Multiple gamma functions and derivatives of L-functions at non-positive integers

by

Marat Rovinsky

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Author ....
Department of Mathematics
May 3, 1996

Certified by ....
Alexander Goncharov
Professor of Mathematics
Thesis Supervisor

Accepted by ....
David A. Vogan
Chairman, Departmental Committee on Graduate Students

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ABSTRACT

A pattern of a "coordinate-free approach to multiple gamma functions is introduced. Derivatives of certain L-functions at non-positive integers are expressed in terms of values of multiple gamma functions at rational points.

A commutative associative binary operation, "the finite convolution", is introduced on some class of holomorphic functions. The logarithm of a multiple gamma function becomes the finite convolution of the function log u and a polynomial in u in an appropriate space of analytic functions. This operation seems to be useful for investigating arithmetic properties of some analytic functions.

Thesis supervisor: Alexander Goncharov
Title: Associate Professor of Mathematics

Contents

Multiple gamma functions were introduced and studied by E.W.Barnes at the end of the last century (cf. [Ba]). It became clear since the time of Barnes, that similar to the case of Dedekind ζ-function, multiple gamma functions are "gamma-factors" of functional equations for Selberg zeta-functions (for the case of double gamma function see [Vi], in general [K1]).

On the other hand, multiple gamma functions appear naturally as characteristic "polynomials" of certain differential operators of first order (cf. [Den] for the case of the usual gamma function). In the context of regularized determinants it is possible to define these characteristic "polynomials" correctly and relate them to the Riemann (or Hurwitz) ζ-function.

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1. Notation and conventions

$B_n(x)$ are (resp., $B_n = B_n(0)$) Bernoulli polynomials (resp., Bernoulli numbers) defined by the power series

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n,$$

(resp.,

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.)$$

$\zeta(s, u)$ is the Hurwitz zeta function, defined for $\Re s > 1$ and $\Re u > 0$ by absolutely convergent series $\zeta(s, u) = \sum_{n=0}^{\infty} (n + u)^{-s}$. In fact $\zeta(s, u)$ as a function of s is meromorphic in $\mathbb{C}$, has a simple pole at $s = 1$, and $\zeta(1 - n, u) = -\frac{B_n(u)}{n}$ for all positive integers n.

$\zeta'(s, u)$ denotes its derivative with respect to the variable s.

$\zeta(s)$ is the Riemann zeta function, $\zeta(s) = \zeta(s, 1) = \sum_{n=1}^{\infty} n^{-s}$.

$L(s, \chi)$ is the Dirichlet L-function of a (multiplicative) character χ, defined for $\Re s > 1$ by absolutely convergent series $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$.

$\mathrm{Li}_n(z)$ is the n-logarithm, defined for $|z| < 1$ by the power series $\sum_{k=1}^{\infty} \frac{z^k}{k^n}$.

The Gothic letters are reserved for operators. In particular, $\mathfrak{T}$ will denote a locally unipotent linear operator on the space of polynomials, or on a completion with respect to a topology on it (e.g., the linear operator $f(x) \longmapsto f(x + 1)$).

In order to avoid confusion between action of a linear operator $\mathfrak{D}$ on a function f and compositions the operator $\mathfrak{D}$ with the operator of multiplication by the function f, let us write them as $\mathfrak{D}f$ and $\mathfrak{D} \cdot f$, respectively.

2. Definitions

2.1. **Gamma functions.** It is well-known that the classical Γ function satisfies some very nice properties, such as

- easy behavior under translations by an integer: $\Gamma(u+1) = u\Gamma(u)$;
- Gauß multiplication formula: $\Gamma(u) = (2\pi)^{-\frac{m-1}{2}} m^{u-1/2} \prod_{k=0}^{m-1} \Gamma\left(\frac{u+k}{m}\right)$;
- $\Gamma(u)$ is a meromorphic function of order 1 of maximal type and all its poles are non-positive integers;
- $\Gamma(u)$ has integer values at positive integer points;
- The complement formula holds:

$$\Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin \pi u}.$$

The aim of this section is to define a homomorphism Γ from the additive group of the ring of polynomials with integer values at integer points $\mathcal{R}$ to the multiplicative group of meromorphic functions with properties similar to those of $\Gamma(u)$. More precisely, we would like the fraction $\Gamma(P)(u+1)/\Gamma(P)(u)$ to be equal to a product of some powers of u and of $\Gamma(Q)(u)$ for polynomials Q of degree less than the degree of P. As a "first approximation" to this homomorphism we can try

$$\widetilde{\Gamma} : P \longmapsto \widetilde{\Gamma}_P(u) = \exp \int_1^u P(t) d\log\Gamma(t).$$

Then

$$\begin{aligned} \widetilde{\Gamma}_P(u+1) &= \exp \int_1^2 P(t) d\log\Gamma(t) \cdot \exp \int_1^u P(t+1) d\log\Gamma(t+1) \\ &= \widetilde{\Gamma}_P(2) \cdot \widetilde{\Gamma}_{\mathfrak{T}P}(u) \cdot \exp \int_1^u P(t+1)\frac{dt}{t}, \end{aligned}$$

where $\mathfrak{T}P(t) = P(t+1)$.

In order to improve this "approximate" definition and make the above formula nice, consider a polynomial correction of the integral in this definition. This appendix is given by a linear operator $\mathfrak{Q}$ on the space of real polynomials shifting degrees by one and given by the formula

$$\mathfrak{Q} = (\mathfrak{V} - Id) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\zeta'(-n) + \zeta(-n, u) \sum_{m=1}^{n} \frac{1}{m} \right) \frac{d^n}{du^n},$$

where $\mathfrak{V}$ is the operator of value at 1, and Id is the identity operator.

Definition 2.1. *The homomorphism $\Gamma : P \longmapsto \Gamma_P$ assigns to any polynomial P with integer values in integer points, a meromorphic function defined by*

$$\Gamma_P(u) = \exp\left[\mathfrak{Q}P(u) + \int_1^u P(t) d\log\Gamma(t)\right]. \tag{1}$$

It will be explained below (cf. Lemma 3.4), why it is a reasonable and in a sense the only reasonable definition of multiple gamma functions.

Remarks. 1. This definition coincides with a definition of Γ_{n+1} given by Barnes when the polynomial $P(t) = \binom{n-t}{n}$ is chosen. It will be shown in Corollary 3.3 that

$$\Gamma_{n+1}(u+1) = \Gamma_n(u)\Gamma_{n+1}(u)$$

for all natural n. Clearly, $\Gamma_1(u)$ is the usual gamma function.

2. The definition of the multiple gamma function above is given to make it a meromorphic function. However, as it was noticed by Don Zagier, the log multiple gamma functions are more natural. In particular, there is no "global" presentation of the multiple gamma functions as a "period" (like

$$\Gamma(u) = \int_0^\infty e^{-t} t^u \frac{dt}{t},$$

and that was the reason, why we did not mention this property among those of the usual gamma function). Another point is that in their connection with various L-series, the multiple gamma functions appear only in logarithmized form. Moreover, unlike the (exponential) gamma functions, the log multiple gamma functions $\log \Gamma_P(u)$ can be p-adically interpolated to analytic p-adic functions (cf. [C-N], [I]). One more argument can be found in Section 4 below, where a polynomial $P \in \mathcal{R}$ is replaced by any entire function of order less than one.

2.2. **The "sine functions" and the polylogarithms.** Once a gamma function is defined, one can symmetrize its definition to get the "multiple sine" function

$$\Lambda_P(u) = \Gamma_P(u)\Gamma_{P^*}(1-u) = \exp\left[\mathfrak{Z}P(u) - \pi \int_{1/2}^{u} P(t) \cot \pi t dt\right], \tag{2}$$

where $P^*(t) = P(1-t)$ and $\mathfrak{Z}$ is a polynomial operator,

$$d/du \cdot \mathfrak{Z} = -2 \sum_{n=0}^{\infty} \frac{\zeta'(-2n)}{(2n)!} \frac{d^{2n+1}}{du^{2n+1}}.$$

These functions are closely related to polylogarithms.

Classically, the polylogarithm is defined in the unit disc $|z| < 1$ by the power series

$$\mathrm{Li}_n(z) = \sum_{k=0}^{\infty} \frac{z^k}{k^n}.$$

Using the obvious identity

$$\mathrm{Li}_{n-1}(z) = z \frac{d}{dz} \mathrm{Li}_n(z),$$

one can get an analytic continuation of $\mathrm{Li}_n(z)$ by means of an iterated integral. For any path $\gamma_z : [0,1] \longrightarrow \mathbb{C}\backslash\{0,1\}$, joining the origin and the variable point z, i.e., $\gamma_z(0) = 0$ and $\gamma_z(1) = z$

$$\mathrm{Li}_n(z) = \int_{\gamma_z} \underbrace{\frac{dz}{z} \circ \cdots \circ \frac{dz}{z}}_{n-1 \text{ time}} \circ \frac{dz}{1-z} = \int_{0 \le t_1 \le \cdots \le t_n \le 1} \frac{d\gamma_z(t_n)}{\gamma_z(t_n)} \cdots \frac{d\gamma_z(t_2)}{\gamma_z(t_2)} \frac{d\gamma_z(t_1)}{1-\gamma_z(t_1)}.$$

In particular, if the path γ_z does not cross the ray of reals greater than one, then the corresponding branch of the polylogarithm can be proved to be equal to the following integrals

$$\mathrm{Li}_n(z) = \frac{z}{(n-1)!} \int_0^\infty \frac{t^{n-1} dt}{e^t - z} = \frac{z}{(n-1)!} \int_0^1 \frac{(-\log t)^{n-1} dt}{1 - tz}$$

for any z not on the real ray $z > 1$. The polylogarithm extends to an analytic function on the whole $\mathbb{P}^1_{\mathbb{C}}$ with ramification at $0, 1, \infty$.

Lemma 2.2. • *Up to a constant multiple,* Λ_P *coincides with*

$$\exp\left[-\pi i\int_0^u P(t)dt + 3P(u) + \sum_{k=1}^{\infty}\frac{P^{(k-1)}}{(2\pi i)^{k-1}}\mathrm{Li}_k(e^{-2\pi iu})\right], \tag{3}$$

where $\Im u < 0$.

- *Up to an exponential multiple any meromorphic function of the type*

$$\exp\left[\sum_{k=1}^{n} f_k(u)\mathrm{Li}_k\left(e^{-2\pi iu}\right)\right]$$

for some meromorphic functions f_k*, coincides with* $\Lambda_P(u)$ *for a polynomial* $P \in \mathcal{R}$.

Proof.

- For a proof note that logarithmic derivative of (3) is

$$\frac{d}{du}\cdot 3P(u) - \pi iP(u) + \sum_{k=1}^{\infty}\frac{P^{(k)}(u)}{(2\pi i)^{k-1}}\mathrm{Li}_k(e^{-2\pi iu}) - \sum_{k=0}^{\infty}\frac{P^{(k)}(u)}{(2\pi i)^k}\cdot 2\pi i\cdot \mathrm{Li}_k(e^{-2\pi iu})$$

$$= \frac{d}{du}\cdot 3P(u) - \pi iP(u) - 2\pi i\cdot P(u)\frac{e^{-2\pi iu}}{1-e^{-2\pi iu}} = \frac{d}{du}\cdot 3P(u) - \pi P(u)\cot\pi u,$$

and coincides with logarithmic derivative of (2) .

- Consider the logarithmic derivative:

$$\sum_{k=0}^{\infty}\left(f_k'(u) - 2\pi i f_{k+1}(u)\right)\mathrm{Li}_k(e^{-2\pi iu}).$$

It is a meromorphic function. Denote $f_k'(u) - 2\pi i f_{k+1}(u)$ by $g_k(u)$.

Let $g_N(u) \neq 0$ and $g_k(u) = 0$ for all $k > N$. We prove that $g_k(u) = 0$ for all $k > 0$ by induction on N, case $N = 1$ being trivial. Then

$$\sum_{k=1}^{N}\frac{g_k(u)}{g_N(u)}\mathrm{Li}_k(e^{-2\pi iu})$$

is a meromorphic function too. Its derivative

$$-2\pi iu\frac{g_1(u)}{g_N(u)}\frac{1}{e^{2\pi iu}-1} + \sum_{k=1}^{N}\left(\frac{d}{du}\frac{g_k(u)}{g_N(u)} - 2\pi i\frac{g_{k+1}(u)}{g_N(u)}\right)\mathrm{Li}_k(e^{-2\pi iu})$$

should be equal to $-2\pi iug_1(u)/(g_N(u)(\exp(2\pi iu)-1))$ by the induction assumption. Thus

$$\frac{d}{du}\frac{g_k(u)}{g_N(u)} = 2\pi i\frac{g_{k+1}(u)}{g_N(u)}$$

for any $k > 0$. Thus for a polynomial P of degree $N-1$ we get

$$g_k(u) = g_N(u)(2\pi i)^{-k}P^{(k-1)}(u).$$

Thus

$$\sum_{k=1}^{N}(2\pi i)^{-k}P^{(k-1)}(u)\mathrm{Li}_k(e^{-2\pi iu})$$

is, again, meromorphic. But the monodromy of the latter about an integer point m subtracts

$$\sum_{k=1}^{N} \frac{(2\pi i)^{1-k}}{(k-1)!} P^{(k-1)}(u)(2\pi i(u-m))^{k-1} = P(m).$$

Thus, $P(u) = 0$, and $f_k'(u) = 2\pi i f_{k+1}(u)$.

□

3. Properties of the gamma functions

One of the aims of this note is to show that study of derivatives of Dirichlet L-functions is equivalent to study of the values of the multiple gamma functions at rational points. So in all the formulas below the stress will be made on various constants and reducing them to the exponent of the first derivatives of the ζ-function at non-positive integers.

Lemma 3.1. *$\Gamma_P(u)$ is a meromorphic function of order $\deg P + 1$ and infinite type. There is the following decomposition into Hadamard-Weierstraß product*

$$\Gamma_P(u) = e^{\mathfrak{S}P(u)} u^{-P(0)} \prod_{k=1}^{\infty} \left[\left(1 + \frac{u}{k}\right)^{-P(-k)} e^{\left(\frac{u}{k} - \frac{u^2}{2k^2} + \cdots - \frac{(-u)^{(\deg P+1)}}{(\deg P+1)k^{\deg P+1}}\right) P(-k)} \right], \tag{4}$$

where $\mathfrak{S}$ is a polynomial operator such that $\mathfrak{S}P(0) = 0$ and

$$\frac{d}{du}\mathfrak{S}P(u) = \frac{P(0) - P(u)}{u} - \gamma P(u) + u \sum_{k=1}^{\infty} \frac{k^{\deg P} P(u) - (-u)^{\deg P} P(-k)}{(k+u) k^{\deg P+1}}.$$

Proof. Consider the infinite product presentation of $\Gamma(u)$:

$$\frac{1}{\Gamma(u)} = e^{\gamma u} u \prod_{k=1}^{\infty} \left[\left(1 + \frac{u}{k}\right) e^{-\frac{u}{k}} \right].$$

Take the logarithmic derivative to get

$$\frac{\Gamma'(u)}{\Gamma(u)} = -\gamma - \frac{1}{u} + \sum_{k=1}^{\infty} \frac{u}{(u+k)k}, \tag{5}$$

and multiply by $P(u)$. Then we are trying to replace the polynomial multiple $P(u)$ in the kth summand by a constant one, namely by $P(-k)$:

$$\frac{u}{(u+k)k} \cdot P(u) = P(-k) \frac{u}{(u+k)k} + u \cdot \frac{P(u) - P(-k)}{u+k}.$$

After some more work we get the decomposition (4). □

The following is the most powerful property of the gamma functions.

Theorem 3.2. *Gamma functions are related to the Hurwitz ζ-function by the formula*

$$\Gamma_P(u) = \exp \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} P^{(n)}(u) \left(\zeta'(-n, u) - \zeta'(-n) \right) \right]. \tag{6}$$

Proof. For any s with $\Re s > 1$ the series $\sum_{n=0}^{\infty}(n+u)^{-s}$ is absolutely convergent, so one can see the following obvious functional equations:

$$\zeta(s,u) = u^{-s} + \zeta(s,u+1) \text{ and } \zeta'(s,u) = -\log u \cdot u^{-s} + \zeta'(s,u+1), \tag{7}$$

$\zeta'(s,u) = -\sum_{n=1}^{\infty}\log(n+u)(n+u)^{-s}$ and $\partial/\partial u\zeta'(s,u) = s\sum_{n=0}^{\infty}\log(n+u)(n+u)^{-s-1} - \sum_{n=1}^{\infty}(n+u)^{-s-1}$. Thus we get an identity

$$\frac{\partial}{\partial u}\zeta'(s,u) = -\zeta(s+1,u) - s\zeta'(s+1,u). \tag{8}$$

Now take logarithmic derivatives of both sides of (6). Then use the identity (8) and a formula of Lerch

$$\frac{\partial}{\partial u}\zeta'(0,u) = \frac{\Gamma'(u)}{\Gamma(u)}$$

coming from the Bohr-Mollerup theorem (on uniqueness of the function $\Gamma(u)$ such that $\Gamma(u+1) = u\Gamma(u)$, $\Gamma(1) = 1$ and $d^2/du^2 \log\Gamma(u) > 0$ for positive real u).

To see the Lerch formula, specialize (7) to $s = 0$. This gives the identities

$$\zeta'(0,u+1) = \log u + \zeta'(0,u), \text{ and } \exp\zeta'(0,u+1) = u\cdot\exp\zeta'(0,u).$$

The formula (8) above implies

$$\frac{\partial^2}{\partial u^2}\zeta'(s,u) = s(s+1)\zeta'(s+2,u) + (2s+1)\zeta(s+2,u).$$

When $s = 0$ this function is reduced to $\zeta(2,u)$, which is real positive for all real positive u. Thus $\exp(\zeta'(0,u) - \zeta'(0))$ satisfies the assumptions of the Bohr-Mollerup theorem, and coincides with $\Gamma(u)$. □

This Theorem has a number of consequences, as can be seen from

Corollary 3.3. 1. *$\Gamma_P(u)$ admits a presentation as a ζ-regularized product:*

$$\Gamma_P(u) = \exp[-\zeta P(u)]\cdot\prod_{k=0}^{\infty}(u+k)^{-P(-k)},$$

where $\zeta P(u) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\zeta'(-n)P^{(n)}(u)$, and $(u+k)^{-P(-k)}$ is understood as the product of $|P(-k)|$ copies of the multiple $(u+k)^{-sgnP(-k)}$.

Choose a path joining infinity with the origin. Then one can choose a branch of the logarithm and for any sequence of complex numbers $a_1, a_2, a_3, \dots$ not on this path, the series $\zeta_a(s) = \sum_{k=1}^{\infty}a_k^{-s}$ is defined. Suppose that this series is convergent for sufficiently large $\Re s$ and admits an analytic continuation to a simply-connected domain, containing the origin and a point of convergence. Then the ζ-regularized product of the sequence $a_1, a_2, a_3, \dots$ defined as $\exp\zeta_a'(0)$.

2. *The homomorphism Γ commutes with the translation by one in the following way*

$$\Gamma_P(u+1) = u^{P(1)}\Gamma_{\mathfrak{T}P}(u), \tag{9}$$

where $\mathfrak{T}P(t) = P(t+1)$. In particular, for any positive integer m

$$\Gamma_P(m+1) = 1^{P(m)}\cdot 2^{P(m-1)}\cdot\ldots\cdot(m-1)^{P(2)}\cdot m^{P(1)}, \tag{10}$$

or

$$\Gamma_P(m+1) = 1!^{P(m)-P(m-1)}\cdot 2!^{P(m-1)-P(m-2)}\cdot\ldots\cdot(m-1)!^{P(2)-P(1)}\cdot m!^{P(1)} \tag{11}$$

3. *"Gauß multiplication formula"*

$$\Gamma_P(u) = c_P(m,u) m^{\mathfrak{I}P(u)} \prod_{k=0}^{m-1} \Gamma_{\mathfrak{H}_{k,m}P}\left(\frac{u+k}{m}\right), \tag{12}$$

where $\mathfrak{I}P(u) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n(u)}{n!}P^{(n-1)}(u)$[1], $\mathfrak{H}_{k,m}P(t) = P(mt-k)$ *and* $c_P(m,u)$ *is the exponent of a polynomial, the "Gauß exponent". The latter can be expressed via values of derivatives of the Riemann* ζ*-function at non-positive integers:*

$$c_P(m,u) = \exp\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\zeta'(-n)(m^{n+1}-1)P^{(n)}(u)\right].$$

4. *Value at 1/2*

$$\Gamma_P(1/2) = 2^{\sum_{n=1}^{\infty}\frac{B_n}{n!}2^{1-n}P^{(n-1)}(1/2)}\exp\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\zeta'(-n)(2^{-n}-2)P^{(n)}(1/2)\right]. \tag{13}$$

5. *Denote by* $\mathbb{Q}(\mu_m)$ *the m-th cyclotomic extension of the field of rationals. Then the* $\mathbb{Q}(\mu_m)$*-subspaces of* $\mathbb{C}$ *generated by*
   (a) *logarithms of values of gamma functions of order less than n at positive rational points with denominator dividing m;*
   (b) *first derivatives of Dirichlet L-functions with conductor dividing m at integers between* $1-n$ *and 0*

   coincide.

Proofs.

1. By the definition of ζ-regularized product

$$\prod_{k\geq 0}(u+k)^{-P(-k)} = \exp\left[-\sum_{k\geq 0}P(-k)\log(u+k)(u+k)^{-s}\right]\Bigg|_{s=0}.$$

Then from the Taylor formula $\sum_{n=0}^{\infty}\frac{(-u-k)^n}{n!}\frac{d^nP}{du^n} = P(-k)$ we get

$$\prod_{k\geq 0}(u+k)^{-P(-k)} = \exp\left[-\sum_{k,n\geq 0}\frac{(-u-k)^n}{n!}P^{(n)}(u)\log(u+k)(u+k)^{-s}\right]\Bigg|_{s=0}.$$

Summing over k's identifies the product with

$$\exp\left[\sum_{n\geq 0}(-1)^n\frac{P^{(n)}(u)}{n!}\zeta'(s-n,u)\right]\Bigg|_{s=0}.$$

The rest is straightforward.

2. From an obvious identity

$$\zeta'(s,u+1)-\zeta'(s,u) = u^{-s}\log u$$

and theorem 6 we get an identity

$$\log\Gamma_P(u+1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}P^{(n)}(u+1)(\zeta'(-n,u)-\zeta'(-n)+u^n\log u).$$

---

[1] Notice, that $\frac{d}{du}\mathfrak{I}P(u) = P(u)$.

This identity, when combined with the Taylor formula $\sum_{n=0}^{\infty}(-1)^n \frac{P^{(n)}(u+1)}{n!}u^n \cdot \log u = P(1) \cdot \log u$, gives the identity (9).

3. This follows from an obvious identity

$$\sum_{k=0}^{m-1} \zeta'\left(s, \frac{u+k}{m}\right) = m^s(\zeta'(s,u) + \zeta(s,u)\log m).$$

4. Plug $u = 1$ and $m = 2$ into the "Gauß multiplication formula".
5. Apply the formula (6) of the Lemma to $P = (1-2t)^n$ and specialize u to $1/2$. This gives a formula for $\zeta'(-n)$.
   The same formula, applied to $P = (a - mu)^n$ describes $\zeta'(-n, a/m)$ as a linear combination of logarithms of multiple gamma functions at $1/2$ and at a/m, and a rational number. The corollary comes as soon as one notices that any Dirichlet series with conductor, dividing m, is a linear combination of $\zeta(s, a/m)$ for $1 \le a \le m$. □

REMARKS. 1. Strangely enough, differences of the logarithmic derivatives of the gamma function at rationals are algebraic linear combinations of logarithms of algebraic numbers (e.g., [An]). More precisely, for any integers $0 < j < N$

$$\frac{\Gamma'(j/N)}{\Gamma(j/N)} - \frac{\Gamma'(1)}{\Gamma(1)} = \sum_{1 \ne \zeta \in \mu_N} (\zeta^{-j} - 1)\log(1-\zeta) \tag{14}$$

(*Proof.* Using the formula (5), we get

$$\frac{\Gamma'(j/N)}{\Gamma(j/N)} = -\gamma - \frac{N}{j} + \sum_{k=1}^{\infty}\left(\frac{N}{kN} - \frac{N}{j+kN}\right) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{N}{kN} - \frac{N}{j-N+kN}\right).$$

It is not hard to see that

$$\sum_{\zeta \in \mu_N} \zeta^m = \begin{cases} N & \text{if } m \text{ divides } N \\ 0 & \text{otherwise} \end{cases},$$

where μ_N denotes the group of roots of unity of order N, and

$$\sum_{1 \ne \zeta \in \mu_N} (\zeta^m - \zeta^l) = \sum_{\zeta \in \mu_N} (\zeta^m - \zeta^l) = \begin{cases} N & \text{if } N \text{ divides } m \text{ and } N \text{ does not divide } k \\ -N & \text{if } N \text{ does not divide } m \text{ and } N \text{ divides } k \\ 0 & \text{otherwise} \end{cases}.$$

Thus

$$\sum_{k=1}^{\infty}\left(\frac{N}{kN} - \frac{N}{j-N+kN}\right) = \sum_{m=1}^{\infty}\sum_{1 \ne \zeta \in \mu_N} \frac{\zeta^m - \zeta^{m-j}}{m} = \sum_{1 \ne \zeta \in \mu_N} (\zeta^{-j} - 1)\log(1-\zeta).\ \square)$$

2. A formula of Moreno [Mo] suggests that certain more general multiple gamma functions (also introduced by Barnes) are appropriate to deal with second non-trivial coefficients of Taylor series of ζ-functions of totally real number fields at non-positive integers. However, they can be avoided in the case of totally real abelian extensions of $\mathbb{Q}$.

3.1. A universality property of gamma functions.

Lemma 3.4. *For any system of meromorphic functions Γ_n such that*

- $\frac{\Gamma_n(u+1)}{\Gamma_n(u)}$ *is a product of powers of $\Gamma_k(u)$ for $1 \le k < n$ and a power of u;*
- $\Gamma_n(u)$ *is a fraction of entire functions of finite order and it has neither zeroes nor poles in a right half-plane;*
- $\Gamma_n(u)$ *is real for positive real u and $\Gamma_n(1) = 1$,*

there exists a unique system of polynomials P_n with $\Gamma_{P_n} = \Gamma_n$.

Proof. By induction on n, it is easy to see, that the order of pole (or minus order of zero) of Γ_n at a non-positive integer p is the value of a polynomial $P_n(u)$ of degree $n-1$ at p. For $n = 1$ this polynomial is a constant, for $n > 0$ there is a unique polynomial $P_n(u)$ with a fixed value at 0 and such that $P_n(m+1) - P_n(m)$ coincides with the value at m a fixed linear combination of $P_k(u)$ for $1 \le k < n$ and all negative integer m.

Then the fraction

$$\frac{\Gamma_n(u)}{\Gamma_{P_n}(u)}$$

is a periodic meromorphic function of a finite order with no zeroes and poles in a right half-plane, and must be the exponent of a real polynomial of degree $\le n$ and a rational function of $e^{2\pi i u}$ simultaneously. So it is the unity. □

3.2. Properties of sine-functions.

Corollary 3.5. 1. *The homomorphism Λ, when considered as a function on the product $\mathcal{R} \times \mathbb{C}$, is invariant under the action of the subgroup of even numbers, when $\mathbb{Z}$-action is given by the formula $f(P(t), u) \longmapsto f(P(t-1), u+1)$. More precisely,*

$$\Lambda_P(u+1) = (-1)^{P(1)} \Lambda_{\mathfrak{T}P}(u), \tag{15}$$

where $\mathfrak{T}P(t) = P(t+1)$. In the same sense, it is invariant under the involution $f(P(t), u) \longmapsto f(P(1-t), 1-u)$:

$$\Lambda_P(1-u) = \Lambda_{P^*}(u), \tag{16}$$

where $P^*(t) = P(1-t)$.

2. *"Gauß multiplication formula"*

$$\Lambda_P(u) = \exp\left[2\sum_{n=0}^{\infty} \frac{\zeta'(-2n)}{(2n)!}(m^{2n+1} - 1)P^{(2n)}(u)\right] \prod_{k=0}^{m-1} \Lambda_{\mathfrak{H}_{k,m}P}\left(\frac{u+k}{m}\right), \tag{17}$$

where $\mathfrak{H}_{k,m}P(t) = P(mt - k)$. [2]

3. *Value at 1/2*

$$\Lambda_P(1/2) = 2^{-P(1/2)} \exp\left[\sum_{n=0}^{\infty} \frac{2(2^{-2n} - 2)}{(2n)!} \zeta'(-2n) P^{(2n)}(1/2)\right]. \tag{18}$$

4. *Denote by $\mathbb{Q}(\mu_m)$ the m-th cyclotomic extension of the field of rationals. Then the $\mathbb{Q}(\mu_m)$-subspaces of $\mathbb{C}$ generated by*
   (a) *logarithms of values of sine-functions of order less than n at positive rational points with denominator dividing m;*

[2] The functional equation for $\zeta(s)$ implies $2\frac{\zeta'(-2n)}{(2n)!} = (-1)^{n+1}\frac{\zeta(2n+1)}{(2\pi)^{2n}}$.

(b) *"2π-normalized" values of Dirichlet L-functions with conductor dividing m at positive integers not exceeding n*

coincide.

Proof follows immediately from Corollary 3.3. □

3.3. **The complement formula and fields of sine-functions.** Suppose for a moment that we do not know yet what is the function $(\Gamma(u)\Gamma(1-u))^{-1}$. Then we can describe it as the only solution of the differential equation $f'' = -\pi^2 f$, with the initial data $f(0) = 0$ and $f'(0) = 1$. There are no that simple differential equation for the functions Λ_P. But the following claim may be considered as a replacement of the complement formula.

Lemma 3.6. 1. *The function Λ_P is of the same order as Γ_P; if degree of P is even, then the type of Λ_P is a positive integer multiple of $\frac{\pi}{(\deg P+1)!}$.*

2. *Unlike gamma functions, any multiple sine function Λ_P generates over $\mathbb{C}$ (resp., over $\mathbb{Q}$) a differential field Φ_P° of degree of transcendence not more than three (resp., $[(\deg P-1)/2]+5$).*
   If $\deg P > 2$, then differential endomorphisms of the field Φ_P° over $\mathbb{C}$ form a group isomorphic to one-dimensional algebraic torus $\mathbb{C}^\times$, which coincides with the differential Galois group of the field Φ_P° over $\Phi_1 = \mathbb{C}(u, \pi/\sin \pi u)$.
3. *Multiple sine functions of order $\leq d$ generate over $\mathbb{C}$ a differential field Φ_d of degree of transcendence $d+1$.*
   Differential endomorphisms of the field Φ_d form a group isomorphic to a (certainly, non-algebraic) group $(\mathbb{C}^\times)^{d-1} \times \mathbb{Z}$.

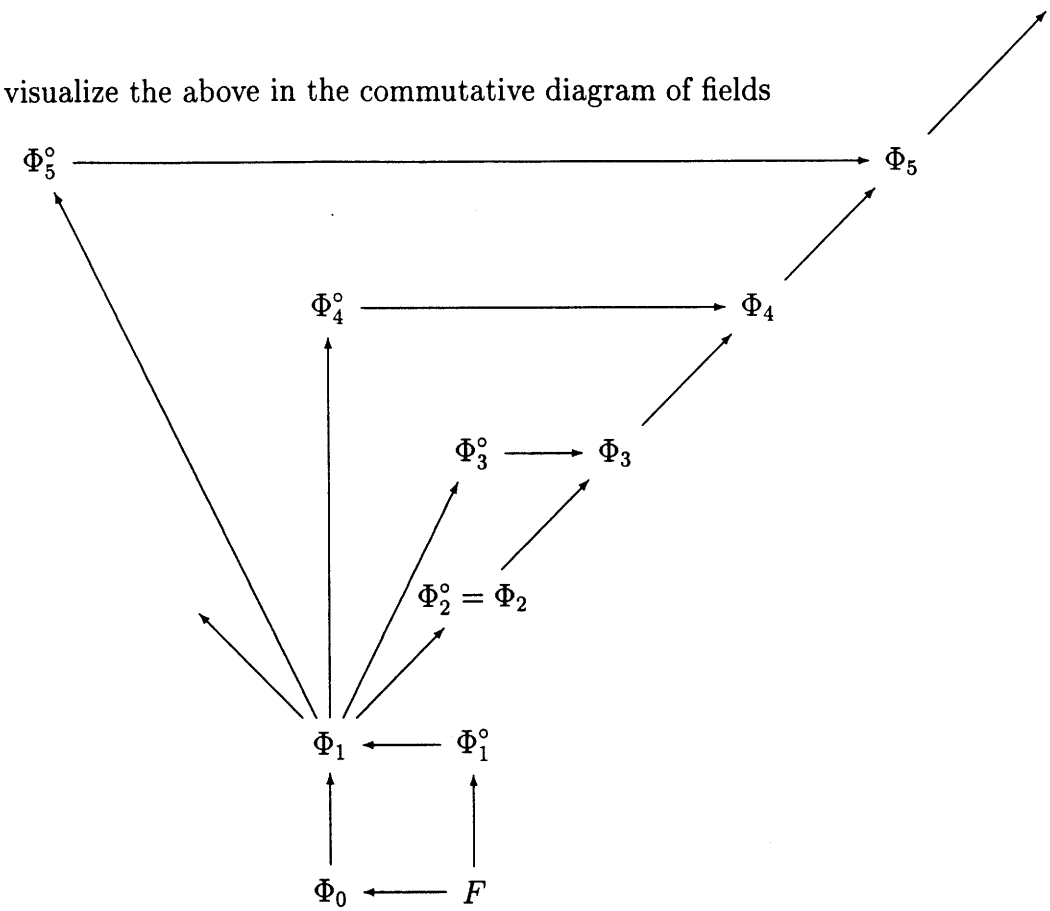
Proof.

1. Suppose that $\Im u < 0$. Then $\mathrm{Li}_k(e^{-2\pi i u})$ is bounded by $\zeta(k)$ for all $k > 1$. Then in the expression (3) the corresponding terms are bounded by a polynomial of degree $\deg P - 2$. The only term left is $\exp(u^{n-1}\log(1-\exp(-2\pi i u)))$. Since $\Im u < 0$, the values of $\log(1-\exp(-2\pi i u))$ lie on the half-strip $\Re z < \log 2$, $|\Im z| < \pi$. Then $|\exp(u^{n-1}\log(1-\exp(-2\pi i u)))| \leq 2^{\Re u^{n-1}} \cdot e^{\pi \Im u^{n-1}}$. Finally, the type of Λ_P coincides with $(\pi/(\deg P+1)) \times$(the leading coefficient of P) (see below).
2. Immediately from the definition

$$\frac{d}{du}\log \Lambda_P(u) = -2\sum_{n=0}^{\infty} \frac{\zeta'(-2n)}{(2n)!} P^{(2n+1)}(u) - \pi P(u)\cot \pi u.$$

   Thus a field, generated by $\Lambda_P(u)$, u, $\pi \cot \pi u$, π^2 and $\zeta'(-2m)$ for all integer m between 0 and $(\deg P-1)/2$, is closed under the differentiation.
3. Since the subfields $\mathbb{C}(u)$ and $\mathbb{C}(\cot \pi u)$ are closed under the differentiation, they should be preserved by any differential endomorphism. Clearly, their endomorphisms form groups isomorphic to $\mathbb{G}_a(\mathbb{C})$ and $\mathbb{G}_m(\mathbb{C})$, respectively.
   The above expansion for the logarithmic derivative of Λ_P shows that differential endomorphisms should *almost* preserve the set of poles of $P(u)\cot \pi u$ ('almost' means upto a finite set). Thus they must act as translations by integers on $\mathbb{C}(u)$.
   □

One can visualize the above in the commutative diagram of fields



where each non-horizontal arrow is a purely transcendental Picard-Vessiot extension of degree one.

The above Lemma contrasts with the case of one-dimensional extensions of fields of constants.

Lemma 3.7. *For a finitely generated differential extension F of degree of transcendence one of an algebraically closed field k of characteristic zero, the monoid of all differential endomorphisms is an algebraic group.*

This group is either a finite group, or connected one-dimensional ($\mathbb{G}_a$, $\mathbb{G}_m$, or an elliptic curve). The case of an affine group corresponds to a Picard-Vessiot extension of the field of constants ($\partial^2 x = 0$, or $\partial x = x$, respectively).

Proof. Choose a smooth projective model X of the field F over the field k. It is defined uniquely. Then a differentiation is presented by a vector field v on X. Differential endomorphisms are mappings of X into itself, preserving the vector field v. Clearly, this implies that all differential endomorphisms are invertible, and condition of preserving the vector field v is algebraic, as far as the group of automorphisms of X is algebraic.

If the genus of the curve X is greater than one, or X is an elliptic curve and a point is marked, or X is the projective line and three points are marked, then its group of automorphisms is finite. Indeed, the consider corresponding punctured curve. It is hyperbolic, in particular, has a canonical (depending only on the complex structure) metric. Thus group of its automorphisms is a closed complex subgroup of a unitary group, hence finite.

Any differentiation on a field defines a vector field on its model. Differentiation preserve this vector field, and poles and zeroes of this vector field play the role of marked points.

The only cases left are an elliptic curve with a translation-invariant vector field, and the projective line with a vector field with either 2 distinct zeroes (then after a change of coordinate, it is $ax\frac{d}{dx}$, so its group of automorphisms is $\mathbb{G}_m$), or with only one zero (then after a change of coordinate, it is $\frac{1}{P(x)}\frac{d}{dx}$, so its group of automorphisms is $\mathbb{G}_a$ for a constant P, and trivial otherwise). □

4. The finite convolution product

The contents and methods of this section either come from, or very close to [DP].

For a set S let $\mathrm{Maps}(\mathbb{N}, S)$ be the set of S-valued functions on the natural numbers. Let for a pair of sets S_1 and S_2, and a group G a binary operation $S_1 \times S_2 \xrightarrow{\mu} G$. In this situation we define the following $\mathrm{Maps}(\mathbb{N}, G)$-valued pairing between $\mathrm{Maps}(\mathbb{N}, S_1)$ and $\mathrm{Maps}(\mathbb{N}, S_2)$, given by 'the finite convolution product':

Definition 4.1. *Let φ be an element of $\mathrm{Maps}(\mathbb{N}, G_2)$ and ψ be an element of $\mathrm{Maps}(\mathbb{N}, G_1)$. Then the pairing*

$$\mathrm{Maps}(\mathbb{N}, S_1) \times \mathrm{Maps}(\mathbb{N}, S_2) \longrightarrow \mathrm{Maps}(\mathbb{N}, G) \text{ is given by } (f * g)(n) = \sum_{k=1}^{n-1} \mu(f(k), g(n-k)), \tag{19}$$

where n is a positive integer.

Lemma 4.2. *For any commutative associative ring A the operation $*$ defines on $\mathrm{Maps}(\mathbb{N}, A)$ a structure of a non-unitary commutative associative A-algebra.*

Proof. Straightforward. □

Definition 4.3. *A set $\mathcal{M}$ of holomorphic functions on the right half-plane $\Re u > 0$ is called* admissible, *if both of the following two conditions are satisfied*

- *The natural mapping $V : \mathcal{M} \longrightarrow \mathrm{Maps}(\mathbb{N}, \mathbb{C})$, assigning to a function $f(u)$ its values at positive integer points $(f(1), f(2), f(3), \dots)$, is injective.*
- *For any f and g in $\mathcal{M}$ the finite convolution $V(f) * V(g)$ lies in the image of $\mathcal{M}$.*

Thus, if a set $\mathcal{M}$ of functions is admissible, the pull-back of $*$ under mapping V defines the finite convolution $*_{\mathcal{M}}$ on the set $\mathcal{M}$ itself.

To construct first non-trivial example of an admissible algebra we need the following claim, essentially due to [DP].

Theorem 4.4. *The operator*

$$\int_u^{u+1} : f(u) \longmapsto \int_u^{u+1} f(t)dt$$

acting on the space of all entire functions of order less than one (i.e., functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $|f(z)| \leq ae^{b\cdot|z|^\rho}$ for some $a > 0$, $b > 0$ and $0 \leq \rho < 1$) is invertible and preserves the order of any function.

Proof. Let $f(u)$ be an entire function of order less than $\rho < 1$. Let $f(u) = \sum_{n=0}^{\infty} a_n u^n$ be its Taylor series at the origin. Then

$$\overline{\lim}_{n\to\infty} n^{1/\rho} |a_n|^{1/n} = 0.$$

(For a proof of this claim we refer to [Sh].)

Consider the series $\sum_{n=0}^{\infty} \frac{B_n(u)}{n!} t^n = \frac{te^{ut}}{e^t - 1}$. Since the function $\frac{te^{ut}}{e^t-1}$ is regular for $|t| < 2\pi$ and has a pole at $t = 2\pi i$, the radius of convergence of the above series is 2π. Hence, the upper limit of $\left|\frac{B_n(u)}{n!}\right|^{1/n}$ is $1/(2\pi)$, when n tends to infinity and u is fixed.

Now consider a series[3]

$$\frac{d/du}{e^{d/du} - 1} f(u) = \sum_{n=0}^{\infty} a_n B_n(u). \tag{20}$$

Since

$$\overline{\lim}_{n\to\infty} \left(|a_n B_n(u)|\right)^{1/n} = \overline{\lim}_{n\to\infty} \left(n |a_n|^{1/n}\right) \cdot \left(\frac{|B_n(u)|}{n!}\right)^{1/n} \cdot \left(\frac{n!}{n^n}\right)^{1/n} = 0,$$

the series $\sum_{n=0}^{\infty} a_n B_n(u)$ converges absolutely, defining an operator $\frac{d/du}{e^{d/du}-1}$ on the space of entire functions of order less than one. The component-wise integration

$$\int_u^{u+1} B_n(t)dt = u^n$$

shows that both compositions $\int_u^{u+1} \cdot \frac{d/du}{e^{d/du}-1}$ and $\frac{d/du}{e^{d/du}-1} \cdot \int_u^{u+1}$ are the identity.

The rest is a rather technical estimate of the order of the function $\frac{d/du}{e^{d/du}-1} f(u)$. We again refer to [Sh] for a proof of the following claim.

If the sequence $n^{1/\rho}|b_n|^{1/n}$ is bounded then the function $g = \sum_{n=0}^{\infty} b_n z^n$ is of order $\leq \rho$.

We rewrite (20) as a Taylor series and estimate its coefficients as follows.

$$\frac{d/du}{e^{d/du} - 1} f(u) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} B_{n-k} u^k = \sum_{k=0}^{\infty} \frac{u^k}{k!} \left(\sum_{n=k}^{\infty} a_n n! \frac{B_{n-k}}{(n-k)!}\right).$$

The sequences $n^{1/\rho}|a_n|^{1/n}$, $\left(\frac{|B_n|}{n!}\right)^{1/n}$, and $\left(\frac{n!}{n^n}\right)^{1/n}$ are bounded. Let α, β and 1 be the corresponding upper bounds. Then

$$\left|\frac{B_{n-k}}{(n-k)!} n! a_n\right| \leq \beta^{n-k} \frac{\alpha^n}{n^{n(1/\rho-1)}} \text{ and } \left|\sum_{n\geq k} \frac{B_{n-k}}{(n-k)!} n! a_n\right| \leq \beta^{-k} \sum_{n\geq k} \frac{(\alpha\beta)^n}{n^{n(1/\rho-1)}}.$$

Let $\delta = \log(\alpha\beta)$ and n is big enough to satisfy $\log n - \frac{\delta\rho}{1-\rho} + 1 > 1$. Then

$$\left|\sum_{n\geq k} \frac{B_{n-k}}{(n-k)!} n! a_n\right| \leq \beta^{-k} \sum_{n\geq k} (\log n - \frac{\delta\rho}{1-\rho} + 1) \frac{(\alpha\beta)^n}{n^{n(1/\rho-1)}}.$$

The last sum can be dominated by the integral

$$\beta^{-k} \int_{k-1}^{\infty} (\log x - \frac{\delta\rho}{1-\rho} + 1) \frac{(\alpha\beta)^x}{x^{x(1/\rho-1)}} = \frac{\beta^{-k}}{1/\rho - 1} \cdot \frac{(\alpha\beta)^{k-1}}{(k-1)^{(k-1)(1/\rho-1)}}$$

[3] It is not hard to see, that as operators on the space of polynomials, $\frac{e^{d/du}-1}{d/du} = \int_u^{u+1}$

for k sufficiently big. Finally,

$$\left|\sum_{n\geq k}\frac{B_{n-k}}{(n-k)!}n!a_n\right| \leq \frac{\beta^{-k}}{1/\rho-1}\cdot\frac{(\alpha\beta)^{k-1}}{(k-1)^{(k-1)(1/\rho-1)}}$$

for k sufficiently big, so

$$\left|\sum_{n\geq k}\frac{B_{n-k}}{(n-k)!}n!a_n\right|^{1/k} \leq \frac{\beta^{-1}}{(1/\rho-1)^{1/k}}\frac{(\alpha\beta)^{1-1/k}}{(k-1)^{(1-1/k)(1/\rho-1)}} = \frac{\alpha}{k}+o(\frac{1}{k}),$$

as k tends to infinity, and

$$\overline{\lim}_{k\to\infty}k^{1/\rho}\left|\frac{1}{k!}\sum_{n\geq k}\frac{B_{n-k}}{(n-k)!}n!a_n\right|^{1/k} \leq \alpha e.$$

Thus, the order of $d/du/(e^{d/du}-1)f(u)$ is at most ρ. Since the operator $\int_u^{u+1}$ does not increase the orders, we see that both $d/du/(e^{d/du}-1)f(u)$ and $\int_u^{u+1}$ preserve the orders. □

Let $\mathcal{A}n^{<1}$ be the algebra of holomorphic functions on the universal cover of the complement of $\mathbb{C}$ to a discrete set of branch points of order less than one. We say that the order of an analytic function f is less than one, when there is a positive $\rho<1$ such that

$$|f(\gamma_z)| \leq ae^{b(\text{length of }\gamma_z)^\rho}$$

for any path in the complement of $\mathbb{C}$ to the set of branch points, joining a fixed point with the variable point z. Here "length" means the hyperbolic length, if f has at least 2 points of ramification. If f has only 1 point of ramification then the "length" is the euclidean length, induced by the universal cover $\mathbb{C}\xrightarrow{\exp}\mathbb{C}^\times$.

Lemma 4.5. • *The space of all entire functions of order less than one is admissible.*

- *The additive group of $\mathcal{R}$ with a multiplicative structure given by $*$ is isomorphic to the ideal in $\mathbb{Z}[t]$, generated by t. In particular, integer $*$-powers of 1*

$$1^{*^n} = \binom{t-1}{n-1} \tag{21}$$

generate $\mathcal{R}$ as an abelian group.

- $$(P*Q)(t) = \sum_{m=0}^{\infty}\frac{(-1)^m B_{m+1}(t)+B_{m+1}}{(m+1)!}\frac{d^m}{dt^m}\left(P(t)Q(u-t)\right)\Bigg|_{u=t}. \tag{22}$$

- $$(P*Q)(t) = \int_1^t\frac{d/d\tau}{e^{d/d\tau}-1}\left(P(t-\tau)Q(\tau)\right)d\tau. \tag{23}$$

Proof.

- Number of zeroes of a function of order less than one cannot grow linearly, as it does when that function has zeroes at all natural points. This proves injectivity of the operator V.

To verify the second condition of admissibility we use the formula (23) to be proved later. Due to Theorem 4.4, the operator $\frac{d/d\tau}{e^{d/d\tau}-1}$ is defined and preserves the order of the function.

- Let $P(t) = \sum_{m\geq 0} p_m t^m$ and $Q(t) = \sum_{m\geq 0} q_m t^m$. Then

$$(P * Q)(n) = \sum_{l,m\geq 0} p_l q_m \sum_{k=1}^{n-1} k^m (n-k)^l = \sum_{k=1}^{n-1} \sum_{l,m\geq 0} p_l q_m \sum_{r=1}^{l} (-1)^r \binom{l}{r} n^{l-r} k^{r+m}$$

$$= \sum_{l,m\geq 0} p_l q_m \sum_{r=0}^{l} (-1)^r \binom{l}{r} n^{l-r} \frac{B_{r+m+1}(n) - B_{r+m+1}(1)}{r+m+1}$$

$$= \sum_{r,m\geq 0} (-1)^r \frac{q_m}{r!} \frac{B_{r+m+1}(n) - B_{r+m+1}(1)}{r+m+1} \sum_{l\geq r} p_l \frac{l!}{(l-r)!} n^{l-r}$$

$$= \sum_{m\geq r\geq 0} (-1)^r \binom{m}{r} Q^{(m-r)}(0) \frac{B_{m+1}(n) - B_{m+1}(1)}{(m+1)!} P^{(r)}(n).$$

-

$$\frac{d}{dt} \cdot \left[\sum_{m=0}^{\infty} \frac{(-1)^m B_{m+1}(t) + B_{m+1}}{(m+1)!} \frac{d^m}{dt^m} \right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m B_{m+1}(t) + B_{m+1}}{(m+1)!} \frac{d^{m+1}}{dt^{m+1}} + \sum_{m=0}^{\infty} (-1)^m \frac{B_m(t)}{m!} \frac{d^m}{dt^m}$$

$$= \sum_{m=0}^{\infty} \frac{B_m}{m!} \frac{d^m}{dt^m} = \frac{d/dt}{e^{d/dt} - 1}$$

□

To see how do the multiple gamma functions fit into this scheme, we need the existence of the finite convolution product in a slightly more general situation. To be precise, we should talk about logarithm of the multiple gamma functions rather than the multiple gamma functions, since orders of the latter are at least one. We consider the algebra of analytic functions with ramification in a discrete set of branch points in the left half-plane with certain growth restrictions.

Lemma 4.6. • *The algebra of analytic functions generated by logarithms of entire functions of finite order with no zeroes in the right half-plane, by their derivatives and (iterated) integrals, is admissible.*

- *In the space of holomorphic functions on a sector as above*

$$\log \Gamma_P(u) = \log u * P(u). \tag{24}$$

Proof. To verify injectivity of the operator of values at all natural numbers, suppose that there is a function $f(u)$ on the right half-plane of order less than unity (i.e., $|f(u)| \leq a \cdot e^{b|u|^\rho}$ for some non-negative real $\rho < 1$) vanishing at all positive integer points and such that it does *not* tends to zero on the real positive ray[4]. Then there is a sequence of real positive numbers (a_n) such that $|f(a_n)| > e^{-\alpha n^c} > 0$ for some $\alpha > 0$, $0 < c < 1$ and $a_n > n$.

[4]or, if it does, than not faster than the exponential function.

Then the Jensen formula gives

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(a_n+a_ne^{i\varphi})|d\varphi \geq \log|f(a_n)| + \sum_{|k-a_n|<a_n, k\in\mathbb{Z}} \log\frac{a_n}{|k-a_n|}.$$

The sum above is greater than $\sum_{k=1}^{[a_n]}\log\frac{a_n}{k}\sim a_n$. On the other hand, the left hand side grows only as a_n^ρ, when n tends to infinity, hence we get a contradiction.

This implies that the operator $\int_u^{u+1}$ is injective. Indeed, any element φ of its kernel should be periodic (derivative of $\int_u^{u+1}\varphi(t)dt$ is trivial). In particular, it defines an entire function on the whole complex plane. Moreover, $\varphi(t)=\psi(\exp(2\pi iu))$ for some entire function ψ, and cannot be of polynomial growth. □

5. An L-function

Let $Q(t)$ be a monic polynomial of degree m with roots in the left half-plane. Generalizing [Va] (compare, however, with [C-N]), let us introduce the following L-series:

Definition 5.1. *For a polynomial $P(t)$*

$$L_P(s,Q)=\sum_{n=0}^{\infty}\frac{P(-n)}{Q(n)^s}. \tag{25}$$

The following accounts some basic properties of $L_P(s,Q)$.

Lemma 5.2. 1. *$L_P(s,Q)$ is $\mathbb{C}$-linear with respect to P.*

2. *For a positive integer v*

$$L_{P(t)}(s-v,Q)=L_{P(t)Q(-t)^v}(s,Q).$$

3. *For any μ with a non-negative real part*

$$L_{\mathfrak{T}_\mu P}(s,Q)=L_P(s,\mathfrak{T}_\mu Q)+(P*Q^{-s})(\mu)+Q(0)^{-s}P(\mu). \tag{26}$$

4. *The function $L_P(s,Q)$ admits an analytic continuation to the universal cover of the variety $\{(s,Q)\in\mathbb{C}\times\mathbb{A}^m_{\mathbb{C}}|Q(k)\neq 0$ for $k\in\{0,1,2,3,\dots\}\}$ with simple poles along the divisors $s=-k/m$, where k is a positive integer not divided by m, and*

$$s\in\left\{\frac{1}{m},\frac{2}{m},\dots,\frac{\deg P+1}{m}\right\}.$$

5. *For any positive integer n and a polynomial P the function $L_P(1-n,Q)$ is a polynomial of degree $\deg P+m(n-1)+1$. Its coefficients are rational linear combinations of the coefficients of P.*

6. *For any positive integer n*

$$L'_P(1-n,Q)=L'_{P(t)Q(-t)^{n-1}}(0,Q),$$

and (compare with [Va], Prop. 3.1)

$$L'_P(0,Q)-\sum_{i=1}^m\log\Gamma_P(u_i)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\zeta'(-n)\sum_{i=1}^m P^{(n)}(u_i)+\sum_{i=1}^m \star P(u_i)$$

$$+\frac{1}{2m}\sum_{n=1}^{\infty}\frac{P^{(n)}(1)}{n!}\frac{(-1)^n}{(n+1)!}\frac{d^{n+1}}{dt^{n+1}}\log^2(t^mQ(1/t-1))\bigg|_{t=0}, \tag{27}$$

where $Q(-u_i) = 0$ *and* $\star P(u) = \sum_{n=1}^{\infty} \frac{(u-1)^{n+1}}{(n+1)!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) P^{(n)}(1)$.

7. *If* $\deg P < m-1$ *then the series (25) is absolutely convergent at* $s = 1$ *and*
$$L_P(1,Q) = -\sum_{i=1}^{m} \left[\frac{P(u_i)}{Q'(-u_i)} \frac{\Gamma'(u_i)}{\Gamma(u_i)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot n!} \left(B_n(u_i) - (u_i - 1)^n\right) \frac{P^{(n)}(u_i)}{Q'(-u_i)} \right],$$
where $Q(-u_i) = 0$.

Remark. For a detailed analysis of the quadratic term in (27), the 'determinant anomaly', see [KV].

Proof.

1. Straightforward.
2. Straightforward.
3. If μ is a positive integer then
$$L_{\mathfrak{T}_\mu P}(s,Q) = \frac{P(\mu)}{Q(0)^s} + \sum_{n=0}^{\mu-1} \frac{P(-n)}{Q(n)^s} + \sum_{n \geq 0} \frac{P(\mu - n)}{Q(n+\mu)^s},$$
and Lemma 4.6 implies the rest.
4. Consider the Taylor series of a function $L_l(s,Q) = \sum_{n=1}^{\infty} n^l Q(n)^{-s}$ at $Q(t) = t^m$. More explicitly, let $Q(n) = (n+u_1)\cdots(n+u_m)$ then
$$Q(n)^{-s} = n^{-ms} \sum_{k_1,\ldots,k_m=0}^{\infty} \binom{-s}{k_1} \cdots \binom{-s}{k_m} (u_1/n)^{k_1} \cdots (u_m/n)^{k_m}.$$
Thus we get an analytic continuation for $\Re s < (l+1)/m$ as a series
$$L_l(s,Q) = \sum_{k_1,\ldots,k_m=0}^{\infty} \binom{-s}{k_1} \cdots \binom{-s}{k_m} u_1^{k_1} \cdots u_m^{k_m} \zeta(ms + k_1 + \cdots + k_m - l)$$
for any non-negative integer l. Linearity implies the rest.
5. Straightforward from the above expansion for $L_l(s,Q)$. (Note that values of the Riemann ζ-function are rational at non-positive integers.)
6. The first identity comes from the second property.
For the second one use the Taylor series of $L_l(s,Q)$ at $u_1 = \cdots = u_m = 0$ above and a well-known identity
$$\frac{d}{ds}\left(s\zeta(s+1)\right)\Big|_{s=0} = -\Gamma'(1) = \gamma$$
(coming from the functional equation of the Riemann ζ-function).
Then we get
$$L_l'(0,Q) = m\zeta'(-l) + \sum_{\substack{k=1 \\ k \neq l+1}}^{\infty} \frac{(-1)^k}{k} \zeta(k-l)(u_1^k + \cdots + u_m^k) + (-1)^{l+1}\gamma \frac{u_1^{l+1} + \cdots + u_m^{l+1}}{l+1}$$
$$+ \frac{(-1)^{l+1}}{m}(1 + \cdots + 1/l)\frac{u_1^{l+1} + \cdots + u_m^{l+1}}{l+1} + \frac{(-1)^{l+1}}{m} \sum_{1 \leq i < j \leq m} \sum_{k=1}^{l} \frac{u_i^k}{k} \cdot \frac{u_j^{l+1-k}}{l+1-k}.$$
Now compare $L_l'(0,Q)$ with $\sum_{i=1}^{m} \log \Gamma_P(u_i + 1)$ for $P(t) = (1-t)^l$.

The definition of $\Gamma_P(u)$ once combined with a well-known Taylor expansion

$$\log\Gamma(u+1) = -\gamma u + \sum_{m=2}^{\infty}(-1)^m\zeta(m)\frac{u^m}{m}$$

gives

$$\frac{d}{du}\log\Gamma_P(u) = \frac{d}{du}\mathfrak{Q}P(u) - \gamma P(u) + \sum_{m=1}^{\infty}(-1)^{m+1}\zeta(m+1)(u-1)^{m}P(u).$$

Finally, we identify $L'_{(1-t)^l}(0, Q(t+1))$ with

$$L'_l(0,Q) = \sum_{i=1}^{m}\log\Gamma_{(1-t)^l}(u_i+1) + (-1)^l\left(1+\frac{1}{2}+\cdots+\frac{1}{l}\right)\frac{u_1^{l+1}+\cdots+u_m^{l+1}}{l+1}$$

$$+\sum_{k=0}^{l}(-1)^k\binom{l}{k}\zeta'(k-l)(u_1^k+\cdots+u_m^k) + \frac{(-1)^{l+1}}{2m}\sum_{1\le i,j\le m}\sum_{k=1}^{l}\frac{u_i^k}{k}\cdot\frac{u_j^{l+1-k}}{l+1-k},$$

and $L'_P(0,Q) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}P^{(l)}(1)L'_l(0,Q(t-1))$.

7. The convergence is straightforward.
To compute the value of $L_P(1,Q)$, consider a one-parameter variation of the series (25), namely, $L_P(s, Q+\tau)$. Clearly,

$$\frac{\partial}{\partial\tau}L'_P(s,Q+\tau) = -sL'_P(s+1,Q+\tau) - L_P(s+1,Q+\tau).$$

Now specialize s and τ to 0. We get

$$\left.\frac{\partial}{\partial\tau}L'_P(0,Q+\tau)\right|_{\tau=0} = -L_P(1,Q).$$

Then use the formula (27):

$$\frac{\partial}{\partial\tau}L'_P(0,Q+\tau) = \sum_{i=1}^{m}\frac{\Gamma'_P(u_i(\tau))}{\Gamma_P(u_i(\tau))}u'_i(\tau) + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\zeta'(-n)\sum_{i=1}^{m}P^{(n+1)}(u_i(\tau))\cdot u'_i(\tau)$$

$$+\sum_{i=1}^{m}u'_i(\tau)\cdot\frac{\partial}{\partial\tau}\star P\left(u_i(\tau)\right)$$

$$\left.+\frac{1}{2m}\sum_{n=1}^{\infty}\frac{P^{(n)}(1)}{n!}\frac{(-1)^n}{(n+1)!}\frac{d^{n+1}}{dt^{n+1}}\frac{\partial}{\partial\tau}\log^2\left(t^m(Q(1/t-1)+\tau)\right)\right|_{t=0}.$$

Under the assumption on degree of the polynomial P, the latter sum is trivial, since

$$\frac{\partial}{\partial\tau}\log^2\left(t^m(Q(1/t-1)+\tau)\right) = \left(\frac{2\log\left(t^m(Q(1/t-1)+\tau)\right)}{t^m\left(Q(1/t-1)+\tau\right)}\right)t^m.$$

Clearly, $u'_i(\tau) = 1/Q'(-u_i(\tau))$, in particular, $u'_i(0) = 1/Q'(-u_i)$. We use the formula (29) below to find

$$\frac{\partial}{\partial\tau}\star = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{(u-1)^m}{m\cdot m!}\frac{d^m}{du^m}.$$

Finally we get the desired identity. □

The operator $\star$ admits more descriptions:

Lemma 5.3. 1. *Integral formula:*

$$\star\varphi(u) = \frac{1}{2}\int_1^u \int_1^u \frac{\varphi(v)-\varphi(t)}{v-t}dtdv = \int_1^u \int_1^v \frac{\varphi(v)-\varphi(t)}{v-t}dtdv. \tag{28}$$

2.

$$\star = \sum_{m=1}^{\infty}(-1)^{m+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)\frac{(u-1)^{m+1}}{(m+1)!}\frac{d^m}{du^m}. \tag{29}$$

3.

$$\star = \sum_{m=1}^{\infty}(-1)^{m+1}\star\binom{u-1}{m}(1-\mathfrak{T}^{-1})^m, \tag{30}$$

where $\star$ *can actually be replaced by any operator* $\star = \sum_{n=1}^{\infty} A_n \frac{(u-1)^{n+1}}{(n+1)!}\frac{d^n}{du^n}$ *such that* $\star = \sum_{n=1}^{\infty}(-1)^{n+1}A_n\frac{(u-1)^{n+1}}{(n+1)!}\frac{d^n}{dt^n}\Big|_{t=1}$.

Proof.

1. Replace $\varphi(v)$ and $\varphi(t)$ by their Taylor series at $v=1$ and $t=1$ respectively. This proves the first identity.
2. Replace $\varphi(v)$ and $\varphi(t)$ in the integral by their Taylor series at $v=u$ and $t=u$ respectively. This proves the second identity.
3. To prove the third one, notice that $\mathfrak{T} = \exp d/du$, so d^m/du^m and $(1-\mathfrak{T}^{-1})^m$ form two bases for the same space of "pro-differential" operators with constant coefficients. Then $\star = \sum_{m=1}^{\infty} P_m(u)(1-\mathfrak{T}^{-1})^m$ for some polynomials $P_m(u)$. Denote d/du by x and $1-\mathfrak{T}^{-1}$ by N. Then

$$P_m(u) = \sum_{n=1}^{\infty} A_n \frac{(u-1)^{n+1}}{(n+1)!}\mathrm{res}_{N=0}\frac{x^n dN}{N^{m+1}}.$$

Notice, that $\mathrm{res}_{N=0}\frac{x^n dN}{N^{m+1}} = (-1)^n\frac{d^n}{dt^n}\mathrm{res}_{N=0}\frac{e^{-x(t-1)}dN}{N^{m+1}}$. By the assumption on $\star$ above

$$P_m(u) = -\star\,\mathrm{res}_{N=0}\frac{(e^{-x})^{u-1}dN}{N^{m+1}}.$$

Since $N = 1-e^{-x}$, the residue is $(-1)^m\binom{u-1}{m}$ and $P_m(u) = (-1)^{m+1}\star\binom{u-1}{m}$. □

Appendix A. p-adic log multiple gamma functions

This appendix is to show, how the above methods can work in the p-adic situation. In fact, many of the questions considered above become simpler. For instance, the a much more wide class of functions than entire ones is admissible (note, however, that the set of naturals is not discrete anymore).

First some explanations, why do we make any difference between the (multiple) gamma functions the log (multiple) gamma functions. An argument of the gamma function is a multiplicative character, and the gamma function itself is defined as an integral of the product of this multiplicative character with a fixed additive character over a cycle.

The log gamma function is somewhat dual. In a sense, it is an integral of the product of a pair of homomorphisms from the multiplicative group to the multiplicative and additive groups of a field over a cycle.

Since the positive integers are dense in the ring $\mathbb{Z}_p$, the finite convolution of may be interpreted in terms of multiplication of formal series as follows

$$t\sum_{n=1}^{\infty}(f*g)(n)t^n = \sum_{n=1}^{\infty} f(n)t^n \cdot \sum_{n=1}^{\infty} g(n)t^n.$$

The following lemma may be found in [Ko]. For a convenience, we give a proof.

Lemma A.1. *The operator "$\int_u^{u+1}$" "remains" defined and "becomes" invertible on the space of power series convergent on the unit disc $\{|u|_p \le 1\} \subset \mathbb{C}_p$. The inversion is given by the formula*

$$f \longmapsto \lim_{m\longrightarrow\infty} p^{-m}\sum_{k=0}^{p^m-1} f(u+k).$$

Proof. The operator "$\int_u^{u+1}$" is defined by a series of operators: $\int_u^{u+1} = \sum_{m=0}^{\infty}\frac{1}{(m+1)!}\frac{d^m}{du^m}$. We need to check the convergence for some class of functions.

$$\sum_{m=0}^{\infty}\frac{1}{(m+1)!}\frac{d^m}{du^m}f(u) = \sum_{m=0}^{\infty}\frac{1}{(m+1)!}\frac{n!}{(n-m)!}\sum_{n=0}^{\infty}a_n u^{n-m}$$

$$= \sum_{n=0}^{\infty}u^n\sum_{m=0}^{\infty}\binom{n+m}{n}\frac{a_{n+m}}{m+1} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m+1}a_{n+m}\frac{u^n}{n} + \sum_{m=0}^{\infty}\frac{a_m}{m+1}.$$

The operator

$$f \longmapsto p^{-m}\sum_{k=0}^{p^m-1} f(u+k)$$

sends the monomial u^n to $\frac{B_{n+1}(u+p^m)-B_{n+1}(u)}{p^m(n+1)}$, that p-adically tends to $B_n(u)$, when m tends to infinity. Since $B_n(u) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}u^k$, one can try to prolong this operation to some power series in u by

$$\sum_{n=0}^{\infty}a_n u^n \longmapsto \sum_{n=0}^{\infty}a_n B_n(u) = \sum_{k=0}^{\infty}u^k\sum_{n=k}^{\infty}a_n\binom{n}{k}B_{n-k}.$$

The only problem is to find some conditions to make series of constants convergent. It is well-known that the Bernoulli numbers are p-adically bounded for all $p \neq \infty$. This can be seen as follows.

$$\frac{B_n(p^N)-B_n}{n} = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}B_{n-k}p^{Nk} = 1^{n-1}+\cdots+(p^N-1)^{n-1}.$$

Let N be big enough. Then by the non-archimedean triangle inequality

$$|p^N B_{n-1}|_p = |1^{n-1}+\cdots+(p^N-1)^{n-1}|_p$$

We reduce the problem to estimates of some geometric progressions:

$$\sum_{k=1}^{p^N-1} k^{n-1} = \sum_{m=1}^{p^{N-1}} p^{N-m} \sum_{\substack{(k,p)=1 \\ 0<k<p^m}} k^{n-1}.$$

The multiplicative group $\mathbb{Z}_p^\times$ is topologically cyclic. Let θ be its generator. Then for any m

$$\sum_{\substack{(k,p)=1 \\ 0<k<p^m}} k^{n-1} \equiv \sum_{k=0}^{p^{m-1}(p-1)-1} \theta^{k(n-1)} (\mathrm{mod} p^m) \equiv \frac{1-\theta^{p^{m-1}(p-1)(n-1)}}{1-\theta^{n-1}} (\mathrm{mod} p^m).$$

Since

$$\mathrm{ord}_p \left(\frac{1-\theta^{p^{m-1}(p-1)(n-1)}}{1-\theta^{n-1}} \right) = \begin{cases} m & \text{if } p-1 \text{ does not divide } n-1 \\ m-1 & \text{if } p-1 \text{ divides } n-1 \end{cases},$$

we get

$$|p^N B_{n-1}|_p = \begin{cases} p^{-N} & \text{if } p-1 \text{ does not divide } n-1 \\ p^{1-N} & \text{if } p-1 \text{ divides } n-1 \end{cases}$$

Finally, $|B_{n-1}|_p \leq p$.

Thus, whenever the sequence of coefficient (a_n) tends to zero, the series's $\sum_{n=k}^{\infty} a_n \binom{n}{k} B_{n-k}$ are convergent and tend to zero, when k tends to infinity.

□

The p-adic log multiple gamma functions were introduced, probably, by P.Cassou-Noguès.

The asymptotic decomposition of the log multiple gamma function at infinity "becomes" just a convergent series, and may be used to define the p-adic multiple gamma function. In particular, for the logarithmic derivative of the usual gamma function

$$\frac{{}_{(p)}\Gamma'(u)}{{}_{(p)}\Gamma(u)} \sim \log u - \sum_{k=1}^{\infty} (-1)^k \frac{B_k}{k} u^{-k}.$$

(*Proof.*

$$\frac{d^2}{du^2} \log \Gamma(u+1) = \frac{d}{du} \frac{\Gamma'(u+1)}{\Gamma(u+1)} = \frac{\Gamma(u+1)\Gamma''(u+1) - \Gamma'(u+1)^2}{\Gamma(u+1)^2}$$

$$= \left. \frac{\Gamma(u+1)\Gamma'(u+\alpha+1) - \Gamma(u+\alpha+1)\Gamma'(u+1)}{\alpha \Gamma(u+\alpha+1)^2} \right|_{\alpha=0} = -\frac{d}{du} \left. \left(\frac{\Gamma(u+1)\Gamma(\alpha)}{\Gamma(u+\alpha+1)} \right) \right|_{\alpha=0}$$

$$= -\frac{d}{du} B(u+1,\alpha) \Big|_{\alpha=0} = -\frac{d}{du} \int_0^1 t^u (1-t)^{\alpha-1} dt \Big|_{\alpha=0} = -\int_0^1 \log t \cdot t^u (1-t)^{\alpha-1} \Big|_{\alpha=0} dt$$

$$= -\int_0^1 \frac{\log t \cdot t^u dt}{1-t} = \int_0^\infty \frac{t e^{-ut} dt}{e^t - 1} \sim \sum_{n=0}^{\infty} \frac{B_n}{n!} \int_0^\infty t^n e^{-ut} dt = \sum_{n=0}^{\infty} B_n u^{-n-1}.$$

Then integrate to get the asymptotic for $\frac{d}{du} \log \Gamma(u+1)$.)

This series is convergent, when considered as a function on the open subset $|u|_p > 1$ of $\mathbb{C}_p$. Under the assumption $|u|_p > 1$, one can therefore define (a first approximation to) the p-adic log multiple gamma function as

$$\widetilde{\log\Gamma}_P(u) = \log u \cdot \mathfrak{I}P(u) - \int_0^u \frac{1}{t}\int_0^t P(\tau)d\tau dt$$

$$+ \sum_{n\in\mathbb{Z},n\neq 0}(-1)^n\left(\sum_{m\geq\max(-n,0)}(-1)^{m+1}\frac{B_{m+n+1}}{m+n+1}\cdot\frac{P^{(m)}(0)}{m!}\right)\frac{u^{-n}}{n}, \tag{31}$$

where $\mathfrak{I}P(u) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n(u)}{n!}P^{(n-1)}(u)$. Then, to modified definition 2.1 for the p-adic case, we need to find a polynomial operator $\mathfrak{Q}$ such that the function $\log\Gamma_P(u) = \mathfrak{Q}P(u) + \widetilde{\log\Gamma}_P(u)$ satisfies the property corresponding to (9). Let

$$\varphi_m(u) = \sum_{n\geq -m,n\neq 0}(-1)^{n+m+1}\frac{B_{m+n+1}}{m+n+1}\cdot\frac{u^{-n}}{n}.$$

Then

$$\varphi_m(u+1) = \sum_{n\geq -m,n\neq 0}(-1)^{n+m+1}\frac{B_{m+n+1}}{m+n+1}\cdot\frac{u^{-n}}{n}\sum_{k\geq 0}\binom{-n}{k}u^{-k}$$

$$= \sum_{n\geq -m}(-1)^{n+m+1}\left(\sum_{0\leq k\leq m+n,k\neq n}\binom{n-1}{k}\frac{B_{m+n-k+1}}{(n-k)(m+n-k+1)}\right)u^{-n}$$

$$= \sum_{n\geq -m,n\neq 0}(-1)^{n+m+1}\left(\sum_{0\leq k\leq m+n,k\neq n}\binom{n}{k}\frac{B_{m+n-k+1}}{m+n-k+1}\right)\frac{u^{-n}}{n} + \sum_{k=1}^{m}(-1)^{k+m}\frac{B_{m-k+1}}{k(m-k+1)}.$$

Now we try to find a more convenient formula for the coefficients $C_{n,m} = \sum_{k=0}^{n}\binom{n}{k}\frac{B_{m+k+1}}{m+k+1}$ for positive n. Clearly, $C_{n,m}$ is the coefficient of t^n in the power series

$$n!\cdot e^t\cdot\frac{d^m}{dt^m}\left(\frac{1}{e^t-1}-\frac{1}{t}\right).$$

Note, that $e^t\cdot d^m/dt^m = (d/dt-1)^m\cdot e^t$, so

$$C_{n,m} = \frac{d^n}{dt^n}\cdot(\frac{d}{dt}-1)^m e^t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)\Bigg|_{t=0}$$

$$= \frac{d^n}{dt^n}\cdot(\frac{d}{dt}-1)^m\left(\frac{1}{1-e^{-t}}-\frac{e^t}{t}\right)\Bigg|_{t=0}$$

$$= \sum_{k=0}^{\infty}\frac{d^n}{dt^n}\cdot(\frac{d}{dt}-1)^m t^k\left((-1)^{k+1}\frac{B_{k+1}}{(k+1)!}-\frac{1}{(k+1)!}\right)\Bigg|_{t=0}$$

$$= \sum_{k=n}^{\infty}(-1)^{n+m+1}\binom{m}{k-n}\left(\frac{B_{k+1}}{k+1}-(-1)^{k+1}\frac{1}{k+1}\right)$$

$$= \sum_{k=0}^{m}(-1)^{n+m+1}\binom{m}{k}\left(\frac{B_{k+n+1}}{k+n+1}+(-1)^{k+n}\frac{1}{k+n+1}\right)$$

$$= \sum_{k=0}^{m} (-1)^{n+m+1} \binom{m}{k} \frac{B_{k+n+1}}{k+n+1} + (-1)^{m+1} \frac{m!n!}{(m+n)!}.$$

There is a natural short exact sequence

$$0 \longrightarrow \mathbb{C}_p(1) \longrightarrow B_{DR}^{+}/F^2 B_{DR}^{+} \longrightarrow \mathbb{C}_p \longrightarrow 0,$$

where B_{DR}^{+} is the Fontaine's ring of periods. Following Colmez, it may be defined as the completion of the field $\overline{K}$ with respect to a topology, given by the basis $p^n \mathcal{O}_{\overline{K}}^{(m)}$ of neighborhoods of 0, where $\mathcal{O}_K$-algebras $\mathcal{O}_{\overline{K}}^{(m)}$ are defined inductively by

$$\mathcal{O}_{\overline{K}}^{(m)} = \operatorname{Ker}\left(\mathcal{O}_{\overline{K}}^{(m-1)} \xrightarrow{d} \Omega^1_{\mathcal{O}_{\overline{K}}^{(m-1)}/\mathcal{O}_K} \otimes \mathcal{O}_{\overline{K}}\right),$$

and $\mathcal{O}_{\overline{K}}^{(0)} = \mathcal{O}_{\overline{K}}$. Clearly, if the ring of integers $\mathcal{O}_L$ in a field L is a subring of $\mathcal{O}_{\overline{K}}^{(m)}$, then so is $\mathcal{O}_L^{ur}$. In particular, the ring of integers in the maximal unramified extension of K is a subring of $\mathcal{O}_{\overline{K}}^{(m)}$ for all m. Then the formula (14) and presence on the logarithm in the last formula suggests that the natural range of the values of the logarithmic derivative of the gamma function should be at least $B_{DR}^{+}/F^2 B_{DR}^{+}$.

A.1. **The adelic log multiple gamma function.** In the adelic case the series (31) is convergent nowhere, so we must choose another approach to define the log multiple gamma function. Instead, we interpret the formula

$$\log \Gamma_P(n+1) = \sum_{k=1}^{n} \log k \cdot P(n-k+1).$$

It is well-known that the logarithm on a field k can be interpreted as follows. Consider a canonical map

$$k^{\times} \longrightarrow \mathrm{H}^1(\mathrm{Gal}(k^{ab}/k), \widehat{\mathbb{Z}}(1)), \ \alpha \longmapsto (\sigma \longmapsto (\sigma\alpha_n/\alpha_n)),$$

where $\alpha_1 = \alpha$ and $\alpha_{mn}^n = \alpha_m$. Clearly, the cohomology class is independent of ambiguity in the choice of roots of α. Let $\mathbb{W} = \prod_p W(\overline{\mathbb{F}_p})$,

$$\mathbb{W}[[\widehat{\mathbb{Z}}(1)]] = \varprojlim_N \mathbb{W}[\mu_N],$$

where the limit is taken under the maps induced by the standard inverse system $\widehat{\mathbb{Z}}(1)$, and an element $t \in \mathbb{W}[[\widehat{\mathbb{Z}}(1)]]$ is the difference of the unit in $\mathbb{W}$ and a topological generator of $\widehat{\mathbb{Z}}(1)$. The Dieudonné-Dwork short exact sequence (e.g., [An]):

$$0 \longrightarrow \widehat{\mathbb{Z}}(1) \longrightarrow \left(1 + t\mathbb{W}[[\widehat{\mathbb{Z}}(1)]]\right)^{\times} \longrightarrow t\mathbb{W}[[\widehat{\mathbb{Z}}(1)]] \longrightarrow 0, \tag{32}$$

where the first map is identical, and the second one is given by the identification of the sequence (32) with the sequence

$$0 \longrightarrow (1+t)^{\widehat{\mathbb{Z}}} \longrightarrow (1 + t\mathbb{W}[[t]])^{\times} \xrightarrow{\lambda} t\mathbb{W}[[t]] \longrightarrow 0,$$

where $\lambda : H(t) \longmapsto \log H(t) - p^{-1} \log H^{\phi}((1+t)^p - 1)$ and ϕ is the Frobenius on $W(\overline{\mathbb{F}_p})$ acting on the coefficients.

The homomorphism Log gives rise to a linear map

$$\text{Maps}(\mathbb{N},\widehat{\mathbb{Z}})[\mathbb{N}] \longrightarrow \text{H}^1(\text{Gal}(k^{ab}/k),\widehat{\mathbb{Z}}(1)),\ (\varphi,n)\longmapsto \sum_{k=1}^{n-1}\text{Log}k\cdot\varphi(n-k).$$

Clearly,

$$\text{Log}\Gamma(\varphi,n+1)=\varphi(1)\text{Log}(n)+\text{Log}\Gamma(\mathfrak{T}\varphi,n).$$

Appendix B. Arithmetic Todd genus (according to [GSZ])

This appendix is devoted to applications of the multiple gamma function to arithmetic (or, if one wishes, differential) geometry. The well-known arithmetic Riemann–Roch theorem of H.Gillet and C.Soulé asserts that the Grothendieck–Riemann–Roch theorem remains valid in the arithmetic case, if one modifies the notion of characteristic classes, and the notion of the Todd genus. The arithmetic Todd genus itself was found by D.Zagier. We try to simplify his calculations, but essentially just repeat them.

Let me remind some of definitions from [GSZ] (see also [F]).

Let X be a quasi-projective flat scheme over **Spec** $(\mathbb{Z})$. If E is a locally free sheaf over X, denote by E_∞ the associated holomorphic vector bundle over the complex variety $X_\infty = X(\mathbb{C})$.

Definition B.1. *The arithmetic K-group $\widehat{K}_0(X)$ is a quotient ring of direct sum of the ring generated by vector bundles E with a smooth hermitian metric h on the associated holomorphic vector bundle E_∞ over X_∞ invariant under complex conjugation, and the algebra of differential forms on X_∞ skew-invariant under the simultaneous action of complex conjugation on X_∞ and on the coefficients. The relations are* $[(E_1,h_1)]-[(E_2,h_2)]+[(E_3,h_3)]-\widetilde{\text{ch}}(\mathcal{E})=0$ *for all exact sequences*

$$\mathcal{E}:0\longrightarrow E_1\longrightarrow E_2\longrightarrow E_3\longrightarrow 0,$$

where the secondary Chern class $\widetilde{\text{ch}}(\mathcal{E})$ *is any solution of the equation*

$$dd^c\widetilde{\text{ch}}(\mathcal{E})=\text{ch}(E_1,h_1)-\text{ch}(E_2,h_2)+\text{ch}(E_3,h_3).$$

Multiplication is defined as follows:

$$[(E_1,h_1)]\cdot[(E_2,h_2)]=[(E_1\otimes E_2,h_1\otimes h_2)],$$
$$[(E,h)]\cdot\omega=[\text{ch}'(E,h)\wedge\omega],$$
$$[\omega_1]\cdot[\omega_2]=\left[\frac{\partial\overline{\partial}}{\pi i}\omega_1\wedge\omega_2\right].$$

Definition B.2. *The arithmetic Chow group $\widehat{CH}^*(X)$ is generated by pairs (Z,g) of a subvariety Z on X and a "Green current" g for the corresponding cycle on X_∞. This means that $dd^cg+\delta_Z$ is a smooth form.*

Multiplication is defined as follows:

$$(Z_1,g_{Z_1})\cdot(Z_1,g_{Z_1})=(Z_1\cdot Z_2,g_{Z_1}\cdot\delta_{Z_2}+\delta_{Z_1\cdot Z_2}\cdot g_2-\frac{\partial\overline{\partial}}{\pi i}g_{Z_1}\cdot g_{Z_2}).$$

One can define a homomorphism of the rings $\widehat{\text{ch}}:\widehat{K}_0(X)\longrightarrow\widehat{CH}^*(X)$, that coincides with ch when projected to $CH^*(X)$.

There is a way to define a canonical hermitian metric on the direct images of any hermitian vector bundles (cf. [F]) for any proper map. In particular, for an acyclic line bundle this metric coincides with the L^2-metric multiplied by the exponent of the analytic torsion. This gives rise the direct image homomorphism $Rf_* : \widehat{K}_0(X) \longrightarrow \widehat{K}_0(Y)$ for any proper map $f : X \longrightarrow Y$.

The arithmetic Riemann-Roch theorem states that for any smooth proper morphism $f : X \longrightarrow Y$ of projective arithmetic schemes with metrized fibers the following diagram commutes

$$\begin{array}{ccc} \widehat{K}_0(X) & \xrightarrow{Rf_*} & \widehat{K}_0(Y) \\ {\scriptstyle \widehat{\mathrm{ch}}(\cdot)\widehat{\mathrm{Td}}^R(T_{X/Y})}\downarrow & & \downarrow{\scriptstyle \widehat{\mathrm{ch}}(\cdot)} \\ \widehat{CH}^*(X) & \xrightarrow{f_*} & \widehat{CH}^*(Y), \end{array}$$

where $R(x) = \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \left(2\zeta'(-m) + \zeta(-m) \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right)\right) \frac{x^m}{m!}$.

The proof is parallel to the proof of Grothendieck-Riemann-Roch Theorem.

All details may be found in the lectures of Faltings [F], except for an explicit calculation of the arithmetic Todd genus. We try to evaluate it on the projective spaces.

B.1. **Chern character of the direct image of the trivial hermitian line bundle on a projective space.** We consider a projective space $\mathbb{P}^n$, the trivial line bundle $\mathcal{O}_{\mathbb{P}^n}$ with the trivial hermitian metric on it, and compare two compositions in the diagram above: $\widehat{\mathrm{ch}}\left(Rf_*\left(\mathcal{O}_{\mathbb{P}^n}, 1\right)\right)$ and $f_*\left(\widehat{\mathrm{Td}}^R\left(\mathcal{T}_{\mathbb{P}^n}\right)\right)$, where f maps $\mathbb{P}^n$ to the point.

The left hand side of the Riemann-Roch theorem is

$$\widehat{\mathrm{ch}}\left(Rf_*\left(\mathcal{O}_{\mathbb{P}^n}, 1\right)\right) = -\log n! + \sum_{q \geq 0} (-1)^{q+1} q \zeta_q'(0),$$

where $\zeta_q(s) = \zeta_{\Delta_q}(s)$ and $\Delta_q = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is the Laplacian acting on the space of differential forms of bidegree $(0, q)$ on $\mathbb{P}^n$.

Ikeda and Taneguchi calculate in [IT] the spectrum of the Laplace operator Δ_q. They get $k(k + n + 1 - q)$ for all integer $k \geq q$ as eigenvalues with multiplicities $d_{n,q}(k) + d_{n,q+1}(k)$ and $d_{n,0}(k) = 0$, given by the formula

$$d_{n,q}(k) = q \binom{n}{q} \left(\frac{1}{k} + \frac{1}{k + n + 1 - q}\right) \binom{k+n}{n} \binom{k+n-q}{n}.$$

Denote by $S_{n,r}(t)$ such a polynomial that $d_{n,n+1-r}(k) = S_{n,r}(1 - k)$. Notice that $S_{n,r}(t)$ is a polynomial of degree $2n - 1$ and $S_{n,r}(r + 2 - t) = -S_{n,r}(t)$. So,

$$S_{n,r}(t) = (-1)^n r \binom{n}{r} \left(\frac{1}{1-t} + \frac{1}{r+1-t}\right) \binom{t-2}{n} \binom{r-t}{n} \quad \text{and}$$

$$\widehat{\mathrm{ch}}\left(Rf_*\left(\mathcal{O}_{\mathbb{P}^n}, 1\right)\right) = -\log n! + \sum_{q \geq 0} (-1)^{q+1} q \zeta_q'(0) = -\log n! + \sum_{r=1}^{n} (-1)^{n+r} L'_{S_{n,r}}(0, r+1).$$

By Lemma 5.2

$$L'_{S_{n,r}}(0,r+1) = \log\Gamma_{S_{n,r}}(r+1) - 2\sum_{\substack{m\geq 1\\ m \text{ odd}}} \frac{\zeta'(-m)}{m!} S_{n,r}^{(m)}(1) + \frac{1}{2}\star S_{n,r}(r+1),$$

Due to the lemma 2.4.1(ii,iii) and 2.4.2 of [GSZ] this equals to

$$\frac{1}{2}\sum_{r=1}^{n}(-1)^{r+n}\sum_{m=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)\frac{r^{m+1}}{(m+1)!}S_{n,r}^{(m)}(1)$$

$$-\text{ coefficient of } x^n \text{ in } 2(n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1}\sum_{\substack{m\geq 1\\ m \text{ odd}}}\frac{\zeta'(-m)}{m!}x^m.$$

$$\sum_{r=1}^{n}(-1)^{r+n}r^m S_{n,r}(t) = \sum_{r=1}^{n}(-1)^r r^{m+1}\binom{n}{r}\left(\frac{1}{1-t}+\frac{1}{r+1-t}\right)\binom{t-2}{n}\binom{r-t}{n}$$

$$= \frac{1}{1-t}\binom{t-2}{n}\sum_{r=1}^{n}(-1)^r r^{m+1}\binom{n}{r}\binom{r-t}{n} + \binom{t-2}{n}\sum_{r=1}^{n}(-1)^r r^{m+1}\binom{n}{r}\frac{1}{r+1-t}\binom{r-t}{n}$$

$$= \binom{t-2}{n}[\frac{1}{1-t}(Td/dT)^{m+1}\frac{T^{n+t}}{n!}d^n/dT^n\left(T^{-t}(1-T)^n\right)\Big|_{T=1}$$

$$+\int_0^1 T^{-t}(Td/dT)^{m+1}\frac{T^{n+t}}{n!}d^n/dT^n\left(T^{-t}(1-T)^n\right)dT].$$

Let me introduce some series of bilinear operations on a class of functions:

$$\varphi\star_n\psi = -\frac{1}{2}\sum_{r=0}^{n}(-1)^r\binom{n}{r}T^{-r-1}\left\{(1-T^{-1})[\varphi(r-x)\psi(x)]\right\}^*.$$

$$L_\lambda^{-1} = \sum_{|I|\geq 0}\frac{\Gamma(\lambda)}{\Gamma(\lambda+|I|+1)}(-x)^I\frac{\partial^{|I|}}{\partial x^I}. \tag{33}$$

B.2. **The direct image of the arithmetic Todd genus of a projective space.** RHS: The following lemma might be of use.

Lemma B.3. • *For any pair of a polynomial and a formal power series φ,ψ.*

$$\varphi(d/dx)\psi(x)\Big|_{x=0} = \psi(d/dx)\varphi(x)\Big|_{x=0}.$$

• *For any power series $\sum_{m=0}^{\infty}a_m x^m$ the coefficient of x^n in another power series*

$$\left(\frac{x}{1-e^{-x}}\right)^{n+1}\left(\sum_{m=0}^{\infty}a_m x^m\right) \text{ is equal to } \sum_{m=0}^{n}a_m\frac{d^m}{du^m}\binom{u+n}{n}\Big|_{u=0}.$$

Proof. The first is obvious.

For the second one let us find the residue of $\frac{e^{uz}dz}{(1-e^{-z})^{n+1}}$ at the origin. After a change of coordinate $y = 1 - e^{-z}$ we get

$$\operatorname{res}_{z=0}\frac{e^{uz}dz}{(1-e^{-z})^{n+1}} = \operatorname{res}_{y=0}\frac{(1-y)^{-u-1}dy}{y^{n+1}} = \binom{u+n}{n}.$$

Then consider the derivatives at $u = 0$ of the both sides of the above equality. So

$$\operatorname{res}_{z=0}\frac{z^k dz}{(1-e^{-z})^{n+1}} = \frac{d^k}{du^k}\binom{u+n}{n}\bigg|_{u=0}.$$

Finally, the coefficient of x^n in

$$\left(\frac{x}{1-e^{-x}}\right)^{n+1}\left(\sum_{m=0}^{\infty} a_m x^m\right) \text{ is } \sum_{m=0}^{n} a_m \frac{d^m}{du^m}\binom{u+n}{n}\bigg|_{u=0}.$$

□

$$f_*\left(\widehat{\mathrm{Td}}^R(\mathcal{T}_{\mathbb{P}^n})\right) = f_*\left(\widehat{\mathrm{Td}}(\mathcal{T}_{\mathbb{P}^n})\right) - \int_{X(\mathbb{C})} \mathrm{Td}(\mathcal{T}_{\mathbb{P}^n}) R(\mathcal{T}_{\mathbb{P}^n})$$
$$= f_*\left(\widehat{\mathrm{Td}}(\mathcal{T}_{\mathbb{P}^n})\right) - \operatorname{res}_{x=0}\left[R(x)(n+1)\left(1-e^{-x}\right)^{-n-1} dx\right].$$

Denote the last residue by R_n.

Following [GSZ], one calculates $\widehat{\mathrm{Td}}(\mathcal{T}_{\mathbb{P}^n})$ by the method of [BC]. Let $\mathbb{P}^n = \mathbb{P}(V)$ is the projectivization of an $(n+1)$-dimensional space V. There is a canonical exact sequence

$$\mathcal{E}_n : 0 \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}(1) \longrightarrow \mathcal{T}_{\mathbb{P}(V)} \longrightarrow 0 \qquad (34)$$

where all bundles are endowed with the standard metrics, induced by an hermitian form on V. Then

$$f_*\left(\widehat{\mathrm{Td}}^R(\mathcal{T}_{\mathbb{P}(V)})\right) = t_n + \widetilde{Td}_n - R_n,$$

where

$$R_n = \text{coefficient of } x^n \text{ in } (n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1} R(x),$$

$$t_n = \text{coefficient of } x^{n+1} \text{ in } \left(\frac{x}{1-e^{-x}}\right)^{n+1} = (n+1)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - n,$$

$$\widetilde{Td}_n = \text{coefficient of } x^n \text{ in } \left(\frac{x}{1-e^{-x}}\right)^{n+1} \sum_{m=1}^{\infty} \zeta(-m)\frac{x^m}{m!\cdot m}.$$

The secondary Todd class of the exact sequence $\mathcal{E}_n$ can be computed as follows. $\widetilde{\mathrm{Td}}(\mathcal{E}_n)$ Denote by

$$s_n = -\text{ coefficient of } x^n \text{ in } (n+1)\left(\frac{x}{1-e^{-x}}\right)^{n+1} \sum_{\substack{m\geq 1 \\ m \text{ odd}}} \zeta(-m)\cdot\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)\frac{x^m}{m!}.$$

It is proved in the appendix by Zagier to [GSZ] that

$$\frac{1}{2}\sum_{r=1}^{n}(-1)^{r+n}\sum_{m=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)\frac{r^m}{m!}S^{(m)}_{n,r}(1) = s_n + t_n + \widetilde{Td}_n,$$

verifying the Riemann-Roch theorem for the projective spaces.

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