

# Spin Structure of Nucleon in the Asymptotic Limit

by  
Jian Tang

Submitted to the Department of Physics  
in partial fulfillment of the requirements for the degree of  
Master of Science in Physics  
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**SCIENCE**

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## Abstract

In analogy to the Altarelli-Parisi equation for the quark and gluon helicity contributions to the nucleon spin, we derive an evolution equation for the quark and gluon orbital angular momenta. The solution of the combined equations yields the asymptotic fractions of the nucleon spin carried by quarks and gluons:  $J_q = \frac{1}{2}3n_f/(16+3n_f)$  and  $J_g = \frac{1}{2}16/(16+3n_f)$ , respectively, where  $n_f$  is the number of active quark flavors. These are identical to the well-known asymptotic partitions of the nucleon momentum between quark and gluon contributions. We also discussed phenomenological implications of our results, in particular, about the size of the quark orbital angular momentum.

Thesis Supervisor: Xiangdong Ji

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Dedicated to My Family  
for Their Love and Supports

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# Chapter 1

## Introduction

Particle physics and nuclear physics are always among the frontier to explore the mysteries of the nature. e.g. how the matter constituents and how these constituents interact with each other. and to satisfy human being's curiosities. Over the past thirty years, a lot of progress and success have been achieved in particle physics and nuclear physics. Experimentally, six quarks( $u, d, c, s, t, b$ ) and six leptons( $e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau$ ) are found to be the basic if not ultimate units to constituent the matter around us. Theoretically, two beautiful theories have been established to describe the interactions between these particles. One is the electroweak theory, which describes the electromagnetic interaction(mediated by photon) and weak interaction(mediated by  $W$  and  $Z$  bosons) completely. The other is Quantum Chromodynamics(QCD), which describes the color strong interaction (mediated by the gluons) between the quarks, which is the main force to drive quarks to be bound together.

However, there still exist a lot of problems challenging particle physics and nuclear physics communities, especially those in the strong interaction. QCD has been proved to be an asymptotic theory, which means that the interaction is to be weak when the interaction energy is very high. We can do perturbative calculation in this high energy region. Unfortunately, due to the color confinement, the quarks and gluons are always bound in bound states, i.e. baryons( $qqq$ ) and mesons( $\bar{q}q$ ), which typical mass scale is about a few  $GeV$ . In such energy region, the interactions between quarks are so strong that perturbative methods cannot be used here. Basically,

we need to solve the QCD equation of motion exactly, which is very hard to do due to the complexities of this theory. This situation even makes it hard to explain some basic properties of the hadrons using the fundamental field theories. Among them the most challenging is the spin problem of the nucleon.

According to the naive quark-parton model, the nucleon is composed of three constituent quarks. The macroscopic properties of the nucleon are attributed to the microscopic properties of these constituents. For example, each of these three quark contributes  $\frac{1}{3}$  of the mass of the nucleon, the spin of the nucleon is just a combination of these spin- $\frac{1}{2}$  quarks( two spin-up and one spin-down, for instance). It is a very simple but very beautiful picture. However, this simple beautiful picture was broken into pieces when a few years ago the European Muon Collaboration(EMC)[1] shocked the particle physics and nuclear physics communities by claiming that little or none of the nucleon spin can be attributed to the spins of its three constituent quarks. This result has been verified by several further rounds of experiments by EMC and other groups at CERN and SLAC[2], though the result has shifted somewhat: Quark spins appear to just account for 20 – 30% of the spin of the nucleon. More surprising is that the sea of the quark-antiquark pairs in the nucleon is strongly polarized in the direction opposite to the nucleon's spin. And this sea turns out to contains a surprisingly large admixture of polarized strange quarks.

So, where does the nucleon really gets its spin? EMC experiment indicated that much of it must lie elsewhere. Shortly after EMC result came out, Alteralli et al.[3] and Carlitz et al.[4] pointed out that there is a gluon polarization contribution to the nucleon spin due to the famous axial current triangle anomaly[5]. The small EMC experimental value of the quark contribution can be interpreted due to large cancellation between quark contribution and the anomalous contribution, which would required a rather large and positive gluon contribution at the EMC energy[6]. There have been a lot of discussions and arguments about this and aroused so many controversies[7].

Based on QCD, the right structure of the nucleon is that it is the bound state of three current quarks surrounded by a very complicated quark-antiquark and gluon sea. The spin of the nucleon should be coming from four pieces: the quark helicity

contribution, the gluon helicity contribution, the quark orbital angular momentum contribution and the gluon orbital angular momentum contribution. Quarks and gluons have spin, they should naturally have contributions to the nucleon spin. Furthermore, quarks and gluons are moving relativistically in nucleon, they have orbital angular momenta, which should play important role in nucleon spin problem[8]. The question now turns to how much each of these four pieces contributes to the spin of the nucleon. Especially, these four pieces depend on the renormalization scale due to renormalization, it is very interesting to determine how they evolve with the renormalization scale.

In this thesis, we get a evolution equation to show how these four pieces evolve with energy, in analogy to the famous Altarelli-Parisi equation[9]. By solving this equation, we shall show that in the asymptotic limit, the fractions of the nucleon spin carried by quarks and gluons are identical to the well-known asymptotic partitions of the nucleon momentum between quark and gluon contributions[10]. We also discussed phenomenological implications of our results, in particular, about the size of the quark orbital angular momentum. This thesis is mainly based on the paper Ji, Hoodbhoy and I published in Physical Review Letters[11].

The outline of this thesis is as follows: To make this thesis more self-dependent and readable, in chapter II we will review some basics about the deep inelastic scattering(DIS) process, structure functions, Ellis-Jaffe sum rule[12], EMC experiment results and more. In some sense , this chapter is more like the continuation of this introduction. In Chapter III, we shall discuss the angular momentum operator in QCD, we shall show our derivation of the evolution equation and discuss its physical applications. We will conclude this thesis in the Chapter IV.

# Chapter 2

## The $g_1$ problem

In this chapter, we shall review some theoretical backgrounds for this thesis.

### 2.1 Deep Inelastic Scattering(DIS)

#### 2.1.1 Basic Variables

Deep inelastic scattering(DIS) is the archetype for hard processes in QCD: a lepton(e.g. electron, muon or neutrino) with very high energy scatters off a target hadron(practically a nucleon) with a large quantities of invariant squared-four-momentum transferring from lepton to hadron. Because we are mainly interested in the experiments with polarized targets, we will devote ourselves to charged lepton scattering off a polarized nucleon, in which the dominant reaction mechanism is electromagnetism and one photon-exchange is a good approximation.(see, e.g. [13, 14, 15])

We consider the process shown in Fig. (2-1). The initial lepton with momentum  $k$  and energy  $E$  exchange a photon of momentum  $q$  with a target with momentum  $P$ . The outgoing lepton's momentum is  $k'$  and its energy  $E'$ . Two very useful invariant variables are defined as below:

$$q^2 \equiv (k - k')^2 = -4EE' \sin^2\left(\frac{\theta}{2}\right) = -Q^2 \quad (2.1)$$

$$\nu \equiv P \cdot q = M(E - E') , \quad (2.2)$$

where the lepton mass has been neglected.  $\theta$  is the scattering angle illustrated in Fig. (2-1). Unless otherwise noted,  $E, E', \theta$  refer to the target rest frame, in which one has

$$P_\mu = (M, \vec{0}), \quad k_\mu = (E, \vec{k}), \quad k'_\mu = (E', \vec{k}'). \quad (2.3)$$

*Bjorken* limit is where  $Q^2$  and  $\nu$  both go to infinity with the ration,  $x \equiv Q^2/2\nu$  fixed.  $x$  is the *Bjorken* scaling variable. Since the invariant mass of the hadronic final state is larger than or equal to that of the target,  $(P + q)^2 \geq M^2$ , one has kinetic limit for  $x$ :  $0 < x \leq 1$ .

### 2.1.2 Cross Section and Hadronic Tensor

The differential cross-section for *inclusive* scattering ( $eP \rightarrow e'X$ , in which hadronic final states  $X$  are not observed) is given by:

$$d\sigma = \frac{1}{J} \frac{d^3 k'}{2E'(2\pi)^3} \sum_X \prod_{i=1}^{n_X} \int \frac{d^3 p_i}{(2\pi)^3 2p_{i0}} |T|^2 (2\pi)^4 \delta^4(P + q - \sum_{i=1}^{n_X} p_i). \quad (2.4)$$

The flux factor for the incoming nucleon and electron is denoted by  $J = 4P \cdot k$ , which is equal to  $J = 4ME$  in the rest frame of the nucleon. The sum runs over all unobserved hadronic final states  $X$ . Each hadronic final state consists of  $n_X$  particles

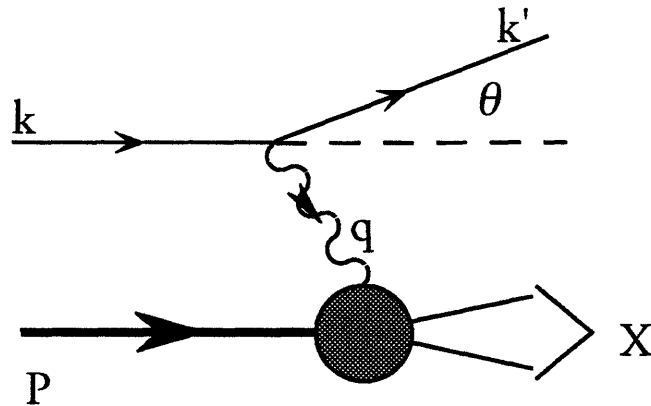


Figure 2-1: Kinematics of lepton-hadron scattering in the target rest frame

with momenta  $p_i$  ( $\sum_{i=1}^{n_X} p_i \equiv p_X$ ). The scattering amplitude is given by

$$\mathcal{T} = e^2 \bar{u}(k', s') \gamma^\mu u(k, s) \frac{1}{q^2} \langle X | J_\mu(0) | PS \rangle, \quad (2.5)$$

where  $J_\mu$  is the hadronic electromagnetic current.  $|X\rangle$  represents the unobserved final hadronic states.  $|PS\rangle$  represents the nucleon state, which is normalized covariantly as follows,

$$\langle P'S | PS \rangle = 2P^0 (2\pi)^3 \delta^3(\vec{P} - \vec{P}'), \quad (2.6)$$

$s$  is the lepton spin vector which is normalized to  $s^2 = -m^2$ , while  $S$  is the nucleon spin vector normalized to  $S^2 = -M^2$  and  $S \cdot P = 0$ .

The polarized differential cross-section can be written in terms of a leptonic ( $l^{\mu\nu}$ ) and a hadronic ( $W^{\mu\nu}$ ) tensor as follows (see Fig. (2-2)):

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2}{Q^4} \left( \frac{E'}{ME} \right) l^{\mu\nu} W_{\mu\nu}, \quad (2.7)$$

where  $\alpha = e^2/4\pi$  is the electromagnetic fine structure constant. The leptonic tensor  $l^{\mu\nu}$  is given by the square of the elementary spin 1/2 current (summed over final spins):

$$\begin{aligned} l^{\mu\nu} &= \sum_{s'} \bar{u}(k, s) \gamma^\mu u(k', s') \bar{u}(k', s') \gamma^\nu u(k, s) \\ &= 2(k'^\mu k^\nu + k'^\nu k^\mu) - 2g^{\mu\nu} k \cdot k' + 2i\epsilon^{\mu\nu\lambda\sigma} q_\lambda s_\sigma, \end{aligned} \quad (2.8)$$

where we have used  $u(k, s) \bar{u}(k, s) = \not{k}(1 + \gamma_5 \not{s})/2$ . The hadronic tensor  $W_{\mu\nu}$  is given by

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{4\pi} \sum_X \prod_{i=1}^{n_X} \int \frac{d^3 p_i}{(2\pi)^3 2p_{i0}} \langle PS | J^\mu | X \rangle \langle X | J^\nu | PS \rangle (2\pi)^4 \delta^4(P + q - \sum_{i=1}^{n_X} p_i) \\ &= \frac{1}{4\pi} \int d^4 \xi e^{iq \cdot \xi} \langle PS | [J^\mu(\xi), J^\nu(0)] | PS \rangle. \end{aligned} \quad (2.9)$$

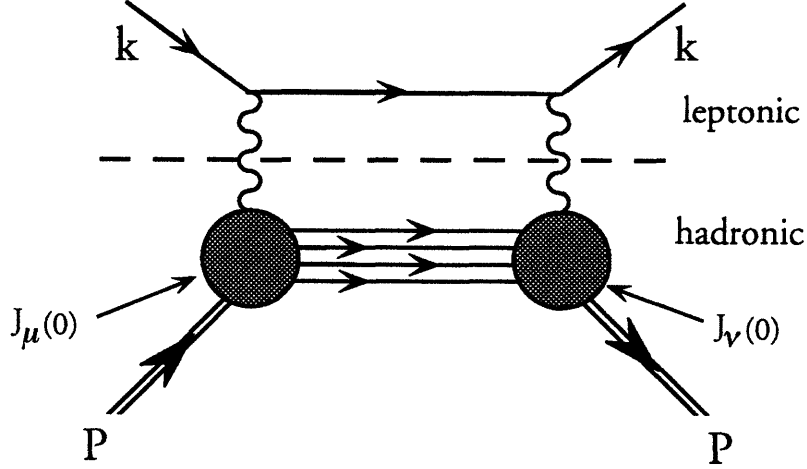


Figure 2-2: The squared amplitude  $\mathcal{T}$  for electron-hadron scattering can be separated into a leptonic tensor  $l^{\mu\nu}$  and a hadronic tensor  $W^{\mu\nu}$ .

In order to get Eq. (2.9) we have written the  $\delta$  function as an exponential,

$$(2\pi)^4 \delta^4(P) = \int d^4\xi e^{i\xi \cdot P}, \quad (2.10)$$

and used the completeness,

$$\sum_X \prod_{i=1}^{n_X} \int \frac{d^3 p_i}{(2\pi)^3 2p_{i0}} |X\rangle \langle X| = 1, \quad (2.11)$$

and the translation invariance,

$$e^{i\xi \cdot (P - p_X)} \langle PS | J^\mu(0) | X \rangle = \langle PS | J^\mu(\xi) | X \rangle. \quad (2.12)$$

Note that another term

$$\frac{1}{4\pi} \int d^4\xi e^{iq \cdot \xi} \langle PS | J^\nu(0) J^\mu(\xi) | PS \rangle \quad (2.13)$$

has been subtracted to convert the current product into a commutator. It is easy to check that this new term vanishes for  $q^0 > 0 (E > E')$  which is the the case for physical lepton scattering from a stable target. The reason is because this term has a



$\delta$  function:  $\delta^4(q - P + p_X)$ , where  $p_X \equiv \sum_{i=1}^{n_X} p_i$ , when sandwiching the completeness condition Eq. (2.11) between two currents,

$$\begin{aligned} & \int d^4\xi e^{iq\xi} \langle PS | J^\nu(0) J^\mu(\xi) | PS \rangle \\ &= \sum_X \prod_{i=1}^{n_X} \int \frac{d^3 p_i}{(2\pi)^3 2p_{i0}} \langle PS | J^\nu(0) | X \rangle \langle X | J^\mu(0) | PS \rangle (2\pi)^4 \delta^4(q - P + p_X). \end{aligned} \quad (2.14)$$

However, we can show that  $q - P + p_X \neq 0$ . The reason is the following: Assume that  $q - P + p_X = 0$  holds. Then we have  $(P - q)^2 = p_X^2$  which leads to  $q^0 = (M^2 + q^2 - p_X^2)/(2M)$  in the target rest frame (laboratory frame). According to the physical region conditions  $p_X^2 \geq M^2$  and  $q^2 \leq 0$ , we find that  $q^0 \leq 0$ . This contradicts the original assumption  $q^0 > 0$ .

[*Note:*  $W^{\mu\nu}$  can be related to the imaginary part of the forward virtual Compton scattering amplitude  $T$  using the optical theorem: (see, for instance, [15, 16])

$$2\pi W^{\mu\nu} = \text{Im} T^{\mu\nu} \quad (2.15)$$

with

$$T^{\mu\nu} = i \int d^4\xi e^{iq\xi} \langle PS | T J^\mu(\xi) J^\nu(0) | PS \rangle. \quad (2.16)$$

This relation will enable us to employ the Operator Product Expansion (OPE) [17] to discuss the perturbative properties of  $W^{\mu\nu}$ . However, in this thesis we will not go further along this direction.]

### 2.1.3 Structure Functions

While the leptonic tensor is known completely,  $W^{\mu\nu}$ , which describes the internal structure of the nucleon, depends on the non-perturbative strong interaction dynamics, about which unfortunately we know little. Using Lorentz covariance, gauge invariance, parity conservation in electromagnetism and standard discrete symmetries of the strong interactions,  $W^{\mu\nu}$  can be parameterized in terms of four scalar dimensionless structure functions  $F_1(x, Q^2)$ ,  $F_2(x, Q^2)$ ,  $g_1(x, Q^2)$  and  $g_2(x, Q^2)$ . They depend

only on the two invariants  $Q^2$  and  $\nu$ , or alternatively on  $Q^2$  and the dimensionless *Bjorken* variable  $x$ ,

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1 + \left[\left(P^\mu - \frac{\nu}{q^2} q^\mu\right) \left(P^\nu - \frac{\nu}{q^2} q^\nu\right)\right] \frac{F_2}{\nu} - i\epsilon^{\mu\nu\lambda\sigma} q_\lambda \left(\frac{S_\sigma}{\nu} (g_1 + g_2) - \frac{q \cdot S P_\sigma}{\nu^2} g_2\right), \quad (2.17)$$

where  $S^\sigma$  is the polarization vector of the nucleon ( $S^2 = -M^2$ ),  $P \cdot S = 0$ .  $S^\sigma$  is a pseudo-vector. Note that in Eq. (2.17) we have explicitly  $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$ , which is the consequence of the conservation of the electromagnetic current,  $\partial_\mu J^\mu = 0$ .

These structure functions  $\{F_1, F_2, g_1, g_2\}$  become functions only of the dimensionless ratio  $x = Q^2/2\nu$ , up to logarithms, in the Bjorken limit,

$$\begin{aligned} F_1(Q^2, \nu) &\rightarrow F_1(x, \ln Q^2), & F_2(Q^2, \nu) &\rightarrow F_2(x, \ln Q^2) \\ g_1(Q^2, \nu) &\rightarrow g_1(x, \ln Q^2), & g_2(Q^2, \nu) &\rightarrow g_2(x, \ln Q^2) \end{aligned} \quad (2.18)$$

This is the so-called *Bjorken* scaling.

## 2.2 Ellis-Jaffe Sum Rule

In this section we will focus on the structure function  $g_1(x)$ , which is related to the polarized quark distribution functions  $\{q_\uparrow(x), q_\downarrow(x), \bar{q}_\uparrow(x), \bar{q}_\downarrow(x)\}$  as follows

$$g_1(x) = \frac{1}{2} \sum_q e_q^2 [q_\uparrow(x) - q_\downarrow(x) + \bar{q}_\uparrow(x) - \bar{q}_\downarrow(x)] \quad (2.19)$$

where  $e_q$  is the quark electric charge measured in terms of the electron electric charge,  $\uparrow$  and  $\downarrow$  mean quark's helicity parallel and antiparallel to the nucleon's, respectively.

Much of the interest in the polarized structure function  $g_1$  is due to its relation to axial vector current matrix elements:

$$\langle PS | A_\alpha^q | PS \rangle |_{\mu^2} \equiv \langle PS | \bar{q} \gamma_\alpha \gamma_5 q | PS \rangle |_{\mu^2} = \Delta q(\mu^2) S_\alpha \quad (2.20)$$

where the label  $\mu$  refers to the mass scale at which the axial vector current operator is renormalized, and

$$\Delta q \equiv \int_0^1 dx [q_\uparrow(x) - q_\downarrow(x) + \bar{q}_\uparrow(x) - \bar{q}_\downarrow(x)] \quad (2.21)$$

where the renormalization scale dependence  $\mu$  is implicit. In naive parton model, the integrals of the  $g_1$  structure functions for proton

$$\Gamma_1^p(Q^2) \equiv \int_0^1 dx g_1^p(x, Q^2) \quad (2.22)$$

are related to the combinations of the  $\Delta q$

$$\Gamma_1^p(Q^2) = \frac{1}{2} \left( \frac{4}{9} \Delta u(Q^2) + \frac{1}{9} \Delta d(Q^2) + \frac{1}{9} \Delta s(Q^2) \right). \quad (2.23)$$

The perturbative QCD corrections to the above relation have been calculated,

$$\begin{aligned} \Gamma_1^p(Q^2) &= \frac{1}{2} \left( \frac{4}{9} \Delta u(Q^2) + \frac{1}{9} \Delta d(Q^2) + \frac{1}{9} \Delta s(Q^2) \right) \\ &\times \left( 1 - \frac{\alpha_s(Q^2)}{\pi} + \mathcal{O}(\alpha_s^2) \right) + \mathcal{O} \left( \frac{\Lambda^2}{Q^2} \right), \end{aligned} \quad (2.24)$$

where the power series in  $\alpha_s(Q^2)$  represents a perturbative calculation of the coefficient function of the axial vector currents[18, 19] and the  $\mathcal{O}(\Lambda^2/Q^2)$  terms are higher-twist corrections.

The  $g_1^p$  sum rule is interesting because some of combinations of  $\Delta u$ ,  $\Delta d$  and  $\Delta s$  can be independently measured, or inferred from data on nucleon and hyperon  $\beta$ -decays. The octet axial currents which mediate Gamow-Teller transitions between octet baryons are defined by

$$A_\mu^a \equiv \bar{\psi} \gamma_\mu \gamma_5 T^a \psi, \quad (2.25)$$

with  $\text{tr} T^a T^b = \frac{1}{2} \delta^{ab}$ . The nucleon matrix element of the  $A_\mu^3$  determines the combination  $\Delta u - \Delta d$ ;

$$(\Delta u - \Delta d) S_\mu = 2 \langle PS | A_\mu^3 | PS \rangle = g_A S_\mu = (F + D) S_\mu, \quad (2.26)$$

where  $g_A$  is the nucleon axial vector charge which can be measured in neutron  $\beta$ -decay.  $F + D$  is its parameterization in terms of SU(3) invariant matrix elements for the axial current in the hyperon semileptonic decay, which is valid in the SU(3) symmetry limit. SU(3) symmetry is also responsible for determining the other linearly independent combination,

$$(\Delta u + \Delta d - 2\Delta s)S_\mu = 2\sqrt{3}\langle PS|A_\mu^8|PS\rangle = (3F - D)S_\mu, \quad (2.27)$$

The third combination,  $\Delta\Sigma$ , is related to the flavor singlet current operator  $A_\mu^0$  as follows

$$\Delta\Sigma(\mu^2)S_\mu \equiv (\Delta u + \Delta d + \Delta s)S_\mu = \langle PS|A_\mu^0|PS\rangle |_{\mu^2} \quad (2.28)$$

with

$$A_\mu^0 \equiv \sum_{f=1}^3 \bar{\psi}_f \gamma_\mu \gamma_5 \psi_f. \quad (2.29)$$

We can rewrite  $g_1^p$  sum rule in Eq. (2.24) in terms of  $F$ ,  $D$ , and  $\Delta\Sigma$  as below

$$\Gamma_1^p(Q^2) = \frac{1}{18} (3F + D + 2\Delta\Sigma(Q^2)) \left( 1 - \frac{\alpha_s(Q^2)}{\pi} + \mathcal{O}(\alpha_s^2) \right) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right). \quad (2.30)$$

Combining Eqs. (2.26), (2.27), and (2.28) together, we have

$$\begin{aligned} \Delta u(Q^2) - \Delta d(Q^2) &= F + D, \\ \Delta u(Q^2) + \Delta d(Q^2) - 2\Delta s(Q^2) &= 3F - D, \\ \Delta u(Q^2) + \Delta d(Q^2) + \Delta s(Q^2) &= \Delta\Sigma(Q^2). \end{aligned} \quad (2.31)$$

Note that  $\Delta\Sigma$  depends upon the renormalization scale  $\mu^2$  because the flavor singlet current  $A_\mu^0$  is not conserved even when quarks are massless, due to the triangle anomaly[5]

$$\partial^\mu A_\mu^0 = \frac{n_f \alpha_s}{2\pi} \text{Tr} F \tilde{F}, \quad (2.32)$$

where  $n_f$  is flavor number.

It is not possible to derive sum rule for  $\Gamma_1^p$  without supplementary assumptions.

In 1973, J. Ellis and R. Jaffe assumed that  $\Delta s = 0$  on the grounds that very possibly there were a negligible number of strange quarks in the nucleon wave function -OZI rule[20]. With this assumption, it was estimated that

$$\int_0^1 dx g_1^p(x, Q^2) = \frac{1}{18}(4\Delta u + \Delta d)(1 - \alpha_s/\pi + \mathcal{O}(\alpha_s^2)) = 0.17 \pm 0.01. \quad (2.33)$$

This is the so-called Ellis-Jaffe sum rule[12]. It based on two main assumptions: SU(3) symmetry and  $\Delta s = 0$

## 2.3 EMC Experiment and Spin Crisis

In 1987 the EMC reported a measurement of  $g_1^p(x, Q^2)$ . The quantity they measured is the spin asymmetry

$$A_1 \equiv \frac{\mu^\uparrow p^\uparrow - \mu^\uparrow p^\downarrow}{\mu^\uparrow p^\uparrow + \mu^\uparrow p^\downarrow} \quad (2.34)$$

in  $\mu - p$  scattering, where the terms denote cross-sections and the arrows  $\uparrow$  and  $\downarrow$  denote polarizations along the beam direction.  $A_1$  is related to  $g_1$  by

$$g_1(x, Q^2) \simeq \frac{A_1(x)F_1(x, Q^2)}{1 + R(x, Q^2)} \quad (2.35)$$

where  $R(x, Q^2)$  is the ratio of longitudinal to transverse virtual photon cross-section, which can be taken from parameterizations of unpolarized scattering data, and  $F_1(x, Q^2)$  is the unpolarized structure function

$$F_1(x) = \frac{1}{2} \sum_q e_q^2 [q_\uparrow(x) + q_\downarrow(x) + \bar{q}_\uparrow(x) + \bar{q}_\downarrow(x)]. \quad (2.36)$$

Note that in general  $A_1$  should depend on  $Q^2$ . However, because this quantity is the ration as defined in Eq. (2.34), which seems experimentally to have only small dependence on  $Q^2$ . In particular, a recent analysis by the E143 collaboration[21] sees no significant  $Q^2$  dependence in  $A_1$  for  $Q^2 \geq 1\text{GeV}^2$  and the  $Q^2$  dependence seen at lower  $Q^2$  is compatible with the magnitude of the higher-twist correction. Therefore

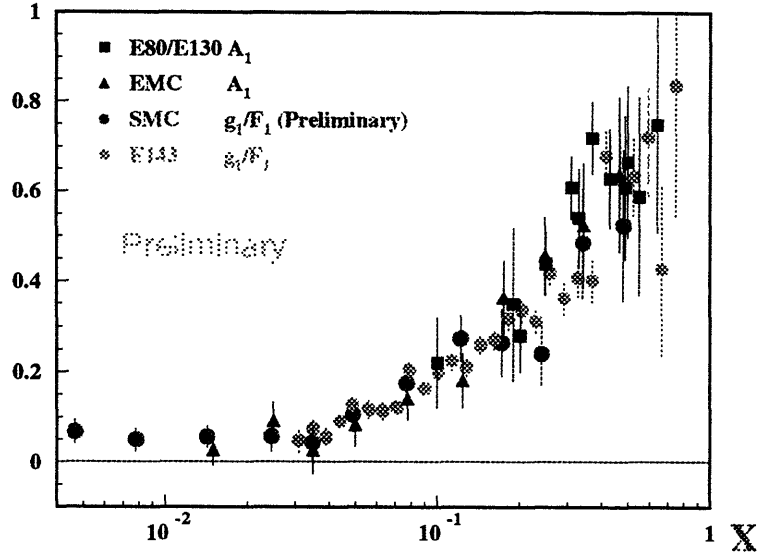


Figure 2-3: A recent compilation of data on  $A_1$  from SMC

experiments often assume that  $A_1$  is a function of  $x$  only (Fig. (2-3))<sup>1</sup>.

EMC extrapolated their measured structure function from  $x \simeq 0.02$  down to  $x = 0$ , and published a result for the first moment of  $g_1(x)$ ,

$$\Gamma_1^p = \int_0^1 dx g_1^p(x) = 0.126 \pm 0.010 \pm 0.015, \quad (2.37)$$

at energy  $Q^2 = 10.7 \text{ GeV}^2$ . It is significantly smaller than the prediction of Ellis-Jaffe sum rule.

Fitting  $F$  and  $D$  to the hyperon semileptonic decays, and using the errors determined from the fit gives [7]

$$F = 0.47 \pm 0.004, \quad D = 0.81 \pm 0.003. \quad (2.38)$$

Plugging Eqs. (2.37) and (2.38) into Eq. (2.30), we find at EMC energy (about  $10.7 \text{ GeV}^2$ )

$$\Delta\Sigma \equiv \Delta u + \Delta d + \Delta s = 0.13 \pm 0.19 \quad (2.39)$$

<sup>1</sup>Figs. (2-3), (2-4), (2-7) are extracted from Ref. [24]

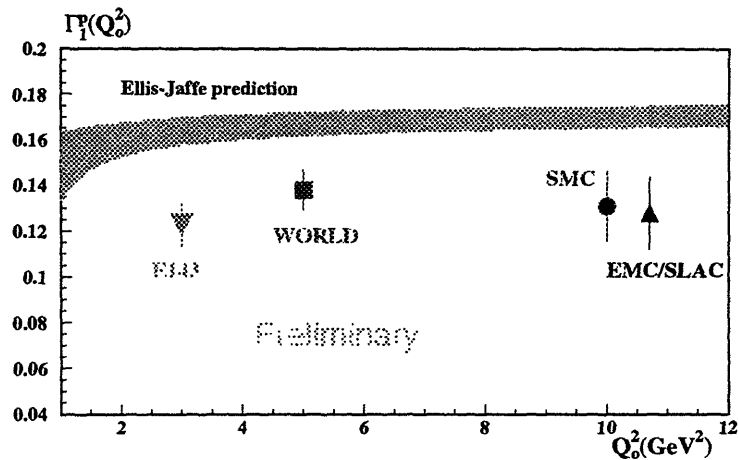


Figure 2-4: Comparison of data with the assumption  $\Delta s = 0$  from SMC.

or equivalently,

$$\Delta u = 0.78 \pm 0.10, \quad \Delta d = 0.50 \pm 0.10, \quad \Delta s = -0.20 \pm 0.11, \quad (2.40)$$

by solving Eqs. (2.31).

These results have shocked the particle community and aroused so much interest. Because according to the Ellis-Jaffe ansatz,  $\Delta s = 0$ [12], it would have predicted

$$\Delta\Sigma = 3F - D = 0.60 \pm 0.12. \quad (2.41)$$

The EMC data are about two standard deviations away from the prediction by Ellis-Jaffe sum rule. According to EMC data it seems that we can not attribute the nucleon's spin to the spins of the quarks. Much of the nucleon's spin must lie elsewhere. This is the so-called "spin crisis". Since EMC announced their results, there have been further more accurate experiments done by EMC, SMC(at CERN), E142,

E143(at SLAC)[2]. The comparison of the data with Ellis-Jaffe sum rule is shown in

Experiments	target	x	$Q^2[\text{GeV}^2]$
E130	p	0.180-0.7	3.5-10
EMC	p	0.010-0.7	1.5-70
SMC	p	0.003-0.70	1.0-60
E143	p	0.029-0.8	1.3-10
SMC-93	d	0.006-0.6	1.0-30
SMC-95	d	0.003-0.7	1.0-60
E143	d	0.029-0.8	1.0-30
E142	n	0.030-0.6	1.0-10

Table 2.1: Current experiments at CERN and SLAC.

Fig. (2-4). From the figure, we can see clearly the deviation of Ellis-Jaffe sum rule from the experimental data. The values of  $\Delta\Sigma$  at  $Q^2 = 3\text{GeV}^2$  extracted from each experiment are shown in Fig. (2-5) <sup>2</sup>. The average value at  $Q^2 = 3\text{GeV}^2$  is[22]

$$\Delta\Sigma = \Delta u + \Delta d + \Delta s = 0.27 \pm 0.04 \quad (2.42)$$

and

$$\Delta u = 0.82 \pm 0.03, \quad \Delta d = -0.44 \pm 0.03, \quad \Delta s = -0.11 \pm 0.03. \quad (2.43)$$

where the more recent  $F + D = 1.2573 \pm 0.0028$  and  $F/D = 0.575 \pm 0.016$  have been used.

One may also get a feeling for the expected range of  $\Delta\Sigma$  and  $\Delta s$  by plotting the results for these two observables extracted from from each of the existing experiments, as shown in Fig. (2-6).

There have been a lot of theoretical issues raised by this unexpected result. Here are some comments regarding some of these issues:

- Violation of  $SU(3)$  symmetry for octet axial charges could affect the evaluation of the sum rule. As we have pointed out,  $SU(3)$  symmetry plays an important role in determining  $\Delta u$ ,  $\Delta d$  and  $\Delta s$ (through the determina-

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<sup>2</sup>Figs. (2-5) and (2-6) are extracted from Ref. [22]



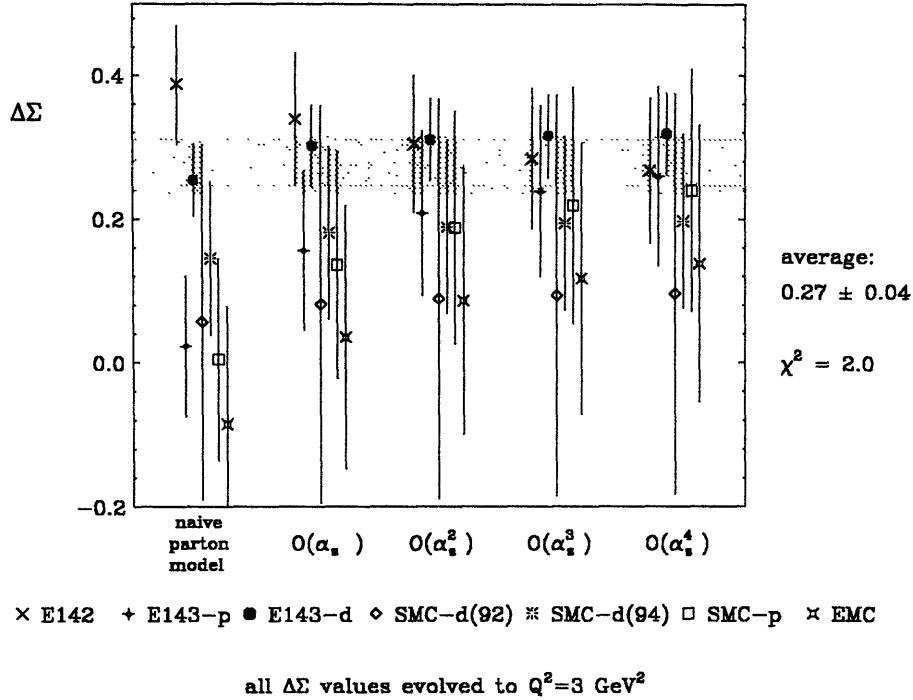


Figure 2-5: The values of  $\Delta\Sigma(Q^2 = 3\text{GeV}^2)$  extracted from each experiment, plotted as the increasing order of QCD perturbation theory used in obtaining  $\Delta\Sigma$  from the data.

tion of  $F$  and  $D$ ). However, we know that actually  $SU(3)$  symmetry is broken badly. In fact, there are large corrections to  $SU(3)$  symmetry relations from the symmetry breaking quark mass matrix, which has already been noted for the hyperon masses and for hyperon nonleptonic decays[23]. These large corrections may change the result significantly. Some authors even claimed that the  $SU(3)$  violation effects may be sufficient to give  $\Delta s = 0$ [25]. Direct measurement of  $\Delta s$  in  $\nu p$  elastic scattering as proposed at Los Alamos would possibly settle this issue[26].

- Uncertainty of the extrapolation from the lowest measured  $x$  ( $\sim 0.02$ ) to  $x = 0$ . Unanticipated behavior of the functions  $xg_1(x, Q^2)$  as  $x \rightarrow 0$  could affect the sum rules. Close and Roberts have pointed out that it is not hard to invent a low- $x$  behavior singular enough to accommodate  $\Delta s = 0$ [27]. The latest low- $x$  data from SMC(see Fig. 2-7) suggest tantalizing deviations from expected behavior. Unfortunately it is hard to obtain

much guidance from the data with such large error bars.

- The possible non-perturbative evolution of  $\Delta\Sigma$ . As we have noted,  $\Delta\Sigma$  does depend on the mass scale at which it is measured. It is possible that  $\Delta\Sigma$  is large at the confinement scale ( $\mu \sim 1\text{GeV}$ ), but somehow evolves away to a very small value at EMC energy range[28]. As we will see, the evolution of  $\Delta\Sigma$  is very slow at perturbative domain. However, its evolution near the confinement scale is not clear, which depends on non-perturbative dynamics.
- Ambiguity of the definition of  $\Delta\Sigma$  due to the axial current anomaly[5]. Shortly after EMC published their results, Alterelli and Ross[3] and Carlitz et al.[4] pointed out that there is gluon contribution to the proton spin through the axial current anomaly[5]. What the EMC measured is actually the combination  $\Delta q = \Delta q' - \frac{\alpha_s}{2\pi}\Delta g$  with  $\Delta q'$  being identified to the naive quark model expectation and  $\Delta g$  being the integrated gluon helicity distribution. The small experimental value of  $\Delta\Sigma$  can be interpreted due to large cancelation between  $\Delta q'$  and  $\Delta g$ , which would require a rather large  $\Delta g (\simeq 5 \sim 6)$  at the EMC energy. There have been a lot of discussions about this issue[7]. Some suggested that  $\Delta g$  might not be that big as required to explain EMC data[7, 29], even it might not be positive[30]. The E581/704 group at Fermilab has analyzed the data from their  $pp$  and  $p\bar{p}$  experiments for jet production cross section[31]. The results implied that the polarized gluon distribution is limited in size which would favor the models giving smaller  $\Delta g$ . In our paper we showed that the anomaly contribution to  $\Delta\Sigma$  can be exactly canceled by a similar term to the quark orbital angular momentum leaving the total quark contribution to the nucleon spin anomaly free[11]. Further direct measurement of  $\Delta g$  by RHIC, HERMES might settle this issue[32].

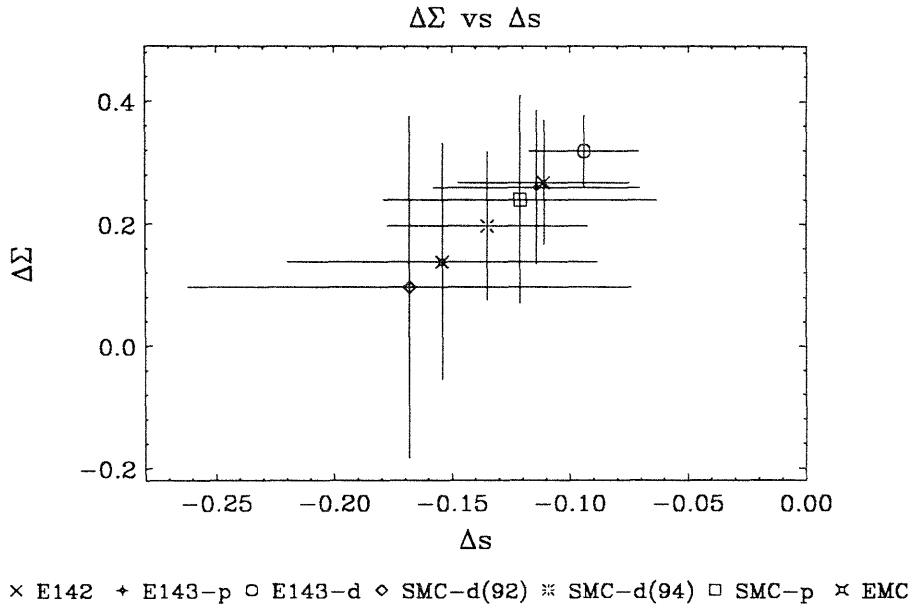


Figure 2-6: The values of  $\Delta\Sigma$  and  $\Delta s$  extracted from each experiment, plotted against each other. All data have been evolved to common  $Q^2 = 3 \text{ GeV}^2$ .

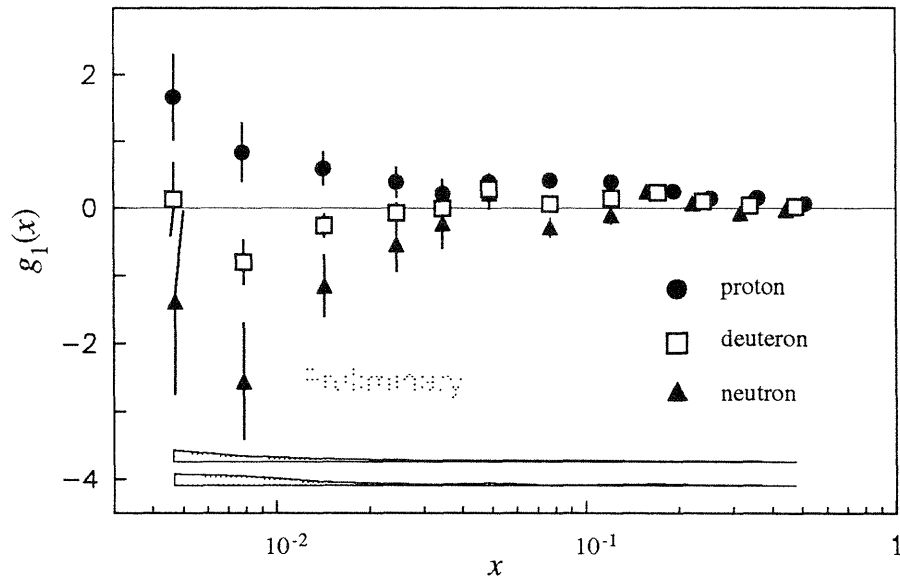


Figure 2-7: Data from SMC on  $g_1$ .

## Chapter 3

# Spin Structure of Nucleon in QCD

In the previous chapter, we have concentrated on the phenomenological analysis of the data on the polarized structure functions presently available. As we have seen, the nucleon's spin cannot be only attributed to the spins of the quarks inside it as people thought in naive constituent quark model in which the quarks are considered as the only constituents of the nucleon. However, in accordance with QCD, nucleon is composed very complicatedly of current quarks(valence quarks and sea quarks) and gluons. The spin of nucleon should come from both quarks and gluons. We consider the angular momentum sum rule in QCD

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma(Q^2) + \Delta g(Q^2) + L_q(Q^2) + L_g(Q^2), \quad (3.1)$$

where  $\Delta\Sigma(Q^2)$ ,  $\Delta g(Q^2)$ ,  $L_q(Q^2)$ ,  $L_g(Q^2)$  represent quark helicity, gluon helicity, quark orbital angular momentum, gluon orbital angular momentum contributions to the nucleon spin, respectively. If using the world average value for  $\Delta\Sigma$  in Eq. (2.42), we can see clearly that the quark spins only contribute about 30% of the nucleon spin. Much of it(about 70%) must come from the other terms in the sum rule as shown in Eq. (3.1). Therefore it is very interesting to determine how much each of these four terms contributes. As we have noted in the previous chapter,  $\Delta\Sigma$  depends on renormalization scale  $Q^2$ , so do the other three terms in Eq. (3.1). It is also very interesting to know how these terms evolve as  $Q^2$ . This will be main topic of this

chapter.

### 3.1 Angular Momentum Operator in QCD

The QCD Lagrangian is as follows

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma_{\mu}D^{\mu} - m)\psi \quad (3.2)$$

where  $D^{\mu} = \partial^{\mu} + A^{\mu}$  with  $A^{\mu} \equiv A^{\mu a}t^a$ , where  $t^a (a = 1, \dots, 8)$  are Gell-Mann color matrices[33].  $\psi$  is the quark field carrying implicit flavor, color and Dirac indices.  $m$  is the quark mass matrix in flavor space. The gluon field strength  $F^{\mu\nu}$  has the usual non-Abelian form

$$F^{\mu\nu a} = \partial^{\mu}A^{\nu a} - \partial^{\nu}A^{\mu a} - gf^{abc}A^{\mu b}A^{\nu c} \quad (3.3)$$

with  $f^{abc}$  the structure constant of the gauge group  $SU(3)_c$ .

In accordance with the Noether theorem, there exists a symmetry generator  $T_{\mu\nu}$  associated with the translational invariance in QCD, which is conserved,  $\partial_{\mu}T^{\mu\nu} = 0$ .  $T^{\mu\nu}$  is called as energy-momentum tensor, which is gauge invariant and symmetric,  $T^{\mu\nu} = T^{\nu\mu}$ . The Noether current associated with Lorentz transformations is a rank-3 tensor constructed entirely from  $T^{\mu\nu}$ [34],

$$M^{\mu\nu\lambda} \equiv x^{\nu}T^{\mu\lambda} - x^{\lambda}T^{\mu\nu} \quad (3.4)$$

It is very easy to verify that  $M^{\mu\nu\lambda}$  is conserved because  $T^{\mu\nu}$  is symmetric and conserved.  $M^{\mu\nu\lambda}$  is gauge invariant and has no totally antisymmetric part,

$$\epsilon_{\alpha\mu\nu\lambda}M^{\mu\nu\lambda} = 0 \quad (3.5)$$

or equivalently

$$M^{\mu\nu\lambda} + M^{\lambda\mu\nu} + M^{\nu\lambda\mu} = 0. \quad (3.6)$$

The generators of Lorentz transformations are defined as

$$J^{\mu\nu} \equiv \int d^3x M^{0\mu\nu}. \quad (3.7)$$

It is very easy to verify that  $J^{\mu\nu}$  is conserved,  $\frac{d}{dt}J^{\mu\nu} = 0$ , and obey the Lie algebra of the Poincaré group

$$\begin{aligned} [P^\mu, P^\nu] &= 0, \\ [J^{\mu\nu}, P^\lambda] &= i(g^{\mu\lambda}P^\nu - g^{\nu\lambda}P^\mu), \\ [J^{\mu\nu}, J^{\lambda\sigma}] &= i(g^{\mu\lambda}J^{\nu\sigma} - g^{\nu\lambda}J^{\mu\sigma} - g^{\mu\sigma}J^{\nu\lambda} + g^{\nu\sigma}J^{\mu\lambda}) \end{aligned} \quad (3.8)$$

where  $P^\mu \equiv \int d^3x T^{0\mu}$  is the generator of the translation group.

The energy-momentum density tensor of QCD is as follows[35]

$$T_{\text{QCD}}^{\mu\nu} = \frac{1}{2}\bar{\psi}i \overleftrightarrow{D}^{(\mu} \gamma^{\nu)}\psi + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\alpha}F_{\alpha}^{\nu}, \quad (3.9)$$

where the covariant derivative  $\overleftrightarrow{D} \equiv \overrightarrow{D} - \overleftarrow{D}$  with  $\overleftarrow{D} = \overleftarrow{\partial} - igA$ . The symmetrization of the indices  $\mu$  and  $\nu$  in the first term is indicated by  $(\mu\nu)$ .

Plugging Eq. (3.9) into Eq. (3.4), a short algebra yields the angular momentum density tensor for QCD as below,

$$\begin{aligned} M_{\text{QCD}}^{\mu\nu\lambda} &= \frac{i}{4}\bar{\psi}x^\nu(\gamma^\mu \overleftrightarrow{D}^\lambda + \gamma^\lambda \overleftrightarrow{D}^\mu)\psi - \frac{i}{4}\bar{\psi}x^\lambda(\gamma^\mu \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\mu)\psi \\ &\quad - \left( x^\nu F^{\mu\alpha}F_{\alpha}^{\lambda} - x^\lambda F^{\mu\alpha}F_{\alpha}^{\nu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}(x^\nu g^{\mu\lambda} - x^\lambda g^{\mu\nu}) \right). \end{aligned} \quad (3.10)$$

Clearly, there is no breakup of the tensor into pieces that one can identify as being due to the quark spin, gluon spin, quark orbital angular momentum, and gluon orbital angular momentum. However, note that there is an arbitrariness in the definition of  $M^{\mu\nu\lambda}$ , that is, in the context of some field theory let  $B^{[\mu\beta][\nu\lambda]}$  be some operator antisymmetric in  $(\mu, \beta)$  and  $(\nu, \lambda)$  ( $B^{[\mu\beta][\nu\lambda]}$  may contain an explicit factor of

the coordinate  $x$ ), then we can define a new tensor by adding a superpotential

$$M'^{\mu\nu\lambda} \equiv M^{\mu\nu\lambda} + \partial_\beta B^{[\mu\beta][\nu\lambda]}, \quad (3.11)$$

which is conserved and antisymmetric in  $\nu$  and  $\lambda$ , the Lorentz generators  $J'^{\mu\nu}$  defined from  $M'^{\mu\nu\lambda}$  are the same as  $J^{\mu\nu}$ ,

$$J'^{\mu\nu} = J^{\mu\nu} + \int d^3x \partial_i B^{[0i][\nu\lambda]} = J^{\mu\nu} \quad (3.12)$$

for field configurations which vanish at spatial infinity. By choosing a suitable superpotential, one obtains[7]

$$\begin{aligned} M^{\alpha\mu\nu} = & \quad i\bar{\psi}\gamma^\alpha(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi + \frac{i}{4}\bar{\psi}\gamma^\alpha[\gamma^\mu, \gamma^\nu]\psi \\ & - F^{\alpha\sigma}(x^\mu\partial^\nu - x^\nu\partial^\mu)A_\sigma - F^{\alpha\mu}A^\nu + F^{\alpha\nu}A^\mu. \end{aligned} \quad (3.13)$$

Now the terms in Eq. (3.13) correspond successfully to the spin and orbital angular momenta of the quarks and gluons.

## 3.2 Angular Momentum Sum Rule

Now we apply the results of the previous section to the nucleon system. According to the definition, a nucleon moving in the  $z$  direction with momentum  $P^\mu$  and helicity  $1/2$  satisfies

$$J^{12}|P+\rangle = \frac{1}{2}|P+\rangle, \quad (3.14)$$

where  $J^{12} = \int d^3x M^{012}$ .

Thus one can write down a spin sum rule,

$$\begin{aligned} \frac{1}{2} &= \langle P+ | J^{12} | P+\rangle / \langle P+ | P+\rangle \\ &= \frac{1}{2}\Delta\Sigma + \Delta g + L_q + L_g, \end{aligned} \quad (3.15)$$

where the matrix elements are defined as ( $\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ ),

$$\begin{aligned}
\Delta\Sigma &= \langle P+ | \hat{S}_{3q} | P+ \rangle = \langle P+ | \int d^3x \bar{\psi} \gamma^3 \gamma_5 \psi | P+ \rangle , \\
\Delta g &= \langle P+ | \hat{S}_{3g} | P+ \rangle = \langle P+ | \int d^3x (E^1 A^2 - E^2 A^1) | P+ \rangle , \\
L_q &= \langle P+ | \hat{L}_{3q} | P+ \rangle = \langle P+ | \int d^3x i\bar{\psi} \gamma^0 (x^1 \partial^2 - x^2 \partial^1) \psi | P+ \rangle , \\
L_g &= \langle P+ | \hat{L}_{3g} | P+ \rangle = \langle P+ | \int d^3x E^i (x^2 \partial^1 - x^1 \partial^2) A^i | P+ \rangle , \quad (3.16)
\end{aligned}$$

where for simplicity we have neglected the normalization of the state.  $E^{ia} \equiv -F^{0ia} = -(\partial^0 A^{ia} - \partial^i A^{0a} - g f^{abc} A^{0b} A^{ic})$  is the color electric field.

It is clear from the above that  $\Delta\Sigma$  and  $\Delta g$  are the quark and gluon helicity contributions to the nucleon spin, and  $L_q$  and  $L_g$  are the quark and gluon orbital angular momentum contributions. Apart from  $\hat{S}_{3q}$ , the other three operators  $\hat{S}_{3g}$ ,  $\hat{L}_{3q}$  and  $\hat{L}_{3g}$  are not manifestly gauge invariant, and thus a decomposition of the nucleon spin is in general gauge-dependent. [Note that X.Ji gives a gauge invariant decomposition of angular momentum operator in his recent paper[36]. However, he proved that the new gauge-invariant operators satisfy the same evolution equation as the old gauge-dependent ones. Therefore, here we will still use the gauge-dependent ones to demonstrate the calculations.] Furthermore, the matrix elements depend on the choice of Lorentz frame. Only in light-front coordinates and light-front gauge [37]  $\Delta g$  is the gluon helicity measured in high-energy scattering processes. We henceforth work in this coordinates and gauge [38] (the index 0 in Eq. (3.7) is now replaced by +).



### 3.3 Light-front Coordinates and Angular Momentum Operators

Light-front coordinates  $(x^+, x^-, \vec{x}_\perp)$  are such a set of coordinates which are related to the usual coordinates  $(x^0, x^1, x^2, x^3)$  as follows

$$\begin{aligned} x^+ &\equiv \frac{1}{\sqrt{2}}(x^0 + x^3) \\ x^- &\equiv \frac{1}{\sqrt{2}}(x^0 - x^3) \\ \vec{x}_\perp &\equiv (x^1, x^2). \end{aligned} \tag{3.17}$$

In this basis, the metric  $g_{\mu\nu}$  has non-zero components,  $g_{+-} = g_{-+} = 1$  and  $g_{ij} = -\delta_{ij}$ . Therefore,  $x \cdot y \equiv g_{\mu\nu} x^\mu y^\nu = x^+ y^- + x^- y^+ - \vec{x}_\perp \cdot \vec{y}_\perp$ .  $x^+$  is to play the role of the time in the light-front formalism, therefore  $P^-$ , the momentum conjugate to  $x^+$ , play the role of the energy, and  $(P^+, \vec{P}_\perp)$  is the three-momentum that specifies the state of a particle. Now the proper normalization condition for a momentum eigenstate is

$$\langle P' S | P S \rangle = 2P^+ (2\pi)^3 \delta^3(\vec{P} - \vec{P}'). \tag{3.18}$$

Consider a nucleon of momentum  $P$  moving in  $z$ -direction. It is helpful to introduce two light-like vectors

$$\begin{aligned} p^\mu &= \frac{\Lambda}{\sqrt{2}}(1, 0, 0, 1) \\ n^\mu &= \frac{1}{\sqrt{2}\Lambda}(1, 0, 0, -1) \end{aligned} \tag{3.19}$$

with  $p^2 = n^2 = 0$  and  $p \cdot n = 1$  and clearly the only non-zero components for  $p$  and  $n$  are  $p^+$  and  $n^-$ . Up to the scale factor  $\Lambda$ , the vectors  $p^\mu$  and  $n^\mu$  function as unit vectors along opposite tangents to the light-cone.  $\Lambda$  actually selects a specific frame. For example  $\Lambda = M/\sqrt{2}$  yields the nucleon rest frame while  $\Lambda \rightarrow \infty$  selects

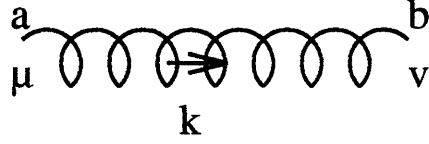


Figure 3-1: Feynman digram for gluon propagator in light-front coordinate frame.

the infinite momentum frame. We can express  $P$  using  $p^\mu$  and  $n^\mu$  as follows

$$P^\mu = p^\mu + \frac{M^2}{2} n^\mu. \quad (3.20)$$

Choose two unit vectors in the transverse dimensions  $\hat{e}_1 = (0, 1, 0, 0)$  and  $\hat{e}_2 = (0, 0, 1, 0)$ , then  $p$ ,  $n$ ,  $\hat{e}_1$  and  $\hat{e}_2$  form a complete set of basis. Any vector  $k^\mu$  can be expanded using these vectors as follows,

$$k^\mu = p^\mu k \cdot n + n^\mu k \cdot p + k_\perp^\mu. \quad (3.21)$$

Especially, the vector  $q^\mu$  in deep inelastic scattering can be expanded as

$$q^\mu = \frac{1}{M^2} \left( \nu - \sqrt{\nu^2 + M^2 Q^2} \right) p^\mu + \frac{1}{2} \left( \nu + \sqrt{\nu^2 + M^2 Q^2} \right) n^\mu, \quad (3.22)$$

in the Bjorken limit it simplifies to

$$\lim_{Bj} q^\mu \sim \left( \nu + \frac{1}{2} M^2 x \right) n^\mu - x p^\mu + \mathcal{O} \left( \frac{1}{Q^2} \right). \quad (3.23)$$

The gluon propagator in light-front gauge  $A^+ = 0$ (Fig. (3-1)) is

$$D_{\mu\nu}^{ab}(k) = \frac{i\delta_{ab}}{k^2 + i\epsilon} \left[ -g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n} \right]. \quad (3.24)$$

In light-front coordinates the angular momentum operator is defined as below

$$J^{12} = \int d^3x M^{+12}, \quad (3.25)$$

where  $d^3x \equiv d^2x_\perp dx^-$ . Therefore we have expressions for  $\Delta\Sigma$ ,  $\Delta g$ ,  $L_q$  and  $L_g$  in

light-front coordinates and light-front gauge( $A^+ = 0$ ),

$$\begin{aligned}
\Delta\Sigma &= \langle P+ | \int d^3x \bar{\psi} \gamma^+ \gamma_5 \psi | P+ \rangle , \\
\Delta g &= \langle P+ | \int d^3x (A^1 \partial^+ A^2 - A^2 \partial^+ A^1) | P+ \rangle , \\
L_q &= \langle P+ | \int d^3x i \bar{\psi} \gamma^+ (x^1 \partial^2 - x^2 \partial^1) \psi | P+ \rangle , \\
L_g &= \langle P+ | \int d^3x \partial^+ A^\alpha (x^2 \partial^1 - x^1 \partial^2) A_\alpha | P+ \rangle ,
\end{aligned} \tag{3.26}$$

where once again for simplicity we have neglected the normalization of the state.

As we have mentioned before, the individual operators in  $J^{12}$  are not conserved charges, and hence their matrix elements are generally divergent[15]. They can be renormalized in a scheme and the renormalization introduces a scale dependence. The scale dependence of these operators satisfies renormalization group equation and generally they will mix with each other under renormalization

$$\frac{d}{dt} \begin{pmatrix} \Delta\Sigma \\ \Delta g \\ L_q \\ L_g \end{pmatrix} = \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} & \Gamma_{qL_q} & \Gamma_{qL_g} \\ \Gamma_{gq} & \Gamma_{gg} & \Gamma_{gL_q} & \Gamma_{gL_g} \\ \Gamma_{L_qq} & \Gamma_{L_qg} & \Gamma_{L_qL_q} & \Gamma_{L_qL_g} \\ \Gamma_{L_gq} & \Gamma_{L_gg} & \Gamma_{L_gL_q} & \Gamma_{L_gL_g} \end{pmatrix} \begin{pmatrix} \Delta\Sigma \\ \Delta g \\ L_q \\ L_g \end{pmatrix} \tag{3.27}$$

where

$$\Gamma_a \equiv \begin{pmatrix} \Gamma_{qq} & \Gamma_{qg} & \Gamma_{qL_q} & \Gamma_{qL_g} \\ \Gamma_{gq} & \Gamma_{gg} & \Gamma_{gL_q} & \Gamma_{gL_g} \\ \Gamma_{L_qq} & \Gamma_{L_qg} & \Gamma_{L_qL_q} & \Gamma_{L_qL_g} \\ \Gamma_{L_gq} & \Gamma_{L_gg} & \Gamma_{L_gL_q} & \Gamma_{L_gL_g} \end{pmatrix} \tag{3.28}$$

is the *anomalous dimension matrix*, which will be determined to the leading order in the next sections,  $t \equiv \ln Q^2 / \Lambda_{\text{QCD}}^2$ .

### 3.4 Evolution of Spin: Altarelli-Parisi Equation

In their famous paper in 1977, Altarelli and Parisi derived a bunch of master equations for the  $Q^2$  dependence of polarized quark and gluon densities[9]

$$\begin{aligned}\frac{d}{dt}\Delta q^i(x,t) &= \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ \Delta q^i(y,t)\Delta P_{qq}\left(\frac{x}{y}\right) + 2\Delta g(y,t)\Delta P_{gq}\left(\frac{x}{y}\right) \right], \\ \frac{d}{dt}\Delta g(x,t) &= \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ \sum_{i=1}^{n_f} \Delta q^i(y,t)\Delta P_{gq}\left(\frac{x}{y}\right) + \Delta g(y,t)\Delta P_{gg}\left(\frac{x}{y}\right) \right].\end{aligned}\tag{3.29}$$

where  $\Delta q^i(x,t) \equiv q_{\uparrow}^i(x,t) - q_{\downarrow}^i(x,t) + \bar{q}_{\uparrow}^i(x,t) - \bar{q}_{\downarrow}^i(x,t)$  is the polarized quark density for flavor  $i$ , and  $\Delta g(x,t) \equiv g_{\uparrow}(x,t) - g_{\downarrow}(x,t)$  is the polarized gluon density,  $t \equiv \ln Q^2/\Lambda_{\text{QCD}}^2$ .  $(\alpha_s/2\pi)\Delta P_{qq}(z)$  is the probability density per unit  $t$  at order  $\alpha$  of finding a polarized quark inside a polarized quark with fraction  $z$  of the longitudinal momentum of the parent quark,  $(\alpha_s/2\pi)\Delta P_{gq}(z)$  is the probability density per unit  $t$  at order  $\alpha$  of finding a polarized gluon inside a polarized quark with fraction  $z$  of the longitudinal momentum of the parent quark,  $(\alpha_s/2\pi)\Delta P_{qg}(z)$  is the probability density per unit  $t$  at order  $\alpha$  of finding a polarized quark inside a polarized gluon with fraction  $z$  of the longitudinal momentum of the parent gluon,  $(\alpha_s/2\pi)\Delta P_{gg}(z)$  is the probability density per unit  $t$  at order  $\alpha$  of finding a polarized gluon inside a polarized gluon with fraction  $z$  of the longitudinal momentum of the parent gluon. In the leading-log approximation[9]

$$\begin{aligned}\Delta P_{qq}(z) &= C_F \left[ \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(z-1) \right], \\ \Delta P_{gq}(z) &= C_F \frac{1-(1-z)^2}{z}, \\ \Delta P_{qg}(z) &= \frac{1}{2}[1-(1-z)^2], \\ \Delta P_{gg}(z) &= 3 \left[ (1+z^4) \left( \frac{1}{z} + \frac{1}{(1-z)_+} \right) - \frac{(1-z)^3}{z} + \frac{\beta_0}{6}\delta(z-1) \right]\end{aligned}\tag{3.30}$$

with  $\beta_0 = 11 - 2n_f/3$  and  $C_F = 4/3$ . The distribution  $1/(1-z)_+$  is defined as

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} \equiv \int_0^1 dz \frac{f(z) - f(1)}{(1-z)_+}. \quad (3.31)$$

Integrate Eqs. (3.29) over  $x$  from 0 to 1 and sum over quark flavor  $i$ , one gets

$$\begin{aligned} \frac{d}{dt} \Delta \Sigma(t) &= \frac{\alpha_s(t)}{2\pi} (A_{qq} \Delta \Sigma(t) + A_{qg} \Delta g(t)), \\ \frac{d}{dt} \Delta g(t) &= \frac{\alpha_s(t)}{2\pi} (A_{gq} \Delta \Sigma(t) + A_{gg} \Delta g(t)) \end{aligned} \quad (3.32)$$

where

$$\begin{aligned} \Delta \Sigma(t) &\equiv \int_0^1 dx \sum_{i=1}^{n_f} \Delta q^i(x, t) = \int dx \sum_{i=1}^{n_f} [q_{\uparrow}^i(x, t) - q_{\downarrow}^i(x, t) + \bar{q}_{\uparrow}^i(x, t) - \bar{q}_{\downarrow}^i(x, t)], \\ \Delta g(t) &\equiv \int_0^1 dx \Delta g(x, t) = \int dx [g_{\uparrow}(x, t) - g_{\downarrow}(x, t)], \end{aligned} \quad (3.33)$$

and

$$\begin{pmatrix} A_{qq} & A_{qg} \\ A_{gq} & A_{gg} \end{pmatrix} = \int_0^1 dx \begin{pmatrix} \Delta P_{qq}(x) & 2n_f \Delta P_{qg}(x) \\ \Delta P_{gq}(x) & \Delta P_{gg}(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{3}{2} C_F & \frac{\beta_0}{2} \end{pmatrix}, \quad (3.34)$$

Combine Eqs. (3.32) and (3.34), one gets

$$\frac{d}{dt} \begin{pmatrix} \Delta \Sigma \\ \Delta g \end{pmatrix} = \frac{\alpha_s}{2\pi} \begin{pmatrix} 0 & 0 \\ \frac{3}{2} C_F & \frac{\beta_0}{2} \end{pmatrix} \begin{pmatrix} \Delta \Sigma \\ \Delta g \end{pmatrix}, \quad (3.35)$$

where the square matrix on the right-hand side is called AP splitting matrix.

From Eq. (3.35) we can write down solutions for the quark and gluon helicities[3, 40] using  $\alpha_s(t) = 4\pi/(\beta_0 t)$  to the leading order[39]

$$\begin{aligned} \Delta \Sigma(t) &= \text{const}, \\ \Delta g(t) &= -\frac{4\Delta \Sigma}{\beta_0} + \frac{t}{t_0} \left( \Delta g(t_0) + \frac{4\Delta \Sigma}{\beta_0} \right). \end{aligned} \quad (3.36)$$

The second equation exhibits the famous behavior of the gluon helicity: increasing

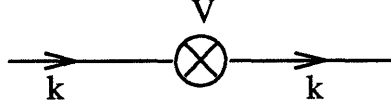


Figure 3-2: Vertex for quark spin operator, where  $V$  represents the vertex  $\gamma^+\gamma_5$ .

like  $\ln Q^2$  as  $Q^2 \rightarrow \infty$ . The coefficient of the term depends on the special combination of initial quark and gluon helicities, which is likely positive at low- momentum scales according to the recent experimental data on  $\Delta\Sigma$ [1, 2] and theoretical estimates for  $\Delta g$ [41]. It is very hard to understand why the gluon helicity has such kind of “strange” behavior without considering the orbital angular momentum contribution. We will show that the large gluon helicity at large  $Q^2$  is canceled by an equally large, but negative, gluon orbital angular momentum. However, instead of devoting ourselves to discussing the evolution of the orbital angular momenta directly, as a warm-up, we will first re-derive the AP equation, i.e. compute the first two-row elements in Eq. (3.28),  $\Gamma_{qq}$ ,  $\Gamma_{qg}$ ,  $\Gamma_{gq}$  and  $\Gamma_{gg}$ . Because we will just calculate the anomalous dimension matrix elements, we will for simplicity use the free particle(quark, gluon) state (free plane wave) to do the calculations.

### 3.4.1 Quark Spin

First of all, let's recall the expression for  $\Delta\Sigma$  in light-front coordinates

$$\Delta\Sigma = \frac{\langle P+ | \int d^3x \bar{\psi}(x) \gamma^+ \gamma_5 \psi(x) | P+ \rangle}{\langle P+ | P+ \rangle}. \quad (3.37)$$

The normalization is  $\langle P+ | P'++ \rangle = 2P^+ (2\pi)^3 \delta^3(\vec{P} - \vec{P}')$ . Using the translation invariant relation for any operator  $\hat{O}(x)$

$$\hat{O}(x) = e^{i\hat{P}\cdot x} \hat{O}(0) e^{-i\hat{P}\cdot x} \quad (3.38)$$

we have

$$\Delta\Sigma = \frac{1}{2P^+} \langle P+ | \bar{\psi}(0) \gamma^+ \gamma_5 \psi(0) | P+ \rangle, \quad (3.39)$$

where the summation over quark flavors is implicit. Eq. (3.39) is actually consistent with Eq. (2.28). From Eq. (3.39), we get Feynman vertex for quark spin operator (shown in Fig. (3-2))

$$V = \gamma^+ \gamma_5 \quad (3.40)$$

### Quark Spin In A Polarized Quark State: $\Gamma_{qq}$

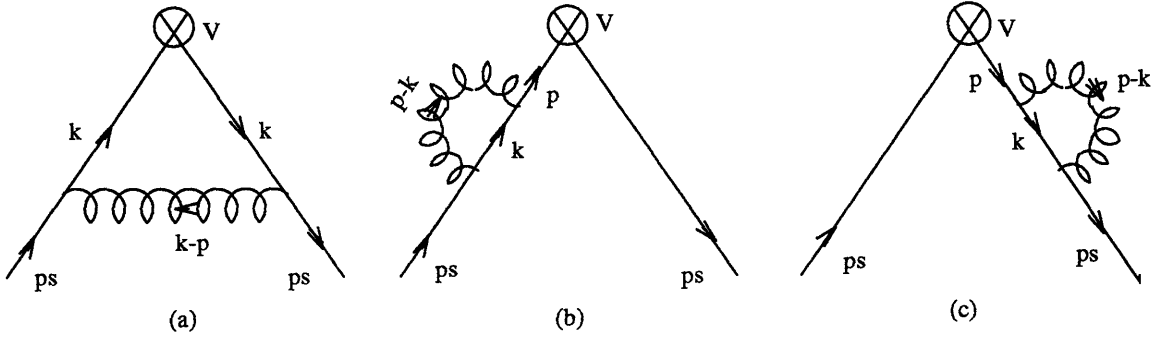


Figure 3-3: Feynman diagrams for calculating  $\Gamma_{qq}$ , where  $V$  represents the vertex  $\gamma^+ \gamma_5$ .

To calculate  $\Gamma_{qq}$ , we choose the state  $|P^+\rangle$  in Eq. (3.39) to be polarized quark state. Let  $T_{qq}^i$  ( $i = a, b, c$ ) represent the contribution from the three diagrams in Fig. (3-3) for each flavor (the masses of the quarks have been ignored, therefore  $P = p$  according to Eq. (3.20))

$$T_{qq} \equiv T_{qq}^a + T_{qq}^b + T_{qq}^c, \quad (3.41)$$

$\Gamma_{qq}$  is the coefficient of  $T_{qq}$ . Using the Feynman rules, we can write down these three pieces as follows

$$\begin{aligned} T_{qq}^a &= \frac{1}{2p^+} \int \frac{d^4 k}{(2\pi)^4} \bar{u}(ps) (-ig\gamma_\nu t^a) \frac{i\cancel{k}}{k^2 + i\epsilon} \gamma^+ \gamma_5 \frac{i\cancel{k}}{k^2 + i\epsilon} (-ig\gamma_\mu t^b) u(ps) \\ &\times \frac{-i\delta^{ab}}{(k-p)^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(k-p)^\mu n^\nu + (k-p)^\nu n^\mu}{(k-p) \cdot n} \right], \end{aligned} \quad (3.42)$$

$$\begin{aligned} T_{qq}^b &= \frac{1}{2p^+} \int \frac{d^4 k}{(2\pi)^4} \bar{u}(ps) \gamma^+ \gamma_5 \frac{i\cancel{p}}{p^2 + i\epsilon} (-ig\gamma_\mu t^a) \frac{i\cancel{k}}{k^2 + i\epsilon} (-ig\gamma_\nu t^b) u(ps) \\ &\times \frac{-i\delta^{ab}}{(p-k)^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(p-k)^\mu n^\nu + (p-k)^\nu n^\mu}{(p-k) \cdot n} \right], \end{aligned} \quad (3.43)$$

$$\begin{aligned}
T_{qq}^c &= \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \bar{u}(ps) (-ig\gamma_\mu t^a) \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma_\nu t^b) \frac{i\not{p}}{p^2 + i\epsilon} \gamma^+ \gamma_5 u(ps) \\
&\times \frac{-i\delta^{ab}}{(p-k)^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(p-k)^\mu n^\nu + (p-k)^\nu n^\mu}{(p-k) \cdot n} \right]. \tag{3.44}
\end{aligned}$$

Actually, while one can use the light-front perturbation theory Feynman rules in [38] to do the calculations more quickly, we think it more clear, though a little bit tedious, to first use ordinary equal-time perturbation theory Feynman rules (see, e.g., Ref. [42]) in light-front gauge  $A^+ = 0$  and use light-front transverse cut-off to regularize the divergences we will encounter. The anomalous dimension is the coefficient of the divergent part. We will calculate  $T_{qq}^a$  detailedly to demonstrate the method we used and just show the results of  $T_{qq}^b$  and  $T_{qq}^c$  afterwards.

Doing some very simple algebra, we end up with the following expression for  $T_{qq}^a$

$$T_{qq}^a = \frac{-ig^2 C_F}{2p^+} \int \frac{d^4k}{(2\pi)^4} \text{tr}[\gamma_\nu \not{k} \gamma^+ \gamma_5 \not{k} \gamma_\mu u(ps) \bar{u}(ps)] \frac{g^{\mu\nu} - \frac{(k-p)^\mu n^\nu + (k-p)^\nu n^\mu}{(k-p) \cdot n}}{(k^2 + i\epsilon)^2 ((k-p)^2 + i\epsilon)}, \tag{3.45}$$

where  $C_F$  is the color factor mentioned before,  $C_F = \sum_{a=1}^8 t^a t^a = 4/3$  [42],  $\text{tr}$  represents the trace over the Dirac matrix. Now one can use the identity  $u(ps) \bar{u}(ps) = (\not{p} + m)(1 + \gamma_5 \not{\beta})/2$ , however, he must be careful when he try to apply the limit  $m \rightarrow 0$  because in light-front coordinates  $s_\mu = p_\mu/m$ , one has  $u(ps) \bar{u}(ps) = (\not{p} + m)(1 + \gamma_5 \not{\beta})/2 = (\not{p} + \gamma_5 \not{p} + m)/2$ . In the limit  $m \rightarrow 0$ , it goes to  $u(ps) \bar{u}(ps) = (\not{p} + \gamma_5 \not{p})/2$  not just  $\not{p}/2$ . Keeping this in mind and using  $p^- = n^+ = 0$ ,  $p^2 = n^2 = 0$ ,  $p \cdot n = 1$ , one gets

$$\begin{aligned}
T_{qq}^a &= \frac{8ig^2 C_F}{2p^+} \int \frac{d^4k}{(2\pi)^4} p^+ \left[ k^+ k^- + \frac{k^+}{k^+ - p^+} (2k^+ k^- - \vec{k}_\perp^2) - \frac{k^2}{2} \right] \\
&\times \frac{1}{(k^2 + i\epsilon)^2 [(k-p)^2 + i\epsilon]} \\
&= \frac{8ig^2 C_F}{(2\pi)^4 2p^+} \int dk^+ dk^- d^2 k_\perp p^+ \left[ k^+ k^- + \frac{k^+}{k^+ - p^+} (2k^+ k^- - \vec{k}_\perp^2) - \frac{k^2}{2} \right] \frac{1}{(2k^+)^2} \\
&\times \frac{1}{(k^- - \frac{\vec{k}_\perp^2}{2k^+} + i\epsilon)^2} \frac{1}{2(k^+ - p^+)} \left[ k^- - \frac{\vec{k}_\perp^2}{2(k^+ - p^+)} - i\epsilon \right] \tag{3.46}
\end{aligned}$$

The integrand for the integration  $\int dk^-$  is like  $F(k^-)/[k^- - \frac{\vec{k}_\perp^2}{2(k^+ - p^+)} - i\epsilon]$  and  $F(k^-) \sim$



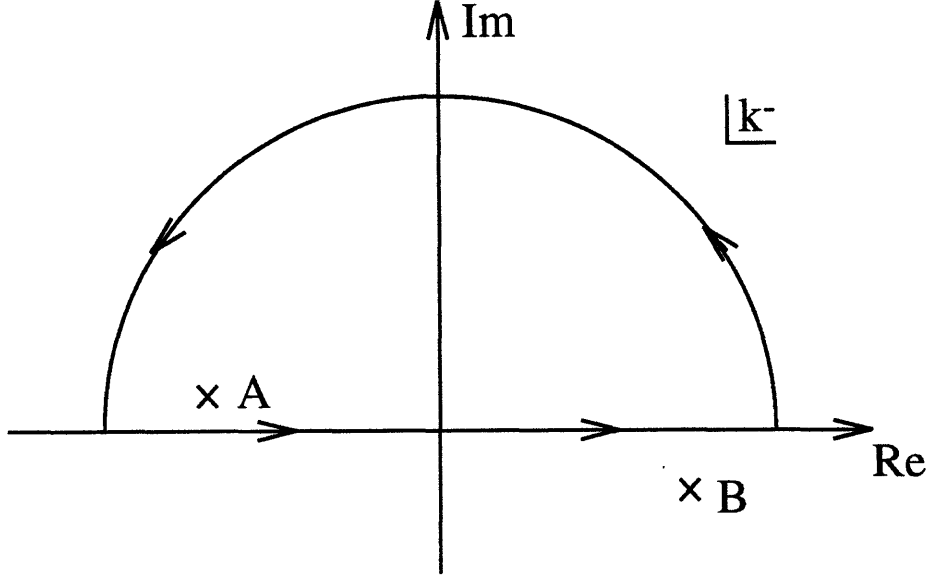


Figure 3-4: Pole positions in complex  $k^-$  plane.  $A$  represents the pole  $\vec{k}_\perp^2/2(k^+ - p^+)$ ,  $B$  the pole  $\vec{k}_\perp^2/2k^+$ .

$1/k^- \rightarrow 0$  as  $k^- \rightarrow \infty$ , therefore the integration along the arc is zero(see Fig. (3-4)).

One can use the residue theorem to do the  $dk^-$  integration, to get

$$T_{qq}^a = 2i\pi \frac{8ig^2 C_F}{(2\pi)^4 4p^+} \int dk^+ d^2 k_\perp \frac{1}{\vec{k}_\perp^2} \left[ \frac{k^+}{p^+} \left( \frac{1}{2} + \frac{p^+}{k^+ - p^+} \right) - \frac{1}{2} \right] \quad (3.47)$$

$$= \frac{\alpha_s}{2\pi} C_F \int_0^1 dx \frac{1+x^2}{1-x} \ln \frac{Q^2}{\mu^2} \quad (3.48)$$

where  $Q^2$  is a transverse momentum cut-off used to regularized the ultraviolet divergence, and  $\mu^2$  is an infrared cut-off which must be large enough so that perturbative QCD is valid,  $\alpha_s \equiv g^2/4\pi$ .  $x = k^+/p^+$  and  $0 < x < 1$  in light-front coordinates[38], therefore  $k^+ - p^+ < 0$ . This fact is also used to get right pole position in Eq. (3.46)(changing  $+i\epsilon$  to  $-i\epsilon$ ). This is important to get right answer.

Using same method one can calculate  $T_{qq}^b + T_{qq}^c$  (actually it is easy to show  $T_{qq}^b = T_{qq}^c$ ). The result is the following

$$T_{qq}^b + T_{qq}^c = \frac{\alpha_s}{2\pi} \int_0^1 dx \left( \frac{3}{2} - \frac{2}{1-x} \right) \ln \frac{Q^2}{\mu^2}. \quad (3.49)$$

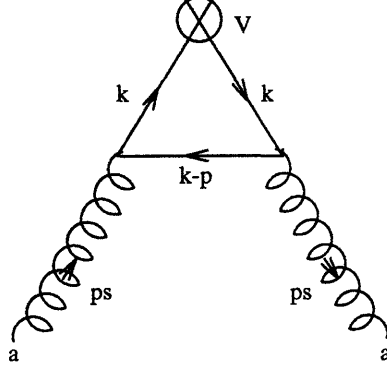


Figure 3-5: Feynman diagram for calculating  $\Gamma_{qg}$ , where  $V$  represents the vertex  $\gamma^+\gamma_5$ ,  $a$  is the color index of the gluon.

Therefore,

$$\begin{aligned}
T_{qq} &= T_{qq}^a + T_{qq}^b + T_{qq}^c \\
&= \frac{\alpha_s}{2\pi} \int_0^1 dx \left( \frac{1+x^2}{1-x} + \frac{3}{2} - \frac{2}{1-x} \right) \ln \frac{Q^2}{\mu^2} \\
&= 0 \ln \frac{Q^2}{\mu^2}.
\end{aligned} \tag{3.50}$$

As mentioned before, the coefficient of the divergent part, here  $\ln(Q^2/\mu^2)$ , gives us the anomalous dimension we want

$$\Gamma_{qq} = 0, \tag{3.51}$$

which is the upper-left corner element in AP splitting matrix.

### Quark Spin In A Polarized Gluon State: $\Gamma_{qg}$

To calculate  $\Gamma_{qg}$ , we now choose the state  $|P+\rangle$  in Eq. (3.39) to be polarized gluon state. Like in the last section, let  $T_{qg}$  represent Feynman amplitude for the Feynman diagram shown in Fig. (3-5),

$$\begin{aligned}
T_{qg} &= \frac{-2n_f}{2p^+} \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ (-ig\gamma^\nu t^a) \frac{i\not{k}}{k^2 + i\epsilon} \gamma^+ \gamma_5 \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma^\mu t^a) \frac{i(\not{k} - \not{p})}{(k-p)^2 + i\epsilon} \right] \\
&\quad \times \epsilon_\mu(ps) \epsilon_\nu^*(ps),
\end{aligned} \tag{3.52}$$

where the color index  $a$  is not being summed over, trace  $\text{tr}$  is acting on the color and Dirac space,  $C_A \equiv \text{tr}(t^a t^a) = 1/2$ ,  $2n_f$  comes from the flavor summation (quarks and antiquarks), minus sign appears because we have a fermion loop here,  $\epsilon_\mu(p_s)$  is the polarization vector of the gluon, which is

$$\epsilon_\mu(p_+) = \left( 0, 0, -\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}i \right) \quad (3.53)$$

for a positive-helicity gluon moving along  $\hat{e}_3$  direction [9, 38]. Finishing the trace over the Dirac matrices, we get

$$T_{gg} = \frac{4g^2}{2p^+} C_A 2n_f \int \frac{d^4 k}{(2\pi)^4} \left[ \epsilon^{\mu\nu\alpha+} k^2 k_\alpha + 2\epsilon^{\mu\nu\alpha\beta} k^+ p_\alpha k_\beta - \epsilon^{\mu\nu\alpha+} k^2 p_\alpha \right] \times \frac{\epsilon_\mu(p_s) \epsilon_\nu^*(p_s)}{(k^2 + i\epsilon)^2 [(k-p)^2 + i\epsilon]^2}, \quad (3.54)$$

Taking into account the fact that only the transverse components of  $\epsilon_\mu$  in Eq. (3.53) are non-zero,  $\mu, \nu$  can only be 1 or 2 so that

$$T_{gg} = \frac{\alpha_s}{2p^+ \pi^3} C_A 2n_f \epsilon^{\mu\nu-+} \int d^2 k_\perp dk^+ dk^- (k^2 k^+ + 2k^+ p^+ k^- - k^2 p^+) \times \frac{\epsilon_\mu(p_+) \epsilon_\nu^*(p_+)}{(k^2 + i\epsilon)^2 [(k-p)^2 + i\epsilon]^2}. \quad (3.55)$$

Now we can eliminate the  $dk^-$  integration by using the residue theorem like in last section. By doing that and some trivial algebra, we end up with the following expression,

$$T_{gg} = i \frac{\alpha_s}{2\pi^2} 2n_f \epsilon^{\mu\nu-+} \epsilon_\mu(p_+) \epsilon_\nu^*(p_+) \int \frac{d^2 k_\perp}{\vec{k}_\perp^2} \int_0^1 \frac{(1-2x)}{2}, \quad (3.56)$$

Plugging the polarization vector in Eq. (3.53), we get

$$\begin{aligned} T_{gg} &= i \frac{\alpha_s}{2\pi^2} 2n_f (\epsilon_1(p_+) \epsilon_2^*(p_+) - \epsilon_2(p_+) \epsilon_1^*(p_+)) \int \frac{d^2 k_\perp}{\vec{k}_\perp^2} \int_0^1 \frac{(1-2x)}{2} \\ &= \frac{\alpha_s}{2\pi} 2n_f \int_0^1 \frac{x^2 - (1-x)^2}{2} \ln \frac{Q^2}{\mu^2} \\ &= 0 \ln \frac{Q^2}{\mu^2}, \end{aligned} \quad (3.57)$$

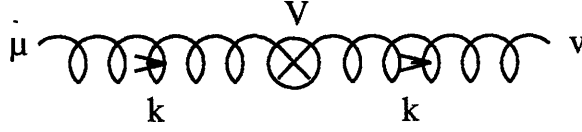


Figure 3-6: Vertex for gluon spin operator, where  $V$  represents the vertex  $2ik^+(g^{\mu 1}g^{\nu 2} - g^{\mu 2}g^{\nu 1})$ .

from which we get

$$\Gamma_{gg} = 0, \quad (3.58)$$

which is the upper-right corner element of the AP splitting matrix.

### 3.4.2 Gluon Spin

Recall the expression for gluon spin

$$\Delta g = \frac{\langle P+ | \int d^3x (A^1(x)\partial^+ A^2(x) - A^2(x)\partial^+ A^1(x)) | P+ \rangle}{\langle P+ | P+ \rangle} \quad (3.59)$$

in light-front gauge ( $A^+ = 0$ ). Using the translation invariance, one gets,

$$\Delta g = \frac{1}{2P^+} \langle P+ | (A^1(0)\partial^+ A^2(0) - A^2(0)\partial^+ A^1(0)) | P+ \rangle, \quad (3.60)$$

from which we get the vertex for gluon spin operator shown in Fig. (3-6)

$$V = 2ik^+(g^{\mu 1}g^{\nu 2} - g^{\mu 2}g^{\nu 1}). \quad (3.61)$$

#### Gluon Spin In A Polarized Quark State: $\Gamma_{gq}$

We now choose the state  $|P+\rangle$  in Eq. (3.60) to be polarized quark state. The Feynman amplitude  $T_{gq}$  in Fig. (3-7) is

$$\begin{aligned} T_{gq} = & \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \bar{u}(ps) (-ig\gamma_\nu t^a) \frac{i(\not{p} - \not{k})}{(p-k)^2 + i\epsilon} 2ik^+ [g_{\mu'1}g_{\nu'2} - g_{\mu'2}g_{\nu'1}] (-ig\gamma_\mu t^a) \\ & \times u(ps) \frac{i^2}{(k^2 + i\epsilon)^2} \left[ -g^{\mu\mu'} + \frac{k^\mu n^{\mu'} + k^{\mu'} n^\mu}{k \cdot n} \right] \left[ -g^{\nu\nu'} + \frac{k^\nu n^{\nu'} + k^{\nu'} n^\nu}{k \cdot n} \right]. \quad (3.62) \end{aligned}$$

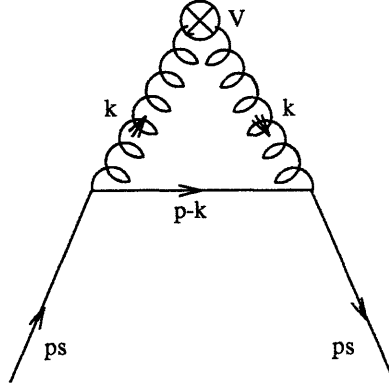


Figure 3-7: Feynman diagram for calculating  $\Gamma_{gq}$ , where  $V$  represents the vertex  $2ik^+(g^{\mu 1}g^{\nu 2} - g^{\mu 2}g^{\nu 1})$ .

Using exactly same methods in the previous two section, we get

$$\begin{aligned}
 T_{gq} &= \frac{\alpha_s}{2\pi} C_F \int_0^1 dx \frac{1 - (1-x)^2}{x} \ln \frac{Q^2}{\mu^2} \\
 &= \frac{\alpha_s}{2\pi} \frac{3}{2} C_F \ln \frac{Q^2}{\mu^2},
 \end{aligned} \tag{3.63}$$

which gives the anomalous dimension

$$\Gamma_{gq} = \frac{\alpha_s}{2\pi} \frac{3}{2} C_F, \tag{3.64}$$

which is exactly the lower-left corner element of the AP splitting matrix.

### Gluon Spin In A Polarized Gluon State: $\Gamma_{gg}$

To the leading-log approximation, the Feynman diagrams for calculating  $\Gamma_{gg}$  is shown in Fig. (3-8).

$$T_{gg} = T_{gg}^a + T_{gg}^b + T_{gg}^c + T_{gg}^d + T_{gg}^e \tag{3.65}$$

where  $T_{gg}^i$  ( $i = a, \dots, e$ ) represents the amplitudes from the five Feynman diagrams shown in Fig. (3-8), respectively, and

$$\begin{aligned}
 T_{gg}^a &= \frac{1}{2p^+} \int \frac{d^4 k}{(2\pi)^4} \epsilon^{\sigma\lambda}(ps) F_{\sigma\rho\nu'}^{adc'}(p, -k, k-p) D^{db'\rho\rho'}(k) 2ik^+ [g_{\rho'2}g_{\mu'1} - g_{\rho'1}g_{\mu'2}] \\
 &\quad \times D^{bb'\mu\mu'}(k) F_{\lambda\nu\mu}^{acb}(-p, p-k, k) D^{cc'\nu\nu'}(k-p) \epsilon^\lambda(ps),
 \end{aligned}$$

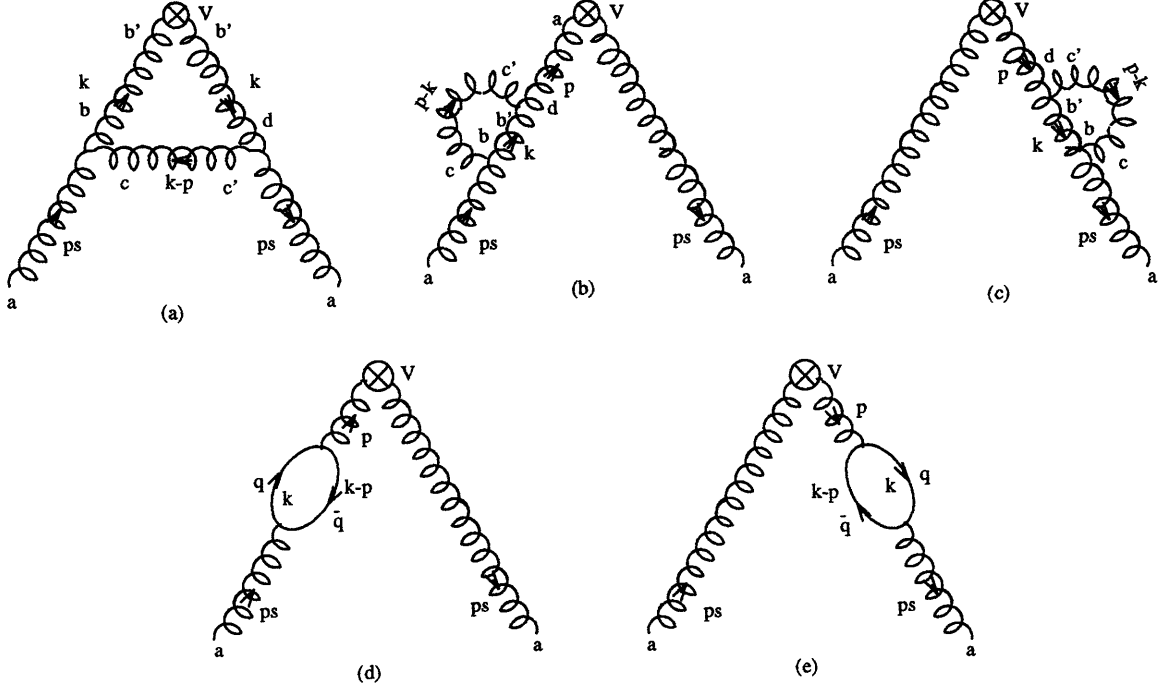


Figure 3-8: Feynman diagrams for calculating  $\Gamma_{gg}$ , where  $V$  represents the vertex  $2ik^+(g^{\mu 1}g^{\nu 2} - g^{\mu 2}g^{\nu 1})$ .

$$T_{gg}^b = \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \epsilon^{\mu*}(ps) 2ip^+ [g_{\mu 2}g_{\mu' 1} - g_{\mu 1}g_{\mu' 2}] D^{ad\rho\mu'}(p) F_{\rho\sigma\lambda}^{dc'b'}(p, k-p, -k) \\ \times D^{cc'\sigma\sigma'}(p-k) D^{bb'\lambda\lambda'}(k) F_{\nu\lambda'\sigma'}^{abc}(-p, p-k, k) \epsilon^\nu(ps),$$

$$T_{gg}^c = \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \epsilon^{\mu*}(ps) F_{\mu\sigma'\lambda'}^{acb}(p, k-p, -k) D^{bb'\lambda\lambda'}(k) D^{cc'\sigma\sigma'}(p-k) \\ \times F_{\rho\lambda\sigma}^{db'c'}(-p, p-k, k) D^{ad\rho\rho'}(p) \\ \times 2ip^+ [g_{\rho' 2}g_{\nu 1} - g_{\rho' 1}g_{\nu 2}] \epsilon^\nu(ps),$$

$$T_{gg}^d = \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \epsilon_\mu^*(ps) 2ik^+ [g^{\mu 2}g^{\mu' 1} - g^{\mu 1}g^{\mu' 2}] D_{\mu'\rho}^{ad}(p) n_f \\ \times (-) \text{tr} \left[ (-ig\gamma^\rho t^d) \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma^\nu t^a) \frac{i(\not{k} - \not{p})}{(k-p)^2 + i\epsilon} \right] \epsilon_\nu(ps),$$

$$T_{gg}^e = \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \epsilon_\mu^*(ps) (-) n_f \text{tr} \left[ (-ig\gamma^\mu t^a) \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma^\rho t^d) \frac{i(\not{k} - \not{p})}{(k-p)^2 + i\epsilon} \right]$$

$$\times D_{\rho\nu}^{ad}(p)2ip^+[g^{\nu'2}g^{\nu1} - g^{\nu'1}g^{\nu2}]\epsilon_\nu(ps) , \quad (3.66)$$

where

$$F_{\lambda\mu\nu}^{abc}(p_1, p_2, p_3) \equiv -gf^{abc} [(p_1 - p_2)_\nu g_{\lambda\mu} + (p_2 - p_3)_\lambda g_{\mu\nu} + (p_3 - p_1)_\mu g_{\nu\lambda}] \quad (3.67)$$

is the triple-gluon interaction vertex,  $f^{abc}$  is the structure constant of  $su(3)$  Lie algebra(see, e.g. [42]),  $D_{\mu\nu}^{ab}(k)$  represents the gluon propagator in Eq. (3.24), and polarized gluon state has been chosen.

Eq. (3.66) is much more complicated because of the triple-gluon vertex. Nevertheless, we can use essentially same method to get

$$\begin{aligned} T_{gg} &= \frac{\alpha_s}{2\pi} \int_0^1 \left[ \frac{1+x^4-(1-x)^3}{x} + \frac{x^4-1}{1-x} + \frac{1}{2} \left( 11 - \frac{2}{3}n_f \right) \right] \ln \frac{Q^2}{\mu^2} \\ &= \frac{\alpha_s}{2\pi} \frac{1}{2} \left( 11 - \frac{2}{3}n_f \right) \ln \frac{Q^2}{\mu^2} . \end{aligned} \quad (3.68)$$

Therefore the lower-right anomalous dimension in the AP splitting matrix is

$$\Gamma_{gg} = \frac{\alpha_s}{2\pi} \frac{1}{2} \left( 11 - \frac{2}{3}n_f \right) . \quad (3.69)$$

So far in this section we have shown how to calculate the AP splitting matrix for quark and gluon helicity[9], which we shall use to derive an equation for leading-log evolution of the quark and gluon orbital angular momentum similar to the AP equation. Before that, we would like to give a physical interpretation to the AP splitting matrix from point of view of conservation of angular momentum and helicity. Consider a parent quark with momentum  $p^\mu = (p^- = 0, p^+, p_\perp = 0)$  and helicity  $+1/2$ , splitting into a daughter gluon of momentum  $k^\mu = (k^-, xp^+, k_\perp)$  and a daughter quark with momentum  $(p-k)^\mu$ . Only 3-momentum is conserved during the splitting. The total probability for the splitting is,  $\int_0^1 dx(1+(1-x)^2)/x$ . [A multiplicative factor  $(\alpha_s/2\pi) \ln Q^2/\mu^2$  is implied when we talk about probability.] Since the quark helicity is conserved at the leading-log, we therefore have the item 0 in the upper-left corner of the AP splitting matrix. The helicity of the daughter gluon can either

be  $+1$  or  $-1$ . The probabilities for both cases are  $\int_0^1 dx 1/x$  and  $\int_0^1 dx(1-x)^2/x$ , respectively. The gluon helicity produced in the splitting is just  $\int_0^1 dx(1-(1-x)^2)/x = (3/2)C_F$ , which is the element in the lower-left corner of the AP splitting matrix. A similar discussion leads to the second columns of the AP splitting matrix. Let us remark here the physical origin for the well-known result that the gluon helicity increases logarithmically in the asymptotic limit [3]. When a gluon splits, there are four possible final states: 1) a quark with helicity  $1/2$  and an antiquark with helicity  $-1/2$ ; 2) a quark with helicity  $-1/2$  and an antiquark with helicity  $1/2$ ; 3) a gluon with helicity  $+1$  and another with helicity  $-1$ ; 4) two gluons with helicity  $+1$ . In the first two processes, there is a loss of the gluon helicity with probability  $(n_f/2) \int_0^1 dx(x^2 + (1-x)^2)$ . In the third process, there is also a loss of the gluon helicity, with probability  $\int_0^1 dx(x^3/(1-x) + (1-x)^3/x)$ . In the last process, there is a gain of the gluon helicity with probability  $\int_0^1 dx 1/(x(1-x))$ . When summed, the gluon helicity has a net gain with probability  $11/2 - n_f/3 = \beta_0/2$  in the splitting. Thus at increasingly smaller distance scales, gluons split consecutively and the helicity builds up logarithmically.



## 3.5 Evolution of Orbital Angular Momenta

The evolution of orbital angular momenta was first studied by Ratcliffe[40]. The result he got is as follows,

$$\frac{d}{dt}L_q = -\frac{1}{2}\frac{d}{dt}\Delta\Sigma, \quad \frac{d}{dt}L_g = -\frac{d}{dt}\Delta g, \quad (3.70)$$

or written in matrix form,

$$\frac{d}{dt} \begin{pmatrix} L_q \\ L_g \end{pmatrix} = \frac{\alpha_s(t)}{2\pi} \begin{pmatrix} 0 & 0 \\ -\frac{3}{2}C_F & -\frac{\beta_0}{2} \end{pmatrix} \begin{pmatrix} \Delta\Sigma \\ \Delta g \end{pmatrix}, \quad (3.71)$$

from which he concluded that orbital angular momentum plays an essential role in the nucleon spin. At the operator level, this means that the orbital angular momentum operators contain leading-twist contributions[7]. We agree with him that orbital angular momentum is important to understanding the nucleon spin. However, the result we got is

$$\frac{d}{dt} \begin{pmatrix} L_q \\ L_g \end{pmatrix} = \frac{\alpha_s(t)}{2\pi} \begin{pmatrix} -\frac{4}{3}C_F & \frac{n_f}{3} \\ \frac{4}{3}C_F & -\frac{n_f}{3} \end{pmatrix} \begin{pmatrix} L_q \\ L_g \end{pmatrix} + \frac{\alpha_s(t)}{2\pi} \begin{pmatrix} -\frac{2}{3}C_F & \frac{n_f}{3} \\ -\frac{5}{6}C_F & -\frac{11}{2} \end{pmatrix} \begin{pmatrix} \Delta\Sigma \\ \Delta g \end{pmatrix}, \quad (3.72)$$

which disagrees with Ratcliffe's. Especially, we have a homogeneous term which is new. (We call the first term on the right-hand side in Eq. (3.72) the *homogeneous* term and the second the *inhomogeneous* term.) In the following we will show the derivation of the evolution equation Eq. (3.72).

### 3.5.1 Off-Forward Matrix Element of Orbital Angular Momentum

Recall the expressions for orbital angular momentum  $L_q$  and  $L_g$  here

$$L_q = \langle P+ | \hat{L}_{3q} | P+ \rangle = \langle P+ | \int d^3x i\bar{\psi}\gamma^+(x^1\partial^2 - x^2\partial^1)\psi | P+ \rangle,$$

$$L_g = \langle P+ | \hat{L}_{3g} | P+ \rangle = \langle P+ | \int d^3x \partial^+ A^\alpha (x^2 \partial^1 - x^1 \partial^2) A_\alpha | P+ \rangle, \quad (3.73)$$

Because of the explicit factors of  $x^\mu$  in the definition of orbital angular momentum operators, their matrix elements are rather more singular than operators which may be more familiar, e.g.  $\Delta\Sigma$ , and more care must be taken with them. To deal with this kind of singularity, we start with less singular off-forward matrix elements and consider their forward limit[7].

Let's first consider the off-forward matrix element of a general local operator  $\hat{O}^\mu \equiv \int d^3x x^\mu \hat{A}(x)$ , which has explicit dependence of  $x$ ,

$$\begin{aligned} \langle p's | \hat{O}^\mu | ps \rangle &= \langle p's | \int d^3x x^\mu \hat{A}(x) | ps \rangle \\ &= \int d^3x x^\mu \langle p's | \hat{A}(x) | ps \rangle. \end{aligned} \quad (3.74)$$

Using the translation invariance of the operator of  $\hat{A}(x) = e^{i\hat{P}\cdot x} \hat{A}(0) e^{-i\hat{P}\cdot x}$ , and trading  $x^\mu$  for  $-i\partial/\partial p'_\mu$ , we have

$$\langle p's | \hat{O}^\mu | ps \rangle = -i \frac{\partial}{\partial p'_\mu} \delta^3(p' - p) \langle p's | \hat{A}(0) | ps \rangle. \quad (3.75)$$

The derivative on the  $\delta$  function means that when the distribution is convoluted with a test function, the derivative will be taken of the test function.

With the above recipe, we can now turn to calculate the splitting matrix for orbital angular momentum operators  $\hat{L}_{3q}$  and  $\hat{L}_{3g}$ . We consider the matrix elements of  $\hat{L}_{3q}$  and  $\hat{L}_{3g}$  in a parton state with non-vanishing transverse momentum. In a bare parton state, we have,

$$\langle p' + | \hat{L}_{3q} | p+ \rangle = 2p^+ (2\pi)^3 \left( -ip'_1 \frac{\partial}{\partial p'_2} + ip'_2 \frac{\partial}{\partial p'_1} \right) \delta^3(p' - p). \quad (3.76)$$

When calculating the matrix element in the composite parton states from the parton splitting, we have

$$\langle p' + | \hat{L}_{3q} | p+ \rangle = 2p^+ (2\pi)^3 \phi^q(p', p) \left( -ip'_1 \frac{\partial}{\partial p'_2} + ip'_2 \frac{\partial}{\partial p'_1} \right) \delta^3(p' - p). \quad (3.77)$$

When the distribution is convoluted with a test function, the derivative will be taken of  $\phi^q(p', p)$  and then of the test function. The first term represents generation of the orbital angular momentum from the parton helicity discussed previously. The second term has the same structure as the basic matrix element in Eq. (3.76) and represents the self-generation of orbital angular momentum in the splitting.

### 3.5.2 Quark Orbital Angular Momentum

The quark orbital angular momentum operator is

$$\hat{L}_{3q} = i \int d^3x \bar{\psi} \gamma^+ (x^1 \partial^2 - x^2 \partial^1) \psi . \quad (3.78)$$

Consider its off-forward matrix element in quark or gluon state

$$\begin{aligned} \langle p' + | \hat{L}_{3q} | p + \rangle &= \int d^3x x^1 \langle p' + | i \bar{\psi}(x) \gamma^+ \partial^2 \psi(x) | p + \rangle \\ &\quad - \int d^3x x^2 \langle p' + | i \bar{\psi}(x) \gamma^+ \partial^1 \psi(x) | p + \rangle \\ &= \int d^3x x^1 e^{i(p'-p) \cdot x} \langle p' + | i \bar{\psi}(0) \gamma^+ \partial^2 \psi(0) | p + \rangle \\ &\quad - \int d^3x x^2 e^{i(p'-p) \cdot x} \langle p' + | i \bar{\psi}(0) \gamma^+ \partial^1 \psi(0) | p + \rangle \\ &= (2\pi)^3 \left[ -i \frac{\partial}{\partial p'_1} \delta^3(p' - p) \langle p' + | i \bar{\psi}(0) \gamma^+ \partial^2 \psi(0) | p + \rangle \right. \\ &\quad \left. + i \frac{\partial}{\partial p'_2} \delta^3(p' - p) \langle p' + | i \bar{\psi}(0) \gamma^+ \partial^1 \psi(0) | p + \rangle \right] , \quad (3.79) \end{aligned}$$

where  $|p+\rangle$  is the quark or gluon state produced in the splitting. When it is convoluted with a test function, we get two terms: one is

$$\lim_{p' \rightarrow p} \left[ i \frac{\partial}{\partial p'_1} \langle p' + | i \bar{\psi}(0) \gamma^+ \partial^2 \psi(0) | p + \rangle - i \frac{\partial}{\partial p'_2} \langle p' + | i \bar{\psi}(0) \gamma^+ \partial^1 \psi(0) | p + \rangle \right] ,$$

which represents the generation of orbital angular momentum from the parton or gluon helicities; the other is

$$-\frac{1}{p_i} \langle p + | i \bar{\psi}(0) \gamma^+ \partial^i \psi(0) | p + \rangle ,$$

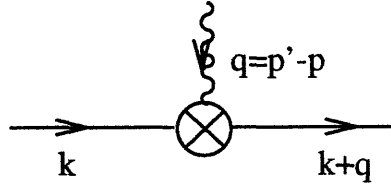


Figure 3-9: Vertex for quark orbital angular momentum operator.

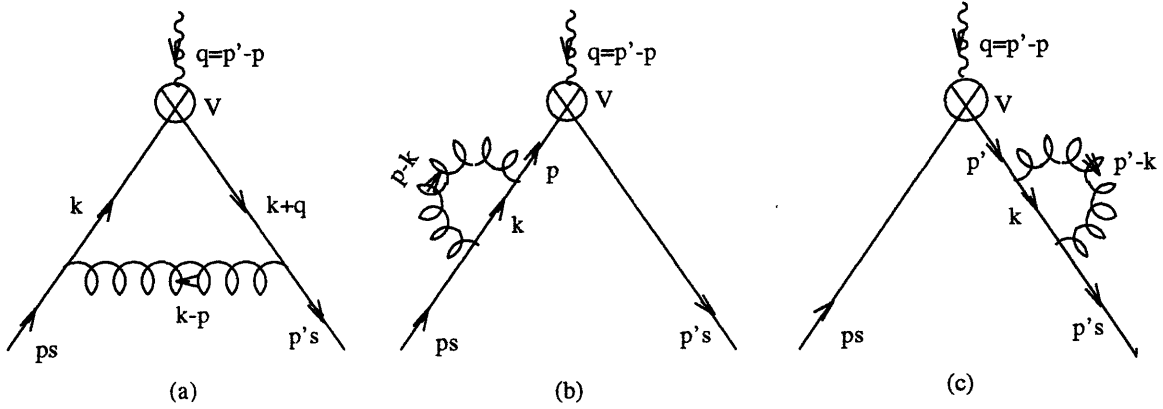


Figure 3-10: Feynman diagrams for calculating  $\Gamma_{L_{qq}}$ , where  $V$  represents the vertex  $\gamma^+ k^i$ .

which represents the self-generation of orbital angular momentum in the splitting.

Considering the matrix element  $\langle p' + | i\bar{\psi}(0)\gamma^+\partial^i\psi(0) | p + \rangle$  ( $i = 1, 2$ ), we can get Feynman rule for the vertex shown in Fig. (3-9),

$$V = \gamma^+ k^i \quad (3.80)$$

### Quark Orbital Angular Momentum In A Polarized Quark State: $\Gamma_{L_{qq}}$

The Feynman amplitude for calculating  $\Gamma_{L_{qq}}$  (shown in Fig. (3-10)) can be written as

$$\tilde{T}_{L_{qq}}^i = \tilde{T}_{L_{qq}}^{i,a} + \tilde{T}_{L_{qq}}^{i,b} + \tilde{T}_{L_{qq}}^{i,c} \quad (3.81)$$

$\Gamma_{L_{qq}}$  is the coefficient of the divergent part of the following quantity

$$T_{L_{qq}} = \lim_{p' \rightarrow p} \left( i \frac{\partial}{\partial p'_1} \tilde{T}_{L_{qq}}^2 - i \frac{\partial}{\partial p'_2} \tilde{T}_{L_{qq}}^1 \right) \quad (3.82)$$

Using the Feynman rules, we can write down these three pieces in Eq. (3.81) as follows

$$\begin{aligned}\tilde{T}_{Lqq}^{i,a} &= \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \bar{u}(p'+) (-ig\gamma_\nu t^a) \frac{i(\not{k} + \not{q})}{(k+q)^2 + i\epsilon} (\gamma^+ k^i) \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma_\mu t^b) \\ &\quad \times u(p+) \frac{-i\delta^{ab}}{(k-p)^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(k-p)^\mu n^\nu + (k-p)^\nu n^\mu}{(k-p) \cdot n} \right],\end{aligned}\tag{3.83}$$

$$\begin{aligned}\tilde{T}_{Lqq}^{i,b} &= \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \bar{u}(p'+) (\gamma^+ p^i) \frac{i\not{p}}{p^2 + i\epsilon} (-ig\gamma_\mu t^a) \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma_\nu t^b) u(p+) \\ &\quad \times \frac{-i\delta^{ab}}{(p-k)^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(p-k)^\mu n^\nu + (p-k)^\nu n^\mu}{(p-k) \cdot n} \right],\end{aligned}\tag{3.84}$$

$$\begin{aligned}\tilde{T}_{Lqq}^{i,c} &= \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \bar{u}(p'+) (-ig\gamma_\mu t^a) \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma_\nu t^b) \frac{i\not{p}'}{p'^2 + i\epsilon} (\gamma^+ p^i) u(p+) \\ &\quad \times \frac{-i\delta^{ab}}{(p'-k)^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(p'-k)^\mu n^\nu + (p'-k)^\nu n^\mu}{(p'-k) \cdot n} \right].\end{aligned}\tag{3.85}$$

To simplify the calculations, we will choose a specific frame so that  $p_x = 0$  and  $p_y = 0$  (we change notations  $p^1 \rightarrow p_x$ ,  $p^2 \rightarrow p_y$ ), in which  $\tilde{T}_{qq}^{i,b}$  and  $\tilde{T}_{qq}^{i,c}$  will be zero because they are proportional to  $p_x$  or  $p_y$ . Nevertheless,  $p'$  must have transverse momentum.

Because the momenta of the initial and final states are not equal,  $p \neq p'$ , we have to use explicit expression for the spinors instead of rearranging them to just do trace over Dirac matrix,

$$u(k+) = \frac{1}{\sqrt{2}\sqrt{\sqrt{2}k^+}} \begin{pmatrix} \sqrt{2}k^+ \\ k_x + ik_y \\ \sqrt{2}k^+ \\ k_x + ik_y \end{pmatrix}\tag{3.86}$$

in light-front coordinates[38]. Furthermore, because only transverse components of  $p'$  matter in this calculation and we will let  $p' = p$  in the end, we can let  $q^\pm = p'^\pm - p^\pm = 0$

in the calculations for simplicity.

Plugging Eq. (3.86) into Eq. (3.85) and doing some tedious but easy algebra, we get

$$\begin{aligned}
T_{L_{qq}} &\equiv \lim_{p' \rightarrow p} \left( i \frac{\partial}{\partial p'_x} \tilde{T}_{L_{qq}}^2 - i \frac{\partial}{\partial p'_y} \tilde{T}_{L_{qq}}^1 \right) \\
&= \frac{g^2 C_F}{2p^+ (2\pi)^4} \int d^2 k_\perp dk^+ dk^- \frac{-4i \vec{k}_\perp^2 (p^+ + k^+)}{(k^2 + i\epsilon)^2 [(k-p)^2 + i\epsilon]}, \quad (3.87)
\end{aligned}$$

which represents generation of the orbital angular momentum from the quark helicity.

Using residue theorem to do the  $dk^-$  integration, trading  $k^+$  for  $xp^+$ , and finishing  $d^2 k_\perp$  integration, we end up with

$$\begin{aligned}
T_{L_{qq}} &= \frac{\alpha_s}{2\pi} C_F \int_0^1 dx \left( (1-x)^2 - 1 \right) \ln \frac{Q^2}{\mu^2} \\
&= -\frac{2}{3} C_F \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2}. \quad (3.88)
\end{aligned}$$

The coefficient of  $\ln Q^2/\mu^2$  gives us the anomalous dimension,

$$\Gamma_{L_{qq}} = -\frac{2}{3} C_F \frac{\alpha_s}{2\pi}. \quad (3.89)$$

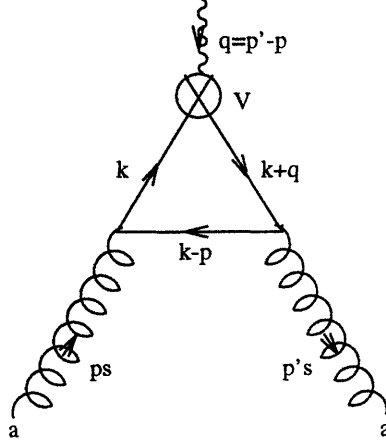


Figure 3-11: Feynman diagrams for calculating  $\Gamma_{L_{qg}}$ , where  $V$  represents the vertex  $\gamma^+ k^i$ .

### Quark Orbital Angular Momentum in A Polarized Gluon State: $\Gamma_{L_{qg}}$

The Feynman amplitude for calculating  $\Gamma_{L_{qg}}$  shown in Fig. (3-11) is

$$\tilde{T}_{L_{qg}}^i = \frac{-2n_f}{2p^+} \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ (-ig\gamma^\nu t^a) \frac{i(\not{k} + \not{q})}{(k+q)^2 + i\epsilon} (\gamma^+ k^i) \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma^\mu t^a) \frac{i(\not{k} - \not{p})}{(k-p)^2 + i\epsilon} \right] \cdot \epsilon_\mu(p^+) \epsilon_\nu^*(p'^+) , \quad (3.90)$$

where color index  $a$  is not summed over,  $2n_f$  comes from the summation of the quark flavors, minus sign appears because of fermion loop, and gluon polarization vector in light-front coordinates is as follows[38],

$$\begin{aligned} \epsilon_\mu(p^+) &= \left( 0, 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}i \right) \\ \epsilon_\mu(p'^+) &= \left( 0, -\frac{p'_x + ip'_y}{\sqrt{2}p'^+}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}i \right) . \end{aligned} \quad (3.91)$$

Plugging Eq. (3.91) into Eq. (3.90), we get

$$\begin{aligned} T_{L_{qg}} &= \lim_{p' \rightarrow p} \left( i \frac{\partial}{\partial p'_x} \tilde{T}_{L_{qg}}^2 - i \frac{\partial}{\partial p'_y} \tilde{T}_{L_{qg}}^1 \right) \\ &= \frac{g^2 C_A}{2p^+ (2\pi)^4} n_f \int d^2 k_\perp dk^+ dk^- \frac{-4i\vec{k}_\perp^2 \left( p^+ + 2\frac{k^+ k^-}{p^+} \right)}{(k^2 + i\epsilon)^2 [(k-p)^2 + i\epsilon]} \\ &= \frac{\alpha_s}{2\pi} C_A n_f \int_0^1 dx (1-x)(1+2x^2) \ln \frac{Q^2}{\mu^2} \end{aligned}$$

$$= \frac{n_f \alpha_s}{3} \frac{1}{2\pi} \ln \frac{Q^2}{\mu^2}. \quad (3.92)$$

Therefore, we get the anomalous dimension

$$\Gamma_{L_{qg}} = \frac{n_f \alpha_s}{3} \frac{1}{2\pi}. \quad (3.93)$$

### 3.5.3 Gluon Orbital Angular Momentum

The gluon orbital angular momentum operator in light-front coordinates can be written as follows,

$$\begin{aligned} \hat{L}_{3g} &= \int d^3x F^{+\alpha} (x^2 \partial^1 - x^1 \partial^2) A_\alpha \\ &= \int d^3x \partial^+ A^\alpha [x^1 (-\partial^2) - x^2 (-\partial^1)] A_\alpha. \end{aligned} \quad (3.94)$$

Apply the same arguments as in quark orbital angular momentum, we will get the Feynman rule (Fig. (3-12)) for the off-forward matrix element  $\langle p'+ | \partial^+ A^\alpha(0) (-\partial^i) A_\alpha(0) | p+ \rangle$  ( $i = 1, 2$ ),

$$V = [-k^i (k+q)^+ - (k+q)^i k^+] g^{\mu\nu}. \quad (3.95)$$

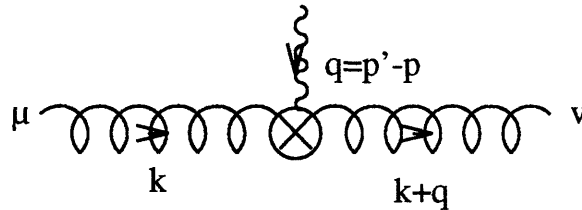


Figure 3-12: Vertex for gluon orbital angular momentum operator.

#### Gluon Orbital Angular Momentum In A Polarized Quark State: $\Gamma_{L_{gq}}$

The Feynman amplitude for calculating  $\Gamma_{L_{gq}}$  shown in Fig. (3-13) is

$$\tilde{T}_{L_{gq}}^i = \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \bar{u}(p'+) (-ig\gamma_\nu t^a) \frac{i(\not{p} - \not{k})}{(p-k)^2 + i\epsilon} [-k^i (k+q)^+ - (k+q)^i k^+] (-ig\gamma_\mu t^a)$$



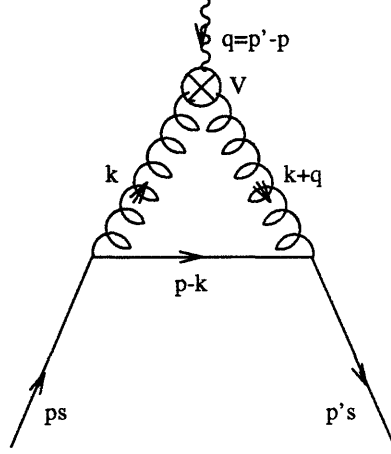


Figure 3-13: Feynman diagrams for calculating  $\Gamma_{L_{gq}}$ , where  $V$  represents the vertex  $-(k+q)^+k^i - k^+(k+q)^i$ .

$$\begin{aligned}
& \times u(p^+) \frac{i}{(k+q)^2 + i\epsilon} \left[ -g^{\mu\alpha} + \frac{(k+q)^\mu n^\alpha + (k+q)^\alpha n^\mu}{(k+q) \cdot n} \right] \\
& \times \frac{i}{k^2 + i\epsilon} \left[ -\delta_\alpha^\nu + \frac{k^\nu n_\alpha + k_\alpha n^\nu}{k \cdot n} \right]. \tag{3.96}
\end{aligned}$$

Using same procedures as those in the previous sections, we get

$$\begin{aligned}
T_{L_g L_q} &= \lim_{p' \rightarrow p} \left( i \frac{\partial}{\partial p'_x} \tilde{T}_{L_{gq}}^2 - i \frac{\partial}{\partial p'_y} \tilde{T}_{L_{gq}}^1 \right) \\
&= -\frac{g^2 C_F}{2p^+ (2\pi)^4} \int d^2 k_\perp dk^+ dk^- \frac{-4i \vec{k}_\perp^2 (1 - 2\frac{p^+}{k^+})}{(k^2 + i\epsilon)^2 [(k-p)^2 + i\epsilon]} \\
&= -\frac{\alpha_s}{2\pi} C_F \int_0^1 (1-x)(2-x) \ln \frac{Q^2}{\mu^2} \\
&= -\frac{5}{6} C_F \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2}, \tag{3.97}
\end{aligned}$$

from which we get the anomalous dimension  $\Gamma_{L_{gq}}$  as follows

$$\Gamma_{L_{gq}} = -\frac{5}{6} \frac{\alpha_s C_F}{2\pi}. \tag{3.98}$$

**Gluon Orbital Angular Momentum In A Polarized Gluon State:  $\Gamma_{L_{gg}}$**

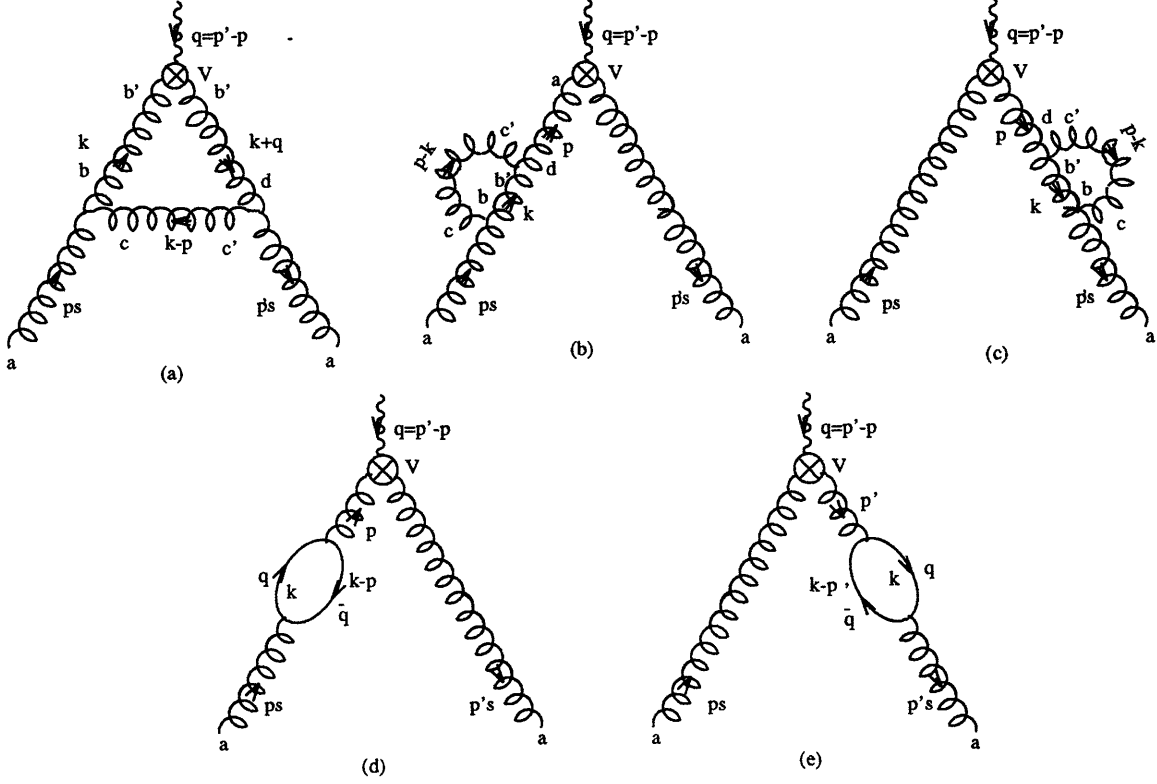


Figure 3-14: Feynman diagrams for calculating  $\Gamma_{L_g q}$ , where  $V$  represents the vertex  $-(k+q)^+ k^i - (k+q)^i k^+$ .

The Feynman amplitudes for calculating  $\Gamma_{L_g q}$  shown in Fig. (3-14) are

$$\tilde{T}_{L_g g}^i = \tilde{T}_{L_g g}^{ia} + \tilde{T}_{L_g g}^{ib} + \tilde{T}_{L_g g}^{ic} + \tilde{T}_{L_g g}^{id} + \tilde{T}_{L_g g}^{ie} \quad (3.99)$$

where

$$\begin{aligned} \tilde{T}_{L_g g}^{ia} &= \frac{1}{2p^+} \int \frac{d^4 k}{(2\pi)^4} \epsilon^{\sigma*}(p^+) F_{\sigma\rho\nu'}^{adc'}(p, -(k+q), k-p) D^{db'\rho\rho'}(k+q) \\ &\quad \times [-k^i(k+q)^+ - (k+q)^i k^+] D^{bb'\rho'\mu}(k) \\ &\quad \times F_{\lambda\nu\mu}^{acb}(-p, p-k, k) D^{cc'\nu\nu'}(k-p) \epsilon^\lambda(p^+) , \end{aligned}$$

$$\begin{aligned} \tilde{T}_{L_g g}^{ib} &= \frac{1}{2p^+} \int \frac{d^4 k}{(2\pi)^4} \epsilon_\mu^*(p^+) [-p^i p'^+ - p^+ p'^i] D^{ad\rho\mu}(p) F_{\rho\sigma\lambda}^{dc'b'}(p, k-p, -k) \\ &\quad \times D^{cc'\sigma\sigma'}(p-k) D^{bb'\lambda\lambda'}(k) F_{\nu\lambda'\sigma'}^{abc}(-p, p-k, k) \epsilon^\nu(p^+) , \end{aligned}$$

$$\begin{aligned}
\tilde{T}_{L_{gg}}^{ic} &= \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \epsilon^{\mu*}(p'+) F_{\mu\sigma'\lambda'}^{acb}(p', k-p', -k) D^{bb'\lambda\lambda'}(k) D^{cc'\sigma\sigma'}(p'-k) \\
&\quad \times F_{\rho\lambda\sigma}^{db'b'c'}(-p', p'-k, k) D^{ad\rho\rho'}(p) \\
&\quad \times [-p^i p'^+ - p^+ p'^i] \epsilon^\nu(p+) ,
\end{aligned}$$

$$\begin{aligned}
\tilde{T}_{L_{gg}}^{id} &= \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \epsilon^{\mu*}(p'+) [-p^i p'^+ - p^+ p'^i] D_{\mu\rho}^{ad}(p) n_f \\
&\quad \times (-) \text{tr} \left[ (-ig\gamma^\rho t^d) \frac{i\cancel{k}}{k^2 + i\epsilon} (-ig\gamma^\nu t^a) \frac{i(\cancel{k} - \not{p})}{(k-p)^2 + i\epsilon} \right] \epsilon_\nu(p+) ,
\end{aligned}$$

$$\begin{aligned}
\tilde{T}_{L_{gg}}^{ie} &= \frac{1}{2p^+} \int \frac{d^4k}{(2\pi)^4} \epsilon_\mu^*(p'+) (-) n_f \text{tr} \left[ (-ig\gamma^\mu t^a) \frac{i\cancel{k}}{k^2 + i\epsilon} (-ig\gamma^\rho t^d) \frac{i(\cancel{k} - \not{p})}{(k-p)^2 + i\epsilon} \right] \\
&\quad \times D_{\rho\nu}^{ad}(p) [-p^i p'^+ - p^+ p'^i] \epsilon^\nu(p+) , \tag{3.100}
\end{aligned}$$

Although these equations are complicated, we can use same procedures as those used previously to get

$$\begin{aligned}
T_{L_{gg}} &= \lim_{p' \rightarrow p} \left( i \frac{\partial}{\partial p'_x} \tilde{T}_{L_{gg}}^2 - i \frac{\partial}{\partial p'_y} \tilde{T}_{L_{gg}}^1 \right) \\
&= -\frac{11}{2} \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} . \tag{3.101}
\end{aligned}$$

The anomalous dimension is

$$\Gamma_{L_{gg}} = -\frac{11}{2} \frac{\alpha_s}{2\pi} . \tag{3.102}$$

### 3.5.4 Self-Generation of the Orbital Angular Momentum

So far in the previous sections we have calculated the inhomogeneous terms  $\Gamma_{L_{qq}}$ ,  $\Gamma_{L_{gq}}$ ,  $\Gamma_{L_{gq}}$  and  $\Gamma_{L_{gg}}$ , we now turn to calculate the homogeneous terms  $\Gamma_{L_q L_q}$ ,  $\Gamma_{L_q L_g}$ ,  $\Gamma_{L_g L_q}$  and  $\Gamma_{L_g L_g}$ , namely, to discuss self-generation of the orbital angular momentum.

**Self-generation of gluon orbital angular momentum in a polarized quark state:**  $\Gamma_{L_g L_q}$

All we need to calculate is the  $\phi(p, p)$  function in Eq. (3.77). The Feynman diagram is very similar to Fig. (3-3) except that the vertex is replaced by Eq. (3.95) (but now  $q = 0$ ), and momentum  $p$  must have transverse components.<sup>1</sup>

$$T_{L_g L_q}^i = \frac{1}{2p^+} \int \frac{d^4 k}{(2\pi)^4} \bar{u}(ps) (-ig\gamma_\nu t^a) \frac{i(\not{p} - \not{k})}{(p-k)^2 + i\epsilon} (-2k^i k^+) (-ig\gamma_\mu t^a) u(ps) \\ \times \frac{i^2}{(k^2 + i\epsilon)^2} \left[ -g^{\mu\mu'} + \frac{k^\mu n^{\mu'} + k^{\mu'} n^\mu}{k \cdot n} \right] \left[ -g^{\nu\nu'} + \frac{k^\nu n^{\nu'} + k^{\nu'} n^\nu}{k \cdot n} \right] \quad (3.103)$$

where  $T_{L_g L_q}^i = -p^i \phi_{gq}(p, p)$ .

Doing some simple algebra, eliminating  $dk^-$  integration by residue theorem, and replacing  $k^+$  with  $xp^+$ , we end up with

$$T_{L_g L_q}^i = -\frac{g^2 C_F}{8\pi^3} \int_0^1 dx \left[ \int d^2 k_\perp k^i \frac{1 + (1-x)^2}{x(\vec{k}_\perp - x\vec{p}_\perp)^2} \right]. \quad (3.104)$$

Making the shift  $\vec{k}_\perp \rightarrow \vec{k}'_\perp = \vec{k}_\perp - x\vec{p}_\perp$ , we get

$$T_{L_g L_q}^i = -\frac{g^2 C_F}{8\pi^3} \int d^2 k_\perp \frac{k^i}{\vec{k}_\perp^2} \int_0^1 dx \frac{1 + (1-x)^2}{x} \\ - \frac{g^2 C_F}{8\pi^3} p^i \int d^2 k_\perp \frac{1}{\vec{k}_\perp^2} \int_0^1 dx \frac{1 + (1-x)^2}{x}. \quad (3.105)$$

The first term in Eq. (3.105) will be vanishing because the integrand is odd when changing  $k^i$  to  $-k^i$ , therefore, we have

$$T_{L_g L_q}^i = -\frac{g^2 C_F}{8\pi^3} p^i \int d^2 k_\perp \frac{1}{\vec{k}_\perp^2} \int_0^1 dx \frac{1 + (1-x)^2}{x} \\ = -\frac{4}{3} C_F \frac{\alpha_s}{2\pi} p^i \ln \frac{Q^2}{\mu^2}. \quad (3.106)$$

---

<sup>1</sup>Physical picture is very clear here: if a free particle(plane wave) is moving along  $\hat{e}_3$ -direction(no transverse momentum), there is no orbital angular momentum with respect to that direction.

Therefore

$$\phi_{gq}(p, p) = \frac{4}{3} C_F \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2}, \quad (3.107)$$

from which we get the anomalous dimension

$$\Gamma_{L_g L_q} = \frac{4}{3} C_F \frac{\alpha_s}{2\pi}. \quad (3.108)$$

**Self-generation of quark orbital angular momentum in a polarized gluon state:**  $\Gamma_{L_q L_g}$

The Feynman diagram is very similar to Fig. (3-5) except that the vertex is replaced by Eq. (3.80) (but now  $q = 0$ ), and momentum  $p$  must have transverse components.

$$T_{L_q L_g}^i = \frac{-2n_f}{2p^+} \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ (-ig\gamma^\nu t^a) \frac{i\not{k}}{k^2 + i\epsilon} \gamma^+ k^i \frac{i\not{k}}{k^2 + i\epsilon} (-ig\gamma^\mu t^a) \frac{i(\not{k} - \not{p})}{(k-p)^2 + i\epsilon} \right] \times \epsilon_\mu(p+) \epsilon_\nu^*(p+), \quad (3.109)$$

where the color index  $a$  is not being summed over. Now instead of using explicit expression for  $\epsilon_\mu(p+)$  in Eq. (3.91),  $\epsilon_\mu(p+) \epsilon_\nu^*(p+)$  can be replaced by

$$\begin{aligned} \epsilon_\mu(p+) \epsilon_\nu^*(p+) &= \frac{1}{2} \sum_\lambda \epsilon_\mu(p\lambda) \epsilon_\nu^*(p\lambda) \\ &= \frac{1}{2} \left( -g_{\mu\nu} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n} \right). \end{aligned} \quad (3.110)$$

Doing some trivial algebra, doing  $dk^-$  integration, shifting  $\vec{k}_\perp$  to  $\vec{k}'_\perp = \vec{k}_\perp - x\vec{p}_\perp$ , and trading  $k^+$  for  $xp^+$ , we end up with,

$$T_{L_q L_g}^i = -\frac{\alpha_s}{2\pi} p^+ 2n_f C_A \int_0^1 dx x [(1-x)^2 + x^2] \ln \frac{Q^2}{\mu^2}, \quad (3.111)$$

therefore,

$$\begin{aligned} \phi_{gq}(p, p) &\equiv -\frac{1}{p^+} T_{L_q L_g}^i \\ &= \frac{1}{3} n_f \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2}, \end{aligned} \quad (3.112)$$

which gives us the anomalous dimension  $\Gamma_{L_q L_q}$

$$\Gamma_{L_q L_q} = \frac{n_f \alpha_s}{3 \cdot 2\pi} \quad (3.113)$$

Similar discussions will yield

$$\begin{aligned} \Gamma_{L_q L_q} &= -\frac{4}{3} C_F \frac{\alpha_s}{2\pi} \\ \Gamma_{L_g L_g} &= -\frac{n_f \alpha_s}{3 \cdot 2\pi} . \end{aligned} \quad (3.114)$$

### 3.5.5 Evolution Equation of Angular Momentum

Plugging all the anomalous dimensions we got into Eq. (3.28), we get the evolution equation to leading-log order of quark and gluon angular momenta as follows

$$\frac{d}{dt} \begin{pmatrix} \Delta\Sigma \\ \Delta G \\ L_q \\ L_g \end{pmatrix} = \Gamma_a \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{3}{2} C_F & \frac{\beta_0}{2} & 0 & 0 \\ -\frac{2}{3} C_F & \frac{n_f}{3} & -\frac{4}{3} C_F & \frac{n_f}{3} \\ -\frac{5}{6} C_F & -\frac{11}{2} & \frac{4}{3} C_F & -\frac{n_f}{3} \end{pmatrix} \begin{pmatrix} \Delta\Sigma \\ \Delta G \\ L_q \\ L_g \end{pmatrix} \quad (3.115)$$

The solution of this evolution equation can be obtained straightforwardly,

$$\begin{aligned} \Delta\Sigma(t) &= \text{const} , \\ \Delta g(t) &= -\frac{4\Delta\Sigma}{\beta_0} + \frac{t}{t_0} \left( \Delta g_0 + \frac{4\Delta\Sigma}{\beta_0} \right) \\ L_q(t) &= -\frac{1}{2}\Delta\Sigma + \frac{1}{2} \frac{3n_f}{16 + 3n_f} + (t/t_0)^{-2(16+3n_f)/(9\beta_0)} \left( L_q(0) + \frac{1}{2}\Delta\Sigma - \frac{1}{2} \frac{3n_f}{16 + 3n_f} \right) , \\ L_g(t) &= -\Delta g(t) + \frac{1}{2} \frac{16}{16 + 3n_f} + (t/t_0)^{-2(16+3n_f)/(9\beta_0)} \left( L_g(0) + \Delta g(0) - \frac{1}{2} \frac{16}{16 + 3n_f} \right) . \end{aligned} \quad (3.116)$$

Given a composition of the nucleon spin at some initial scale  $Q_0^2$ , the above equations yield the spin composition at any other perturbative scales in the leading-log approximation. From the expression for  $L_g(t)$ , it is clear that that the large gluon helicity at large  $Q^2$  is canceled by an equally large, but negative, gluon orbital angular

momentum.

Neglecting the sub-leading terms at large  $Q^2$ , we get,

$$\begin{aligned} J_q &= L_q + \frac{1}{2}\Delta\Sigma = \frac{1}{2} \frac{3n_f}{16 + 3n_f}, \\ J_g &= L_g + \Delta g = \frac{1}{2} \frac{16}{16 + 3n_f}. \end{aligned} \quad (3.117)$$

Thus partition of the nucleon spin between quarks and gluons follows the well-known partition of the nucleon momentum [10] ! Mathematically, one can understand this from the expression for the QCD angular momentum density  $M^{\mu\alpha\beta} = T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha$ . When  $M^{\mu\alpha\beta}$  and  $T^{\alpha\beta}$  are each separated into gluon and quark contributions, the anomalous dimensions of the corresponding terms are the same because they have the same short distance behavior.

It is interesting to speculate phenomenological consequences of this asymptotic partition of the nucleon spin. Assuming, as found in the case of the momentum sum rule[43], that the evolution in  $Q^2$  is very slow, then the above partition may still be roughly correct at low momentum scales, say,  $Q^2 \sim 3 \text{ GeV}^2$ . If this is the case, from the experimentally measured  $\Delta\Sigma$  we get an estimate of the quark orbital contribution at these scales,

$$L_q = 0.05 \sim 0.15. \quad (3.118)$$

To find a separation of the gluon contribution into spin and orbit parts, we need to know  $\Delta g$ , which shall be measurable in the future[44]. However, if  $Q^2$  variation is rapid, the asymptotic result implies nothing about the low  $Q^2$  spin structure of the nucleon. Unfortunately, no one knows yet how to measure  $L_q$  to determine the role of  $Q^2$  variation. In his recent paper, X.Ji related angular momenta of the nucleon to some form factors in the so-called deep virtual Compton scattering process[36]. He pointed out that some informations about the spin structure, especially orbital angular momenta, could be probed in this kind of process.

# Chapter 4

## Conclusions

To summarize, in this thesis we discussed the spin problem of the nucleon system, which has aroused so much interest in particle and nuclear community. To make this thesis more complete and readable, in chapter II, we gave a brief introduction to the so-called proton spin crisis aroused by the EMC experiments. We treated it as an introduction of some theoretical backgrounds. Then in chapter III, we mainly discussed the spin structure of nucleon in QCD. we derived an evolution equation for the quark and gluon orbital angular momenta in QCD. In the asymptotic limit, the solution of the evolution equation indicates that the quark and gluon contributions to the nucleon spin are the same as their contributions to the nucleon momentum. Furthermore, both the gluon orbital and helicity contributions grow logarithmically at large  $Q^2$  and with opposite sign, so their sum stays finite. We also discussed phenomenological implications of our results, in particular, about the size of the quark orbital angular momentum if, as found in the case of the momentum sum rule[43], the evolution in  $Q^2$  is very slow so that the partition we got in Eq. (3.117) may still be roughly correct at low momentum scales. Using the experimental value for  $\Delta\Sigma$  we get an estimate of the quark orbital contribution at the experimental energy scale,  $L_q = 0.05 \sim 0.15$ . However, if  $Q^2$  variation is rather rapid found by experiments in the future, the asymptotic result here implies nothing about the low  $Q^2$  spin structure of the nucleon. Nevertheless, orbital angular momentum does play very important role in nucleon spin, as we show here.



Nucleon spin crisis, on one hand, presents us a problem to be solved , on the other hand, provide us a special window to probe the inner structure of the hadron and explore the non-perturbative aspects of QCD. Nucleon spin problem have been one of the most active , exciting, and challenging problems in particle and nuclear physics. With more experimental results and theoretical developments coming out in the near future, it will be more and more exciting. I am very lucky and grateful to working with Professor Ji on such an interesting and exciting subject.

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