33}$$

This c is a number of y 's and dx 's, for example for a term $y^{i_1} \ldots y^{i_p} dx^{j_1} \ldots dx^{j_q}$ this number $c = p + q$. Let us put

$$r_1 = \delta^* \nabla^2$$

then indeed:

$$[\delta, r_1] = \delta(r_1) = \delta(\delta^* \nabla^2) = \nabla^2$$

and also $\delta^* r_1 = 0$.

b) Uniqueness.

Let $r_1' = r_1 + \alpha$, such that $\delta^*\alpha = 0$. Then $\alpha = \delta^*\beta$ for some β. Hence, $\delta\delta^*\beta = 0$, because $\delta(r_1 + \alpha) = \delta r_1$.

From (33) we get that $\beta = \delta^*\delta\beta$ and $\alpha = \delta^*\beta = \delta^*(\delta^*\delta\beta) = 0$

Degree 1. $[\delta, r_2] = [\nabla, r_1]$ gives the equation on the operator r_2.

a) Existence. Again we show

$$[\delta, [\nabla, r_1]] = [[\delta, \nabla], r_1] - [\nabla, [\delta, r_1]] = 0$$

b) Uniqueness. $r_2 = \delta^*([\nabla, r_1])$ similar to the previous one.

From these equations one finds the Fedosov connection:

$$D = \delta + \nabla + \delta^{-1}\nabla^2 + \delta^{-1}\{\nabla, \delta^{-1}\nabla^2\} + \dots$$

A.2. **Flat sections and the $*$–product.** The Fedosov connection is flat and its flat sections are in one-to-one correspondence with series in $\hbar$ with functional coefficients. There is a map:

$$C^\infty(M)[[\hbar]] \to \mathcal{A}^0(W)$$

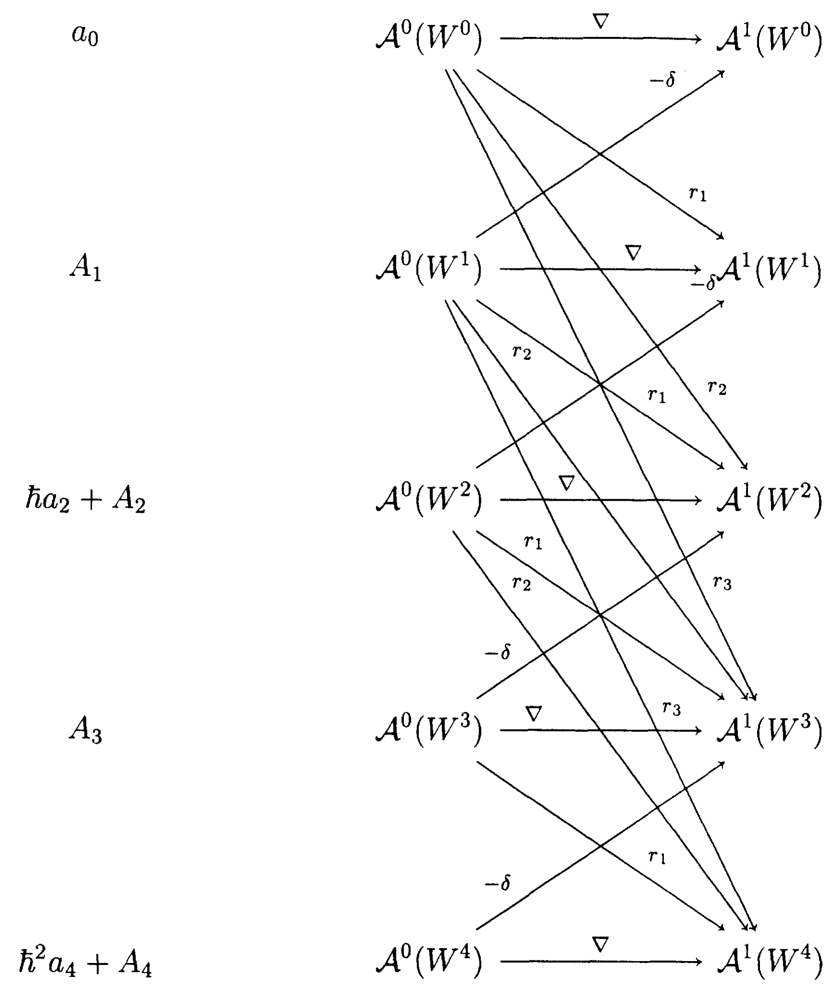
Given a series:

$$a = a_0 + \hbar a_2 + \hbar^2 a_4 + \dots$$

one can find uniquely the corresponding flat section of the Weyl algebra bundle

$$A = a + A_1 + A_2 + A_3 + \dots$$

so that $\delta^{-1}(A - a) = 0$.



All r_i actually kill a_0 so the equations are as follows

1. $\nabla a_0 - \delta A_1 = 0$
2. $\nabla A_1 - \delta(\hbar a_2 + A_2) = 0$
3. $r_1 A_1 + \nabla(\hbar a_2 + A_2) - \delta A_3 = 0$
4. $r_2 A_1 + r_1(\hbar a_2 + A_2) + \nabla A_3 - \delta(\hbar^2 a_4 + A_4) = 0$
5. $r_3 A_1 + r_2(\hbar a_2 + A_2) + r_1 A_3 + \nabla(\hbar^2 a_4 + A_4) - \delta A_5 = 0$

Let dx^i be a local frame in $\mathcal{T}^*M$. Then let the corresponding generators in W be $\{e^i\}$. Then A_i are of the form $A_{k_1 \dots k_i} e^{k_1} \dots e^{k_i}$. the symplectic connection ∇ locally can be written as:

$$\nabla = dx^j \frac{\partial}{\partial x^j} + \frac{i}{2\hbar}[\Gamma_{jkl} dx^j e^k e^l, \ ]$$

For $a = a_0$ one gets:

1. $A_1 = \delta^{-1}\nabla a = \delta^{-1} da = \partial_l a e^l$
2. $A_2 = \delta^{-1}\nabla A_1 = \delta^{-1}\nabla(\partial_l a e^l) = \{\partial_k\partial_l a + \Gamma_{kl}^{j}\partial_j a\} e^k e^l$
3. and so on.

One can deduce first terms in the $*$–product of two functions $\mathbf{a}, \mathbf{b} \in C^\infty(M)$:

$$\mathbf{a} * \mathbf{b} = \mathbf{ab} - \frac{i\hbar}{2}\omega^{ij}\partial_i\mathbf{a}\partial_j\mathbf{b} - \hbar^2(\partial_i\partial_j\mathbf{a} + \Gamma_{ij}^{l}\partial_l\mathbf{a})\omega^{im}\omega^{jn}(\partial_m\partial_n\mathbf{b} + \Gamma_{mn}^{k}\partial_k\mathbf{b}) + \dots$$

A.3. **Examples of deformation quantization of symplectic manifolds.** The procedure of deformation quantization requires calculations which are not obvious and sometimes do not give nice formulas. However in few cases one can calculate explicitly the $*$–product for particular manifolds. The first trivial example is the quantization of $\mathbb{R}^{2n}$. Let $\{x^1, \dots, x^{2n}\}$ be a local coordinate system at some point $x \in \mathbb{R}^{2n}$. The Darboux symplectic form in these coordinates is

$$\omega = dx^i \wedge dx^{i+n}, \quad 1 \le i \le n$$

The standard symplectic form and the trivial connection gives an algebra of pseudodifferential operators in $\mathbb{R}^{2n}$. Namely, calculations from the last two sections show that for the trivial symplectic connection:

$$\nabla = d$$

one gets Fedosov connection:

$$D = d - \delta \quad \text{or in coordinates} \quad D = dx^i\left(\frac{\partial}{\partial x^i} - \frac{\partial}{\partial e^i}\right).$$

Flat section of such connection corresponding to a function $\mathbf{a}$ under the quantization map is as follows

$$A = \mathbf{a} + e^i\frac{\partial \mathbf{a}}{\partial x^i} + e^i e^j \frac{\partial^2 \mathbf{a}}{\partial x^i \partial x^j} + \cdots$$

We see that it gives a formula for Taylor decomposition of a function $\mathbf{a}$ at a point x. Then the $*$–product of two flat sections is given by the formula (2). It is easy to deduce that for two functions $\mathbf{a}$ and $\mathbf{b}$ the $*$–product is

$$\begin{aligned} \mathbf{a} * \mathbf{b} &= \exp\{-i\hbar\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^{n+k}}\}\mathbf{a}(y)\mathbf{b}(z)|_{y=z=x} \\ &= \mathbf{ab} - i\hbar\frac{\partial \mathbf{a}}{\partial x^k}\frac{\partial \mathbf{b}}{\partial x^{n+k}} - \frac{\hbar^2}{2}\left(\frac{\partial^2 \mathbf{a}}{\partial x^k \partial x^l}\right)\left(\frac{\partial^2 \mathbf{b}}{\partial x^{n+k}\partial x^{n+l}}\right) + \cdots . \end{aligned} \tag{34}$$

So it is clear that if we map $C^\infty(\mathbb{R}^{2n})$ to differential operators on $\mathbb{R}^n$ $*$–product gives exactly the product of differential symbols.

Remark A.1. The same scheme actually works for any cotangent bundle $\mathcal{T}^*M$ with the canonical symplectic form – the quantized algebra of functions on $\mathcal{T}^*M$ is isomorphic to the algebra of differential operators on M. See for example [18], also [15],[24] for applications to index theorems.

Explicit formulas for quantization of Kähler manifolds were given in [19]. Tamarkin ([26]) showed that for symmetric Kähler manifolds the Fedosov connection can be constructed to have only three summands, it has no terms of degree higher than 1, i.e.

$$D = \delta + \nabla + r_1$$

This gives compact formulas for the $*$-product on such manifolds as the 2–sphere, any projective space $\mathbb{C}P^n$, flag manifolds and Grassmanians. for z, v, w being holomorphic coordinates at 0. The case of $\mathbb{C}P^n$ was also considered in ([5]).

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