Multiplicity-free Hamiltonian actions and existence of invariant Kähler structure

by

Christopher Thomas Woodward

A.B. in Mathematics and Physics, Harvard College, 1991

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1996

Authc ....
Department of Mathematics
May 3, 1996

Certified by ....
Victor Guillemin
Professor of Mathematics
Thesis Supervisor

Accepted by.. ....
David Vogan
Chairman, Departmental Committee on Graduate Students

Multiplicity-free Hamiltonian actions and existence of invariant Kähler structure

by

Christopher Thomas Woodward

Submitted to the Department of Mathematics
on May 3, 1996, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Mathematics

Abstract

As for toric varieties, with any projective spherical variety is associated a convex polytope, and any facet of this polytope is defined by a prime divisor stable under a Borel subgroup [4]. In this paper we use the moment map to prove, for certain smooth projective spherical varieties, two characterizations of the facets that are defined by divisors stable under the full group action. As a corollary we get a necessary criterion for certain symplectic manifolds with multiplicity-free Hamiltonian group actions to admit invariant compatible Kähler structures. In cases when the group acting is $SO(5)$, we prove that the criterion is sufficient as well as necessary, and show that the existence of a compatible Kähler structure invariant under the action of a maximal torus implies that there exists a compatible Kähler structure invariant under the action of $SO(5)$.

Thesis Supervisor: Victor Guillemin
Title: Professor of Mathematics

Acknowledgments

If it takes a village to raise a child, then it takes an even bigger village to raise a child with a Ph.D.! My thanks to the many people without whom this thesis would never have been written. Here I can mention only a few of them: First, my parents Bob and Mary Woodward always encouraged me and sacrificed to make every opportunity available. They have been and always will be my most important teachers. My dissertation adviser Victor Guillemin has been incredibly supportive. Beyond his brilliant mathematical advice and direction, his spirit of enthusiasm gave my years in graduate school a sense of fun that I will always remember. The idea of the project carried out here, as well as the first steps, are due to him (partly in collaboration with R. Souza.) I was also lucky to have excellent teachers of symplectic geometry in Yael Karshon, Eugene Lerman, Eckhard Meinrenken, Reyer Sjamaar, and Sue Tolman. The goal of this project has been to compare two different viewpoints (symplectic and algebraic) and I owe a lot to Friedrich Knop for patiently explaining the algebraic perspective and contributing many important ideas. He, David Vogan and Allen Knutson tried hard to read the thesis and made many helpful suggestions. The help of Giuseppe Castellacci, Colin Ingalls, Tony Pantev, Karen Smith, and Michael Thaddeus with algebraic geometry was much appreciated.

Behind the scenes an important source of happiness has been my relationship with Neepa Maitra and her family, whose love and support helped me write. Also my sister Rebecca, who came to Boston at just the right time; my former housemates Mathieu Gagné and David Metzler; and my friends Sheila Kannappan, Brodie Fischer, and Malcolm and Laura Quinn were always there.

Finally I want to thank my fellow graduate students Ana Canas da Silva, Edith Mooers, Richard Stone, and many others I have run out of room to mention, for making graduate school at M.I.T. so enjoyable, and my old advisers Mark McConnell and Andrew Lesniewski who encouraged me to believe in my mathematical ability and go on to graduate school.

Contents

1 Introduction

Most of the standard examples of Hamiltonian actions arise from projective varieties in the following way: Let K be a compact connected Lie group acting on projective n-space $\mathbf{P}^n$ via a unitary representation $K \to U(n+1)$. If one gives $\mathbf{P}^n$ a symplectic structure via the Fubini-Study form, then the action of K on $\mathbf{P}^n$ is Hamiltonian, and any smooth K-invariant sub-variety $M \subset \mathbf{P}^n$ inherits the structure of a Hamiltonian K-manifold from $\mathbf{P}^n$. Naturally one wants to know what class of Hamiltonian actions arise in this way, and to phrase a related question, what class of Hamiltonian actions admit an invariant, compatible Kähler structure.

One expects the answers to these questions to depend on the "degree of symmetry" of the symplectic manifold in question. For instance, by the results of Thurston, McDuff, and Gompf and Mrowka (see e.g. [13],[14]), we know many examples of symplectic manifolds without group actions that admit no compatible Kähler structure. On the other hand, as observed by Kostant and Souriau, transitive Hamiltonian actions of compact groups are coadjoint orbits, and therefore Kähler. Coadjoint orbits are examples of *multiplicity-free Hamiltonian actions*, which are a class of symplectic manifolds with very high symmetry. Multiplicity-free torus actions were studied by Delzant (under the name *completely integrable torus actions*) who showed that, under certain assumptions, each of these actions admits an invariant compatible Kähler structure.

The idea of this paper is to study the existence of invariant Kähler structures on a class of Hamiltonian actions which in a previous paper [40] we more or less classified: multiplicity-free actions whose moment maps are transversal to a Cartan subalgebra (which we call *transversal multiplicity-free actions.*) Under certain assumptions, these actions are in one-to-one correspondence with a collection of convex polytopes which we call *reflective Delzant polytopes.* The correspondence is given by assigning to a compact connected Hamiltonian K-manifold M with moment map $\Phi : M \to \mathfrak{k}^*$ its *Kirwan (or moment) polytope*, which is the intersection

$$\Delta = \Phi(M) \cap \mathfrak{t}^*_+$$

of $\Phi(M)$ with a closed positive Weyl chamber $\mathfrak{t}^*_+$. In [41] we show that not all of these actions admit invariant, compatible Kähler structures.

The main result of this paper is a necessary criterion, in terms of the polytope Δ, for certain transversal multiplicity-free actions to admit invariant compatible Kähler structures. Let Δ be a convex polytope in $\mathfrak{t}^*_+$, and let $V(\Delta)$ denote the set of inward pointing normal vectors to facets of Δ. (The elements of this set are unique up to multiplication by positive scalars.) We say that a facet $F \subset \Delta$ with inward-pointing normal vector v_F is *negative* if v_F lies in $-\mathfrak{t}_+$. For each simple root $\alpha \in \mathfrak{t}^*$, let $H_\alpha \subset \mathfrak{t}^*$ denote the hyperplane defined by $(\alpha, \cdot) = 0$, where $(\cdot, \cdot)$ is an invariant inner product on $\mathfrak{k}$. Our criterion is

Theorem 1.1 *Let K be a compact connected Lie group with center Z, and let M be a com-*

pact, connected, transversal, multiplicity-free Hamiltonian K-manifold, with discrete principal isotropy subgroup. Let $\Delta \subset \mathfrak{t}_+^*$ denote the Kirwan polytope of M, and assume that $\Delta \cap H_\alpha$ is non-empty for all simple roots α. If M admits a compatible invariant Kähler structure, then a facet F is non-negative if and only if F contains $\Delta \cap H_\alpha$ for some simple root α. Furthermore, the number of non-negative facets is less than or equal to 2 rank K/Z.

Since many Hamiltonian actions do not satisfy this criterion, the Theorem implies that the symplectic category is much larger than the Kähler category in this highly symmetric situation.

The idea of proof is to show that after perturbing the symplectic form, and replacing the complex structure, we can assume that M is a projective K-variety. By an observation of Brion [4], any smooth multiplicity-free projective K-variety is *spherical*; that is, if G denotes the complexification of K, then a Borel subgroup $B \subset G$ has a Zariski-open orbit. In this case, each facet F of the polytope Δ corresponds to a (not necessarily unique) B-stable prime divisor in M. The main idea is to identify the B-stable divisors that are not G-stable in two different ways: [1]

Theorem 1.2 *Let M be a smooth projective K-variety satisfying the assumptions of Theorem 1.1. A facet $F \subset \Delta$ corresponds to a B-stable prime divisor that is not G-stable if and only if*

1. *F contains $\Delta \cap H_\alpha$ for some simple root α; if and only if*
2. *F is non-negative.*

We should emphasize that because of the assumptions this result does not apply to many spherical varieties. However, Theorem 1.2 does apply to several well-known examples that arise in representation theory, such as the flag variety $GL(n+1, \mathbb{C})/B$ under the action of $GL(n, \mathbb{C})$ (Gelfand-Zetlin system) and similarly the generalized flag variety of $SO(n+1, \mathbb{C})$

[1] Another way of identifying these divisors begins with the knowledge of the generic stabilizer $H \subset G$. However, because we want to deduce results about the existence of complex structures, we will always work without knowledge of H.

under the action of $SO(n, \mathbb{C})$ (at least for a generic projective embedding.) Other examples will be given later.

The main ingredients in the proof of characterization (1) are Brion's expression for the polytope associated to a line bundle over a spherical variety [4], and the stability of the transversality condition under perturbation. Characterization (2) is derived from Knop's definition of the little Weyl group of a G-variety [25]. The bound in Theorem 1.1 follows from the fact that for a simple, reflective polytope Δ and simple root α, there are most two facets $F_\pm$ of Δ satisfying $F_\pm \supset \Delta \cap H_\alpha$.

In the second half of the paper we prove a sufficient criterion for the existence of an invariant Kähler structure. For certain actions of $SO(5)$, we show that our existence and non-existence results combine to give a complete answer, and that our criterion is equivalent to a criterion for the Kählerizability of Hamiltonian torus actions due to S. Tolman [39]. The main result is

Theorem 1.3 *Let K be $SO(5)$ with maximal torus $T \subset K$, and M a Hamiltonian K-manifold satisfying the assumptions in Theorem 1.1, and assume that M is torsion-free. (This term will be defined in the next section.) Then the following conditions are equivalent:*

1. *M admits a compatible K-invariant Kähler structure.*

2. *M admits a compatible T-invariant Kähler structure.*

3. *The Kirwan polytope Δ of M satisfies the necessary criterion in Theorem 1.1.*

In particular, in this case the criterion in Theorem 1.1 is sufficient as well as necessary.

We hope that the results of this paper might be extended in several ways. First, under the assumptions of Theorem 1.2 one might hope to identify the generic stabilizer H, and perhaps even compute the colored fan from the polytope. This would give alternative proofs to many of the results in this paper. In particular, one would like to show that any two invariant complex structures compatible with the symplectic form are related by an equivariant complex automorphism. One would also like to know whether the criterion in Theorem 1.1 is sufficient for actions of higher rank groups.

2 The definition of a spherical variety

Let G be a connected complex reductive group, with Borel subgroup $B \subset G$. A G-variety X is called *spherical* if B has a Zariski open orbit.

Recall that to any connected G-variety X and G-line bundle L we can associate a convex set $\Delta(L) \subset \mathfrak{t}^*_+$ as follows. For any dominant weight $\lambda \in \mathfrak{t}^*_+$ we denote by V_λ the corresponding irreducible representation. (Here we assume that we have chosen the chamber $\mathfrak{t}^*_+$ so that it contains the weight of any B-eigenvector in a finite-dimensional representation of G.) Following Brion [4] we define

$$\Delta(L) = \overline{\{\mu \mid V_{n\mu} \subseteq H^0(L^n), \text{ some } \ n \in \mathbb{Z}_+\}}. \tag{1}$$

One can show, by tensoring highest-weight sections, that the set $\Delta(L)$ is convex. Now let $K \subset G$ be a maximal compact subgroup with maximal torus T as before. If X is smooth and compact, $\omega \in \Omega^{1,1}(X)$ is an invariant positive form representing the first Chern class of L, and Φ is a moment map for the action of K on (M, ω) then up to a central constant $\Delta(L)$ equals the Kirwan polytope $\Phi(X) \cap \mathfrak{t}^*_+$. This is an easy consequence of "quantization commutes with reduction" (Guillemin-Sternberg multiplicity formula) [36, 34].[2]

If X is spherical, then it follows from work of Brion [5] that the facets of $\Delta(L)$ are defined by B-stable prime divisors in X:[3] For any B-module V let $V^{(B)}$ be the set of non-zero B-eigenvectors in V. For any element $v \in V^{(B)}$, we denote by $\chi(v) : B \to \mathbb{C}^*$ the associated character of B, which we identify with a weight in $\mathfrak{t}^*$. Let $\mathbb{C}(X)$ denote the field of rational functions on X. The set of B-eigenfunctions $\mathbb{C}(X)^{(B)}$ has the structure of an abelian group under multiplication, and since X is spherical, we have an exact sequence

$$\mathbb{C}^* \to \mathbb{C}(X)^{(B)} \xrightarrow{\chi} \mathfrak{t}^*.$$

[2]Similar results should hold for non-compact X, provided the map Φ is proper.
[3]This was pointed out to me by F. Knop.

The image $\Lambda = \chi(\mathbf{C}(X)^{(B)})$ is a lattice in $\mathfrak{t}^*$. The *rank* of X is the dimension of Λ.

Now let $\mathcal{D}(X)$ denote the set of B-stable prime divisors in X. Each element $D \in \mathcal{D}(X)$ defines a valuation

$$v_D : \mathbf{C}(X) \to \mathbf{Z}$$

measuring the order of vanishing of any rational function at D. By restriction to $\mathbf{C}(X)^{(B)}$, the divisor D defines an element of $\mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z})$ which we also denote by v_D.

Remark 2.1 The map $\mathcal{D}(X) \to \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z})$ given by $D \mapsto v_D$ is not in general injective, e.g. for $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $G = \mathrm{SL}(2, \mathbf{C})$.

Let $\mathbf{C}(L)$ denote the space of rational sections of L, and assume that $H^0(L) \subset \mathbf{C}(L)$ is non-trivial. Since $H^0(L)$ is locally finite [28, p. 67], there exists a B-eigensection $\sigma \in H^0(L)^{(B)}$ which defines an isomorphism

$$\mathbf{C}(X)^{(B)} \cong \mathbf{C}(L)^{(B)}, \quad f \mapsto f \otimes \sigma.$$

For any element $D \in \mathcal{D}(X)$ we denote by $v_D(\sigma)$ the order of vanishing of σ at D. An element $f \otimes \sigma \in \mathbf{C}(L)^{(B)}$ is a global section if and only if

$$v_D(f) + v_D(\sigma) \geq 0 \text{ for all } D \in \mathcal{D}(X).$$

One sees that

$$\begin{aligned} \Delta(L) &= \chi(\sigma) + \{x \in \Lambda \otimes_{\mathbf{Z}} \mathbf{R} \mid v_D(x) \geq -v_D(\sigma), \text{for all } D \in \mathcal{D}(X)\} && (2) \\ &= \{x \in \Lambda \otimes_{\mathbf{Z}} \mathbf{R} \mid v_D(x) \geq -v_D(\sigma) + v_D(\chi(\sigma)) \text{ for all } D \in \mathcal{D}(X)\}. && (3) \end{aligned}$$

Since X is spherical, the set $\mathcal{D}(X)$ is finite so that (3) expresses $\Delta(L)$ as a finite intersection of half-spaces. It follows that if F is a facet of $\Delta(L)$ then there exists a divisor $D \in \mathcal{D}(X)$

such that $F = \Delta \cap H_D$ where

$$H_D = \{x \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \mid v_D(x) = -v_D(\sigma) + v_D(\chi(\sigma))\}. \tag{4}$$

We call any D with $F = \Delta \cap H_D$ a *divisor corresponding to F.*

Remark 2.2 There are two important differences from the toric case:

1. The divisor D is not necessarily unique, and
2. Not every set of the form (4) is a facet, even if the line bundle L defines a projective embedding of X. See the example in Section 8, and also Lemma 5.3.

2.1 An example from representation theory

To get an idea of how these definitions work in practice, we give in the simplest case Brion's proof of "Pieri's formula" (see [6, Section 2].)

Theorem 2.3 *Let $G = \mathrm{GL}(n, \mathbb{C})$, let V denote the standard representation of G, and $S(V^*)$ denote the representation of G on polynomials on V. For any dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, $\lambda_i \geq \lambda_{i+1}$ for $i = 1, \ldots, n-1$, let V_λ denote the irreducible representation of G with highest weight λ. Then*

$$S(V^*) \otimes V_\lambda = \bigoplus_{\substack{\mu=(\mu_1,\ldots,\mu_n) \\ \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n}} V_\mu.$$

Let B (resp. B_+) denote the Borel subgroup of lower (resp. upper) triangular matrices in G. Let X be the variety $V \times G/B_+$ and L the line bundle $\pi_2^* L_\lambda$, where $L_\lambda = G \times_{B_+} \mathbb{C}_\lambda$ is the line bundle over G/B_+ with $H^0(L_\lambda) = V_\lambda$, and $\pi_2 : V \times G/B_+ \to G/B_+$ is the projection onto the second factor. The global sections $H^0(L)$ of L form the representation $S(V^*) \otimes V_\lambda$.

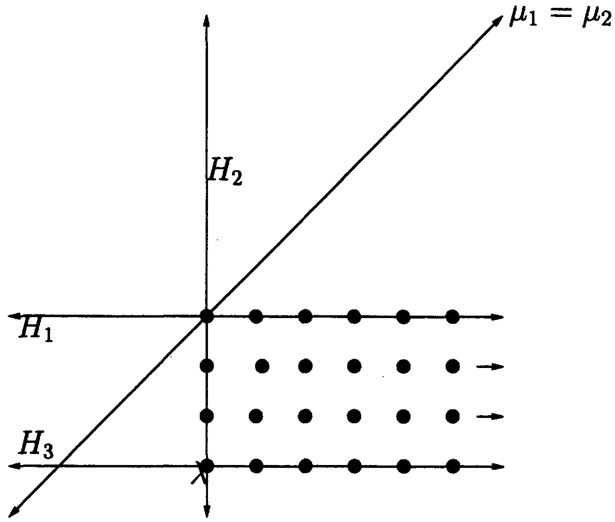


Figure 1: The case $n = 2$

We will consider the simple case $n = 2$, in which $X \cong \mathbf{C}^2 \times \mathbf{P}^1$. In this case there are three B-stable prime divisors. Let (z, w) where $z = (z_1, z_2)$ and $w = [w_1, w_2]$ be coordinates for $\mathbf{C}^2 \times \mathbf{P}^1$. The three B-stable divisors are

$$D_1 = \{z_1 = 0\}, \quad D_2 = \{w_1 = 0\}, \quad \text{and} \quad D_3 = \{z \in w\}.$$

Only the divisor D_3 is G-stable. One checks that the corresponding valuation vectors $v_i \in \mathfrak{t}$ are

$$v_1 = (1, 0), \quad v_2 = (0, -1), \quad \text{and} \quad v_3 = (0, 1)$$

and that the B-eigenvector $\sigma \in H^0(L)^{(B)}$ with $\chi(\sigma) = \lambda$ is given in homogeneous coordinates by

$$\sigma(z, w) = w_1^{\lambda_1} w_2^{\lambda_2}.$$

We have

$$v_1(\sigma) = v_3(\sigma) = 0, \quad \text{and} \quad v_2(\sigma) = \lambda_1 - \lambda_2$$

which implies that

$$\begin{aligned}\chi(H^0(L)^{(B)}) &= \{\mu \in \mathbb{Z}^2 \mid v_1(\mu) \geq \lambda_1,\ v_2(\mu) \geq -\lambda_1,\ v_3(\mu) \geq \lambda_2\} \\ &= \{\mu \in \mathbb{Z}^2 \mid \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2\}\end{aligned}$$

as required. The hyperplanes

$$H_i = \{\mu \in \mathbb{R}^2 \mid v_i(\mu) = -v_i(\sigma) + v_i(\chi(\sigma))\}, \qquad i = 1, 2, 3$$

are drawn in Figure 1.

One can prove Theorem 2.3 for any n in a similar way. To get a compact example one could replace $X = V \times G/B_+$ with $\mathbb{P}(V \oplus \mathbb{C}) \times G/B_+$. The flag variety $GL(n+1, \mathbb{C})/B_+(n+1)$ is also a G-equivariant compactification of X.

3 Multiplicity-free Hamiltonian actions

In this section we will review the symplectic geometry needed to prove Theorems 1.1 and 1.2. A complex representation V of a compact connected Lie group K is called *multiplicity-free* if each irreducible representation appears at most once in V, or equivalently, if the algebra $\mathrm{End}_K(V)$ of K-equivariant endomorphisms is abelian. A Hamiltonian K-manifold M is called *multiplicity-free* if the set $C^\infty_K(M)$ of K-invariant smooth functions forms an abelian Poisson algebra [17]. That is, for any $f_1, f_2 \in C^\infty_K(M)$ we have $\{f_1, f_2\} = 0$. Let G be the connected complex reductive group that is the complexification of K. Brion has noted that

Proposition 3.1 (Brion [4]) *Let $M \subset \mathbb{P}^N$ be a smooth projective G-variety. Then M is a multiplicity-free K-manifold if and only if M is a spherical G-variety.*

There is an important, alternative definition of multiplicity-free in terms of *symplectic reduction.* Recall that if M is a Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$, then for each coadjoint orbit $Kx \subset \mathfrak{k}^*, x \in \mathfrak{k}^*$ there is an associated *Marsden-Weinstein* *symplectic reduced space* M_x defined by

$$M_x = \Phi^{-1}(Kx)/K.$$

If x is a regular value of Φ then M_x is a symplectic orbifold.

Proposition 3.2 (Sjamaar) *A compact connected Hamiltonian K-manifold is multiplicity-free if and only if M_x is a point for any $x \in \mathfrak{k}^*$. If the principal isotropy subgroup is discrete, then M is multiplicity-free if and only if* $\dim M = \dim K + \operatorname{rank} K$.

That this holds not only for generic x is a consequence of the fact that singular reductions have the structure of "stratified symplectic spaces" as in [37]. See also [40, Proposition A.1]. By Proposition 3.2, a compact connected Hamiltonian K-manifold M is multiplicity-free if and only if the map Φ induces a homeomorphism $M/K \cong \Delta$. (See [4] for the proof when M is a projective K-variety.) The decomposition of M/K into orbit-types is related to the face decomposition of Δ:

Lemma 3.3 (Delzant) *Let M be a compact, connected multiplicity-free K-manifold with discrete principal isotropy subgroup, and $F \subset \Delta$ an open face contained in the interior $(\mathfrak{t}^*_+)^\circ$ of $\mathfrak{t}^*_+$. Then the Lie algebra $\mathfrak{k}_m$ of the isotropy subgroup K_m of any point $m \in \Phi^{-1}(F)$ equals the annihilator $F^\circ \subset \mathfrak{t}$ of F. Furthermore, if the principal isotropy subgroup is trivial then K_m is connected.*

For a proof see [12] or [40, Lemma 3.2]. In particular, a face $F \subset \Delta \cap (\mathfrak{t}^*_+)^\circ$ is a vertex of Δ if and only if $\Phi^{-1}(F)$ is a T-fixed point.

Remark 3.4 Describing the orbit types of points in $\Phi^{-1}(F) \subset \partial\mathfrak{t}^*_+$ is in general an open problem. For the transversal case, see [40, Theorem 7.2].

Note that if M is a connected Hamiltonian G-manifold, then the principal isotropy subgroup (which in general is only defined up to conjugacy) is fixed by the choice of maximal torus and positive chamber $\mathfrak{t}^*_+$. Indeed, there exists a face σ of $\mathfrak{t}^*_+$ such that $\Phi^{-1}(K\sigma)$ is connected and dense (see [30, Theorem 3.7]), and we define the *principal isotropy subgroup* of M to be the isotropy subgroup K_x of any point x in the principal orbit-type stratum for the action of K_σ on $\Phi^{-1}(\sigma)$. Since K_x contains the semisimple part of K_σ (see [30, Remark 3.10]) K_x is independent of the choice of x. The following conjecture has been proved in many cases [11, 23, 12, 40]:

Conjecture 3.5 (The Multiplicity-free or Delzant Conjecture) *Let M_1 and M_2 be compact connected multiplicity-free Hamiltonian K-manifolds with the same Kirwan polytope and the same principal isotropy subgroup. Then M_1 and M_2 are equivariantly symplectomorphic.*

Remark 3.6 A. Knutson points out that the statement fails if one allows M_1 and M_2 to have singularities. Also, F. Knop has observed that a multiplicity-free Hamiltonian K-manifold may admit invariant complex structures that are not equivariantly isomorphic. The simplest examples are the $SL(2,\mathbb{C})$-spherical varieties $\mathbb{P}^1 \times \mathbb{P}^1$ and $SL(2,\mathbb{C}) \times_B \mathbb{P}^1$ which are $SU(2)$-equivariantly symplectomorphic for suitable choice of symplectic forms.

If the conjecture holds, it suggests that geometric properties of multiplicity-free actions should be "translatable" into the language of convex polytopes. We call a Hamiltonian K-manifold *transversal* if the moment map is transversal to a Cartan subalgebra. There is a characterization of transversality in terms of isotropy subgroups (see [40, Lemma 2.2]):

Lemma 3.7 (Guillemin-Souza) *Let M be a Hamiltonian K-manifold, and $\sigma \subset \mathfrak{t}^*_+$ any face of the positive chamber. Then Φ is transversal to $\mathfrak{t}^*$ at $\Phi^{-1}(\sigma)$ if and only if for any $x \in \sigma$ the semisimple part (K_x, K_x) of K_x acts locally freely on $\Phi^{-1}(x)$, or equivalently, if Φ is transversal to σ.*

The main result of [40] is that transversality has the following description in terms of convex polytopes: For each $x \in \Delta$ we denote by $H(x)$ (resp. $V(x)$) the set of hyperplanes in $\mathfrak{t}^*$ intersecting Δ in facets meeting x (resp. inward pointing normal vectors to these facets.) The polytope Δ is called *simple* if $V(x)$ is linearly independent, for all $x \in \Delta$. We call Δ *reflective* if

1. the set $H(x)$ is W_x-invariant, for all $x \in \Delta$, and

2. the intersection $\Delta \cap \partial\mathfrak{t}^*_+$ is a union of faces of codimension at least 2.

Here W denotes the Weyl group of $T \subset K$, and W_x the isotropy subgroup of x.

Theorem 3.8 *[40] Let M be a compact, connected multiplicity-free K-manifold with Kirwan polytope Δ and discrete principal isotropy subgroup. If M is transversal then Δ is simple and reflective.*

In fact the converse is also true, but at this point our proof is too technical to publish.

For the proof of the bound in Theorem 1.1 we will need the following combinatorial result on reflective simple polytopes. For any convex polytope Δ, let $H(\Delta)$ denote the set of hyperplanes intersecting Δ in facets.

Proposition 3.9 *[40, Proposition 5.1] Let $\Delta \subset \mathfrak{t}^*_+$ be a simple reflective convex polytope and α a simple root such that Δ meets the hyperplane H_α. Then there are exactly two elements $H_\pm \in H(\Delta)$ such that $H_\pm$ contains $\Delta \cap H_\alpha$. The corresponding normal vectors $v_\pm \in \mathfrak{t}$ satisfy $(v_\pm, \alpha) > 0$, and the intersection $\Delta \cap H_+ \cap H_-$ equals $\Delta \cap H_\alpha$. Any other element of $H(\Delta)$ meeting $H_\alpha \cap \Delta$ intersects H_α transversally.*

Finally in Section 8 we will need the following definitions. We call a transversal Hamiltonian K-manifold *torsion-free* if (K_x, K_x) acts freely on $\Phi^{-1}(x)$ for all $x \in \Delta$. If (K_x, K_x) is simply-connected, we call Δ *Delzant* at x if $V(x)$ extends to a basis of the lattice $\exp^{-1}(\mathrm{Id})$, where $\exp : \mathfrak{t} \to T$ is the exponential map. Otherwise, we say that Δ is Delzant at x if the conditions in [40, Remark 10.2] hold. We call Δ Delzant if Δ is Delzant at all $x \in \Delta$.

Theorem 3.10 *A compact connected transversal multiplicity-free K-manifold with trivial principal isotropy is torsion-free if and only if Δ is Delzant.*

This is proved in the case $\pi_1((K_x, K_x)) = \{1\}$ in [40, Theorem 6.2].

Theorem 3.11 *[40] The map $M \to \Delta(M)$ induces a bijection between compact connected transversal torsion-free multiplicity-free K-manifolds with trivial principal isotropy and reflective, Delzant polytopes.*

In particular, Delzant's conjecture 3.5 holds for these actions.

4 Algebraization

The main goal of this section is to show that in the proof of Theorem 1.1, we can assume that M has the structure of a smooth projective variety. Proposition 4.4 is also used in the proof of Theorem 1.2. Our first result in this direction is

Proposition 4.1 *Let (M, ω) be a compact Hamiltonian K-manifold and J an invariant compatible Kähler structure. Suppose that the fixed points of a maximal torus T are isolated. Then there exists a perturbation $\overline{\omega}$ of ω, an integer $n \in \mathbb{N}$, and an invariant compatible Kähler structure $\overline{J}$ such that $(M, n\overline{\omega}, \overline{J})$ embeds in projective space.*

Proof - Since M admits a $\mathbb{C}^*$-action with isolated fixed points, then by the results of Carrell-Liebermann [10] or Carrell-Sommese the cohomology $H^{i,j}(M)$ vanishes unless $i = j$. In particular, $H^{2,0}(M) \cong H^{0,2}(M)$ vanishes, so there exists an invariant perturbation of $\overline{\omega}$ of ω such that $\overline{\omega} \in \Omega^{1,1}(M)$ and $[\overline{\omega}] \in H^2(M, \mathbb{Q})$. Since $\overline{\omega} \in \Omega^{1,1}$, the pair $(\overline{\omega}, J)$ defines an invariant Kähler structure on M. Let $n_1 \in \mathbb{Z}$ be an integer such that $[n_1\overline{\omega}] \in H^2(M, \mathbb{Z})$. Let L be a holomorphic metric line bundle with invariant connection and curvature $\overline{\omega}$ [15, p. 149]. By the Kodaira embedding theorem, there exists an integer $n_2 \in \mathbb{N}$ such that the sections of L^{n_2} give an equivariant embedding $i : M \to \mathbb{P}^N$ of (M, J) in projective N-space. Let ω_{FS} denote the Fubini-Study 2-form. Unfortunately, $i^*\omega_{FS}$ will not usually equal $n_1 n_2 \overline{\omega}$. However, the metrics $i^*\omega_{FS}(\cdot, J\cdot)$ and $n_1 n_2 \overline{\omega}(\cdot, J\cdot)$ are positive definite. If ω_t is the invariant closed 2-form defined by

$$\omega_t = t i^* \omega_{FS} + (1-t) n_1 n_2 \overline{\omega}$$

then for $t \in [0,1]$ the metric $\omega_t(\cdot, J\cdot)$ is also positive definite, and so ω_t is symplectic for $t \in [0,1]$. Furthermore, w_0 is cohomologous to w_1, so by Moser isotopy (see e.g. [33, p. 91]) there exists a K-equivariant symplectomorphism

$$\varphi : (M, i^*\omega_{FS}) \cong (M, n_1 n_2 \overline{\omega}).$$

Defining $\overline{J} = \varphi^* J$ and $n = n_1 n_2$ completes the proof. □

To apply this result to Kähler multiplicity-free actions, we need to note that if M is a compact multiplicity-free K-manifold, then any maximal torus $T \subset K$ acts with isolated fixed points. Indeed, let M_T denote the T-fixed point set. Let $m \in M_T$ and let $N \subseteq M_T$ be the connected component of M_T containing m. The image $\Phi(N)$ lies in $\mathfrak{t}^*$, by equivariance of Φ, and because N is a smooth connected symplectic submanifold on which T acts trivially, in fact $\Phi(N)$ equals $\Phi(m)$. Since M is multiplicity-free,

$$\Phi^{-1}(\Phi(m)) \cong K_{\Phi(m)}/K_m$$

and it is known that the fixed point set of the action of T on any K-homogeneous space is finite.[4] In case M is transversal, one can argue alternatively that $\Phi(M_T) \subset \mathfrak{t}^*_{\text{reg}}$ and so M_T is discrete by Delzant's Lemma 3.3.

Combining with 4.1 and 3.1 we have proved that

Corollary 4.2 *Any compact connected multiplicity-free K-manifold which admits an invariant Kähler structure admits (after a perturbation) the structure of a projective spherical variety.*

Remark 4.3 These arguments work only if M is compact. For results in the general case, see [21].

We want to check that certain properties and invariants of the Hamiltonian K-manifold M are invariant under perturbation.

[4]In fact the fixed point set has a transitive action of W, as was pointed out to me by F. Knop.

Proposition 4.4 *Let (M, ω) be a compact connected Hamiltonian K-manifold, and $t \mapsto \omega_t$ an (affine) linear map of $\mathbb{R}^k$ into the space of closed, K-invariant 2-forms on M, with $\omega_0 = \omega$. There exists a linear map $t \mapsto \Phi_t$ such that each $\Phi_t : M \to \mathfrak{k}^*$ is a moment map for the action of K on (M, ω_t). There also exists a neighborhood, U, of $0 \in \mathbb{R}^n$ such that for $t \in U$*

1. *the form ω_t is symplectic.*

2. *If Φ_0 is transversal to $\mathfrak{t}^*$, then*

   (a) *Φ_t is transversal to $\mathfrak{t}^*$;*

   (b) *if $\Phi_0(M)$ meets a face $\sigma \subset \mathfrak{t}^*_+$ then $\Phi_t(M)$ does; and*

   (c) *if M is multiplicity-free and has discrete principal isotropy then $V(\Delta_t) = V(\Delta_0)$.*

Proof - The existence of Φ_t follows from the discussion on [2, p.23], by which the contraction $\imath(X_M)\omega_t$ is exact for any $X \in \mathfrak{k}$ and $t \in \mathbb{R}^k$. It follows that for any t there exists a map $\Phi_t : M \to \mathfrak{k}^*$ such that $\imath(X_M)\omega_t = d\langle \Phi, X \rangle$. The map Φ_t may be made equivariant by [20, Section 26]. To construct a linear map $t \mapsto \Phi_t$, choose a basis $t_1, \ldots, t_n$ for $\mathbb{R}^n$, construct Φ_{t_i} as above, and define Φ_t for arbitrary t by linearity.

Statement (1) follows from the compactness of M, and the fact that the set of non-degenerate 2-forms on $T_m M$ is open, for any $m \in M$. Statement 2(a) follows from a similar argument. By Lemma 3.7 Φ_0 is transversal to σ, and so any perturbation of Φ_0 also meets σ, which shows 2(b).

The proof of 2(c) uses the correspondence between lines perpendicular to facets of $\Delta \cap (\mathfrak{t}^*_+)^\circ$ with one-dimensional isotropy subgroups of T acting on

$$Y_+ := \Phi^{-1}((\mathfrak{t}^*_+)^\circ).$$

For any subset $S \subseteq M$ let $I(S) \subset \mathbf{P}(\mathfrak{t})$ denote the set

$$I(S) = \{\mathfrak{t}_s \mid s \in S, \dim T_s = 1\}$$

of Lie algebras of 1-dimensional isotropy subgroups of T acting on S. Let

$$N(\Delta) = \{H^\circ \in \mathsf{P}(\mathfrak{t}) \mid H \in H(\Delta), H \cap \Delta \cap (\mathfrak{t}_+^*)^\circ \neq \emptyset\}$$

denote the set of normal subspaces to hyperplanes meeting Δ in interior facets. By Delzant's Lemma 3.3,

$$I(Y_+) = N(\Delta). \tag{5}$$

The following lemma shows that one can replace Y_+ in (5) by a small neighborhood of Y_+ in $\Phi^{-1}(\mathfrak{k}^*_{\text{reg}})$. If M is transversal, even more is true: one can replace Y_+ by a neighborhood of Y_+ in M.

Lemma 4.5 *Let M be a compact connected multiplicity-free K-manifold with polytope Δ and discrete principal isotropy subgroup. Then*

1. *for a sufficiently small neighborhood U of Y_+ in $\Phi^{-1}(\mathfrak{k}^*_{\text{reg}})$, we have $I(U) = I(Y_+) = N(\Delta)$, and*

2. *if Φ is transversal to $\mathfrak{t}^*$ then for a sufficiently small neighborhood V of $\Phi^{-1}(\mathfrak{t}^*_+)$ in M we have $I(V) = I(Y_+) = N(\Delta)$.*

Proof - Let m be a point in U with $\dim \mathfrak{t}_m = 1$ and $k \in K$ an element such that $km \in Y_+$. By definition of Y_+ we have that $T_{km} = kT_mk^{-1}$ is contained in T. Let v be a non-zero vector in $\mathfrak{t}_{km}$. Then $\text{Ad}(k)v \in \mathfrak{t}$ which implies that $\text{Ad}(k)v \in Wv$. Since $\Phi^{-1}(\mathfrak{k}^*_{\text{reg}})$ is isomorphic to $K \times_T Y_+$, we can assume that U is of the form $U_K Y_+$, where $U_K \subset K$ is a neighborhood of T such that $U_K \cap N(T) = T$. Thus $\text{Ad}(k)v = v$ and $\mathfrak{t}_{km} = \mathfrak{t}_m$ which shows (1).

Now assume that Φ is transversal to $\mathfrak{t}^*$. Let $\sigma \subset \mathfrak{t}^*_+$ be a face and let K_σ denote the isotropy subgroup of any point in σ, K_σ^{ss} its semisimple part, $\mathfrak{k}_\sigma^{ss}$ the Lie algebra of K_σ^{ss}, and $\pi_\sigma : \mathfrak{k}^* \to (\mathfrak{k}_\sigma^{ss})^*$ the projection. A simple check shows that Φ is transversal to $\mathfrak{k}^*_\sigma$, so that $\Phi^{-1}(\mathfrak{k}^*_\sigma)$ is a smooth submanifold of M. Lemma 3.7 implies that the restriction of Φ to

$\Phi^{-1}(\mathfrak{k}_\sigma^*)$ is transversal to the center z_σ^* of $\mathfrak{k}_\sigma^*$. That is, the map

$$\pi_\sigma \circ \Phi|_{\Phi^{-1}(\mathfrak{k}_\sigma^*)}$$

is transversal to 0. Since $\pi_\sigma \circ \Phi|_{\Phi^{-1}(\mathfrak{k}_\sigma^*)}$ is K_σ-equivariant, its fibers are K_σ-diffeomorphic in a neighborhood of $(\pi_\sigma \circ \Phi)^{-1}(0)$.

Let m be a point in V such that $\dim \mathfrak{t}_m = 1$, and $k \in K$ be such that $\Phi(km) \in \mathfrak{t}_+^*$. Then, as before, $\mathfrak{t}_{km} = w\mathfrak{t}_m$ for some $w \in W$. We can assume that V is a union of sets of the form $V_\sigma Y_\sigma \cong V_\sigma \times_{K_\sigma} Y_\sigma$, where

1. V_σ is a neighborhood of K_σ in K such that $V_\sigma \cap N(T) = K_\sigma \cap N(T)$,
2. Y_σ is a K_σ-invariant neighborhood of a point $x \in \Phi^{-1}(\sigma)$ in $\Phi^{-1}(\mathfrak{k}_\sigma^*)$,
3. the fibers of the map $(\pi_\sigma \circ \Phi)|_{Y_\sigma}$ are K_σ-equivariantly diffeomorphic, and
4. the image $\Phi(Y_\sigma) \cap \mathfrak{t}_+^*$ meets only those faces H_α such that H_α contains σ.

Suppose that m lies in $V_\sigma Y_\sigma$, so that w has a representative $k' \in K_\sigma$. The point $k'km$ lies in Y_σ and $t_{k'km}$ equals t_m. By definition of Y_σ there exists a point $m' \in Y_\sigma \cap Y_+$ such that

$$\mathfrak{t}_{m'} = \mathfrak{t}_{k'km} = \mathfrak{t}_m$$

as required. □

Now we finish the proof of 2(c). Let

$$(Y_+)_t := \Phi_t^{-1}((\mathfrak{t}_+^*)^\circ),$$

let U_t be a neighborhood of $(Y_+)_t$ in $\Phi_t^{-1}(\mathfrak{k}_{reg}^*) = K(Y_+)_t$ and V_t a neighborhood of $(Y_+)_t$ in M. Since $(Y_+)_0$ contains only a finite number of orbit-types, we can choose a compact subset $Z \subset (Y_+)_0$ such that $I(Z) = I((Y_+)_0)$. For t sufficiently small, $Z \subset U_t$ and so by (1) of the Lemma,

$$N(\Delta_0) = I(Z) \subseteq I(U_t) = N(\Delta_t). \tag{6}$$

On the other hand, for t sufficiently small $(Y_+)_t \subset V_0$ and so by (2) of the Lemma

$$N(\Delta_t) = I((Y_+)_t) \subseteq I(V_0) = N(\Delta_0). \tag{7}$$

Since the polytopes Δ_t are reflective $\mathbf{R}V(\Delta_t) = N(\Delta_t)$ so by (6) and (7) $\mathbf{R}V(\Delta_t) = \mathbf{R}V(\Delta_0)$. That is, given any facet F of Δ we can find a facet F_t of Δ_t such that $F^\circ = F_t^\circ$, and vice-versa. By taking t small we can assume that F_t is close to F, and in this case F_t and F must have the same normal vector.

Remark 4.6 Using 2(c) one can simplify the proof of Proposition 4.2 in [40].

5 Review of the Luna-Vust classification

In this section we review the Luna-Vust classification theorem and related results of Brion. In fact, the results in this section are not needed for the proofs of Theorems 1.1 or 1.2, but they provide an important background for the results about the little Weyl group covered in the next section.

Let G be a connected complex reductive group and X a spherical G-variety. Let $x \in X$ be any point in the open B-orbit, and $H = G_x$ its isotropy subgroup. Then X is an equivariant embedding of the homogeneous space G/H. The Luna-Vust theory classifies such embeddings by a combinatorial invariant of X called the *colored fan.* Let $\mathcal{D}(X)$ denote the set of B-stable prime divisors in X. For each G-orbit $Y \subset X$, let $C_Y \subset \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Q})$ be the cone on the vectors v_D, for $D \in \mathcal{D}(X)$ containing Y. Let E_Y denote the set

$$E_Y = \{D \in \mathcal{D}(G/H) \mid \overline{D} \supset Y\}.$$

The pair

$$C_Y^c = (C_Y, E_Y)$$

is the *colored cone* associated to Y. The set

$$\mathcal{F}(X) = \{C_Y^c \mid Y \subset X\}$$

is the *colored fan* of X. Note that if $D \subset X$ is a G-stable divisor, then the corresponding valuation v_D is G-invariant. Let $\mathcal{V}^G \subseteq \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$ denote the image of the set of G-invariant discrete valuations (with rational values) on $\mathbb{C}(G/H) \cong \mathbb{C}(X)$. It is a convex cone containing $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}) \cap -\mathfrak{t}_+$ (see [24, Corollary 5.3].) A pair (C, E) with $C \subset \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$ and $E \subset \mathcal{D}(G/H)$ is called a *colored cone* if

1. C is generated by vectors v_D with $D \in E$, together with finitely many elements of $\mathcal{V}^G$.

2. The interior C° of C intersects $\mathcal{V}^G$.

A colored cone (C, E) is called *strictly convex* if C is strictly convex, and $v_D \neq 0$ for $D \in E$. A colored cone (C', E') is called a *face* of (C, E) if C' is a face of C, and $E' = \{D \in E \mid v_D \in C'\}$. A *colored fan* is a non-empty finite set $\mathcal{F}$ of colored cones such that

1. If $C^c \in \mathcal{F}$, then every face of C^c lies in $\mathcal{F}$.

2. For every $v \in \mathcal{V}^G$ there is at most one $(C, E) \in \mathcal{F}$ such that $v \in C^\circ$.

A colored fan $\mathcal{F}$ is called *strictly convex* if each colored cone in $\mathcal{F}$ is strictly convex. For the following see also [24, Theorem 3.3].

Theorem 5.1 (Luna-Vust [31]) *The map $X \mapsto \mathcal{F}(X)$ induces a bijection between isomorphism classes of embeddings and strictly convex colored fans.*

Define

$$C(X) = \bigcup_{(C,E) \in \mathcal{F}} C.$$

The variety X is complete if and only if $C(X)$ contains $\mathcal{V}^G$ [24, p.12].

For projective spherical varieties, the colored fan is related to the fan of the polytope of the hyperplane bundle. Recall that if Δ is a convex polytope, its associated fan $\mathcal{F}(\Delta)$ is the set of dual cones to faces of Δ. Here the *dual cone* to a face F' of Δ is the cone generated by normal vectors v_F to facets F of Δ containing F'. The following results are due, in a somewhat different form, to Brion [4].

Lemma 5.2 *Let X be a projective spherical G-variety of maximal rank with polytope Δ, generic stabilizer H, and colored fan $\mathcal{F}$. Then the set $\mathcal{C}(X)$ of cones C_1 such that $(C_1, E_1) \in \mathcal{F}$ for some $E_1 \subset \mathcal{D}(G/H)$ equals the set $\mathcal{F}(\Delta, \mathcal{V}^G)$ of cones C_2 in $\mathcal{F}(\Delta)$ such that $C_2^\circ \cap \mathcal{V}^G$ is non-empty.*

For the proof we will need the following Lemma.

Lemma 5.3 *Let X be a projective spherical G-variety with polytope $\Delta = \Delta(L)$, where L is the hyperplane bundle, $D \in \mathcal{D}(X)$ a divisor corresponding to F, $H_D \subset \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ the hyperplane defined by D as in Equation (4), and $Y \subset X$ a G-orbit. Let $\Delta_{\overline{Y}} = \Delta_{\overline{Y}}(L)$ denote the polytope of the restriction of L to Y as in Equation (1). Then D contains Y if and only if H_D contains $\Delta_{\overline{Y}}$.*

Proof - This follows from Brion's [6, Theorem p.409]. Alternatively, by Equation (4) H_D contains $\Delta_{\overline{Y}}$ if and only if any element s of $H^0(L^n)^{(B)}$ zero on D also is also zero on Y, for any $n \in \mathbf{N}$. By [24, Corollary 1.7] (where we let v_0 be a valuation with center Y), this holds if and only if any global section s of L^n vanishing on D vanishes on Y, for any $n \in \mathbf{N}$; that is, Y is contained in D. □

Proof of Lemma 5.2 - First, note that $\Delta_{\overline{Y}}$ is a face of Δ. Indeed, the locus of vanishing of a section $s \in H^0(L^n)^{(B)}$ is the union of $D \in \mathcal{D}(X)$ such that $v_D(s) > 0$. Since $\mathcal{D}(X)$ is finite, s does not vanish identically on Y if and only if $v_D(s) = 0$ for every $D \in \mathcal{D}(X)$ containing Y. Therefore,

$$\Delta_{\overline{Y}} = \Delta \cap \bigcap_{D \supset Y} H_D.$$

It follows from Lemma 5.3 and Equation (3) that the dual cone to $\Delta_{\overline{Y}}$ equals C_Y, which implies that $\mathcal{C}(X)$ is contained in $\mathcal{F}(\Delta, \mathcal{V}^G)$. Conversely, since X is complete, $\mathcal{C}(X)$ must contain $\mathcal{V}^G$ and so any $C_2 \in \mathcal{F}(\Delta, \mathcal{V}^G)$ of maximal dimension must be contained in $\mathcal{C}(X)$.

Any other cone in $\mathcal{F}(\Delta, \mathcal{V}^G)$ is the face of a cone $C_2 \in \mathcal{F}(\Delta, \mathcal{V}^G)$ of maximal dimension and so contained in $\mathcal{C}(X)$. $\Box$

Remark 5.4 The proof of Lemma 5.2 shows that if X is a projective spherical variety with polytope Δ and G-orbit Y, then $\Delta_{\overline{Y}}$ is a face of Δ. If X is smooth, transversal, and maximal rank, then the face $\Delta_{\overline{Y}}$ intersects the interior $(\mathfrak{t}_+^*)^\circ$ of the positive chamber. Indeed, let $\sigma \subset \mathfrak{t}_+^*$ be a face of maximal dimension intersecting $\Phi(Y)$, so that Y is locally contained in $\Phi^{-1}(\sigma)$. Let $y \in Y \cap \Phi^{-1}(\sigma)$ be any point; then the tangent space to the orbit $(K_\sigma, K_\sigma)y$ lies in T_yY, since Y is K-invariant, but also lies in the symplectic orthogonal to T_yY, by definition of the moment map. By Lemma 3.7, $T_y((K_\sigma, K_\sigma)y)$ is of positive dimension, which contradicts that Y is a complex and therefore symplectic submanifold of X.

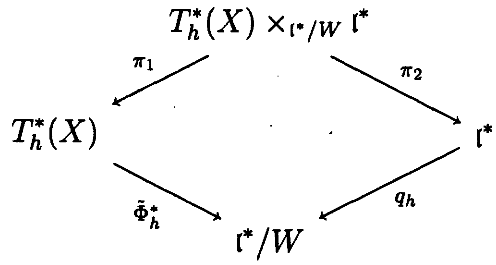
6 Little Weyl groups and collective functions

In this section we will make an application of Brion and Knop's theory of the little Weyl group of a projective G-variety to the smoothness of invariant collective functions. This application was suggested by Knop [26]. For the following, see Brion [7] or Knop [25, Theorem 1.3].

Theorem 6.1 (Brion, Knop) *Let G be a connected complex reductive group, X a spherical G-variety and $\mathcal{V}^G(X) \subseteq \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$ the cone of invariant valuations. There is a finite reflection subgroup $W_X \subseteq W$ such that $\mathcal{V}^G(X)$ is a fundamental domain of W_X acting on* $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$.

Brion and Knop's definitions of W_X are quite different, and one of the main results of [25] is that the definitions agree. For simplicity we present Knop's definition of W_X in the case that X is maximal rank. The variety X need not be spherical. Recall that the *rank* of a G-variety is the dimension of the lattice of characters $\chi(\mathbb{C}(X)^{(B)})$.

Let T_h^*X denote the bundle of holomorphic cotangent vectors. The action of G on X induces a holomorphic moment map $\Phi_h : T_h^* \to \mathfrak{g}^*$. Let $\mathfrak{l} \subset \mathfrak{g}$ be the complexification of the real Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$. Composing with the quotient map $q_h : \mathfrak{g}^* \to \mathfrak{g}^*//G \cong \mathfrak{l}^*/W$ we get a morphism $\tilde{\Phi}_h^* : T_h^* \to \mathfrak{t}^*/W$. We form the fiber product



Because X is maximal rank, one can show that the inverse image $\pi_2^{-1}(\mathfrak{t}^*_{\mathrm{reg}})$ is non-empty, and that the morphism π_1 is a ramified cover with generic fiber W. The fiber product $T_h^*X \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$ may have several irreducible components, which are closures of the components of $\pi_2^{-1}(\mathfrak{t}^*_{\mathrm{reg}})$ and are permuted transitively by W. By [25, p. 317] there is a distinguished component $\hat{T}_h^*$ of $T_h^*X \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$, called the *polarized tangent bundle.* The *little Weyl group* $W_X \subseteq W$ is the set of elements $w \in W$ such that $w\hat{T}_h^* = \hat{T}_h^*$.

Now let M be a Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$, and let $\tilde{\Phi} : M \to \mathfrak{t}^*_+$ be the composition of Φ with the quotient map $q : \mathfrak{k}^* \to \mathfrak{t}^*_+$ which assigns to any $x \in \mathfrak{k}^*$ the unique point of intersection $Kx \cap \mathfrak{t}^*_+$. A *collective function* on M is a function of the form $\Phi^* f$, for some continuous function f on $\mathfrak{k}^*$. A K-invariant collective function can be written $\tilde{\Phi}^* f$. Our application of the little Weyl group is the following theorem which was suggested by Knop [26].

Theorem 6.2 *Let M be a smooth projective K-variety of maximal rank, i.e., with Kirwan polytope of maximal dimension. If $f \in C^\infty(\mathfrak{t}^*)^{W_M}$ is a W_M-invariant smooth function, then the function $\tilde{\Phi}^* f$ is smooth.*

Remark 6.3 If $f \in \mathbf{R}[\mathfrak{t}^*]^W$ then this follows from Chevalley's theorem.

Proof - First, we consider the case when $f \in \mathbf{R}[\mathfrak{t}^*]^{W_M}$ is polynomial. Let $\alpha = \partial \ln |z|^2$ be the Fubini-Study 1-form on $L\backslash 0$, the geometric realization of the pull-back of the hyperplane bundle, minus its zero section. The variety $L\backslash 0$ is a spherical $G \times \mathbf{C}^*$ variety, and the Weyl groups of the complex maximal torus $T_{\mathbf{C}}$ in G and $T_{\mathbf{C}} \times \mathbf{C}^*$ in $G \times \mathbf{C}^*$ are naturally identified.

Under this identification, the little Weyl groups $W_{L\backslash 0}$ and W_M are isomorphic [27, p.11]. Let $\pi : L\backslash 0 \to M$ denote projection onto the base.

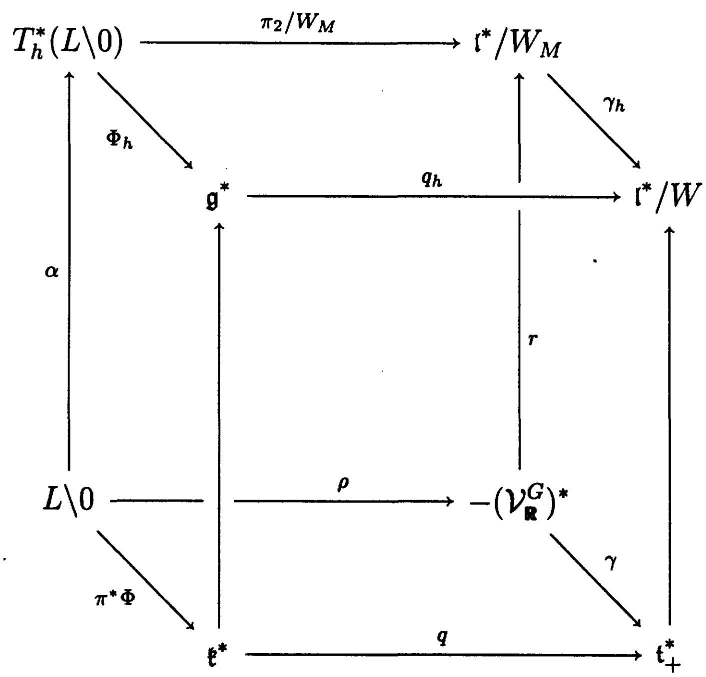
The restriction $\pi_1|_{\hat{T_h^*}(L\backslash 0)}$ is a ramified cover with generic fiber W_M, and the quotient $\hat{T_h^*}(L\backslash 0)/W_M$ equals $T_h^*(L\backslash 0)$. Indeed, since π_1 is W_M-invariant, its restriction to $\hat{T_h^*}(L\backslash 0)$ induces an affine birational morphism $\hat{T_h^*}(L\backslash 0)/W_M \to T_h^*(L\backslash 0)$. This map has finite fibers and normal target space, and is therefore an isomorphism by Zariski's Main Theorem. Hence $\pi_2|_{\hat{T_h^*}}$ induces a morphism

$$\pi_2/W_M : T_h^*(L\backslash 0) \to \mathfrak{t}^*/W_M \times \mathbb{C}^* \to \mathfrak{t}^*/W_M.$$

The last map is just projection onto the first factor. Let $f_h \in \mathbb{C}[\mathfrak{t}^*]^{W_M}$ be the analytic continuation of f, so that $f = f_h|_{\mathfrak{t}^*}$. We write f_h as a polynomial $h \in \mathbb{C}[\mathfrak{t}^*/W_M]$ in the generators of $\mathbb{C}[\mathfrak{t}^*]^{W_M}$, so that f_h is the pullback of h by the quotient map map $\mathfrak{t}^* \to \mathfrak{t}^*/W_M$. The function

$$\alpha^*(\pi_2/W_M)^* h$$

is a smooth function on $L\backslash 0$ and we claim it equals $\pi^*\tilde{\Phi}^* f$. This follows from the commutativity of the following diagram.



In the top square we have left off extra factors of $\mathbb{C}$; for example, Φ_h denotes the holomorphic moment map for the action of $G \times \mathbb{C}^*$ on $L\backslash 0$, composed with projection onto $\mathfrak{g}^*$. The notation $(\mathcal{V}_{\mathbb{R}}^G)^*$ denotes the fundamental domain of W_M acting on $\mathfrak{t}^*$, containing $-\mathfrak{t}_+^*$. Note that $\mathfrak{k}^*$ denotes $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{k}, \mathbb{R})$, while $\mathfrak{g}^*$ denotes $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$. Since $\mathfrak{g}$ is isomorphic to $\mathfrak{k} \oplus i\mathfrak{k}$, analytic continuation defines an inclusion $\mathfrak{k}^* \to \mathfrak{g}^*$. The map r is the restriction of $\mathfrak{t}^* \to \mathfrak{t}^*/W_M$ to $-(\mathcal{V}_{\mathbb{R}}^G)^*$. The map ρ is defined as follows. Let $\hat{\pi}_1$ denote the restriction of π_1 to $\hat{T}_h^*(L\backslash 0)$. Since $\Phi_h(\alpha(L\backslash 0))$ is contained in $\mathfrak{k}^*$, we have that

$$\pi_2(\hat{\pi}_1{}^{-1}(\alpha(L\backslash 0))) \subset \mathfrak{t}^*.$$

Since π_2 is W_M-equivariant, we can quotient by the action of W_M to obtain a continuous map

$$\alpha(L\backslash 0) \to -(\mathcal{V}_{\mathbb{R}}^G)^*$$

whose composition with α we define to be ρ. By definition $\rho \circ r$ equals $\alpha \circ \pi_2 / W_M$. The map γ_h is the polynomial map obtained by expressing a set of generators of $\mathbb{C}[\mathfrak{t}^*]^W$ in terms of a set of generators of $\mathbb{C}[\mathfrak{t}^*]^{W_M}$. The map γ is the restriction of q to $-(\mathcal{V}^G_{\mathbb{R}})^*$.

Lemma 6.4 *The restriction of γ to $\rho(L\backslash 0)$ is the identity.*

Proof - We first show that there exists an element $w \in W$ such that $\gamma(x) = wx$ for all $x \in \rho(L\backslash 0)$. Recall the

Lemma 6.5 *(see e.g. [30]) Let M be a compact connected Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$ such that $\Phi^{-1}(\mathfrak{k}^*_{\text{reg}})$ is non-empty. Then $\Phi^{-1}(\mathfrak{k}^*_{\text{reg}})$ is connected and dense.*

By the Lemma, $\pi^{-1}\Phi^{-1}(\mathfrak{k}^*_{\text{reg}})$ is connected and dense. Hence, the image $\rho(\pi^{-1}\Phi^{-1}(\mathfrak{k}^*_{\text{reg}}))$ is connected and therefore contained in some chamber $w\ (\mathfrak{t}^*_+)^\circ$ in $\mathfrak{t}^*_{\text{reg}}$. By continuity, $\gamma(x) = wx$ for all $x \in \rho(L\backslash 0)$.

To see that $w = \text{Id}$ we need to invoke Knop's definition of $\hat{T}^*_h$. We will use freely the notation developed in his paper [25]. Let $\eta \in (\mathfrak{t}^*_+)^\circ$ be a generic point in Δ and let $l \in \pi^{-1}\Phi^{-1}(\eta)$ be a point in its inverse image. It suffices to show that $\rho(l) = \eta$. In the notation of [25, p.5], there exists an element $\bar{b} \in A^s_r$ such that $\chi_{D(\bar{b})} = \eta$. The image $\hat{\psi}(l, \bar{b}) \in \hat{T}^*_h(L\backslash 0)$ is therefore

$$\hat{\psi}(l, \bar{b}) = (\psi^*(l, \bar{b}), \eta)$$

and

$$\Phi_h(\psi^*(l, \bar{b})) \in \eta + \mathfrak{b}$$

where $\mathfrak{b}$ is the Lie algebra of B. Now let

$$\zeta_t = t\alpha_l + (1-t)\psi^*(l, \bar{b})$$

for $t \in [0,1]$. The image $\Phi_h(\zeta_t)$ lies in $\eta + \mathfrak{b}$, and in particular the coadjoint orbit of η. Therefore the path

$$(\zeta_t, \eta), \qquad t \in [0,1]$$

lies in the fiber product $T_h^*(L\backslash 0) \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$, and so (α_l, η) is contained in $\hat{T}_h^*(L\backslash 0)$. It follows that $\rho(\alpha_l) = \eta$, as required. □

By Lemma 6.4 the pullback $\pi^*\tilde{\Phi}^* f$ equals $\rho^* f$. By commutativity of the diagram above we have

$$\pi^*\tilde{\Phi}^* f = \rho^* f = \alpha^*(\pi_2/W_M)^* h$$

as claimed.

Now let $f \in C^\infty(\mathfrak{t}^*)^{W_M}$ be any smooth W_M-invariant function. By a theorem of G. Schwarz [35], f can be written as a smooth function of the generators of the W_M-invariant polynomials on $\mathfrak{t}^*$, so the result follows from the previous case. □

Theorem 6.6 *Let M be a smooth projective K-variety of maximal rank with moment map $\Phi : M \to \mathfrak{t}^*$. Suppose that Φ is transversal to a face $\sigma \subset \mathfrak{t}_+^*$, and let W_σ be the Weyl group of T in K_σ. Then $W_\sigma \subset W_M$.*

Proof - Let $W' \subseteq W$ be the subgroup of W generated by reflections contained in both W_σ and W_M. It suffices to show that W' equals W_σ. Let $\sigma^\perp \subset \mathfrak{t}^*$ be the subspace perpendicular to σ. First, we show that $\mathbf{R}[\sigma^\perp]^{W'}$ is contained in $\mathbf{R}[\sigma^\perp]^{W_\sigma}$. Let $\pi_\sigma : \mathfrak{t}^* \to \sigma^\perp$ denote orthogonal projection onto $\sigma^\perp$, and let f be any element of $\mathbf{R}[\sigma^\perp]^{W'}$.

The following argument, provided by E. Lerman, shows that the pullback $\tilde{\Phi}^*\pi_\sigma^* f$ is smooth at any point $m \in \Phi^{-1}(\sigma)$. Let $\rho \in C^\infty(\mathfrak{t}^*)$ be a cutoff function supported near $\Phi(m)$, with $\rho = 1$ in a neighborhood of $\Phi(m)$. Let

$$h = \sum_{w \in W} w^*(\rho \cdot \pi_\sigma^* f)$$

which is a W-invariant smooth function equal to $\pi_\sigma^* f$ near $\Phi(m)$. By Theorem 6.2 $\tilde{\Phi}^* h$ is smooth at $\Phi(m)$, which shows that $\tilde{\Phi}^*\pi_\sigma^* f$ is.

Since Φ is transversal to σ, we can choose a submanifold $U \subset \Phi^{-1}(\mathfrak{t}_\sigma^*)$ such that Φ is a diffeomorphism on U and $\Phi(U)$ meets σ transversally at $\Phi(m)$. The function $q^*\pi_\sigma^* f$ is therefore smooth on $\Phi(U)$, and since the restriction of $q^*\pi_\sigma^* f$ to $\mathfrak{t}_\sigma^*$ is locally constant on the fibers of π_σ^* near $\Phi(m)$, this restriction is smooth at $\Phi(m)$. Let V be a small K_σ-invariant neighborhood of $\Phi(m)$ in $\mathfrak{k}_\sigma^*$. Since $KV \subset \mathfrak{k}*$ is isomorphic to $K \times_{K_\sigma} V$ and $q^*\pi_\sigma^* f$ is K-invariant, it follows that $q^*\pi_\sigma^* f$ is smooth at $\Phi(m)$.

We claim that f is W_σ-invariant. Let $R(\sigma)$ be the set of simple roots perpendicular to σ. For any $\alpha \in r(\sigma)$, let $r_\alpha \in W_\sigma$ denote the corresponding reflection. The function $q^*\pi_\sigma^* f$ is even with respect to r_α, and therefore

$$(D_\alpha^n q^* \pi_\sigma^* f)(\Phi(m)) = (D_\alpha^n \pi_\sigma^* f)(\Phi(m)) = 0$$

for $n \in \mathbb{N}$ odd. Since f is polynomial, this shows that f is itself even with respect to r_α, for each $\alpha \in R(\sigma)$, and therefore f is W_σ-invariant.

That W' equals W_σ is now a simple application of the theory of finite reflection groups. By [22, Theorem 3.9] the order $|W_{\mathrm{FR}}|$ of any finite reflection subgroup $W_{\mathrm{FR}} \subset W$ is the product of degrees of generators of

$$\mathbb{C}[\mathfrak{t}^*]^{W_{\mathrm{FR}}} \cong \mathbb{R}[\mathfrak{t}^*]^{W_{\mathrm{FR}}} \otimes_{\mathbb{R}} \mathbb{C}.$$

It follows that $|W_\sigma| = |W'|$ and since W_σ contains W', the two groups are equal. □

Corollary 6.7 *Let M be a smooth projective transversal K-variety with polytope Δ of maximal dimension that meets the hyperplane H_α for each simple root α. Then $W_M = W$.*

7 Proof of the criterion

Characterization (1) in Theorem 1.2 follows from

Theorem 7.1 *Let G be a connected complex reductive group, and M a smooth projective transversal spherical G-variety of maximal rank, with moment polytope Δ such that $\Delta \cap H_\alpha$ is non-empty for all simple roots α. Let $F \subset \Delta$ be a facet and $D \in \mathcal{D}(M)$ a corresponding*

B-stable divisor, and assume that the action of G lifts to the line bundle $[D]$.[5] *Let $\sigma_D \in H^0([D])^{(B)}$ be the canonical section, and $\chi(D) = \chi(\sigma_D)$ the corresponding character of B. Then for any simple root α, the pairing $(\chi(D), \alpha)$ is non-zero if and only if*

$$F \supset \Delta \cap H_\alpha. \tag{8}$$

In particular D is G-stable if and only if (8) holds for no α.

Proof - The proof follows from considering the variation of the moment polytope as L varies by multiples of $[D]$. Let $\omega_D \in \Omega^{1,1}(M)$ be a curvature form of $[D]$, and let $\epsilon < 0$ be a rational number sufficiently close to zero so that $\omega_\epsilon = \omega + \epsilon\omega_D$ is symplectic. By taking a sufficiently high multiple of ω_ϵ, we can assume that ϵ is integral. Let L_ϵ be the line bundle $L + \epsilon[D]$.

First we consider the effect of the perturbation in terms of the description (3). If σ is a B-eigensection of the hyperplane bundle, then $\sigma \otimes \sigma_D^\epsilon$ is a B-eigensection of L_ϵ. By (3)

$$\Delta(L_\epsilon) = \chi(\sigma \otimes \sigma_D^\epsilon) + \{x \in \mathfrak{t}^* \mid v_{D'}(x) \geq -v_{D'}(\sigma \otimes \sigma_D^\epsilon), \text{ for all } D' \in \mathcal{D}(X)\}.$$

Since $v_{D'}(\sigma_D) = 1$ if $D' = D$, and $v_{D'}(\sigma_D) = 0$ otherwise, we have that

$$\Delta(L_\epsilon) = \epsilon\chi(D) + \widetilde{\Delta_\epsilon}$$

where

$$\widetilde{\Delta_\epsilon} = \{x \in \Delta \mid v_D(x) \geq -v_D(\sigma) + v_D(\chi(\sigma)) + \epsilon\}.$$

On the other hand, $\omega_\epsilon \in \Omega^{1,1}(M)$ represents the first Chern class of L_ϵ, so up to a central constant $\Delta(L_\epsilon)$ equals the Kirwan polytope $\Delta_\epsilon = \Phi_\epsilon(M) \cap \mathfrak{t}_+^*$ where Φ_ϵ is a moment map Φ_ϵ for the action of K on (M, ω_ϵ). By Proposition 4.4 the polytope Δ_ϵ meets H_α, for ϵ sufficiently small. Therefore,

$$\min_{x \in \widetilde{\Delta_\epsilon}} (x, \alpha) = -\epsilon(\chi(D), \alpha)$$

[5]This is always possible up to finite cover of G [28].

and so $(\chi(D), \alpha)$ vanishes if and only if $\widetilde{\Delta_\epsilon}$ meets H_α. But this happens exactly when equation (8) holds: If $F \supset \Delta \cap H_\alpha$, then since $\widetilde{\Delta_\epsilon}$ does not contain F, the polytope $\widetilde{\Delta_\epsilon}$ does not meet H_α. On the other hand, suppose that there exists a point $x \in \Delta \cap H_\alpha - F$. Then for ϵ sufficiently small, $\widetilde{\Delta_\epsilon}$ contains x as well. □

Example 7.2 Let $K = SO(4)$, and M is a generic coadjoint orbit of $SO(5)$, on which K acts via the inclusion $SO(4) \to SO(5)$ given by $A \to \text{diag}(A, 1)$. With respect to the standard basis for $\mathfrak{t} \subset \mathfrak{k}$, the polytope Δ of M equals $[\lambda, \mu] \times [-\lambda, \lambda]$ for $\lambda, \mu \in \mathbf{R}$ such that $0 < \lambda < \mu$. Let F be the top facet of Δ, and D a divisor corresponding to F. By the above argument, $\chi(D)$ is proportional to $(-1, 1)$. See Figure 2.

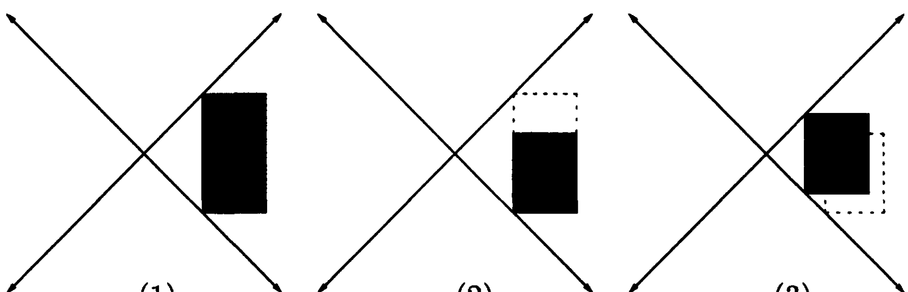


Figure 2: The polytopes (1) Δ (2) $\widetilde{\Delta_\epsilon}$ and (3) Δ_ϵ

Proof of Theorem 1.2 - Let $F \subset \Delta$ be a facet. By Theorem 7.1, it suffices to show that F corresponds to a G-stable divisor D if and only if $v_F \in -\mathfrak{t}_+$. By Corollary 6.7, $W_M = W$, so $\mathcal{V}^G(M) = -\mathfrak{t}_+$ and if v_F does not lie in $-\mathfrak{t}_+$ then D cannot be G-stable. On the other hand, if D is not G-stable then by Theorem 7.1 F contains $\Delta \cap H_\alpha$ for some simple root α. By Proposition 3.9, $(v, \alpha) > 0$, so $v \notin -\mathfrak{t}_+$. □

Proof of Theorem 1.1 - By Corollary 4.2 and Proposition 4.4, we can assume that M is a smooth projective K-variety, and the result follows from Theorem 1.2. □

Remark 7.3 The bound in Theorem 1.1 is not in general sharp, because a facet F may contain $\Delta \cap H_\alpha$ for more than one simple root α. See Figure 2.

8 Example: Blow-ups of a product of coadjoint orbits of $SO(5)$

In this section we describe an example: symplectic blow-ups of a product M of coadjoint orbits of $SO(5)$. The Hamiltonian K-manifold M contains exactly two symplectic K-orbits, and we can symplectically blow-up either one. Depending on which orbit we choose, the blow-up admits (resp. does not admit) an invariant, compatible Kähler structure.

Let $K \subset SO(5)$ and $T \subset K$ the standard choice of maximal torus. The usual choice for a basis for $\mathfrak{t}$ gives isomorphisms $\mathfrak{t} \cong \mathbf{R}^2$ and

$$\mathfrak{t}^*_+ = \{(x, y) \in \mathbf{R}^2 \mid 0 \le y \le x\}.$$

Let λ, μ be positive real numbers, and define

$$\overline{\lambda} = (\lambda, \lambda) \quad \text{and} \quad \overline{\mu} = (\mu, 0)$$

so that $\overline{\lambda}, \overline{\mu}$ lie in the boundary $\partial\mathfrak{t}^*_+$. Let $\Theta_\lambda, \Theta_\mu \subset \mathfrak{so}(5)^*$ be the coadjoint orbits through $\overline{\lambda}$ (resp. $\overline{\mu}$.) Let

$$M = \Theta_\lambda \times \Theta_\mu$$

denote the product, with the diagonal action of K, which has moment map $\Phi : M \to \mathfrak{k}^*$ given by

$$\Phi(v, w) = v + w.$$

Theorem 8.1 *The Hamiltonian K-manifold M is multiplicity-free with trivial principal isotropy subgroup, and its polytope Δ equals*

$$\Delta = \{(x, y) \in \mathbf{R}^2 \mid y \le \lambda \le x \text{ and } x - y \le \mu \le x + y\}.$$

If μ does not equal λ or 2λ then Φ is transversal to $\mathfrak{t}^*$.

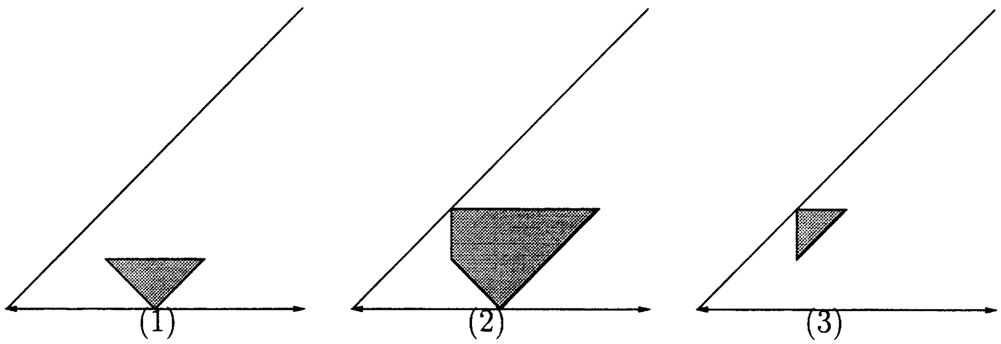


Figure 3: The polytopes Δ for (1) $\mu \geq 2\lambda$, (2)$\lambda \leq \mu \leq 2\lambda$, and (3) $\mu \leq \lambda$.

Proof - The fixed point set M_T equals

$$\begin{aligned} M_T &= W\overline{\lambda} \times W\overline{\mu} \\ &= \{((\pm\lambda, \pm\lambda), (\pm\mu, 0)), ((\pm\lambda, \pm\lambda), (0, \pm\mu))\} \end{aligned}$$

so that

$$\Phi(M_T) = \{(\pm\lambda \pm \mu, \pm\lambda), (\pm\lambda, \pm\lambda \pm \mu)\}$$

and

$$\begin{aligned} \Phi(M_T) \cap \mathfrak{t}_+^* \quad &= \{(\mu \pm \lambda, \lambda)\} && \text{if } \mu \geq 2\lambda \\ &= \{(\lambda, \mu - \lambda), (\lambda + \mu, \lambda)\} && \text{if } \lambda \leq \mu \leq 2\lambda \\ &= \{(\lambda, \lambda - \mu), (\lambda + \mu, \lambda)\} && \text{if } \mu \leq \lambda \end{aligned}$$

For simplicity we will consider only the case $\lambda < \mu < 2\lambda$. The weights of T on $T_{(\overline{\lambda},\overline{\mu})}M$ are the negative roots of $\mathfrak{k}$, and the negative roots $(-1,0), (-1,-1)$ which annihilate neither $\overline{\lambda}$ nor $\overline{\mu}$ appear with multiplicity two. Since for any $m \in Y_+$

$$T_m M \cong T_m Y_+ \oplus (\mathfrak{k}/\mathfrak{t})^*$$

the weights of T on $T_{(\lambda,\mu)}Y_+$ are $(-1,0)$ and $(-1,-1)$. It follows that $\Phi(Y_+) = \Delta \cap (\mathfrak{t}_+^*)^\circ$ is locally the cone on the vectors $(-1,0)$ and $(-1,-1)$. By similar arguments, near $(\lambda, \mu - \lambda)$ the polytope Δ equals the cone on $(0,1)$ and $(1,-1)$. If M is any Hamiltonian K-manifold, and $x \in (\mathfrak{t}^*_+)^\circ$ is a vertex of Δ, then $\Phi(x) \subset M_T$. Therefore, $(\lambda + \mu, \lambda)$ and $(\lambda, \mu - \lambda)$ are the only vertices of Δ lying in $(\mathfrak{t}^*_+)^\circ$. By the description of the local cones, the only possible additional vertices are $\overline{\lambda}$ and $\overline{\mu}$.

Since the weights $(-1,0)$ and $(-1,-1)$ of T on $T_{(\overline{\lambda},\overline{\mu})}Y_+$ are a lattice basis, the map

$$T \to \mathrm{Aut}\ (\mathrm{T}_{(\overline{\lambda},\overline{\mu})}\mathrm{Y}_+)$$

is injective, so the principal isotropy subgroup of T acting on Y_+, which equals the principal isotropy subgroup of K acting on M, is trivial.

The assertion on transversality follows from Delzant's list of local models [12], and can also be verified directly.

8.1 Symplectic blow-ups of M

We will define symplectic blow-ups as a special case of Lerman's symplectic cuts [29]. Let M be a Hamiltonian K-manifold, $\mu : M \to \mathbf{R}$ a K-invariant continuous function, and $a \in \mathbf{R}$ a real number such that in a neighborhood U of $\mu^{-1}(a)$, the function μ is a moment map for a Hamiltonian circle action. Let $M_a = \mu^{-1}(a)/S^1$ be the Marsden-Weinstein reduced space at a, and let $M_{>a} = \mu^{-1}(a, \infty)$. Then the disjoint union

$$M_{\geq a} := M_a \cup M_{>a}$$

is called the *symplectic cut* of M at a. If S^1 acts freely on $\mu^{-1}(a)$, then $M_{\geq a}$ has the structure of a smooth Hamiltonian K-manifold as follows: Define $\nu : M \times \mathbf{C} \to \mathbf{R}$ by

$$\nu(m, z) = \mu(m) - |z|^2/2$$

so that ν is a moment map for the diagonal action of S^1 on $U \times \mathbb{C}$ (where $\mathbb{C}$ has the opposite symplectic form). Let

$$U_{\geq a} = \nu^{-1}(a)/S^1$$

be the symplectic reduction of $U \times \mathbb{C}$ at a. Then

$$U_{\geq a} \cong U_a \cup U_{>a}$$

and the map $\varphi : U_{>a} \to M_{>a}$ given by inclusion defines an equivariant symplectomorphism $U_{>a} \cong \varphi(U_{>a})$. Let $M_{\geq a}$ be the union of $U_{\geq a}$ and $M_{>a}$ modulo the identification of $U_{>a}$ with $\varphi(U_{>a})$.

In case X is the minimum of μ, and S^1 acts on the normal bundle of X with weight one, then for $\epsilon > 0$ small $M_{a+\epsilon}$ is a symplectic blow-up of M along the symplectic submanifold X ([29],[32]). We will need one further fact:

Proposition 8.2 *(see [38, 18, 40]) Let M be a Hamiltonian K-manifold with moment map $\Phi : M \to \mathfrak{k}^*$. The composition $\tilde{\Phi} : M \to \mathfrak{t}_+^*$ of Φ with the quotient map is a moment map for the K-equivariant action of T on KY_+, which equals the usual action of T on Y_+.*

This is a consequence of the functoriality of symplectic induction, in the sense that the Hamiltonian action of T on Y_+ induces a K-equivariant action of T on the *symplectic induced space* $K \times_T Y_+$. We call this densely-defined, K-equivariant action of T the *Thimm action.* We now come to the main result of this section:

Theorem 8.3 *The Hamiltonian K-manifold $M = \Theta_\lambda \times \Theta_\mu$ contains two symplectic K-orbits: Km_1 and Km_2 where $m_1 = (\overline{\lambda}, \overline{\mu})$ and $m_2 = ((\lambda, -\lambda), (0, \mu))$. Only the symplectic blow-up of Km_1 admits an invariant compatible Kähler structure.*

Proof - For any Hamiltonian K-manifold M, an orbit $Km \subset M$ is symplectic if and only if $K_m = K_{\Phi(m)}$. If M is transversal and multiplicity-free, then $\tilde{\Phi}(x)$ can be a symplectic orbit only if $x \in (\mathfrak{t}_+^*)^\circ$ by Lemma 3.7, and then x must be a vertex of Δ, by Lemma 3.3.

Now let $v_1 = (-1,0)$ and $v_2 = (2,1)$, and for $i = 1,2$ let $S^1_i = \exp(\mathbf{R}v_i)$ be the corresponding one-parameter subgroups, and let $\mu_i = \langle\tilde{\Phi}, v_i\rangle$. Since

$$(v_1, (-1,0)) = (v_1, (-1,1)) = 1$$

$$(v_2, (0,1)) = (v_2, (1,-1)) = 1$$

the Thimm action of S^1_i on KY_+ has weight one on the tangent space $T_{m_i}Y_+$, and therefore on the normal bundle to Km_i. Let

$$Bl^i_\epsilon(M) = M_{\geq(\Phi(m_i),v_i)+\epsilon}$$

be the corresponding symplectic blow-ups, which have polytopes (see Figure 4)

$$\Delta^i_\epsilon = \{x \in \Delta \mid (v_i, x) \geq (\Phi(m_i), v_i) + \epsilon\}.$$

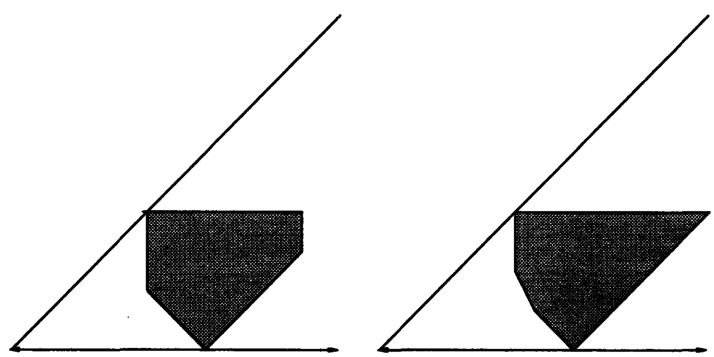


Figure 4: The polytopes Δ^1_ϵ and Δ^2_ϵ

The polytope Δ^2_ϵ fails the bound in Theorem 1.1, so $Bl^2_\epsilon(M)$ admits no invariant compatible Kähler structure. On the other hand, Km_1 is a subvariety, since, if $P_\lambda, P_\mu \subset G$ are parabolics such that

$$\Theta_\lambda \cong G/P_\lambda \qquad \Theta_\mu \cong G/P_\mu$$

then the isotropy group G_{m_1} equals $P_\lambda \cap P_\mu = B$, so $Gm_1 = Km_1$. By the equivalence of Kähler and symplectic blow-ups of subvarieties, $Bl^1_\epsilon(M)$ admits an invariant compatible Kähler structure. (See [32] and, for another argument, the next section.)

9 Existence results

The main result of this section is a sufficient criterion for a multiplicity-free action to admit a compatible invariant Kähler structure (Theorem 9.7), assuming Delzant's Conjecture 3.5. First, we review a few more topics from the theory of spherical varieties.

9.1 Local structure theory

We recall Knop's version [26, Theorem 2.3] of the local structure theorem of Brion-Luna-Vust. Let G be a connected complex reductive group, X be a normal G-variety, and D a B-stable Cartier divisor, which we assume for simplicity is effective. The divisor D induces a line bundle $[D]$ with canonical section σ, and we assume that the action of G lifts to $[D]$. (This is always possible after taking a finite cover of G.) The *parabolic subgroup* $P[D]$ of D is the normalizer of the line $\mathbb{C}\sigma$, and the *character* $\chi(D)$ of D is the character of the action on $\mathbb{C}\sigma$. One has a morphism

$$\psi_D : X \backslash D \to \mathfrak{g}^* \ , \ x \mapsto l_x \text{ where } \ l_x(\xi) = \frac{\xi \, \sigma}{\sigma} \ (x).$$

Theorem 9.1 (Knop) *Let X be a normal G-variety with effective B-stable divisor D. Then the image of ψ_D is the $P[D]$-orbit through χ_D, and if we set $\Sigma = \psi_D^{-1}(\chi_D)$ and $L = G_{\chi_D}$ then there is an isomorphism*

$$X \backslash D \cong P[D] \times_L \Sigma.$$

Typically one uses the local structure theorem to obtain information about X near a G-orbit Y, and so one wants to choose a D not containing Y, but containing enough B-stable prime divisors so that L is as small as possible. In the case that X is a smooth transversal projective spherical variety of maximal rank there is a particularly good choice of D. By Proposition 3.9 and Theorem 1.2, for any simple root α there are two divisors $D_\pm$ such that $H_{D_\pm}$ contains $\Delta \cap H_\alpha$. Not both $D_\pm$ contain Y, since $H_{D_+} \cap H_{D_-} \cap \Delta = H_\alpha \cap \Delta$ and by Remark 5.4 $\Delta_{\overline{Y}}$ is not contained in H_α. Therefore, for any α there is exists a divisor $D_{\alpha,Y} \in \mathcal{D}(G/H)$ such that $\overline{D_{\alpha,Y}}$ does not contain Y, and $(v_{D_{\alpha,Y}}, \alpha) > 0$. Define an effective B-divisor by

$$D_Y = \sum_\alpha n_\alpha \overline{D_{\alpha,Y}}. \tag{9}$$

For some choice of $n_\alpha \in \mathbb{N}$, the divisor D_Y has character $\chi(D_Y) \in \mathfrak{t}^*_{\text{reg}}$, so that $P[D_Y] = B$. By Theorem 9.1 there is an isomorphism $X \backslash D_Y \cong B \times_{T_{\mathbf{C}}} \Sigma$. Since X is spherical the variety Σ is a toric variety, and $Y \cap \Sigma$ is a $T_{\mathbf{C}}$-orbit in Σ. Furthermore, there is a one-to-one correspondence between B-stable divisors in $X \backslash D$ and $T_{\mathbf{C}}$-stable divisors in Σ, and the cones $C_{Y \cap \Sigma}$ and C_Y are equal. By the smoothness criterion for toric varieties, we have the following result, which is a special case of Brion's criterion for smoothness in [8].

Corollary 9.2 *Let X be a smooth transversal spherical projective variety with polytope Δ_X of maximal dimension such that $\Delta_X \cap H_\alpha \neq \emptyset$ for all simple roots α. Then for any G-orbit $Y \subset X$, the set of valuations $v_D \in C_Y$ such that $D \supset Y$ form part of a lattice basis.*

9.2 Line bundles over spherical varieties

We now review several results of Brion [6] on line bundles over spherical varieties. Let $d = \sum_{D \in \mathcal{D}(X)} n_D D$ be a B-stable divisor and $Y \subset X$ a G-orbit. For each divisor $D \in D(X)$ containing Y, set $l_{d,Y}(v_D) = n_D$. If d is Cartier, then $l_{d,Y}$ extends to a linear map $l_{d,Y} : C_Y \to \mathbb{Q}$ and these maps patch together to form a piecewise-linear map $l_d : C(X) \to \mathbb{Q}$. In the case X is a projective variety and d is a B-stable hyperplane section, the function l_d has a simple expression in terms of Δ:

Lemma 9.3 *Let X be a projective spherical G-variety with polytope Δ and d any B-stable hyperplane section. Then for any $v \in C(X)$ we have $l_d(v) = -\min_{x \in \Delta} v(x) + v(\chi(d))$.*

Proof - Suppose that $v \in C_Y$ for some G-orbit $Y \subset X$ and let $D \in \mathcal{D}(X)$ be a divisor containing Y. Note that if σ is the canonical B-eigensection of $[d]$ then $l_d(v_D) = v_D(\sigma)$. By Lemma 5.3 H_D meets Δ so by (3)

$$l_d(v_D) = -\min_{x \in \Delta}(v_D(x)) \ + v_D(\chi(d)).$$

Since l_d is linear on C_Y, the same equation holds with v_D replaced by v. □

As for toric varieties, the association $d \mapsto l_d$ is functorial in the sense that

Lemma 9.4 (Brion [9]) *Let G/H be a spherical homogeneous variety, and $\varphi : X_1 \to X_2$ be a morphism of embeddings of G/H. Then $l_{\varphi^* d}$ is the restriction of l_d to $C(X_1) \subset C(X_2)$.*

Proof - By [6, Section 2] we can assume that X_2 is *simple* (i.e. contains a unique closed G-orbit) and that

$$d = [\phi] \ + \sum_{D \in \mathcal{D}(G/H)} n_D \overline{D}$$

where $\phi \in \mathbb{C}(X_2)^{(B)}$ is a rational function, and $n_D = 0$ if D contains a G-orbit in X. If D does not contain a G-orbit, then $\varphi^* D = D$, since D does not contain the exceptional locus of φ. Therefore,

$$\varphi^* d \ - \sum_{D \in \mathcal{D}(G/H)} n_D \, \overline{D} \ = \ [\varphi^* \phi] \ = \sum_{D' \in \mathcal{D}(X_1)} v_{D'}(\varphi^* \phi) D'$$

so if $D' \in \mathcal{D}(X_1)$ contains a G-orbit then the coefficient of D' in $\varphi^* d$ is $v_{D'}(\varphi^* \phi) = l_d(v_{D'})$ as required. □

9.3 Existence theorems

Recall that a fan $\mathcal{F}_2$ is a *subdivision* of a fan $\mathcal{F}_1$ if any cone in $\mathcal{F}_2$ is contained in a cone in $\mathcal{F}_1$. We say that a fan $\mathcal{F}$ is *rational* if any cone $C \in \mathcal{F}$ is spanned by vectors that are rational with respect to the lattice $\exp^{-1}(\mathrm{Id}) \subset \mathfrak{t}$. If a convex polytope Δ has rational fan, then there is a canonical choice of $V(\Delta)$: we can require that each $v \in V(\Delta)$ is a primitive lattice vector.

Theorem 9.5 *Let X_1 be a projective spherical G-variety with polytope Δ_1, and $\Delta_2 \subset \Delta_1$ a convex polytope with rational fan $\mathcal{F}(\Delta_2)$ such that*

1. *$H(\Delta_1)$ is contained in $H(\Delta_2)$,*
2. *$V(\Delta_2)$ is contained in $V(\Delta_1) \cup \mathcal{V}^G(X_1)$, and*
3. *$\mathcal{F}(\Delta_2)$ is a subdivision of $\mathcal{F}(\Delta_1)$.*

Then there exists a spherical variety X_2 such that $\mathcal{C}(X_2) = \mathcal{F}(\Delta_2, \mathcal{V}^G)$ and a morphism $\varphi : X_2 \to X_1$. Furthermore, if X_2 is smooth, and for each $v \in V(\Delta_2) - V(\Delta_1)$ the difference

$$c(v) = \min_{x \in \Delta_1} v(x) - \min_{x \in \Delta_2} v(x)$$

is an integer, then there exists an ample line bundle L_2 over X_2 with polytope $\Delta(L_2) = \Delta_2$.

Remark 9.6 The polytope $\Delta(L)$ of a G-line bundle over a G-variety has rational, but not necessarily integral vertices [4].

Proof of Theorem 9.5 - For any cone $C_2 \in \mathcal{F}(\Delta_2, \mathcal{V}^G)$ let $C_1 \in \mathcal{C}(\Delta_1, \mathcal{V}^G)$ be the cone in $\mathcal{F}(\Delta_1)$ whose interior contains the interior of C_2. By Lemma 5.2, there exists a subset $E_1 \subset \mathcal{D}(G/H)$ such that $(C_1, E_1) \in \mathcal{F}(X_1)$. Let E_2 denote the set of divisors $D \in E_1$ such that $v_D \in C_2$. We claim that (C_2, E_2) is a colored cone. Indeed, suppose that $v \in V(\Delta_2)$ is an extremal vector of C_2 that does not lie in $\mathcal{V}^G$. Then v lies in $V(\Delta_1)$ and C_1. By Lemma 5.2, C_1 is the dual cone to some face F_1 of Δ_1. The vector v is normal to some facet of Δ_1 containing F_1, so that v is extremal in C_1. Since (C_1, E_1) is a colored cone, there exists a divisor $D \in E_1$ such that v_D equals v, and by definition E_2 contains D as required. If we let $\mathcal{F}_2$ be the set of all such pairs (C_2, E_2), then it is straightforward to check that $\mathcal{F}_2$ is a colored fan for G/H. By the Luna-Vust Theorem 5.1 there exists an embedding X_2 of G/H with colored fan $\mathcal{F}_2$ and (see [24, Section 4]) a morphism $\varphi : X_2 \to X_1$.

Now assume that X_2 is smooth. For each $v \in V(\Delta_2) - V(\Delta_1)$ let $D_v \in \mathcal{D}(X_2)$ denote the corresponding G-stable divisor. Let d_1 be a B-stable hyperplane section of X_1, and define

$$\begin{aligned} d_2 &= \varphi^* d_1 + \sum_{v \in V(\Delta_2) - V(\Delta_1)} c(v) D_v \\ &= \sum_{D \in \mathcal{D}(X_2)} n_2(D) D. \end{aligned}$$

Since X_2 is smooth, any Weil divisor is Cartier and so d_2 defines a line bundle $[d_2]$ over X_2. We claim that $\Delta([d_2]) = \Delta_2$.

Suppose that $d_1 = \sum_{D \in \mathcal{D}(X_1)} n_1(D) D$. By Proposition 9.4, for any $v \in V(\Delta_2) - V(\Delta_1)$ we have that

$$n_2(D_v) = l_{d_1}(v) + c(v).$$

By Corollary 9.3

$$l_{d_1}(v) = -\min_{x \in \Delta_1}(v(x)) \; + v(\chi(d_1))$$

and since $\chi(d_1)$ equals $\chi(d_2)$

$$n_2(D_v) = -\min_{x \in \Delta_2}(v(x)) \; + v(\chi(d_2)).$$

It follows that

$$\Delta(L_2) \;=\; \{y \in \Delta_1 \mid v(y) \geq \min_{x \in \Delta_2} v(x) \text{ for all } v \in V(\Delta_2) - V(\Delta_1)\}.$$

Since $H(\Delta_1) \subset H(\Delta_2)$, $\Delta(L_2)$ equals Δ_2 as required.

We now show that L_2 is ample. Let $Y_2 \subset X_2$ be a G-orbit. There exists a section $s \in H^0(L_2)^{(B)}$ non-vanishing on Y_2 if and only if the intersection $\Delta_{\overline{Y}} \cap (\Lambda \cap \chi(d_2))$ is non-empty. Since the vertices of Δ_2 are rational we can choose an integer $n \in \mathbf{N}$ such that $F \cap (\Lambda/n + \chi(d_2))$ is non-empty for any open face $F \subset \Delta_2$. Let $s \in H^0(L_2^n)^{(B)}$ be a section with $\chi(s)$ in the interior of $\Delta_{\overline{Y}}$. By work of Brion [6, Section 2], it suffices to show that $v_D(s) > 0$, for any $D \in \mathcal{D}(X_2)$ not containing Y_2.

Suppose that $v_D(s) = 0$ for some divisor $D \in \mathcal{D}(X_2)$. The vector v_D must lie in C_{Y_2}, since H_D is a supporting hyperplane containing $\Delta_{\overline{Y}}$. If D is G-stable, then D contains Y_2 by [24, Lemma 2.4].

If D is not G-stable, let $D_1 \in \mathcal{D}(X_1)$ be the closure of $D \cap G/H$ in X_1, and H_{D_1} the hyperplane defined by D_1 as in Equation (4). By definition of d_2, $n_1(D_1)$ equals $n_2(D)$ and since $\chi(d_1) = \chi(d_2)$ the hyperplanes H_D and H_{D_1} are the same. Let $Y_1 \subset X_1$ be the G-orbit such that $C^\circ_{Y_1}$ contains $C^\circ_{Y_2}$. It suffices to show that

$$H_{D_1} \supset \Delta_{\overline{Y_1}}, \tag{10}$$

since in this case D_1 contains Y_1 by Lemma 5.3 and so $D \in E_{Y_2}$ by definition. Equation (10) holds if and only if $n_1(D_1) = l_{d_1}(v_{D_1})$. Since $l_{d_1} \geq l_{d_2}$ on $C(X_2)$ and d_1 is ample we have that

$$n_1(D_1) \geq l_{d_1}(v_D) \geq l_{d_2}(v_D) = n_2(D)$$

which, since $n_1(D_1) = n_2(D)$ implies the claim. □

Consider a compact, connected multiplicity-free K-manifold (M, ω_M) with polytope Δ for which Delzant's Conjecture 3.5 applies. To construct a compatible invariant Kähler structure on M, it suffices to construct a compact, connected Kähler multiplicity-free K-manifold M' with the same polytope and principal isotropy.

Theorem 9.7 *Let (M, ω_M) be a transversal, multiplicity-free, compact, connected Hamiltonian K-manifold with trivial principal isotropy and polytope Δ_M. Let (X, ω_X) be a Kähler, transversal, multiplicity-free, compact, connected Hamiltonian K-manifold with trivial principal isotropy and polytope $\Delta_X = \Phi(X) \cap \mathfrak{t}^*_+$ with $\Delta_X \cap H_\alpha$ non-empty for all simple roots α.*

1. *Suppose that $[\omega_M]$ and $[\omega_X]$ are rational and that $\Delta_M \subset \Delta_X$ satisfies (1)–(3) in Theorem 9.5. Then there exists a multiplicity-free, compact, connected, Kähler Hamiltonian K-manifold M_2 with trivial principal isotropy and Kirwan polytope Δ_M.*

2. *In the general case, suppose that for any invariant 2-form ω'_M near ω_M there exists an invariant compatible symplectic form ω'_X on ω_X such that the corresponding polytopes Δ'_M and Δ'_X satisfy (1)-(3) in Theorem 9.5. Then the same conclusion holds.*

Proof of (1) - By taking a sufficiently high multiple of $[\omega_M]$ and $[\omega_X]$, we can assume that X is a projective spherical variety and the $c(v)$'s are integral. Let X_2 be the variety given by Theorem 9.5. To prove that X_2 is smooth, let $Y \subset X_2$ be any G-orbit. The image $\varphi(Y)$ is a G-orbit in X and by Remark 5.4 the face $\Delta_{\overline{\varphi(Y)}}$ intersects the interior $(\mathfrak{t}^*_+)^\circ$. Let $D_{\overline{\varphi(Y)}}$ be the B-stable divisor in (9), and D_Y the B-stable divisor in X_2 defined by taking the closure of each $D_{\alpha,\varphi(Y)}$ in X_2. Since the support of $D_{\overline{\varphi(Y)}}$ does not contain $\varphi(Y)$, the support of D_Y does not contain Y, and D_Y is an effective B-stable divisor with $P[D_Y] = B$. By Theorem 9.1 we have an isomorphism $X_2 \backslash D_Y \cong B \times_{T_{\mathbf{c}}} \Sigma$. The cone $C_{Y\cap\Sigma}$ equals the cone C_Y of Y, which is the dual cone to some face F of Δ_M such that $F \cap (\mathfrak{t}^*_+)^\circ$ is non-empty (since if $F \subset H_\alpha$ then $F \subset H_\pm$ and so $C^\circ_Y \cap -\mathfrak{t}_+ = \emptyset$ which is a contradiction). Since M has trivial principal isotropy, the polytope Δ_M is Delzant at F (see [12]). Hence, the extremal vectors of C_Y form part of a lattice basis, which implies that $Y \cap \Sigma$ consists of smooth points. This shows that X_2 is smooth, so by Theorem 9.5 there exists a Kähler structure on X_2 with polytope Δ_M.

Proof of (2) - Choose a linear K-invariant family of 2-forms $\omega_t \in \Omega^2(M)$, $t \in \mathbf{R}^n$, with $\omega_0 = \omega_M$, such that the cohomology classes

$$\{\partial_i[\omega_t]\}_{i=1}^n$$

span $H^2(M)$. Since $H^2(M, \mathbf{Q})$ is dense in $H^2(M)$, there exist K-invariant symplectic forms

$$\omega_1, \ldots, \omega_n \in \Omega^2_K(M)$$

such that

1. $[\omega_1], \ldots, [\omega_n] \in H^2(M, \mathbf{Q})$,

2. each ω_i lies in the neighborhood U of ω_M in Proposition 4.4, and

3. ω_M is contained in the convex hull of the ω_i's.

Let $\Delta_i, i = 1, \ldots, n$ denote the polytopes of (M, ω_i). Let $c_1, \ldots, c_n$ be such that $\sum c_i = 1$ and

$$\sum_{i=1}^{n} c_i \omega_i = \omega_M.$$

By assumption, the sets $V(\Delta_i)$ are the same, so by the preceding case there exists a single smooth spherical embedding X_2 of G/H and invariant symplectic forms $\omega_{X_2,i}$ such that the Hamiltonian K-manifold $(X_2, \omega_{X_2,i})$ has polytope Δ_i. The form

$$\omega_{X_2} = \sum_{i=1}^{n} c_i \omega_{X_2,i}$$

is symplectic and compatible with J, since the set of such forms is convex. Let Φ_i (resp. $\Phi_{X_2,i}$) denote the moment map for the action of K on (M, ω_i) (resp. $(X_2, \omega_{X_2,i})$) so that

$$\Phi_{X_2} = \sum_{i=1}^{n} c_i \Phi_{X_2,i}$$

is a moment map for the K-action on (X_2, ω_{X_2}). We claim that the polytope

$$\Delta_{X_2} = \Phi_{X_2}(X_2) \cap \mathfrak{t}^*_+$$

equals Δ.

We will prove this using localization. Since (M, ω_i) is a transversal multiplicity-free Hamiltonian K-manifold, for any T-fixed point $m \in M_T$ and any $i \in \{1, \ldots, n\}$, the image $\Phi_i(m)$ lies in the regular part $\mathfrak{t}^*_{\text{reg}}$, and by Delzant's Lemma 3.3 $\Phi_i(m)$ must be a vertex of $w\Delta_i$, for some $w \in W$.

Similarly, the image $\Phi_i(x)$ for any $x \in (X_2)_T$ is a vertex of $w\Delta_i$ for some $w \in \Delta$ and contained in the regular part $\mathfrak{t}^*_{\text{reg}}$. Indeed, the orbit $Gx \subset X_2$ is closed and the polytope Δ_{Gx} is a vertex of Δ. If Δ_{Gx} is contained in a hyperplane H_α then the dual cone, which is generated by normal vectors to facets containing Δ_{Gx}, has interior which meets $-\mathfrak{t}_+$ trivially by Proposition 3.9 which is a contradiction.

Since the classes $[\omega_{X_2,i}]$ are close in $H^2(X_2)$ and $\Phi_{X_2,i}|_{(X_2)_T}$ depends only on the cohomology class of the symplectic form, we can assume that for any $x \in (X_2)_T$, the images $\Phi_{X_2,i}(x)$ are arbitrarily close. Therefore, for any $m \in M_T$, there must exist an element $x \in (X_2)_T$ such that $\Phi_{X_2,i}(x) = \Phi_i(m)$, for each $i = 1, \ldots, n$, so that

$$\begin{aligned}\Phi_{X_2}((X_2)_T) &= \{\sum c_i \Phi_{X_2,i}(x) \mid x \in (X_2)_T\} \\ &= \Phi(M_T).\end{aligned}$$

On the other hand, it is clear that the weights of T at m are the same as the weights of T at x, since, for $\Phi(m) \in (\mathfrak{t}_+^*)^\circ$ these are the edge vectors of the polytope $w\Delta_i$ at the vertex $\Phi_i(m) = \Phi_{X_2,i}(x)$, plus some subset of the roots determined by w. Therefore, by localization (see e.g. [16]) the push-forward measures

$$\Phi_* \omega_M = (\Phi_{X_2})_* \omega_X$$

are equal, which implies that (X_2, ω_{X_2}) and (M, ω_M) have the same Kirwan polytope. □

We now apply our existence theorems to the case $K = SO(5)$.

Theorem 9.8 *Let $K = SO(5)$ and M a compact, connected, torsion-free, transversal, multiplicity-free K-manifold with polytope Δ that meets both codimension 1 faces of $\mathfrak{t}_+^*$. Then M admits an invariant compatible Kähler structure if and only if every non-negative facet contains $\Delta \cap H_\alpha$ for some α.*

Remark 9.9 Similar results hold for other rank 2 groups. For rank greater than 2 the question of sufficiency is open, even for $K = U(3)$.

Proof - By Theorem 9.7 and the Delzant Conjecture 3.5 in the case rank $(G) = 2$ [12] it suffices to show that the spherical variety $X = \Theta_\lambda \times \Theta_\mu$ has a symplectic structure ω_X such that $\Delta \subset \Delta_X$ satisfies (1)–(3) in Theorem 9.7. First we note that Lemma 5.2, Corollary 6.7 and Theorem 8.1 imply that the colored fan of X consists of a single (non-trivial) colored cone (C, E) where C is the cone on the vectors $(-1, 1), (0, -1) \in \mathfrak{t}$. (That is, X is a two-orbit variety.) Let

$$\alpha_1 = (1, -1) \text{ and } \alpha_2 = (0, 1)$$

be the simple roots.

Let λ, μ be real numbers such that

$$\overline{\lambda} = (\lambda, \lambda) = \Delta \cap H_{\alpha_1} \text{ and } \overline{\mu} = (\mu, 0) = \Delta \cap H_{\alpha_2}$$

First, we show that $H(\Delta_{\lambda,\mu}) \subset H(\Delta)$ where $\Delta_{\lambda,\mu}$ denotes the polytope of X. Let $v_{\pm} \in V(\lambda)$ be the normal vectors to facets of Δ meeting $\overline{\lambda}$. By Proposition 3.9, we have $(v_{\pm}, \alpha) < 0$, and since Δ is reflective, we must have

$$v_{\pm} = n\alpha_1 \pm m\beta_1$$

for some $\beta_1 \in H_{\alpha_1}$ and $n, m \in \mathbb{Z}/2$ with $n + m \in \mathbb{Z}$. Since Δ is Delzant,

$$(1, 1), (1, -1) \in \text{span}_{\mathbb{Z}}\{v_{\pm}\}$$

and so $n = m = 1/2$, in which case $V(\overline{\lambda}) = \{(1, 0), (0, -1)\}$. A similar argument (using the complicated definition of Delzant [40, Remark 10.2] in the case (K_x, K_x) is not simply connected) shows that $V(\overline{\mu}) = \{(\pm 1, 1)\}$.

Since Δ satisfies the criterion in Theorem 1.1, $V(\Delta) \subset V(\Delta_{\lambda,\mu})$. Clearly $F(\Delta)$ is a subdivision of $F(\Delta_{\lambda,\mu})$, which completes the proof. □

10 Equivalence to Tolman's criterion in the $SO(5)$ case

In this section we will use a criterion of Tolman [39] to show

Theorem 10.1 *Let M be a compact connected torsion-free transversal multiplicity-free $SO(5)$-space with moment polytope Δ that meets both codimension 1 faces of $\mathfrak{t}^*_+$. If Δ fails the criterion in Theorem 1.1, then M admits no T-invariant Kähler structure.*

Together with Theorem 9.8 this proves Theorem 1.3.

10.1 Tolman's criterion

Let T be a real torus and Y a compact connected Hamiltonian T-space with moment map $\Phi : Y \to \mathfrak{t}^*$. Let Y_T denote the fixed point set. For simplicity, we will assume that the restriction $\Phi|_{Y_T}$ of Φ to Y_T is injective. For any subgroup $H \subset T$ let

$$Y_{(H)} = \{y \in Y \mid T_y = H\}$$

be the corresponding orbit-type stratum. Let χ denote the set of connected components of the $Y'_{(H)}s$. Tolman defines:

$$\text{X-ray}(Y) = \{\Phi(\overline{X}) \mid X \in \chi\}.$$

By the convexity theorem of Atiyah and Guillemin-Sternberg, X-ray(Y) is a finite set of convex polytopes whose vertices lie in $\Phi(Y_T)$.

Theorem 10.2 (Tolman Extendibility Theorem) *Let Y be a compact, connected Hamiltonian T-space with a compatible, invariant Kähler structure. Let y be a point in Y_T and $\alpha_1, \ldots, \alpha_k$ a subset of the weights of T on T_yY such that the cone C on $\alpha_1, \ldots, \alpha_k$ is strictly convex. Then there exists a convex polytope $P \in$ X-ray(Y) such that*

1. *there is a neighborhood U of $\Phi(y)$ such that $P \cap U = C \cap U$, and*

2. *for each face F of P there exists a polytope $P_F \in$ X-ray(Y) of the same dimension as F, containing F.*

Tolman proves the above Theorem by constructing an orbit O of the complex torus $T_{\mathbf{C}}$ such that $P = \Phi(\overline{O})$ has property (1). The other properties follow from a theorem of Atiyah [1, Theorem 2].

Proof of Theorem 10.1 - As in the proof of Theorem 9.7 there exist $\lambda, \mu \in \mathbf{R}$ such that Δ is contained in the polytope $\Delta_{\lambda,\mu}$. Let $V_+(\Delta)$ (resp. $V_-(\Delta)$) denote the normal vectors to facets of Δ appearing clockwise (resp. counterclockwise) between $\overline{\lambda}$ and $\overline{\mu}$.

Case 1: $V_-(\Delta) = V_-(\Delta_{\lambda,\mu}) = \{(1,0),(1,1)\}$. Let x equal $(\lambda, \mu - \lambda)$ and let $y_1, \ldots, y_m$ be the vertices of Δ appearing between $\overline{\lambda}$ and $\overline{\mu}$, moving clockwise. (See Figure 6.)

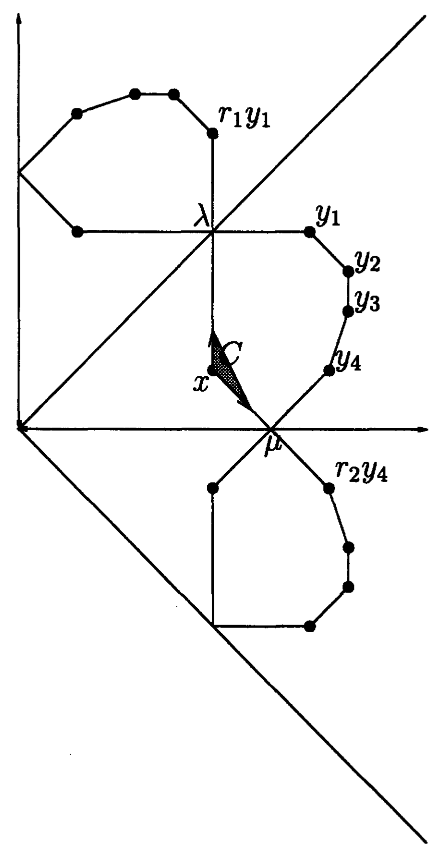


Figure 5: An example with $V^-(\Delta) = V^-(\Delta_{\lambda,\mu})$

Let $\alpha_1 = (1,-1)$ and $\alpha_2 = (0,1)$ be the simple roots and $r_1, r_2 \in W$ the corresponding reflections. Let C be the cone at x on $(0,1)$ and $(1,-1)$, which are the weights of T on

$T_{\Phi^{-1}(x)}Y_+ \subset T_{\Phi^{-1}(x)}M$. Let $P \subset \mathfrak{t}^*$ be the polytope guaranteed by Theorem 10.2, and $p_1, \ldots, p_l \in \mathfrak{t}^*$ the vertices of P, starting with $p_1 = x$ and moving clockwise. Since $r_1 y_1$ is the only element of $\Phi(M_T)$ lying in $x + \mathbf{R}_+(0,1)$, we must have

$$p_2 = r_1 y_1$$

and by the same argument

$$p_l = r_2 y_m.$$

The weights of T on $T_{\Phi^{-1}(r_1 y_1)}M$ are the elements of

$$r_1 N \cup \{(-1,1),(0,-1)\}$$

where N is the set of negative roots. By convexity of P, we must have

$$p_3 \in p_2 + \mathbf{R}_+(1,-1).$$

Similarly,

$$p_{m-1} \in p_m + \mathbf{R}_+(0,1).$$

Since Δ fails the bound in Theorem 1.1, either $v_1 \notin -\mathfrak{t}_+$ or $v_{m+1} \notin -\mathfrak{t}_+$. Assume the latter. If a vertex y_k with $k \neq m-1$ lies in

$$r_2 y_4 + \mathbf{R}_+(0,1)$$

then y_k is contained in a facet $F \subset \Delta$ which lies in the interior $(\mathfrak{t}_+^*)^\circ$ of $\mathfrak{t}_+^*$. By adding to ω a small multiple of the dual class of the submanifold $\tilde{\Phi}(F)$, we can assume that such a vertex does not exist. Hence,

$$p_{l-1} = y_m.$$

By a similar argument, we can assume that no vertex of Δ lies in

$$p_{l-1} + \mathbb{R}_+(-1,1)$$

which implies that

$$p_{l-2} = y_{m-1}.$$

But then v_{m+1} is a normal vector to P lying clockwise between $(-1,-1)$ and $(0,-1)$ - that is, $v \in -\mathfrak{t}_+$ which is a contradiction. The other case is similar.

Case 2: $V_-(\Delta) \neq V_-(\Delta_{\lambda,\mu})$. Let $x_1, \ldots, x_k \in \Delta$ be the vertices of Δ appearing clockwise between $\overline{\mu}$ and $\overline{\lambda}$. (See Figure 7.)

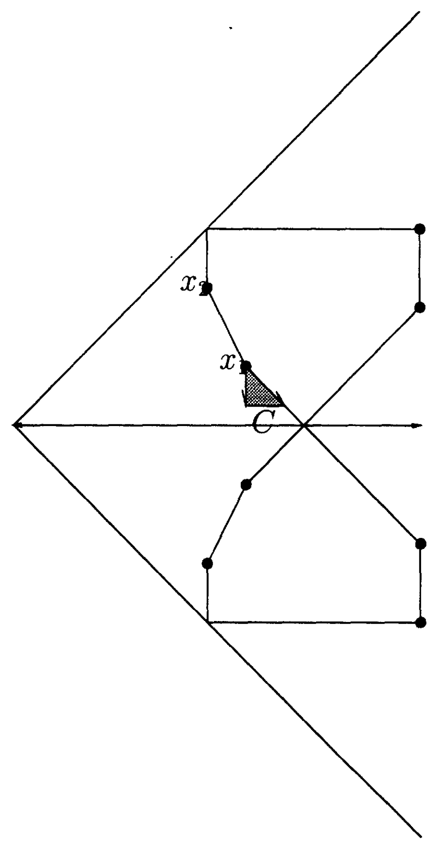


Figure 6: An example with $V^-(\Delta) \neq V^-(\Delta_{\lambda,\mu})$

Let C be the cone at x_1 on $(0, -1)$ and $(1, -1)$. The proof that there is no polytope P satisfying the requirements of Theorem 10.2 is similar to the proof for Case 1, and left to the reader.

References

[1] M. F. Atiyah, Convexity and commuting hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1–15.

[2] M. F. Atiyah and R. Bott, The moment map and equivariant cohomology, *Topology* **23** No. 1 (1984), 1–28.

[3] M. Brion, Quelques propriétés des espaces homogènes sphériques, *Manuscripta Math.* **55** (1986), 191–198.

[4] M. Brion, Sur l'image de l'application moment. In: *Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin*, Lect. Notes Math. **1296**, Springer-Verlag, Heidelberg-Berlin-New-York, 1987.

[5] M. Brion, Groupe de Picard et nombres caractéristiques des variétés sphériques, *Duke Math. Jour.* **58**, No. 2 (1989), 397–424.

[6] M. Brion, Spherical varieties: An introduction, in: *Topological methods in algebraic transformation groups*, (H. Kraft, T. Petrie, and G. Schwarz eds.) Prog. Math. **80**, Birkhäuser, Basel-Boston-Berlin, 1989, 11–26.

[7] M. Brion, Vers une généralisation des espaces symétriques, *J. Algebra* **134** (1990), 115–143.

[8] M. Brion, Sur la géométrie des variétés sphériques, *Comment. Math. Helv.* **66** (1991), 237–262.

[9] M. Brion, January 1996 e-mail to the author.

[10] J.B. Carrell and D. I. Liebermann, Holomorphic vector fields and Kaehler manifolds, *Invent. math.* **21** (1973), 303–309.

[11] T. Delzant, Hamiltoniens périodiques et images convexes de l'application moment, *Bul. Soc. math. France* **116** (1988), 315–339.

[12] T. Delzant, Classification des actions hamiltoniennes complètement intégrables de rang deux, *Ann. Global Anal. Geom.* **8** No. 1 (1990), 87–112.

[13] R. Gompf, A new construction of symplectic manifolds *Ann. of Math.* **142** (1995), 527–595.

[14] R. Gompf and T. Mrowka, Irreducible 4-manifolds need not be complex, *Ann. of Math.* **138** (1993), 61–111.

[15] P. Griffiths and J. Harris, Principles of Algebraic Geometry, John Wiley and Sons, New York, 1978.

[16] V. Guillemin and E. Prato, On Kostant, Heckman, and Steinberg formulas for symplectic manifolds, *Adv. in Math.* **82** No. 2 (1990), 160–179.

[17] V. Guillemin and S. Sternberg, Multiplicity-free spaces, *J. Diff Geom.* **19** (1984), 31–56.

[18] V. Guillemin and S. Sternberg, The Gelfand-Cetlin system and quantization of the complex flag manifolds, *Jour. Funct. Anal.* **52** (1983), 106–128.

[19] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67** (1982), 515–538.

[20] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1984.

[21] A. Huckleberry and T. Wurzbacher, Multiplicity-free complex manifolds, *Math. Ann.* **286** (1990), 261–280.

[22] J. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, Cambridge, 1990.

[23] P. Iglesias, Les $SO(3)$-variétés symplectiques et leur classification en dimension 4, *Bull. Soc. math. France*, **119** (1991), 371–396.

[24] F. Knop, The Luna-Vust theory of spherical embeddings, in: *Proceedings of the Hyderabad Conference on Algebraic Groups, December, 1989*, Madras, Manoj Prakashan, 1991, 225–249.

[25] F. Knop, The asymptotic behavior of invariant collective motion, *Invent. math.* **116** (1994), 309–328.

[26] F. Knop, Lecture at M.I.T., Fall 1994.

[27] F. Knop, Automorphisms, Root Systems, and Compactifications of Homogeneous Varieties, *J. Amer. Math. Soc.* **9** (1996), 153–174.

[28] F. Knop, H. Kraft, D. Luna, and T. Vust, Local properties of algebraic group actions, in: *Algebraische Transformationsgruppen und Invariantentheorie* (H. Kraft, P. Slodowy, T. Springer eds.) DMV-Seminar **13**, Birkhäuser Verlag (Basel-Boston) (1989) 63–76.

[29] E. Lerman, Symplectic cuts, *Math. Research Lett.* **2** (1995), 247–258.

[30] E. Lerman, E. Meinrenken, S. Tolman, and C. Woodward, Non-abelian convexity by symplectic cuts, December 1995 M.I.T. preprint, dg-ga/9603015.

[31] D. Luna and T. Vust, Plongements d'espaces homogénes, *Comment. Math. Helv.* **58** (1983), 186–245.

[32] D. McDuff, Remarks on the uniqueness of symplectic blow-ups, in: *Symplectic Geometry* (ed. D. Salamon) , London Mathematical Society Lecture Note Series **192**, Cambridge University Press, 1993.

[33] D. McDuff and D. Salamon, Introduction to symplectic topology, The Clarendon Press, Oxford University Press, New York, 1995.

[34] E. Meinrenken, Symplectic surgery and the Spin_c-Dirac operator, to appear in *Adv. Math.*, dg-ga/9504002.

[35] G. W. Schwarz, Smooth functions invariant under the action of a compact Lie group, *Topology* **14** (1975), 63–68.

[36] R. Sjamaar, Holomorphic slices, symplectic reduction and multiplicities of representations, *Ann. of Math.* **141** (1995), 87–129.

[37] R. Sjamaar and E. Lerman, Stratified symplectic spaces and reduction, *Ann. of Math.* **134** (1991), 375–422.

[38] A. Thimm, Integrable geodesic flows on homogeneous spaces, *Ergodic Theory and Dynamical Systems* **1** (1981), 495–517.

[39] S. Tolman, Examples of non–Kähler Hamiltonian torus actions, M.I.T. preprint, February 1995.

[40] C. Woodward, The classification of transversal multiplicity-free group actions, *Ann. Global Anal. Geom.* **14** (1996), 3–42.

[41] C. Woodward, Multiplicity-free Hamiltonian actions need not be Kähler, submitted for publication to *Invent. math.*

Dept. of Mathematics Rm 2-229
Massachusetts Institute of Technology
Cambridge, MA 02139
e-mail: woodward@math.mit.edu