# Some subvarieties of the De Concini-Procesi compactification

by

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## Abstract

This thesis is concerned with the intrinsic structure of the De Concini-Procesi compactification of semi-simple adjoint algebraic groups and some relations with other topics of algebraic groups. In chapter 1, we study the closure of the totally positive part of an adjoint group in the group compactification. One result that we obtain is the positive property of the closure with respect to the canonical basis. In addition, we get an explicit description of the closure. As a consequence of the explicit description, the closure admits a cellular decomposition, which was first conjectured by Lusztig.

In chapter 2, we give a new proof of the parametrization of the totally positive part of flag variety which was first proved by Marsh and Rietsch using the generalized Chamber Ansatz. My proof is based on the theory of canonical basis.

The remaining chapters are related to the pieces of the group compactification introduced by Lusztig in the paper "Parabolic character sheaves II". In chapter 3, we study the closure of the unipotent variety in the group compactification, following the previous work of Lusztig and Springer. We show that the closure of the unipotent variety is the union of the unipotent variety itself together with finitely many pieces. By the same method, we also prove a similar result for the closure of arbitrary Steinberg fiber.

In chapter 4, we study the closure of any piece in the group compactification. We show that the closure is a union of some other pieces. We will also discuss the existence of cellular decomposition.

Chapter 1, 3 and 4 of this thesis are roughly based on the papers [H1], [H2] and [H3], in that order. Chapter 2 is based on an unpublished result. Each chapter can be read independently of the others.

Thesis Supervisor: George Lusztig
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# Contents

# Chapter 1

# Total positivity in the De Concini-Procesi Compactification

We study the nonnegative part $\overline{G_{>0}}$ of the De Concini-Procesi compactification of a semisimple algebraic group $G$, as defined by Lusztig. Using positivity properties of the canonical basis and parametrization of flag varieties, we will give an explicit description of $\overline{G_{>0}}$. This answers the question of Lusztig in *Total positivity and canonical bases*, Algebraic groups and Lie groups (ed. G.I. Lehrer), Cambridge Univ. Press, 1997, pp. 281-295. We will also prove that $\overline{G_{>0}}$ has a cell decomposition which was conjectured by Lusztig.

## 1.0. Introduction

Let $G$ be a connected split semisimple algebraic group of adjoint type over $\mathbf{R}$. We identify $G$ with the group of its $\mathbf{R}$-points. In [DP], De Concini and Procesi defined a compactification $\bar{G}$ of $G$ and decomposed it into strata indexed by the subsets of a finite set $I$. We will denote these strata by $\{Z_J \mid J \subset I\}$. Let $G_{>0}$ be the set of strictly totally positive elements of $G$ and $G_{\geqslant 0}$ be the set of totally positive elements of $G$ (see [L1]). We denote by $\overline{G_{>0}}$ the closure of $G_{>0}$ in $\bar{G}$. The main goal of this paper is to give an explicit description of $\overline{G_{>0}}$ (see 1.3.14). This answers the question in [L4, 9.4]. As a consequence, I will prove in 1.3.17 that $\overline{G_{>0}}$ has a cell decomposition which was conjectured by Lusztig.

To achieve our goal, it is enough to understand the intersection of $\overline{G_{>0}}$ with each stratum. We set $Z_{J,\geqslant 0} = \overline{G_{>0}} \cap Z_J$. Note that $Z_I = G$ and $Z_{I,\geqslant 0} = G_{\geqslant 0}$. We define $Z_{J,>0}$ as a certain subset of $Z_{J,\geqslant 0}$ analogous to $G_{>0}$ for $G_{\geqslant 0}$ (see 1.2.6). When $G$ is simply-laced, we will prove in 1.2.7 a criterion for $Z_{J,>0}$ in terms of its image in certain representations of $G$, which is analogous to the criterion for $G_{>0}$ in [L4, 5.4]. As Lusztig pointed out in [L2], although the definition of total positivity was elementary, many of the properties were proved in a non-elementary way, using canonical bases and their positivity properties. Our Theorem 1.2.7 is an example of this phenomenon. As a consequence, we will see in 1.2.9 that $Z_{J,\geqslant 0}$ is the closure of $Z_{J,>0}$ in $Z_J$.

Note that $Z_J$ is a fiber bundle over the product of two flag manifolds. Then understanding $Z_{J,\geqslant 0}$ is equivalent to understanding the intersection of $Z_{J,\geqslant 0}$ with each fiber. In 1.3.5, we will give a characterization of $Z_{J,\geqslant 0}$ which is analogous to the elementary fact that $G_{\geqslant 0} = \cap_{g \in G_{>0}} g^{-1} G_{>0}$. It allows us to reduce our problem to the problem of understanding certain subsets of some unipotent groups. Using the parametrization of the totally positive part of the flag varieties (see [MR]), we will give an explicit description of the subsets of $G$ (see 1.3.7). Thus our main theorem can be proved.

## 1.1 Preliminaries

**1.1.1** Let $G$ be a connected semi-simple algebraic group defined over an algebraically closed field $k$, with a fixed épinglage $(T, B^+, B^-, x_i, y_i; i \in I)$ (see [L1, 1.1]). Let $U^+, U^-$ be the unipotent radicals of $B^+, B^-$. Let $X$ (resp. $Y$) be the free abelian group of all homomorphism of algebraic groups $T \to k^\times$ (resp. $k^\times \to T$) and $\langle , \rangle : Y \times X \to \mathbf{Z}$ be the standard pairing. We write the operation in these groups as addition. For $i \in I$, let $\alpha_i \in X$ be the simple root such that $t x_i(a) t^{-1} = x_i(a)^{\alpha_i(t)}$ for all $a \in k, t \in T$ and let $\alpha_i^\vee \in Y$ be the simple coroot corresponding to $\alpha_i$. We also denote by $\omega_i$ and $\omega_i^\vee$ the corresponding fundamental weight and fundamental coweight. For any root $\alpha$, we denote by $U_\alpha$ the root subgroup corresponding to $\alpha$.

There is a unique isomorphism $\psi : G \xrightarrow{\sim} G^{\text{opp}}$ (the opposite group structure) such that $\psi\big(x_i(a)\big) = y_i(a)$, $\psi\big(y_i(a)\big) = x_i(a)$ for all $i \in I$, $a \in k$ and $\psi(t) = t$, for all $t \in T$.

If $P$ is a subgroup of $G$ and $g \in G$, we write ${}^gP$ instead of $gPg^{-1}$.

For any algebraic group $H$, we denote the Lie algebra of $H$ by $\text{Lie}(H)$ and the center of $H$ by $Z(H)$.

For any variety $X$ and an automorphism $\sigma$ of $X$, we denote the fixed point set of $\sigma$ on $X$ by $X^\sigma$.

For any group, We will write 1 for the identity element of the group.

For any finite set $X$, we will write $|X|$ for the cardinal of $X$.

**1.1.2** Let $N(T)$ be the normalizer of $T$ in $G$ and $\dot{s_i} = x_i(-1)y_i(1)x_i(-1) \in N(T)$ for $i \in I$. Set $W = N(T)/T$ and $s_i$ to be the image of $\dot{s_i}$ in $W$. Then $W$ together with $(s_i)_{i\in I}$ is a Coxeter group.

For $w \in W$, let $\text{supp}(w)$ be the set of simple roots whose associated simple reflections occur in a reduced expression of $w$.

Define an expression for $w \in W$ to be a sequence $\mathbf{w} = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ in $W$, such that $w_{(0)} = 1$, $w_{(n)} = w$ and for any $j = 1, 2, \ldots, n$, $w_{(j-1)}^{-1}w_{(j)} = 1$ or $s_i$ for some $i \in I$. An expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ is called reduced if $w_{(j-1)} < w_{(j)}$ for all $j = 1, 2, \ldots, n$. In this case, we will set $l(w) = n$. It is known that $l(w)$ is independent of the choice of the reduced expression. Note that if $\mathbf{w}$ is a reduced expression of $w$, then for all $j = 1, 2, \ldots, n$, $w_{(j-1)}^{-1}w_{(j)} = s_{i_j}$ for some $i_j \in I$. Sometimes we will simply say that $s_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression of $w$.

For $w \in W$, set $\dot{w} = \dot{s_{i_1}}\dot{s_{i_1}}\cdots \dot{s_{i_n}}$ where $s_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression of $w$. It is well known that $\dot{w}$ is independent of the choice of the reduced expression $s_{i_1}s_{i_2}\cdots s_{i_n}$ of $w$.

Assume that $\mathbf{w} = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ is a reduced expression of $w$ and $w_{(j)} = w_{(j-1)}s_{i_j}$ for all $j = 1, 2, \ldots, n$. Suppose that $v \leqslant w$ for the standard partial order in $W$. Then there is a unique sequence $\mathbf{v}_+ = (v_{(0)}, v_{(1)}, \ldots, v_{(n)})$ such that $v_{(0)} =$

$1, v_{(n)} = v, v_{(j)} \in \{v_{(j-1)}, v_{(j-1)}s_{i_j}\}$ and $v_{(j-1)} < v_{(j-1)}s_{i_j}$ for all $j = 1, 2, \ldots, n$ (see [MR, 3.5]). $\mathbf{v}_+$ is called the positive subexpression of $\mathbf{w}$. We define

$$J^+_{\mathbf{v}_+} = \{j \in \{1, 2, \ldots, n\} \mid v_{(j-1)} < v_{(j)}\},$$
$$J^\circ_{\mathbf{v}_+} = \{j \in \{1, 2, \ldots, n\} \mid v_{(j-1)} = v_{(j)}\}.$$

Then by the definition of $\mathbf{v}_+$, we have $\{1, 2, \ldots, n\} = J^+_{\mathbf{v}_+} \sqcup J^\circ_{\mathbf{v}_+}$.

**1.1.3** Let $\mathcal{B}$ be the variety of all Borel subgroups of $G$. For $B, B'$ in $\mathcal{B}$, there is a unique $w \in W$, such that $(B, B')$ is in the $G$-orbit on $\mathcal{B} \times \mathcal{B}$ (diagonal action) that contains $(B^+, {}^{\dot{w}}B^+)$. Then we write $\mathrm{pos}(B, B') = w$. The following properties follow from the definition and the properties of the Bruhat decomposition.

(1) If $\mathrm{pos}(B_1, B_2) = w$, then $\mathrm{pos}({}^gB_1, {}^gB_2) = w$ for all $g \in G$.

(2) If $\mathrm{pos}(B_1, B_2) = w, \mathrm{pos}(B_2, B_3) = v$ and $l(wv) = l(w) + l(v)$, then we have $\mathrm{pos}(B_1, B_3) = wv$.

(3) If $\mathrm{pos}(B_1, B_2) = w, \mathrm{pos}(B_2, B_3) = s_i$ for some $i \in I$, then $\mathrm{pos}(B_1, B_3) = w$ or $ws_i$.

For any subset $J$ of $I$, let $W_J$ be the subgroup of $W$ generated by $\{s_j \mid j \in J\}$ and let $w_0^J$ be the unique element of maximal length in $W_J$. (We will simply write $w_0^I$ as $w_0$.) Let $W^J$ (resp. ${}^JW$) be the set of minimal length coset representatives of $W/W_J$ (resp. $W_J \backslash W$). For $J, K \subset I$, we write ${}^JW^K$ for ${}^JW \cap W^K$. We denote by $P_J$ the subgroup of $G$ generated by $B^+$ and by $\{y_j(a) \mid j \in J, a \in \mathbf{R}\}$ and denote by $\mathcal{P}^J$ the variety of all parabolic subgroups of $G$ conjugated to $P_J$. It is easy to see that for any parabolic subgroup $P$, $P \in \mathcal{P}^J$ if and only if $\{\mathrm{pos}(B_1, B_2) \mid B_1, B_2$ are Borel subgroups of $P\} = W_J$.

For $P \in \mathcal{P}^J$, $Q \in \mathcal{P}^K$ and $u \in {}^JW^K$, we write $\mathrm{pos}(P, Q) = u$ if there exists $g \in G$, such that ${}^gP = P_J, {}^gQ = {}^{\dot{u}}P_K$.

**1.1.4** For any parabolic subgroup $P$ of $G$, define $U_P$ to be the unipotent radical of $P$ and $H_P$ to be the inverse image of the connected center of $P/U_P$ under $P \to P/U_P$.

If $B$ is a Borel subgroup of $G$, then so is

$$P^B = (P \cap B)U_P.$$

It is easy to see that for any $g \in H_P$, we have ${}^g(P^B) = P^B$. Moreover, $P^B$ is the unique Borel subgroup $B'$ in $P$ such that $\mathrm{pos}(B, B') \in W^J$ (see [L5, 3.2(a)]).

Let $P, Q$ be parabolic subgroups of $G$. We say that $P, Q$ are opposed if their intersection is a common Levi of $P, Q$. (We then write $P \bowtie Q$.) It is easy to see that if $P \bowtie Q$, then for any Borel subgroup $B$ of $P$ and $B'$ of $Q$, we have $\mathrm{pos}(B, B') \in W_J w_0$.

For any subset $J$ of $I$, define $J^* \subset I$ by $\{Q \mid Q \bowtie P \text{ for some } P \in \mathcal{P}^J\} = \mathcal{P}^{J^*}$. Then we have $(J^*)^* = J$. Let $Q_J$ be the subgroup of $G$ generated by $B^-$ and by $\{x_j(a) \mid j \in J, a \in \mathbf{R}\}$. We have $Q_J \in \mathcal{P}^{J^*}$ and $P_J \bowtie Q_J$. Moreover, for any $P \in \mathcal{P}^J$, we have $P =^g P_J$ for some $g \in G$. Thus $\psi(P) =^{\psi(g)^{-1}} Q_J \in \mathcal{P}^{J^*}$.

**1.1.5** In the rest of chapter 1 and chapter 2, we assume that $G$ be a connected semi-simple adjoint algebraic group defined and split over $\mathbf{R}$. We will also identify a real algebraic variety with the set of its $\mathbf{R}$-rational points.

Recall the following definitions from [L1].

For any $w \in W$, assume that $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression of $w$. Define $\phi^{\pm} : R^n_{\geqslant 0} \to U^{\pm}$ by

$$\phi^+(a_1, a_2, \dots, a_n) = x_{i_1}(a_1) x_{i_2}(a_2) \cdots x_{i_n}(a_n),$$
$$\phi^-(a_1, a_2, \dots, a_n) = y_{i_1}(a_1) y_{i_2}(a_2) \cdots y_{i_n}(a_n).$$

Let $U^{\pm}_{w,\geqslant 0} = \phi^{\pm}(R^n_{\geqslant 0}) \subset U^{\pm}$, $U^{\pm}_{w,>0} = \phi^{\pm}(R^n_{>0}) \subset U^{\pm}$. Then $U^{\pm}_{w,\geqslant 0}$ and $U^{\pm}_{w,>0}$ are independent of the choice of the reduced expression of $w$. We will simply write $U^{\pm}_{w_0,\geqslant 0}$ as $U^{\pm}_{\geqslant 0}$ and $U^{\pm}_{w_0,>0}$ as $U^{\pm}_{>0}$.

$T_{>0}$ is the submonoid of $T$ generated by the elements $\chi(a)$ for $\chi \in Y$ and $a \in \mathbf{R}_{>0}$.

$G_{\geqslant 0}$ is the submonoid $U^+_{\geqslant 0} T_{>0} U^-_{\geqslant 0} = U^-_{\geqslant 0} T_{>0} U^+_{\geqslant 0}$ of $G$.

$G_{>0}$ is the submonoid $U^+_{>0} T_{>0} U^-_{>0} = U^-_{>0} T_{>0} U^+_{>0}$ of $G_{\geqslant 0}$.

$\mathcal{B}_{>0}$ is the subset $\{{}^u B^- \mid u \in U^+_{>0}\} = \{{}^u B^+ \mid u \in U^-_{>0}\}$ of $\mathcal{B}$ and $\mathcal{B}_{\geqslant 0}$ is the closure of $\mathcal{B}_{>0}$ in the manifold $\mathcal{B}$.

For any subset $J$ of $I$, $\mathcal{P}^J_{>0} = \{P \in \mathcal{P}^J \mid \exists B \in \mathcal{B}_{>0}, \text{ such that } B \subset P\}$ and $\mathcal{P}^J_{\geqslant 0} = \{P \in \mathcal{P}^J \mid \exists B \in \mathcal{B}_{\geqslant 0}, \text{ such that } B \subset P\}$ are subsets of $\mathcal{P}^J$.

*Remark.* The fact that $\{{}^u B^- \mid u \in U^+_{>0}\} = \{{}^u B^+ \mid u \in U^-_{>0}\}$ was first proved by Lusztig in [L1] using the deep positivity properties of the canonical basis. In the appendix of chapter 2, I will give an elementary proof of this fact.

**1.1.6** For any $w, w' \in W$, define

$$\mathcal{R}_{w,w'} = \{B \in \mathcal{B} \mid \text{pos}(B^+, B) = w', \text{pos}(B^-, B) = w_0 w\}.$$

It is known that $\mathcal{R}_{w,w'}$ is nonempty if and only if $w \leqslant w'$ for the standard partial order in $W$(see [KL]). Now set

$$\mathcal{R}_{w,w',>0} = \mathcal{B}_{\geqslant 0} \cap \mathcal{R}_{w,w'}.$$

Then $\mathcal{R}_{w,w',>0}$ is a connected component of $\mathcal{R}_{w,w'}$ and is a semi-algebraic cell (see [R2, 2.8]). Furthermore, $\mathcal{B} = \bigsqcup_{w \leqslant w'} \mathcal{R}_{w,w'}$ and $\mathcal{B}_{\geqslant 0} = \bigsqcup_{w \leqslant w'} \mathcal{R}_{w,w',>0}$. Moreover, for any $u \in U^+_{w^{-1},>0}$, we have ${}^u\mathcal{R}_{w,w',>0} \subset \mathcal{R}_{1,w',>0}$ (see [R2, 2.2]).

Let $J$ be a subset of $I$. Define $\pi^J : \mathcal{B} \to \mathcal{P}^J$ to be the map which sends a Borel subgroup to the unique parabolic subgroup in $\mathcal{P}^J$ that contains the Borel subgroup. For any $w, w' \in W$ such that $w \leqslant w'$ and $w' \in W^J$, set $\mathcal{P}^J_{w,w'} = \pi^J(\mathcal{R}_{w,w'})$ and $\mathcal{P}^J_{w,w',>0} = \pi^J(\mathcal{R}_{w,w',>0})$. We have $\mathcal{P}^J_{\geqslant 0} = \bigsqcup_{w \leqslant w', w' \in W^J} \mathcal{P}^J_{w,w',>0}$ and $\pi^J \mid_{\mathcal{R}_{w,w',>0}}$ maps $\mathcal{R}_{w,w',>0}$ bijectively onto $\mathcal{P}^J_{w,w',>0}$ (see [R1, Chapter 4, 3.2]). Hence, for any $u \in U^+_{w^{-1},>0}$, we have ${}^u\mathcal{P}^J_{w,w',>0} = \pi^J({}^u\mathcal{R}_{w,w',>0}) \subset \pi^J(\mathcal{P}^J_{1,w',>0})$.

**1.1.7** Define $\pi_T : B^- B^+ \to T$ by $\pi_T(utu') = t$ for $u \in U^-, t \in T, u' \in U^+$. Then for $b_1 \in B^-, b_2 \in B^- B^+, b_3 \in B^+$, we have $\pi_T(b_1 b_2 b_3) = \pi_T(b_1)\pi_T(b_2)\pi_T(b_3)$.

Let $J$ be a subset of $I$. We denote by $\Phi^+_J$ the set of roots that are a linear combination of $\{\alpha_j \mid j \in J\}$ with nonnegative coefficients. We will simply write $\Phi^+_I$ as $\Phi^+$ and we will call a root $\alpha$ positive if $\alpha \in \Phi^+$. In this case, we will simply write

$\alpha > 0$. Define $U_J^+$ to be the subgroup of $U^+$ generated by $\{U_\alpha \mid \alpha \in \Phi_J^+\}$ and ${}'U_J^+$ to be the subgroup of $U^+$ generated by $\{U_\alpha \mid \alpha \in \Phi^+ - \Phi_J^+\}$. Then $U^- \times T \times' U_J^+ \times U_J^+$ is isomorphic to $B^-B^+$ via $(u,t,u_1,u_2) \mapsto utu_1u_2$. Now define $\pi_{U_J^+} : B^-B^+ \to U_J^+$ by $\pi_{U_J^+}(utu_1u_2) = u_2$ for $u \in U^-, t \in T, u_1 \in' U_J^+$ and $u_2 \in U_J^+$. (We will simply write $\pi_{U_I^+}$ as $\pi_{U^+}$.) Note that $U^-T \cdot U^-T'U_J^+ = U^-T'U_J^+$. Thus it is easy to see that for any $a, b \in G$ such that $a, ab \in B^-B^+$, we have $\pi_{U_J^+}(ab) = \pi_{U_J^+}(\pi_{U^+}(a)b)$. Since ${}'U_J^+$ is a normal subgroup of $U^+$, $\pi_{U_J^+}|_{U^+}$ is a homomorphism of $U^+$ onto $U_J^+$. Moreover, we have

$$\pi_{U_J^+}\big(x_i(a)\big) = \begin{cases} x_i(a), & \text{if } i \in J; \\ 1, & \text{otherwise.} \end{cases}$$

Thus $\pi_{U_J^+}(U_{>0}^+) = U_{w_0^J,>0}^+$ and $\pi_{U_J^+}(U_{\geqslant 0}^+) = U_{w_0^J,\geqslant 0}^+$.

Let $U_J^-$ be the subgroup of $U^-$ generated by $\{U_{-\alpha} \mid \alpha \in \Phi_J^+\}$ and ${}'U_J^-$ to be the subgroup of $U^-$ generated by $\{U_{-\alpha} \mid \alpha \in \Phi^+ - \Phi_J^+\}$. Then we define $\pi_{U_J^-} : U^- \to U_J^-$ by $\pi_{U_J^-}(u_1u_2) = u_1$ for $u_1 \in U_J^-, u_2 \in' U_J^-$. (We will simply write $\pi_{U_I^-}$ as $\pi_{U^-}$.) We have $\pi_{U_J^-}(U_{>0}^-) = U_{w_0^J,>0}^-$ and $\pi_{U_J^-}(U_{\geqslant 0}^-) = U_{w_0^J,\geqslant 0}^-$.

**1.1.8** For any vector space $V$ and a nonzero element $v$ of $V$, we denote the image of $v$ in $P(V)$ by $[v]$.

If $(V,\rho)$ is a representation of $G$, we denote by $(V^*,\rho^*)$ the dual representation of $G$. Then we have the standard isomorphism $St_V : V \otimes V^* \xrightarrow{\simeq} \mathrm{End}(V)$ defined by $St_V(v \otimes v^*)(v') = v^*(v')v$ for all $v, v' \in V, v^* \in V^*$. Now we have the $G \times G$ action on $V \otimes V^*$ by $(g_1,g_2) \cdot (v \otimes v^*) = (g_1v) \otimes (g_2v^*)$ for all $g_1, g_2 \in G, v \in V, v^* \in V^*$ and the $G \times G$ action on $\mathrm{End}(V)$ by $\big((g_1,g_2) \cdot f\big)(v) = g_1\big(f(g_2^{-1}v)\big)$ for all $g_1, g_2 \in G, f \in \mathrm{End}(V), v \in V$. The standard isomorphism between $V \otimes V^*$ and $\mathrm{End}(V)$ commutes with the $G \times G$ action. We will identify $\mathrm{End}(V)$ with $V \otimes V^*$ via the standard isomorphism.

## 1.2 The strata of the De Concini-Procesi Compactification

**1.2.1** Let $\mathcal{V}_G$ be the projective variety whose points are the $\dim(G)$-dimensional Lie subalgebras of $\mathrm{Lie}(G \times G)$. For any subset $J$ of $I$, define

$$Z_J = \{(P, Q, \gamma) \mid P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, \gamma = H_P g U_Q, P \bowtie^g Q\}$$

with the $G \times G$ action by $(g_1, g_2) \cdot (P, Q, H_P g U_Q) = \big({}^{g_1}P, {}^{g_2}Q, H_{{}^{g_1}P}(g_1 g g_2^{-1}) U_{{}^{g_2}Q}\big)$.

For $(P, Q, \gamma) \in Z_J$ and $g \in \gamma$, we set

$$H_{P,Q,\gamma} = \{(l + u_1, \mathrm{Ad}(g^{-1})l + u_2) \mid l \in \mathrm{Lie}(P \cap {}^gQ), u_1 \in \mathrm{Lie}(U_P), u_2 \in \mathrm{Lie}(U_Q)\}.$$

Then $H_{P,Q,\gamma}$ is independent of the choice of $g$ (see [L6, 12.2]) and is an element of $\mathcal{V}_G$ (see [L6, 12.1]). Moreover, $(P, Q, \gamma) \to H_{P,Q,\gamma}$ is an embedding of $Z_J \subset \mathcal{V}_G$ (see [L6, 12.2]). We will identify $Z_J$ with the subvariety of $\mathcal{V}_G$ defined above. Then we have $\bar{G} = \bigsqcup_{J \subset I} Z_J$, where $\bar{G}$ is the De Concini-Procesi compactification of $G$ (see [L6, 12.3]). We will call $\{Z_J \mid J \subset I\}$ the strata of $\bar{G}$ and $Z_I$ (resp. $Z_\varnothing$) the highest (resp. lowest) stratum of $\bar{G}$. It is easy to see that $Z_I$ is isomorphic to $G$ and $Z_\varnothing$ is isomorphic to $\mathcal{B} \times \mathcal{B}$.

Set $z_J^\circ = (P_J, Q_J, H_{P_J} U_{Q_J})$. Then $z_J^\circ \in Z_J$ (see 1.4) and $Z_J = (G \times G) \cdot z_J^\circ$.

Since $G$ is adjoint, we have an isomorphism $\chi : T \xrightarrow{\simeq} (\mathbf{R}^*)^I$ defined by $\chi(t) = \big(\alpha_i(t)^{-1}\big)_{i \in I}$. We denote the closure of $T$ in $\bar{G}$ by $\bar{T}$. We have $H_{P_J, Q_J, H_{P_J} U_{Q_J}} = \{(l + u_1, l + u_2) \mid l \in \mathrm{Lie}(P_J \cap Q_J), u_1 \in U_{P_J}, u_2 \in U_{Q_J}\}$. Moreover, for any $t \in Z(P_J \cap Q_J)$, $H_t$ is the subspace of $\mathrm{Lie}(G) \times \mathrm{Lie}(G)$ spanned by the elements $(l, l), (u_1, \mathrm{Ad}(t^{-1})u_1), (\mathrm{Ad}(t)u_2, u_2)$, where $l \in \mathrm{Lie}(P_J \cap Q_J), u_1 \in U_{P_J}, u_2 \in U_{Q_J}$. Thus it is easy to see that $z_J^\circ = \lim_{\substack{t_j = 1, \forall j \in J \\ t_j \to 0, \forall j \notin J}} \chi^{-1}\big((t_i)_{i \in I}\big) \in \bar{T}$.

**Proposition 1.2.2.** *The automorphism $\psi$ of the variety $G$ (see 1.1) can be extended in a unique way to an automorphism $\bar{\psi}$ of $\bar{G}$. Moreover, $\bar{\psi}(P, Q, \gamma) = \big(\psi(Q), \psi(P), \psi(\gamma)\big) \in Z_J$ for $J \subset I$ and $(P, Q, \gamma) \in Z_J$.*

*Proof.* The map $\psi : G \to G$ induces a bijective map $\psi : \text{Lie}(G) \to \text{Lie}(G)$. Moreover, we have $\psi(\text{Ad}(g)v) = \text{Ad}\big(\psi(g)^{-1}\big)\psi(v)$ and $\psi(v+v') = \psi(v)+\psi(v')$ for $g \in G, v, v' \in \text{Lie}(G)$. Now define $\delta : \text{Lie}(G) \times \text{Lie}(G) \to \text{Lie}(G) \times \text{Lie}(G)$ by $\delta(v, v') = \big(\psi(v'), \psi(v)\big)$ for $v, v' \in \text{Lie}(G)$. Then $\delta$ induces a bijection $\bar{\psi} : \mathcal{V}_G \to \mathcal{V}_G$.

Note that for any $g \in G$, we have $H_g = \{(v, \text{Ad}(g)v) \mid v \in \text{Lie}G\}$ and $\bar{\psi}(H_g) = \{(\text{Ad}(\psi(g)^{-1})\psi(v), \psi(v)) \mid v \in \text{Lie}(G)\} = H_{\psi(g)}$. Thus $\bar{\psi}$ is an extension of the automorphism $\psi$ of $G$ into $\mathcal{V}_G$.

Now for any $(P, Q, \gamma) \in Z_J$ and $g \in \gamma$, we have $\psi(P) \in \mathcal{P}^{J^*}, \psi(Q) \in \mathcal{P}^J$ and $\psi(Q) \bowtie^{\psi(g)} \psi(P)$ (see 1.4). Thus $\big(\psi(Q), \psi(P), \psi(\gamma)\big) \in Z_J$. Moreover,

$$\begin{aligned}\bar{\psi}(H_{P,Q,\gamma}) &= \{(\text{Ad}(\psi(g))\psi(l) + \psi(u_2), \psi(l) + \psi(u_1)) \mid l \in \text{Lie}(P \cap {}^gQ), \\ &\qquad\qquad\qquad\qquad u_1 \in \text{Lie}(U_P), u_2 \in \text{Lie}(U_Q)\} \\ &= \{(l + u_2, \text{Ad}(\psi(g)^{-1})l + u_1) \mid l \in \text{Lie}(\psi(Q) \cap^{\psi(g)} \psi(P)), \\ &\qquad\qquad\qquad\qquad u_1 \in \text{Lie}(\psi(U_P)), u_2 \in \text{Lie}(\psi(U_Q))\} \\ &= H_{\psi(Q),\psi(P),\psi(\gamma)}.\end{aligned}$$

Thus $\bar{\psi} \mid_{\bar{G}}$ is an automorphism of $\bar{G}$. Moreover, since $\bar{G}$ is the closure of $G$, $\bar{\psi} \mid_{\bar{G}}$ is the unique automorphism of $\bar{G}$ that extends the automorphism $\psi$ of $G$.

The proposition is proved. □

**1.2.3** For any $\lambda \in X$, set $\text{supp}(\lambda) = \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle \neq 0\}$.

In the rest of the section, I will fix a subset $J$ of $I$ and $\lambda_1, \lambda_2 \in X^+$ with $\text{supp}(\lambda_1) = I - J, \text{supp}(\lambda_2) = J$. Let $(V_{\lambda_1}, \rho_1)$ (resp. $(V_{\lambda_2}, \rho_2)$) be the irreducible representation of $G$ with the highest weight $\lambda_1$ (resp. $\lambda_2$). Assume that $\dim V_{\lambda_1} = n_1, \dim V_{\lambda_2} = n_2$ and $\{v_1, v_2, \dots, v_{n_1}\}$ (resp. $\{v'_1, v'_2, \dots, v'_{n_2}\}$) is the canonical basis of $(V_{\lambda_1}, \rho_1)$ (resp. $(V_{\lambda_2}, \rho_2)$), where $v_1$ and $v'_1$ are the highest weight vectors. Moreover, after reordering $\{2, 3, \dots, n_2\}$, we could assume that there exists some integer $n_0 \in \{1, 2, \dots, n_2\}$ such that for any $i \in \{1, 2, \dots, n_2\}$, the weight of $v'_i$ is of the form $\lambda_2 - \sum_{j \in J} a_j \alpha_j$ if and only if $i \leqslant n_0$.

Define $i_J : G \to P\big(\text{End}(V_{\lambda_1})\big) \times P\big(\text{End}(V_{\lambda_2})\big)$ by $i_J(g) = \Big([\rho_1(g)], [\rho_2(g)]\Big)$. Then since $\lambda_1 + \lambda_2$ is a dominant and regular weight, the closure of the image of $i_J$ in $P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big)$ is isomorphic to the De Concini-Procesi compactification of $G$ (See [DP, 4.1]). We will use $i_J$ as the embedding of $\bar{G}$ into $P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big)$. We will also identify $\bar{G}$ with its image under $i_J$.

**1.2.4** Now with respect to the canonical basis of $V_{\lambda_1}$ and $V_{\lambda_2}$, we will identify $\mathrm{End}(V_{\lambda_1})$ with $gl(n_1)$ and $\mathrm{End}(V_{\lambda_2})$ with $gl(n_2)$. Thus we will regard $\rho_1(g), \rho_1^*(g)$ as $n_1 \times n_1$ matrices and $\rho_2(g), \rho_2^*(g)$ as $n_2 \times n_2$ matrices. It is easy to see that (in terms of matrices) for any $g \in G, \rho_1^*(g) =^t \rho_1(g^{-1})$ and $\rho_2^*(g) =^t \rho_2(g^{-1})$, where ${}^tM$ is the transpose of the matrix $M$. Now for any $g_1, g_2 \in G$, $M_1 \in gl(n_1)$, $M_2 \in gl(n_2)$, $(g_1, g_2) \cdot M_1 = \rho_1(g_1) M_1 \rho_1(g_2^{-1})$ and $(g_1, g_2) \cdot M_2 = \rho_2(g_1) M_2 \rho_2(g_2^{-1})$.

Set $L = P_J \cap Q_J$. Then $L$ is a reductive algebraic group with the épinglage $(T, B^+ \cap L, B^- \cap L, x_j, y_j; j \in J)$. Now let $V_L$ be the subspace of $V_{\lambda_2}$ spanned by $\{v_1', v_2', \ldots, v_{n_0}'\}$ and $I_L = (a_{ij}) \in gl(n_2)$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \in \{1, 2, \ldots, n_0\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $V_L$ is an irreducible representation of $L$ with the highest weight $\lambda_2$ and canonical basis $\{v_1', v_2', \ldots, v_{n_0}'\}$. Moreover, $\lambda_2$ is a dominant and regular weight for $L$. Now set $I_1 = \mathrm{diag}(1, 0, 0, \ldots, 0) \in gl(n_1), I_2 = \mathrm{diag}(1, 0, 0, \ldots, 0) \in gl(n_2)$. Then

$$i_J(z_J^\circ) = \lim_{\substack{t_j = 1, \forall j \in J \\ t_j \to 0, \forall j \notin J}} i_J\Big(\chi^{-1}\big((t_i)_{i \in I}\big)\Big) = \Big([v_1 \otimes v_1^*], [\sum_{i=1}^{n_0} v_i' \otimes v_i'^*]\Big) = \Big([I_1], [I_L]\Big),$$

where $\{{v_1}^*, {v_2}^*, \ldots, {v_{n_1}}^*\}$ (resp. $\{{v_1'}^*, {v_2'}^*, \ldots, {v_{n_2}'}^*\}$) is the dual basis in $(V_{\lambda_1})^*$ (resp. $(V_{\lambda_2})^*$).

**1.2.5** Recall that $\mathrm{supp}(\lambda_1) = I - J$. Thus for any $P \in \mathcal{P}^J$, there is a unique $P$-stable line $L_{\rho_1(P)}$ in $(V_{\lambda_1}, \rho_1)$ and $P \mapsto L_{\rho_1(P)}$ is an embedding of $\mathcal{P}^J$ into $P(V_{\lambda_1})$. Similarly, for any $Q \in \mathcal{P}^{J^*}$, there is a unique $Q$-stable line $L_{\rho_1^*(Q)}$ in $(V_{\lambda_1}^*, \rho_1^*)$ and $Q \mapsto L_{\rho_1^*(Q)}$ is an embedding of $\mathcal{P}^{J^*}$ into $P(V_{\lambda_1}^*)$. It is easy to see $L_{\rho_1(P_J)} = [v_1]$, $L_{\rho_1^*(Q_J)} = [{v_1}^*]$ and

$L_{\rho_1({}^gP)} = \rho_1(g)L_{\rho_1(P)}, L_{\rho_1^*({}^gQ)} = \rho_1^*(g)L_{\rho_1^*(Q)}$ for $P \in \mathcal{P}^J, Q \in \mathcal{P}^{J^*}, g \in G$.

There are projections $p_1 : P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big) \to P\big(\mathrm{End}(V_{\lambda_1})\big)$ and $p_2 : P\big(\mathrm{End}(V_{\lambda_1})\big) \times P\big(\mathrm{End}(V_{\lambda_2})\big) \to P\big(\mathrm{End}(V_{\lambda_2})\big)$. It is easy to see that $p_1 \mid_{Z_J}$, $p_2 \mid_{Z_J}$ commute with the $G \times G$ action and $p_1(z_J^\circ) = [v_1 \otimes v_1{}^*] = [L_{\rho_1(P_J)} \otimes L_{\rho_1^*(Q_J)}]$. Now for any $g_1, g_2 \in G$, we have

$$p_1\big((g_1, g_2) \cdot z_J^\circ\big) = [\rho_1(g_1)L_{\rho_1(P_J)} \otimes \rho_1^*(g_2)L_{\rho_1^*(Q_J)}] = [L_{\rho_1({}^{g_1}P)} \otimes L_{\rho_1^*({}^{g_2}Q)}].$$

In other words, $p_1(z) = [L_{\rho_1(P)} \otimes L_{\rho_1^*(Q)}]$ for $z = (P, Q, \gamma) \in Z_J$.

**1.2.6** Let $\overline{G_{>0}}$ be the closure of $G_{>0}$ in $\bar{G}$. Then $\overline{G_{>0}}$ is also the closure of $G_{\geqslant 0}$ in $\bar{G}$. We have $z_J^\circ \in \overline{G_{>0}}$ (see 1.2.1). Now set

$$Z_{J,\geqslant 0} = Z_J \cap \overline{G_{>0}},$$
$$Z_{J,>0} = \{(g_1, g_2^{-1}) \cdot z_J^\circ \mid g_1, g_2 \in G_{>0}\}.$$

Since $\psi(G_{>0}) = G_{>0}$, we have $\bar{\psi}(\overline{G_{>0}}) = \overline{G_{>0}}$. Moreover, $\bar{\psi}(Z_J) = Z_J$ (see 2.2). Therefore $\bar{\psi}(Z_{J,\geqslant 0}) = Z_{J,\geqslant 0}$. Similarly, $(g_1, g_2^{-1}) \cdot Z_{J,\geqslant 0} \subset Z_{J,\geqslant 0}$ for any $g_1, g_2 \in G_{>0}$. Thus $Z_{J,>0} \subset Z_{J,\geqslant 0}$. Moreover, it is easy to see that $\bar{\psi}(Z_{J,>0}) = Z_{J,>0}$.

Note that for any $u_1, u_4 \in U^-_{>0}, u_2, u_3 \in U^+_{>0}, t, t' \in T_{>0}$, we have

$$\begin{aligned}(u_1 u_2 t, u_3^{-1} u_4^{-1} t') \cdot z_J^\circ &= (u_1 u_2, u_3^{-1} u_4^{-1}) \cdot (P_J, Q_J, H_{P_J} t t' U_{Q_J}) \\ &= (u_1, u_3^{-1}) \cdot \big(P_J, Q_J, H_{P_J} \pi_{U_J^+}(u_2) t t' \pi_{U_J^-}(u_4) U_{Q_J}\big).\end{aligned}$$

Thus

$$\begin{aligned}Z_{J,>0} &= \{(u_1, u_2^{-1}) \cdot (P_J, Q_J, H_{P_J} l U_{Q_J}) \mid u_1 \in U^-_{>0}, u_2 \in U^+_{>0}, l \in L_{>0}\} \\ &= \{(u_1' t, u_2'^{-1}) \cdot z_J^\circ \mid u_1' \in U^-_{>0}, u_2' \in U^+_{>0}, t \in T_{>0}\}.\end{aligned}$$

Moreover, for any $u_1, u_1' \in U^-, u_2, u_2' \in U^+$ and $t, t' \in T$, it is easy to see that $(u_1 t, u_2) \cdot z_J^\circ = (u_1' t', u_2') \cdot z_J^\circ$ if and only if $(u_1 t)^{-1} u_1' t' \in l H_{P_J} \cap B^- \subset l Z(L)$ and

$u_2^{-1}u_2' \in l^{-1}H_{Q_J} \cap U^+ \subset lZ(L)$ for some $l \in L$ , that is, $l \in Z(L)$, $u_1 = u_1', u_2 = u_2'$ and $t \in t'Z(L)$. Thus, $Z_{J,>0} \cong U^-_{>0} \times U^+_{>0} \times T_{>0}/\big(T_{>0} \cap Z(L)\big) \cong R_{>0}^{2l(w_0)+|J|}$.

Now I will prove a criterion for $Z_{J,>0}$.

**Theorem 1.2.7.** *Assume that $G$ is simply-laced. Let $z \in Z_{J,\geqslant 0}$. Then $z \in Z_{J,>0}$ if and only if $z$ satisfies the condition:*

*(*) $i_J(z) = \Big([M_1],[M_2]\Big)$ and $i_J\big(\bar{\psi}(z)\big) = \Big([M_3],[M_4]\Big)$ for some matrices $M_1, M_3 \in gl(n_1)$ and $M_2, M_4 \in gl(n_2)$ with all the entries in $\mathbf{R}_{>0}$.*

*Proof.* If $z \in Z_{J,>0}$, then $z = (g_1, g_2^{-1}) \cdot z_J^\circ$, for some $g_1, g_2 \in G_{>0}$. Assume that $g_1 \cdot v_1 = \sum_{i=1}^{n_1} a_i v_i$ and $g_2^{-1} \cdot v_1^* = \sum_{i=1}^{n_1} b_i v_i^*$. Then for any $i = 1, 2, \dots, n_1$, $a_i, b_i > 0$. Set $a_{ij} = a_i b_j$. Then $p_1(z) = [\rho_1(g_1) I_1 \rho_1(g_2)] = [(a_{ij})]$ is a matrix with all the entries in $\mathbf{R}_{>0}$.

We have $p_2(z) = [\rho_2(g_1) I_L \rho_2(g_2)] = [\rho_2(g_1) I_2 \rho_2(g_2) + \rho_2(g_1)(I_L - I_2)\rho_2(g_2)]$. Note that $\rho_2(g_1) I_2 \rho_2(g_2)$ is a matrix with all the entries in $\mathbf{R}_{>0}$ and $\rho_2(g_1)$, $\rho_2(g_2)$, $(I_L - I_2)$ are matrices with all the entries in $\mathbf{R}_{\geqslant 0}$. Thus $\rho_2(g_1)(I_L - I_2)\rho_2(g_2)$ is a matrix with all its entries in $\mathbf{R}_{\geqslant 0}$. So $\rho_2(g_1) I_L \rho_2(g_2)$ is a matrix with all the entries in $\mathbf{R}_{>0}$.

Similarly, $i_J\big(\bar{\psi}(z)\big) = \Big([M_3],[M_4]\Big)$ for some matrices $M_3, M_4$ with all their entries in $\mathbf{R}_{>0}$.

On the other hand, assume that $z$ satisfies the condition (*). Suppose that $z = (P, Q, \gamma)$ and $L_{\rho_1(P)} = [\sum_{i=1}^{n_1} a_i v_i]$, $L_{\rho_1^*(Q)} = [\sum_{i=1}^{n_1} b_i v_i^*]$. We may also assume that $a_{i_0} = b_{i_1} = 1$ for some integers $i_0, i_1 \in \{1, 2, \dots, n_1\}$.

Set $M = (a_{ij}) \in gL(n_1)$, where $a_{ij} = a_i b_j$ for $i, j \in \{1, 2, \dots, n_1\}$. Then $p_1(z) = [L_{\rho_1(P)} \otimes L_{\rho_1^*(Q)}] = [M]$. By the condition (*) and since $a_{i_0, i_1} = a_{i_0} b_{i_1} = 1$, we have that $M$ is a matrix with all its entries in $\mathbf{R}_{>0}$. In particular, for any $i \in \{1, 2, \dots, n_1\}, a_{i,i_1} = a_i > 0$. Therefore $L_{\rho_1(P)} = [\sum_{i=1}^{n_1} a_i v_i]$, where $a_i > 0$ for all $i \in \{1, 2, \dots, n_1\}$. By [R1, 5.1] (see also [L3, 3.4]), $P \in \mathcal{P}^J_{>0}$. Similarly, $\psi(Q) \in \mathcal{P}^J_{>0}$. Thus there exist $u_1 \in U^-_{>0}, u_2 \in U^+_{>0}$ and $l \in L$, such that $z = (u_1, u_2^{-1}) \cdot (P_J, Q_J, H_{P_J} l U_{Q_J})$.

We can express $u_1, u_2$ in a unique way as $u_1 = u_1' u_1''$, for some $u_1' \in' U_J^-$, $u_1'' \in U_J^-$ and $u_2 = u_2'' u_2'$, for some $u_2' \in' U_J^+$, $u_2'' \in U_J^+$ (see 1.1.7).

Recall that $V_L$ is the subspace of $V_{\lambda_2}$ spanned by $\{v'_1, v'_2, \ldots, v'_{n_0}\}$. Let $V'_L$ be the subspace of $V_{\lambda_2}$ spanned by $\{v'_{n_0+1}, v'_{n_0+2}, \ldots, v'_{n_2}\}$. Then $u\cdot v - v \in V'_L$ and $u \cdot V'_L \subset V'_L$, for all $v \in V_L$, $\alpha \notin \Phi_J^+$ and $u \in U_{-\alpha}$. Thus $u \cdot v - v \in V'_L$ and $u \cdot V'_L \subset V'_L$, for all $v \in V_L$ and $u \in' U_J^-$.

Similarly, let $V_L^*$ be the subspace of $V_{\lambda_2}^*$ spanned by $\{{v'_1}^*, {v'_2}^*, \ldots, {v'_{n_0}}^*\}$ and ${V'_L}^*$ be the subspace of $V_{\lambda_2}^*$ spanned by $\{{v'_{n_0+1}}^*, {v'_{n_0+2}}^*, \ldots, {v'_{n_2}}^*\}$. Then for any $v^* \in V_L^*$ and $u \in' U_J^+$, we have $u \cdot v - v \in {V'_L}^*$ and $u{V'_L}^* \subset {V'_L}^*$.

We define a map $\pi_L : gl(n_2) \to gl(n_0)$ by

$$\pi_L\big((a_{ij})_{i,j\in\{1,2,\ldots,n_2\}}\big) = (a_{ij})_{i,j\in\{1,2,\ldots,n_0\}}.$$

Then for any $u \in' U_J^-, u' \in' U_J^+$ and $M \in gl(n_2)$, we have $\pi_L\big((u,u')\cdot M\big) = \pi_L(M)$. Set $M_2 = \rho_2(u_1 l) I_L \rho_2(u_2)$ and $l' = u''_1 l u''_2 \in L$. Then

$$\begin{aligned}\pi_L(M_2) = \pi_L\Big((u_1, u_2^{-1}) \cdot \big(\rho_2(l) I_L\big)\Big) &= \pi_L\bigg((u'_1, {u'_2}^{-1}) \cdot \Big((u''_1, {u''_2}^{-1}) \cdot \big(\rho_2(l) I_L\big)\Big)\bigg) \\ = \pi_L\Big((u''_1, {u''_2}^{-1}) \cdot \big(\rho_2(l) I_L\big)\Big) &= \pi_L\big(\rho_2(l') I_L\big) = \rho_L(l').\end{aligned}$$

Since $p_2(z) = [M_2]$, $M_2$ is a matrix with all its entries nonzero. Therefore $\rho_L(l') = \pi_L(M_2)$ is a matrix with all its entries nonzero. Thus $l' = l_1 t_1 l_2$, for some $l_1 \in U^- \cap L, l_2 \in U^+ \cap L, t_1 \in T$.

Set $\widetilde{u_1} = u'_1 l_1$ and $\widetilde{u_2} = u'_2 l_2$. Then ${}^{\widetilde{u_1}}P_J = {}^{u_1({u''_1}^{-1} l_1)}P_J = {}^{u_1}P_J$. Similarly, we have ${}^{\widetilde{u_2}^{-1}}Q_J = {}^{u_2^{-1}}Q_J$. So $z = (\widetilde{u_1}, \widetilde{u_2}^{-1}) \cdot (P_J, Q_J, H_{P_J} t_1 U_{Q_J})$.

Now for any $i_0, j_0 \in \{1, 2, \ldots, n_1\}$, define a map $\pi^1_{i_0,j_0} : gl(n_1) \to \mathbf{R}$ by

$$\pi^1_{i_0,j_0}\big((a_{ij})_{i,j\in\{1,2,\ldots,n_1\}}\big) = a_{i_0,j_0}$$

and for any $i_0, j_0 \in \{1, 2, \ldots, n_2\}$, define a map $\pi^2_{i_0,j_0} : gl(n_2) \to \mathbf{R}$ by

$$\pi^2_{i_0,j_0}\big((a_{ij})_{i,j\in\{1,2,\ldots,n_2\}}\big) = a_{i_0,j_0}.$$

Now $z = (\widetilde{u_1} t_1, \widetilde{u_2}^{-1}) \cdot z_J^\circ$ and $\bar\psi(z) = \big(\psi(\widetilde{u_2}) t_1, \psi(\widetilde{u_1})^{-1}\big) \cdot z_J^\circ$.

Set

$$\tilde{M}_1 = \rho_1(\widetilde{u_1}t_1)I_1\rho_1(\widetilde{u_2}), \quad \tilde{M}_3 = \rho_1\big(\psi(\widetilde{u_2})t_1\big)I_1\rho_1\big(\psi(\widetilde{u_1})\big),$$
$$\tilde{M}_2 = \rho_2(\widetilde{u_1}t_1)I_L\rho_2(\widetilde{u_2}), \quad \tilde{M}_4 = \rho_2\big(\psi(\widetilde{u_2})t_1\big)I_1\rho_2\big(\psi(\widetilde{u_1})\big).$$

We have $\widetilde{u_1}\cdot v_1 = \sum_{i=1}^{n_1}\frac{\pi^1_{i,1}(\tilde{M}_1)}{\pi^1_{1,1}(\tilde{M}_1)}v_i$ and $\psi(\widetilde{u_2})\cdot v_1 = \sum_{i=1}^{n_1}\frac{\pi^1_{i,1}(\tilde{M}_3)}{\pi^1_{1,1}(\tilde{M}_3)}v_i$.

Moreover, let $V_0$ be the subspace of $V_{\lambda_2}$ spanned by $\{v'_2, v'_3, \dots, v'_{n_2}\}$ and $V_0{}^*$ be the subspace of $V^*_{\lambda_2}$ spanned by $\{{v'_2}^*, {v'_3}^*, \dots, {v'_{n_2}}^*\}$. Then we have $u\cdot V_0 \subset V_0$, for all $u \in U^-$ and $u'\cdot V_0^* \subset V_0^*$, for all $u' \in U^+$.

Thus for all $i = 1, 2, \dots, n_2$,

$$\begin{aligned}\pi^2_{i,1}(M_2) &= \pi^2_{i,1}\big(\rho_2(\widetilde{u_1}t_1)I_2\rho_2(\widetilde{u_2})\big) + \pi^2_{i,1}\big(\rho_2(\widetilde{u_1}t_1)(I_L - I_2)\rho_2(\widetilde{u_2})\big)\\ &= \pi^2_{i,1}\big(\rho_2(\widetilde{u_1}t_1)I_2\rho_2(\widetilde{u_2})\big).\end{aligned}$$

So $\widetilde{u_1}\cdot v'_1 = \sum_{i=1}^{n_2}\frac{\pi^2_{i,1}(\tilde{M}_2)}{\pi^2_{1,1}(\tilde{M}_2)}v'_i$ and $\psi(\widetilde{u_2})\cdot v'_1 = \sum_{i=1}^{n_2}\frac{\pi^2_{i,1}(\tilde{M}_4)}{\pi^2_{1,1}(\tilde{M}_4)}v'_i$. By [L2, 5.4], we have $\widetilde{u_1}, \psi(\widetilde{u_2}) \in U^-_{>0}$. Therefore to prove that $z \in Z_{J,>0}$, it is enough to prove that $t_1 \in T_{>0}Z(L)$, where $Z(L)$ is the center of $L$.

For any $g \in (U^-, U^+)\cdot\bar{T}$, $g$ can be expressed in a unique way as $g = (u_1, u_2)\cdot t$, for some $u_1 \in U^-$, $u_2 \in U^+$, $t \in \bar{T}$. Now define $\pi_{\bar{T}} : (U^-, U^+)\cdot\bar{T} \to \bar{T}$ by $\pi_{\bar{T}}\big((u_1, u_2)\cdot t\big) = t$ for all $u_1 \in U^-, u_2 \in U^+, t \in \bar{T}$. Note that $(U^-, U^+)\cdot\bar{T} \cap \overline{G_{>0}}$ is the closure of $G_{>0}$ in $(U^-, U^+)\cdot\bar{T}$. Then $\pi_{\bar{T}}\big((U^-, U^+)\cdot\bar{T} \cap \overline{G_{>0}}\big)$ is contained in the closure of $T_{>0}$ in $\bar{T}$. In particular, $\pi_{\bar{T}}(z) = t_1t_J$ is contained in the closure of $T_{>0}$ in $\bar{T}$. Therefore for any $j \in J$, $\alpha_j(t_1) > 0$. Now let $t_2$ be the unique element in $T$ such that

$$\alpha_j(t_2) = \begin{cases}\alpha_j(t_1), & \text{if } j \in J;\\ \alpha_j(t_1)^2, & \text{if } j \notin J.\end{cases}$$

Then $t_2 \in T_{>0}$ and $t_2^{-1}t_1 \in Z(L)$. The theorem is proved. □

*Remark.* Theorem 1.2.7 is analogous to the following statement in [L4, 5.4]: Assume that $G$ is simply laced and $V$ is the irreducible representation of $G$ with the highest weight $\lambda$, where $\lambda$ is a dominant and regular weight of $G$. For any $g \in G$, let $M(g)$ be the matrix of $g : V \to V$ with respect to the canonical basis of $V$. Then for any $g \in G$, $g \in G_{>0}$ if and only if $M(g)$ and $M\big(\psi(g)\big)$ are matrices with all the entries in $\mathbf{R}_{>0}$.

**1.2.8** Before proving Corollary 1.2.9, I will introduce some technical tools.

Since $G$ is adjoint, there exists (in an essentially unique way) $\tilde{G}$ with the épinglage $(\tilde{T}, \tilde{B}^{+}, \tilde{B}^{-}, \tilde{x}_{\tilde{i}}, \tilde{y}_{\tilde{i}}; \tilde{i} \in \tilde{I})$ and an automorphism $\sigma : \tilde{G} \to \tilde{G}$ (over $\mathbf{R}$) such that the following conditions are satisfied.

(a) $\tilde{G}$ is connected semisimple adjoint algebraic group defined and split over $\mathbf{R}$.

(b) $\tilde{G}$ is simply laced.

(c) $\sigma$ preserves the épinglage, that is, $\sigma(\tilde{T}) = \tilde{T}$ and there exists a permutation $\tilde{i} \to \sigma(\tilde{i})$ of $\tilde{I}$, such that $\sigma\big(\tilde{x}_{\tilde{i}}(a)\big) = \tilde{x}_{\sigma(\tilde{i})}(a), \sigma\big(\tilde{y}_{\tilde{i}}(a)\big) = \tilde{y}_{\sigma(\tilde{i})}(a)$ for all $\tilde{i} \in \tilde{I}$ and $a \in \mathbf{R}$.

(d) If $\tilde{i}_1 \neq \tilde{i}_2$ are in the same orbit of $\sigma : \tilde{I} \to \tilde{I}$, then $\tilde{i}_1, \tilde{i}_2$ do not form an edge of the Coxeter graph.

(e) $\tilde{i}$ and $\sigma(\tilde{i})$ are in the same connected component of the Coxeter graph, for any $\tilde{i} \in \tilde{I}$.

(f) There exists an isomorphism $\phi : \tilde{G}^{\sigma} \to G$ (as algebraic groups over $\mathbf{R}$) which is compatible with the épinglage of $G$ and the épinglage $(\tilde{T}^{\sigma}, \tilde{B}^{+\sigma}, \tilde{B}^{-\sigma}, \tilde{x}_p, \tilde{y}_p; p \in \bar{I})$ of $\tilde{G}^{\sigma}$, where $\bar{I}$ is the set of orbit of $\sigma : \tilde{I} \to \tilde{I}$ and $\tilde{x}_p(a) = \prod_{\tilde{i} \in p} \tilde{x}_{\tilde{i}}(a), \tilde{y}_p(a) = \prod_{\tilde{i} \in p} \tilde{y}_{\tilde{i}}(a)$ for all $p \in \bar{I}$ and $a \in \mathbf{R}$.

Let $\lambda$ be a dominant and regular weight of $\tilde{G}$ and $(V, \rho)$ be the irreducible representation of $\tilde{G}$ with highest weight $\lambda$. Let $\overline{\tilde{G}}$ be the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}\}$ in $P\big(\mathrm{End}(V)\big)$ and $\overline{\tilde{G}^{\sigma}}$ be the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}^{\sigma}\}$ in $P\big(\mathrm{End}(V)\big)$. Then since $\lambda$ is a dominant and regular weight of $\tilde{G}$ and $\lambda \mid_{\tilde{T}^{\sigma}}$ is a dominant and regular weight of $\tilde{G}^{\sigma}$, we have that $\overline{\tilde{G}}$ is the De Concini-Procesi compactification of $\tilde{G}$ and $\overline{\tilde{G}^{\sigma}}$ is the De Concini-Procesi compactification of $\tilde{G}^{\sigma}$. Since $\overline{\tilde{G}}$ is closed in $P\big(\mathrm{End}(V)\big)$, $\overline{\tilde{G}^{\sigma}}$ is the closure of $\{[\rho(\tilde{g})] \mid \tilde{g} \in \tilde{G}^{\sigma}\}$ in $\overline{\tilde{G}}$.

We have $\overline{\tilde{G}} = \bigsqcup_{\tilde{J} \subset \tilde{I}} \tilde{Z}_{\tilde{J}} = \bigsqcup_{\tilde{J} \subset \tilde{I}} (\tilde{G} \times \tilde{G}) \cdot \tilde{z}_{\tilde{J}}^{\circ}$ and $\overline{\tilde{G}^{\sigma}} = \bigsqcup_{\tilde{J} \subset \tilde{I}, \sigma \tilde{J} = \tilde{J}} (\tilde{G}^{\sigma} \times \tilde{G}^{\sigma}) \cdot \tilde{z}_{\tilde{J}}^{\circ}$.

Moreover, $\sigma$ can be extended in a unique way to an automorphism $\bar{\sigma}$ of $\overline{\tilde{G}}$. Since $\overline{\tilde{G}}^{\bar{\sigma}} = \bigsqcup_{\tilde{J}\subset\tilde{I},\sigma\tilde{J}=\tilde{J}}(\tilde{Z}_{\tilde{J}})^{\bar{\sigma}}$ is a closed subset of $\overline{\tilde{G}}$ containing $\tilde{G}^{\sigma}$, we have $\overline{\tilde{G}^{\sigma}} \subset \bigsqcup_{\tilde{J}\subset\tilde{I},\sigma\tilde{J}=\tilde{J}}(\tilde{Z}_{\tilde{J}})^{\bar{\sigma}}$.

By the condition (f), there exists a bijection $\phi$ between $\bar{I}$ and $I$, such that $\phi\big(\tilde{x}_p(a)\big) = x_{\phi(p)}(a)$, for all $p \in \bar{I}, a \in \mathbf{R}$. Moreover, the isomorphism $\phi$ from $\tilde{G}^{\sigma}$ to $G$ can be extended in a unique way to an isomorphism $\bar{\phi} : \overline{\tilde{G}^{\sigma}} \to \bar{G}$. It is easy to see that for any $\tilde{J} \subset \tilde{I}$ with $\sigma\tilde{J} = \tilde{J}$, we have $\bar{\phi}\big((\tilde{G}^{\sigma} \times \tilde{G}^{\sigma}) \cdot \tilde{z}^{\circ}_{\tilde{J}}\big) = Z_{\phi\circ\pi(\tilde{J})}$, where $\pi : \tilde{I} \to \bar{I}$ is the map sending element of $\tilde{I}$ into the $\sigma$-orbit that contains it.

**Corollary 1.2.9.** *$Z_{J,\geqslant 0} = \bigcap_{g_1,g_2\in G_{>0}}(g_1^{-1}, g_2) \cdot Z_{J,>0}$ is the closure of $Z_{J,>0}$ in $Z_J$. As a consequence, $Z_{J,\geqslant 0}$ and $\overline{G_{>0}}$ are contractible.*

*Proof.* I will prove that $Z_{J,\geqslant 0} \subset \bigcap_{g_1,g_2\in G_{>0}}(g_1^{-1}, g_2) \cdot Z_{J,>0}$.

First, assume that $G$ is simply laced.

For any $g \in G_{>0}$, $i_J(g) = \Big([\rho_1(g)], [\rho_2(g)]\Big)$, where $\rho_1(g)$ and $\rho_2(g)$ are matrices with all the entries in $\mathbf{R}_{>0}$. Then for any $z \in Z_{J,\geqslant 0}$, we have $i_J(z) = \Big([M_1], [M_2]\Big)$ for some matrices with all the entries in $\mathbf{R}_{\geqslant 0}$. Similarly, $i_J\big(\bar{\psi}(z)\big) = \Big([M_3], [M_4]\Big)$ for some matrices with all their entries in $\mathbf{R}_{\geqslant 0}$.

Note that for any $M'_1, M'_2, M'_3 \in gl(n)$ such that $M'_1, M'_3$ are matrices with all their entries in $\mathbf{R}_{>0}$ and $M'_2$ is a nonzero matrix with all the entries in $\mathbf{R}_{\geqslant 0}$, we have that $M'_1M'_2M'_3$ is a matrix with all the entries in $\mathbf{R}_{>0}$. Thus for any $g_1, g_2 \in G_{>0}$, we have that $(g_1, g_2^{-1}) \cdot z$ satisfies the condition (*) in 1.2.7. Moreover, $(g_1, g_2^{-1}) \cdot z \in Z_{J,\geqslant 0}$. Therefore by 1.2.7, $(g_1, g_2^{-1}) \cdot z \in Z_{J,>0}$ for all $g_1, g_2 \in G_{>0}$.

In the general case, we will keep the notation of 1.2.8. Since the isomorphism $\phi : \tilde{G}^{\sigma} \to G$ is compatible with the épinglages, we have $\phi\big((\tilde{U}^{\pm}_{>0})^{\sigma}\big) = U^{\pm}_{>0}$, $\phi\big((\tilde{T}_{>0})^{\sigma}\big) = T_{>0}$ and $\phi\big((\tilde{G}_{>0})^{\sigma}\big) = G_{>0}$. Now for any $z \in Z_{J,\geqslant 0}$, $z$ is contained in the closure of $G_{>0}$ in $\bar{G}$. Thus $\bar{\phi}^{-1}(z)$ is contained in the closure of $(\tilde{G}_{>0})^{\sigma}$ in $\overline{\tilde{G}^{\sigma}}$, hence contained in the closure of $(\tilde{G}_{>0})^{\sigma}$ in $\overline{\tilde{G}}$. Therefore, $\bar{\phi}^{-1}(z) \in \tilde{Z}_{\tilde{J},\geqslant 0}$, where $\tilde{J} = \pi^{-1} \circ \phi^{-1}(J)$.

For any $\widetilde{g_1}, \widetilde{g_2} \in (\tilde{G}_{>0})^{\sigma}$, we have $(\widetilde{g_1}, \widetilde{g_2}^{-1}) \cdot \bar{\phi}^{-1}(z) = (\widetilde{u_1}\tilde{t}, \widetilde{u_2}^{-1}) \cdot \tilde{z}^{\circ}_{\tilde{J}}$ for some $\widetilde{u_1} \in \tilde{U}^{-}_{>0}, \widetilde{u_2} \in \tilde{U}^{+}_{>0}, \tilde{t} \in \tilde{T}_{>0}$. Since $\bar{\phi}^{-1}(z) \in (\overline{\tilde{G}})^{\bar{\sigma}}$, we have $(\widetilde{g_1}, \widetilde{g_2}^{-1}) \cdot \bar{\phi}^{-1}(z) \in (\tilde{Z}_{\tilde{J},>0})^{\bar{\sigma}}$.

Then

$$\bar{\sigma}\big((\widetilde{u_1}\tilde{t}, \widetilde{u_2}^{-1}) \cdot \tilde{z}^{\circ}_{\tilde{J}}\big) = \big(\sigma(\widetilde{u_1}\tilde{t}), \sigma(\widetilde{u_2}^{-1})\big) \cdot \bar{\sigma}(\tilde{z}^{\circ}_{\tilde{J}}) = \big(\sigma(\widetilde{u_1})\sigma(\tilde{t}), \sigma(\widetilde{u_2}^{-1})\big) \cdot \tilde{z}^{\circ}_{\tilde{J}}$$
$$= (\widetilde{u_1}\tilde{t}, \widetilde{u_2}^{-1}) \cdot \tilde{z}^{\circ}_{\tilde{J}}.$$

Thus $\sigma(\widetilde{u_1}) = \widetilde{u_1}$ and $\sigma(\widetilde{u_2}) = \widetilde{u_2}$. Moreover, $(\tilde{t}, 1) \cdot \tilde{z}^{\circ}_{\tilde{J}} = \big(\sigma(\tilde{t}), 1\big) \cdot \tilde{z}^{\circ}_{\tilde{J}}$, that is, $\tilde{\alpha}_{\tilde{j}}(\tilde{t}) = \tilde{\alpha}_{\tilde{j}}\big(\sigma((\tilde{t})\big) = \tilde{\alpha}_{\sigma(\tilde{j})}(\tilde{t})$ for all $\tilde{j} \in \tilde{J}$, where $\{\tilde{\alpha}_{\tilde{i}} \mid \tilde{i} \in \tilde{I}\}$ is the set of simple roots of $\tilde{G}$. Let $\tilde{t}'$ be the unique element in $\tilde{T}$ such that

$$\tilde{\alpha}_{\tilde{j}}(\tilde{t}') = \begin{cases} \tilde{\alpha}_{\tilde{j}}(\tilde{t}), & \text{if } \tilde{j} \in \tilde{J}; \\ 1, & \text{otherwise .} \end{cases}$$

Then $\tilde{t}' \in (\tilde{T}_{>0})^{\sigma}$ and $(\tilde{t}, 1)\cdot\tilde{z}^{\circ}_{\tilde{J}} = (\tilde{t}', 1)\cdot\tilde{z}^{\circ}_{\tilde{J}}$. Thus $(\widetilde{g_1}, \widetilde{g_2}^{-1})\cdot\bar{\phi}^{-1}(z) = (\widetilde{u_1}\tilde{t}', \widetilde{u_2}^{-1})\cdot\tilde{z}^{\circ}_{\tilde{J}}$. We have

$$\big(\phi(\widetilde{g_1}), \phi(\widetilde{g_2})^{-1}\big) \cdot z = \bar{\phi}\big((\widetilde{g_1}, \widetilde{g_2}^{-1}) \cdot \bar{\phi}^{-1}(z)\big) = \bar{\phi}\big((\widetilde{u_1}\tilde{t}', \widetilde{u_2}^{-1}) \cdot \tilde{z}^{\circ}_{\tilde{J}}\big)$$
$$= \big(\phi(\widetilde{u_1})\phi(\tilde{t}'), \phi(\widetilde{u_2}^{-1})\big) \cdot z^{\circ}_{J} \in Z_{J,>0}.$$

Since $\phi\big((\tilde{G}_{>0})^{\sigma}\big) = G_{>0}$, we have $Z_{J,\geqslant 0} \subset \cap_{g_1,g_2 \in G_{>0}}(g_1^{-1}, g_2) \cdot Z_{J,>0}$.

Note that $(1,1)$ is contained in the closure of $\{(g_1, g_2^{-1}) \mid g_1, g_2 \in G_{>0}\}$. Hence, for any $z \in \cap_{g_1,g_2 \in G_{>0}}(g_1^{-1}, g_2) \cdot Z_{J,>0}$, $z$ is contained in the closure of $Z_{J,>0}$. On the other hand, $Z_{J,\geqslant 0}$ is a closed subset in $Z_J$. $Z_{J,\geqslant 0}$ contains $Z_{J,>0}$, hence contains the closure of $Z_{J,>0}$ in $Z_J$. Therefore, $Z_{J,\geqslant 0} = \cap_{g_1,g_2 \in G_{>0}}(g_1^{-1}, g_2) \cdot Z_{J,>0}$ is the closure of $Z_{J,>0}$ in $Z_J$.

Now set $g_r = \exp\big(r \sum_{i \in I}(e_i + f_i)\big)$, where $e_i$ and $f_i$ are the Chevalley generators related to our épinglage by $x_i(1) = \exp(e_i)$ and $y_i(1) = \exp(f_i)$. Then $g_r \in G_{>0}$ for $r \in \mathbf{R}_{>0}$ (see [L1, 5.9]). Define $f : R_{\geqslant 0} \times Z_{J,\geqslant 0} \to Z_{J,\geqslant 0}$ by $f(r, z) = (g_r, g_r^{-1}) \cdot z$ for $r \in R_{\geqslant 0}$ and $z \in Z_{J,\geqslant 0}$. Then $f(0, z) = z$ and $f(1, z) \in Z_{J,>0}$ for all $z \in Z_{J,\geqslant 0}$. Using the fact that $Z_{J,>0}$ is a cell (see 1.2.6), it follows that $Z_{J,\geqslant 0}$ is contractible.

Similarly, define $f' : R_{\geqslant 0} \times \overline{G_{>0}} \to \overline{G_{>0}}$ by $f'(r, z) = (g_r, g_r^{-1}) \cdot z$ for $r \in R_{\geqslant 0}$ and $z \in \overline{G_{>0}}$. Then $f'(0, z) = z$ and $f'(1, z) \in \bigsqcup_{K \subset I} Z_{K,>0}$ for all $z \in \overline{G_{>0}}$. Note that

$\bigsqcup_{K\subset I} Z_{K,>0} = \big(U^-_{>0}, (U^+_{>0})^{-1}\big)\cdot \bigsqcup_{K\subset I}(T_{>0},1)\cdot z_K^\circ \cong U^-_{>0}\times U^+_{>0}\times \bigsqcup_{K\subset I}(T_{>0},1)\cdot z_K^\circ$ (see 1.2.6). Moreover, by [DP, 2.2], we have $\bigsqcup_{K\subset I}(T_{>0},1)\cdot z_K^\circ \cong R^I_{\geqslant 0}$. Thus $\bigsqcup_{K\subset I} Z_{K,>0} \cong R^{2l(w_0)}_{>0}\times R^I_{\geqslant 0}$ is contractible. Therefore $\overline{G_{>0}}$ is contractible. □

## 1.3 The cell decomposition of $Z_{J,\geqslant 0}$

**1.3.1** For any $P\in \mathcal{P}^J, Q\in\mathcal{P}^{J^*}, B\in\mathcal{B}$ and $g_1\in H_P, g_2\in U_Q, g\in G$, we have $\mathrm{pos}\big(P^B, {}^{g_1 g g_2}(Q^B)\big) = \mathrm{pos}\big({}^{g_1^{-1}}(P^B), {}^{g g_2}(Q^B)\big) = \mathrm{pos}\big(P^B, {}^{g}(Q^B)\big)$. If moreover, $P\bowtie^g Q$, then $\mathrm{pos}\big(P^B, {}^{g}(Q^B)\big) = w w_0$ for some $w\in W_J$ (see 1.4). Therefore, for any $v,v'\in W$, $w,w'\in W^J$ and $y,y'\in W_J$ with $v\leqslant w$ and $v'\leqslant w'$, Lusztig introduced the subset $Z_J^{v,w,v',w';y,y'}$ and $Z_{J,>0}^{v,w,v',w';y,y'}$ of $Z_J$ which are defined as follows:

$$\begin{aligned} Z_J^{v,w,v',w';y,y'} = \{&(P,Q,H_P g U_Q)\in Z_J \mid P\in\mathcal{P}^J_{v,w}, \psi(Q)\in\mathcal{P}^J_{v',w'},\\ &\mathrm{pos}\big(P^{B^+}, {}^{g}(Q^{B^+})\big) = y w_0, \mathrm{pos}\big(P^{B^-}, {}^{g}(Q^{B^-})\big) = y' w_0\}\end{aligned}$$

and

$$Z_{J,>0}^{v,w,v',w';y,y'} = Z_J^{v,w,v',w';y,y'}\cap Z_{J,\geqslant 0}.$$

Then

$$Z_J = \bigsqcup_{\substack{v,v'\in W, w,w'\in W^J, y,y'\in W_J\\ v\leqslant w, v'\leqslant w'}} Z_J^{v,w,v',w';y,y'},$$
$$Z_{J,\geqslant 0} = \bigsqcup_{\substack{v,v'\in W, w,w'\in W^J, y,y'\in W_J\\ v\leqslant w, v'\leqslant w'}} Z_{J,>0}^{v,w,v',w';y,y'}.$$

Lusztig conjectured that for any $v,v'\in W, w,w'\in W^J, y,y'\in W_J$ such that $v\leqslant w, v'\leqslant w'$, $Z_{J,>0}^{v,w,v',w';y,y'}$ is either empty or a semi-algebraic cell. If it is nonempty, then it is also a connected component of $Z_J^{v,w,v',w';y,y'}$.

In this section, we will prove this conjecture. Moreover, we will show exactly when $Z_{J,>0}^{v,w,v',w';y,y'}$ is nonempty and we will give an explicit description of $Z_{J,>0}^{v,w,v',w';y,y'}$.

First, I will prove some elementary facts about the total positivity of $G$.

**Proposition 1.3.2.**

$$\cap_{u\in U^{\pm}_{>0}} u^{-1}U^{\pm}_{>0} = \cap_{u\in U^{\pm}_{>0}} U^{\pm}_{>0}u^{-1} = \cap_{u\in U^{\pm}_{>0}} u^{-1}U^{\pm}_{\geqslant 0} = \cap_{u\in U^{\pm}_{>0}} U^{\pm}_{\geqslant 0}u^{-1} = U^{\pm}_{\geqslant 0},$$
$$\cap_{g\in G_{>0}} g^{-1}G_{>0} = \cap_{g\in G_{>0}} G_{>0}g^{-1} = \cap_{g\in G_{>0}} g^{-1}G_{\geqslant 0} = \cap_{g\in G_{>0}} G_{\geqslant 0}g^{-1} = G_{\geqslant 0}.$$

*Proof.* I will only prove $\cap_{u\in U^+_{>0}} u^{-1}\cdot U^+_{>0} = U^+_{\geqslant 0}$. The rest of the equalities could be proved in the same way.

Note that $uu_1 \in U^+_{>0}$ for all $u_1 \in U^+_{\geqslant 0}, u \in U^+_{>0}$. Thus $u_1 \in \cap_{u\in U^+_{>0}} u^{-1}\cdot U^+_{>0}$. On the other hand, assume that $u_1 \in \cap_{u\in U^+_{>0}} u^{-1}\cdot U^+_{>0}$. Then $uu_1 \in U^+_{>0}$ for all $u \in U^+_{>0}$. We have $u_1 = \lim_{\substack{u\in U^+_{>0}\\ u\to 1}} uu_1$ is contained in the closure of $U^+_{>0}$ in $U^+$, that is, $u_1 \in U^+_{\geqslant 0}$. So $\cap_{u\in U^+_{>0}} u^{-1}\cdot U^+_{>0} = U^+_{\geqslant 0}$. □

For any $v, v' \in W$, $w, w' \in W^J$ such that $v \leqslant w, v' \leqslant w'$, set $Z_J^{v,w,v',w'} = \bigsqcup_{y,y'\in W_J} Z_J^{v,w,v',w';y,y'}$ and $Z_{J,>0}^{v,w,v',w'} = \bigsqcup_{y,y'\in W_J} Z_{J,>0}^{v,w,v',w';y,y'}$. We will give a characterization of $z \in Z_{J,>0}^{v,w,v',w'}$ in 1.3.5.

**Lemma 1.3.3.** *For any $w \in W$, $u \in U^-_{\geqslant 0}$, $\{\pi_{U^+}(u_1u) \mid u_1 \in U^+_{w,>0}\} = U^+_{w,>0}$.*

*Proof.* The following identities hold (see [L1, 1.3]):

(a) $tx_i(a) = x_i\big(\alpha_i(t)a\big)t, ty_i(a) = y_i\big(\alpha_i(t)^{-1}a\big)t$ for all $i \in I, t \in T, a \in \mathbf{R}$.

(b) $y_{i_1}(a)x_{i_2}(b) = x_{i_2}(b)y_{i_1}(a)$ for all $a, b \in \mathbf{R}$ and $i_1 \neq i_2 \in I$.

(c) $x_i(a)y_i(b) = y_i(\frac{b}{1+ab})\alpha_i^\vee(\frac{1}{1+ab})x_i(\frac{a}{1+ab})$ for all $a, b \in \mathbf{R}_{>0}, i \in I$.

Thus $U^+_{w,>0}U^-_{\geqslant 0} \subset U^-_{\geqslant 0}T_{>0}U^+_{w,>0}$ for $w \in W$. So we only need to prove that $U^+_{w,>0} \subset \{\pi_{U^+}(u_1u) \mid u_1 \in U^+_{w,>0}\}$. Now I will prove the following statement:

$$\{\pi_{U^+}\big(u_1y_i(a)\big) \mid u_1 \in U^+_{w,>0}\} = U^+_{w,>0} \quad \text{for } i \in I, a \in \mathbf{R}_{>0}.$$

We argue by induction on $l(w)$. It is easy to see that the statement holds for $w = 1$. Now assume that $w \neq 1$. Then there exist $j \in I$ and $w_1 \in W$ such that $w = s_jw_1$ and $l(w_1) = l(w) - 1$. For any $u'_1 \in U^+_{w,>0}$, we have $u'_1 = u'_2u'_3$ for some $u'_2 \in U^+_{s_j,>0}$ and $u'_3 \in U^+_{w_1,>0}$. By induction hypothesis, there exists $u_3 \in U^+_{w_1,>0}, u' \in U^-$ and $t \in T$ such that $u_3y_i(a) = u'tu'_3$. Since $U^+_{w,>0}U^-_{s_i,>0} \subset U^-_{s_i,>0}T_{>0}U^+_{w,>0}$, we have $u' \in U^-_{s_i,>0}$ and $t \in T_{>0}$.

Now by (a), we have $tu_2't^{-1} \in U^+_{s_j,>0}$. So by (b) and (c), there exists $u_2 \in U^+_{s_j,>0}$ such that $\pi_{U^+}(u_2u') = tu_2't^{-1}$. Thus

$$\pi_{U^+}\big(u_2u_3y_i(a)\big) = \pi_{U^+}\Big((u_2u')\big(u'^{-1}u_3y_i(a)\big)\Big) = \pi_{U^+}\big(\pi_{U^+}(u_2u')u'^{-1}u_3y_i(a)\big)$$
$$= \pi_{U^+}(tu_2't^{-1}tu_3') = \pi_{U^+}(tu_2'u_3') = u_1'.$$

So $u_1' \in \{\pi_{U^+}(u_1y_i(a)) \mid u_1 \in U^+_{w,>0}\}$. The statement is proved.

Now assume that $u \in U^-_{w',>0}$. I will prove the lemma by induction on $l(w')$. It is easy to see that the lemma holds for $w' = 1$. Now assume that $w' \neq 1$. Then there exist $i \in I$ and $w_1' \in W$ such that $l(w_1') = l(w') - 1$ and $w' = s_iw_1'$. We have $u = y_i(a)u'$ for some $a \in \mathbf{R}_{>0}$ and $u' \in U^-_{w_1',>0}$. So

$$\{\pi_{U^+}(u_1u) \mid u_1 \in U^+_{w,>0}\} = \{\pi_{U^+}\big(u_1y_i(a)u'\big) \mid u_1 \in U^+_{w,>0}\}$$
$$= \{\pi_{U^+}\Big(\pi_{U^+}\big(u_1y_i(a)\big)u\Big) \mid u_1 \in U^+_{w,>0}\}$$
$$= \{\pi_{U^+}(u_1'u') \mid u_1' \in U^+_{w,>0}\}.$$

By induction hypothesis, we have

$$\{\pi_{U^+}(u_1u) \mid u_1 \in U^+_{w,>0}\} = \{\pi_{U^+}(u_1'u') \mid u_1' \in U^+_{w,>0}\} = U^+_{w,>0}.$$

□

**Lemma 1.3.4.** *Set* $Z^1_{J,>0} = \{(g_1, g_2^{-1}) \cdot z_J^\circ \mid g_1 \in U^-_{\geqslant 0}T_{>0}, g_2 \in U^+_{\geqslant 0}\}$. *Then*

$$Z_{J,\geqslant 0} = \cap_{u_1 \in U^+_{>0}, u_2^{-1} \in U^-_{>0}} (u_1^{-1}, u_2) \cdot Z^1_{J,>0}. \qquad \text{(a)}$$

$$Z^1_{J,>0} = \bigsqcup_{w_1,w_2 \in W^J} \{({}^{u_1}P_J, {}^{u_2^{-1}}Q_J, u_1H_{P_J}lU_{Q_J}u_2) \mid u_1 \in U^-_{w_1,>0}, \qquad \text{(b)}$$
$$u_2 \in U^+_{w_2,>0}, l \in L_{\geqslant 0}\}$$
$$= \{(P, Q, \gamma) \in Z_{J,\geqslant 0} \mid P = {}^{u_1} P_J, \psi(Q) = {}^{u_2} P_J \textit{ for some } u_1, u_2 \in U^-_{\geqslant 0}\}.$$

*Proof.* (a) By 1.2.9 and 1.3.2, we have

$$
\begin{aligned}
Z_{J,\geqslant 0} &= \cap_{g_1,g_2\in G_{>0}}(g_1^{-1},g_2)\cdot Z_{J,>0} = \cap_{\substack{t_1,t_2\in T_{>0}\\ u_1,u_2\in U^+_{>0},u_3,u_4\in U^-_{>0}}}(u_1^{-1}u_3^{-1}t_1^{-1},u_4u_2t_2)\cdot Z_{J,>0}\\
&= \cap_{u_1\in U^+_{>0},u_4\in U^-_{>0}}(u_1^{-1},u_4)\cdot\cap_{u_2\in U^+_{>0},u_3\in U^-_{>0}}(u_2^{-1},u_3)\cdot\cap_{t_1,t_2\in T_{>0}}(t_1^{-1},t_2)\cdot Z_{J,>0}\\
&= \cap_{u_1\in U^+_{>0},u_4\in U^-_{>0}}(u_1^{-1},u_4)\cdot\cap_{u_2\in U^+_{>0},u_3\in U^-_{>0}}(u_2^{-1},u_3)\cdot Z_{J,>0}\\
&= \cap_{u_1\in U^+_{>0},u_4\in U^-_{>0}}(u_1^{-1},u_4)\cdot\cap_{u_2\in U^+_{>0},u_3\in U^-_{>0}}\big(u_2^{-1}U^-_{>0}T_{>0},(U^+_{>0}u_3^{-1})^{-1}\big)\cdot z_J^\circ\\
&= \cap_{u_1\in U^+_{>0},u_2^{-1}\in U^-_{>0}}(u_1^{-1},u_2)\cdot\Big(\big(U^-_{\geqslant 0}T_{>0},(U^+_{\geqslant 0})^{-1}\big)\cdot z_J^\circ\Big).
\end{aligned}
$$

(b) For any $u\in U^-_{\geqslant 0}, v\in U^+_{\geqslant 0}, t\in T_{>0}$, there exist $w_1,w_2\in W^J, w_3,w_4\in W_J$, such that $u=u_1u_3$ for some $u_1\in U^-_{w_1,>0}$, $u_3\in U^-_{w_3,>0}$ and $v=u_4u_2$ for some $u_2\in U^+_{w_2,>0}$, $u_4\in U^+_{w_4,>0}$. Then $(ut,v^{-1})\cdot z_J^\circ = ({}^{u_1}P_J,{}^{u_2^{-1}}Q_J,u_1H_{P_J}u_3tu_4U_{Q_J}u_2)$. On the other hand, assume that $l\in L_{\geqslant 0}$, then $l=u_3tu_4$ for some $u_3\in U^-_{\geqslant 0}, u_4\in U^+_{\geqslant 0}, t\in T_{>0}$. Thus for any $u_1\in U^-_{\geqslant 0}, u_2\in U^+_{\geqslant 0}$, we have

$$
({}^{u_1}P_J,{}^{u_2^{-1}}Q_J,u_1H_{P_J}lU_{Q_J}u_2)=(u_1u_3t,u_2^{-1}u_4^{-1})\cdot z_J^\circ\in Z^1_{J,>0}.
$$

Therefore,

$$
\begin{aligned}
Z^1_{J,>0} = \bigsqcup_{w_1,w_2\in W^J}\{({}^{u_1}P_J,{}^{u_2^{-1}}Q_J,u_1H_{P_J}lU_{Q_J}u_2)\mid u_1\in U^-_{w_1,>0},\\
u_2\in U^+_{w_2,>0}, l\in L_{\geqslant 0}\}\\
\subset\{(P,Q,\gamma)\in Z_{J,\geqslant 0}\mid P={}^{u_1}P_J,\psi(Q)={}^{u_2}P_J \text{ for some } u_1,u_2\in U^-_{\geqslant 0}\}.
\end{aligned}
$$

Note that $\{{}^uP_J\mid u\in U^-_{\geqslant 0}\}=\bigsqcup_{w\in W^J}\{{}^uP_J\mid u\in U^-_{w,>0}\}$. Now assume that $z=\big({}^{u_1}P_J,{}^{\psi(u_2)^{-1}}Q_J,u_1H_{P_J}lU_{Q_J}\psi(u_2)\big)$ for some $w_1,w_2\in W^J$ and $u_1\in U^-_{w_1,>0}$, $u_2\in U^-_{w_2,>0}$, $l\in L$. To prove that $z\in Z^1_{J,>0}$, it is enough to prove that $l\in L_{\geqslant 0}Z(L)$. By part (a), for any $u_3,u_4\in U^+_{>0}$,

$$
\big(u_3,\psi(u_4)^{-1}\big)\cdot z=\big({}^{u_3u_1}P_J,{}^{\psi(u_4u_2)^{-1}}Q_J,u_3u_1H_{P_J}lU_{Q_J}\psi(u_4u_2)\big)\in Z^1_{J,>0}.
$$

Note that $u_3u_1 = u'_1 t_1 \pi_{U^+}(u_3u_1)$ for some $u'_1 \in U^-_{w_1,>0}, t_1 \in T_{>0}$ and $u_4u_2 = u'_2 t_2 \pi_{U^+}(u_4u_2)$ for some $u'_2 \in U^-_{w_2,>0}, t_2 \in T_{>0}$. So we have that ${}^{u_3u_1}P_J ={}^{u'_1} P_J$, ${}^{\psi(u_4u_2)^{-1}}Q_J ={}^{\psi(u'_2)^{-1}} Q_J$ and

$$u_3u_1 H_{P_J} l U_{Q_J} \psi(u_4u_2) = u'_1 t_1 \pi_{U^+}(u_3u_1) H_{P_J} l U_{Q_J} \psi\big(\pi_{U^+}(u_4u_2)\big) t_2 \psi(u'_2)$$
$$= u'_1 H_{P_J} t_1 \pi_{U^+_J}(u_3u_1) l \psi\big(\pi_{U^+_J}(u_4u_2)\big) t_2 U_{Q_J} \psi(u'_2).$$

Then $t_1 \pi_{U^+_J}(u_3u_1) l \psi\big(\pi_{U^+_J}(u_4u_2)\big) t_2 \in L_{\geqslant 0} Z(L)$. Since $t_1, t_2 \in T_{>0}$, we have that $\pi_{U^+_J}(u_3u_1) l \psi\big(\pi_{U^+_J}(u_4u_2)\big) \in L_{\geqslant 0} Z(L)$ for all $u_3, u_4 \in U^+_{>0}$. By 1.1.8 and 1.3.3,

$$\pi_{U^+_J}(U^+_{>0}u_1) = \pi_{U^+_J}\big(\pi_{U^+}(U^+_{>0}u_1)\big) = \pi_{U^+_J}(U^+_{>0}) = U^+_{w^J_0,>0}.$$

Similarly, we have $\pi_{U^+_J}(U^+_{>0}u_2) = U^+_{w^J_0,>0}$. Thus

$$l \in \cap_{u_3,u_4 \in U^+_{w^J_0,>0}} u_3^{-1} U^+_{w^J_0,\geqslant 0} T_{>0} Z(L) U^-_{w^J_0,\geqslant 0} \psi(u_4)^{-1}$$
$$= U^+_{w^J_0,\geqslant 0} T_{>0} Z(L) U^-_{w^J_0,\geqslant 0} = L_{\geqslant 0} Z(L).$$

The lemma is proved. □

**Proposition 1.3.5.** *Let $z \in Z_J^{v,w,v',w'}$, then $z \in Z_{J,>0}^{v,w,v',w'}$ if and only if for any $u_1 \in U^+_{v^{-1},>0}, u_2 \in U^+_{v'^{-1},>0}$, $\big(u_1, \psi(u_2^{-1})\big) \cdot z \in Z^1_{J,>0}$.*

*Proof.* Assume that $z \in \cap_{u_1 \in U^+_{v^{-1},>0}, u_2 \in U^+_{v'^{-1},>0}} \big(u_1^{-1}, \psi(u_2)\big) Z^1_{J,>0}$. Then we have $z = \lim_{u_1,u_2 \to 1} \big(u_1, \psi(u_2)^{-1}\big) \cdot z$ is contained in the closure of $Z^1_{J,>0}$ in $Z_J$. Note that $Z_{J,>0} \subset Z^1_{J,>0} \subset Z_{J,\geqslant 0}$. Thus by 1.2.9, $Z_{J,\geqslant 0}$ is the closure of $Z^1_{J,>0}$ in $Z_J$. Therefore, $z$ is contained in $Z_{J,\geqslant 0}$.

On the other hand, assume that $z = (P, Q, \gamma) \in Z_{J,>0}^{v,w,v',w'}$. By 1.3.4(a), for any $u_1 \in U^+_{v^{-1},>0}$, $u_2 \in U^+_{v'^{-1},>0}$, we have $\big(u_1, \psi(u_2^{-1})\big) \cdot z \in Z_{J,\geqslant 0}$. Moreover, we have ${}^{u_1}P ={}^{u'_1} P_J$ for some $u'_1 \in U^-_{w,>0}$ (see 1.1.6). Similarly, we have $\psi({}^{\psi(u_2^{-1})}Q) = {}^{u_2}\psi(Q) ={}^{u'_2} P_J$ for some $u'_2 \in U^-_{w',>0}$. By 1.3.4(b), $\big(u_1, \psi(u_2^{-1})\big) \cdot z \in Z^1_{J,>0}$. □

**1.3.6** Now I will fix $w \in W^J$ and a reduced expression $\mathbf{w} = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ of $w$. Assume that $w_{(j)} = w_{(j-1)}s_{i_j}$ for all $j = 1, 2, \ldots, n$. Let $v \leqslant w$ and let $\mathbf{v}_+ = (v_{(0)}, v_{(1)}, \ldots, v_{(n)})$ be the positive subexpression of $\mathbf{w}$.

Define

$$G_{\mathbf{v}_+,\mathbf{w}} = \left\{ g = g_1 g_2 \cdots g_k \middle| \begin{matrix} g_j = y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R} - \{0\}, & \text{if } v_{(j-1)} = v_{(j)} \\ g_j = \dot{s_{i_j}}, & \text{if } v_{(j-1)} < v_{(j)} \end{matrix} \right\},$$

$$G_{\mathbf{v}_+,\mathbf{w},>0} = \left\{ g = g_1 g_2 \cdots g_k \middle| \begin{matrix} g_j = y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R}_{>0}, & \text{if } v_{(j-1)} = v_{(j)} \\ g_j = \dot{s_{i_j}}, & \text{if } v_{(j-1)} < v_{(j)} \end{matrix} \right\}.$$

Marsh and Rietsch have proved that the morphism $g \mapsto^g B^+$ maps $G_{\mathbf{v}_+,\mathbf{w}}$ into $\mathcal{R}_{v,w}$ (see [MR, 5.2]) and $G_{\mathbf{v}_+,\mathbf{w},>0}$ bijectively onto $\mathcal{R}_{v,w,>0}$ (see [MR, 11.3]).

The following proposition is a technical tool needed in the proof of the main theorem.

**Proposition 1.3.7.** *For any $g \in G_{\mathbf{v}_+,\mathbf{w},>0}$, we have*

$$\cap_{u \in U^+_{v^{-1},>0}} \left(\pi_{U_J^+}(ug)\right)^{-1} \cdot U^+_{w_0^J,\geqslant 0} = \begin{cases} U^+_{w_0^J,\geqslant 0}, & \text{if } v \in W^J; \\ \varnothing, & \text{otherwise.} \end{cases}$$

The proof will be given in 1.3.13.

**Lemma 1.3.8.** *Suppose $\alpha_{i_0}$ is a simple root such that $v_1^{-1}\alpha_{i_0} > 0$ for $v \leqslant v_1 \leqslant w$. Then for all $g \in G_{\mathbf{v}_+,\mathbf{w},>0}$ and $a \in \mathbf{R}$, we have $x_{i_0}(a)g = gtg'$ for some $t \in T_{>0}$ and $g' \in \prod_{\alpha \in R(v)} U_\alpha \cdot \left(\dot{v}^{-1} x_{i_0}(a) \dot{v}\right)$, where $R(v) = \{\alpha \in \Phi^+ \mid v\alpha \in -\Phi^+\}$.*

*Proof.* Marsh and Rietsch proved in [MR, 11.8] that $g$ is of the form

$$g = \Big( \prod_{j \in J^\circ_{\mathbf{v}_+}} y_{v_{(j-1)}\alpha_{i_j}}(t_j) \Big) \dot{v}$$

and $v_{(j-1)}\alpha_{i_1} \neq \alpha_{i_0}$, for all $j = 1, 2, \ldots, n$. Thus $g = g_1\dot{v}$ for some

$$g_1 \in \prod_{\alpha\in\Phi^+-\{\alpha_{i_0}\}} U_{-\alpha}.$$

Set $T_1 = \{t \in T \mid \alpha_{i_0}(t) = 1\}$, then $T_1 \prod_{\alpha\in\Phi^+-\{\alpha_{i_0}\}} U_{-\alpha}$ is a normal subgroup of $\psi(P_{\{i_0\}})$. Now set $x = x_{i_0}(a)$, then $xg_1x^{-1} \in B^-$. We may assume that $xg_1x^{-1} = u_1t_1$ for some $u_1 \in U^-$ and $t_1 \in T$. Now $xg = xg_1\dot{v} = (xg_1x^{-1})x\dot{v} = u_1\dot{v}(\dot{v}^{-1}t_1\dot{v})(\dot{v}^{-1}x\dot{v})$. Moreover, by [MR, 11.8], $xg \in gB^+$. Thus $xg = g_1\dot{v}t_2g_2g_3 = g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1})\dot{v}t_2g_3$, for some $t_2 \in T$, $g_2 \in \prod_{\alpha\in R(v)} U_\alpha$ and $g_3 \in \prod_{\alpha\in\Phi^+-R(v)} U_\alpha$. Note that $g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1}), u_1 \in U^-$, $t_2, \dot{v}^{-1}t_1\dot{v} \in T$ and $g_3, \dot{v}^{-1}x\dot{v} \in \prod_{\alpha\in\Phi^+-R(v)} U_\alpha$. Thus $g_1(\dot{v}t_2g_2t_2^{-1}\dot{v}^{-1}) = u_1$, $t_2 = \dot{v}^{-1}t_1\dot{v}$ and $g_3 = \dot{v}^{-1}x\dot{v}$. Note that $g^{-1}x_{i_0}(b)g \in B^+$ for $b \in \mathbf{R}$ (see [MR, 11.8]). We have that $\{\pi_T(g^{-1}x_{i_0}(b)g) \mid b \in \mathbf{R}\}$ is connected and contains $\pi_T\big(g^{-1}x_{i_0}(0)g\big) = 1$. Hence $\pi_T(g^{-1}x_{i_0}(b)g) \in T_{>0}$ for $b \in \mathbf{R}$. In particular, $\pi_T(g^{-1}xg) = t_2 \in T_{>0}$. Therefore $xg = gt_2g'$ with $t_2 \in T_{>0}$ and $g' = g_2g_3 \in \prod_{\alpha\in R(v)} U_\alpha \cdot (\dot{v}^{-1}x\dot{v})$. □

*Remark.* In [MR, 11.9], Marsh and Rietsch pointed out that for any $j \in J^+_{\mathbf{V}_+}$, we have $u^{-1}\alpha_{i_j} > 0$ for all $v_{(j)}^{-1}v \leqslant u \leqslant w_{(j)}^{-1}w$.

**1.3.9** Suppose that $J^+_{\mathbf{V}_+} = \{j_1, j_2, \ldots, j_k\}$, where $j_1 < j_2 < \cdots < j_k$ and $g = g_1g_2\cdots g_n$, where

$$g_j = \begin{cases} y_{i_j}(a_j) \text{ for } a_j \in \mathbf{R}_{>0}, & \text{if } j \in J^\circ_{\mathbf{V}_+}; \\ \dot{s}_{i_j}, & \text{if } j \in J^+_{\mathbf{V}_+}. \end{cases}$$

For any $m = 1, \ldots, k$, define $v_m = v_{(j_m)}^{-1}v$, $g_{(m)} = g_{j_m+1}g_{j_m+2}\cdots g_n$ and $f_m(a) = g_{(m)}^{-1}x_{i_{j_m}}(-a)g_{(m)} \in B^+$ (see [MR, 11.8]). Now I will prove the following lemma.

**Lemma 1.3.10.** *Keep the notation in 1.3.9. Then*

*(a) For any $u \in U^+_{v^{-1},>0}$, $ug = g'tu'$ for some $g' \in U^-_{w,>0}, t \in T_{>0}$ and $u' \in U^+$.*

*(b)* $\pi_{U^+}(U^+_{v^{-1},>0}g) = \{\pi_{U^+}\big(f_k(a_k)f_{k-1}(a_{k-1})\cdots f_1(a_1)\big) \mid a_1, a_2, \ldots, a_k \in \mathbf{R}_{>0}\}$.

*Proof.* I will prove the lemma by induction on $l(v)$. It is easy to see that the lemma holds when $v = 1$. Now assume that $v \neq 1$.

For any $u \in U^+_{v^{-1},>0}$, since ${}^gB^+ \in \mathcal{R}_{v,w,>0}$, we have ${}^{ug}B^+ \in \mathcal{R}_{1,w,>0}$. Thus $ug = g'tu'$ for some $g' \in U^-_{w,>0}, t \in T$ and $u' \in U^+$. Set $y = g_{i_1}g_{i_2}\cdots g_{i_{j_1-1}}$. Note that $y \in U^-_{\geqslant 0}$, we have $uy = y'tu'$ for some $y' \in U^-$, $u' \in U^+_{v^{-1},>0}$ and $t \in T_{>0}$. Hence $\pi_T(ug) = \pi_T(uy\dot{s_{i_{j_1}}}g_{(1)}) = \pi_T(y'tu'\dot{s_{i_{j_1}}}g_{(1)}) \in T_{>0}\pi_T(u'\dot{s_{i_{j_1}}}g_{(1)})$. To prove that $\pi_T(U^+_{v^{-1},>0}g) \subset T_{>0}$, it is enough to prove that $\pi_T(u\dot{s_{i_{j_1}}}g_{(1)}) \in T_{>0}$ for all $u \in U^+_{v^{-1},>0}$.

For any $u \in U^+_{v^{-1},>0}$, we have $u = u_1x_{i_{j_1}}(a)$ for some $u_1 \in U^+_{v^{-1}s_{i_{j_1}},>0}$ and $a \in \mathbf{R}_{>0}$. It is easy to see that $x_{i_{j_1}}(a)\dot{s_{i_{j_1}}}g_{(1)} = \alpha^\vee_{i_{j_1}}(a)y_{i_{j_1}}(a)x_{i_{j_1}}(-a^{-1})g_{(1)}$. Note that $\alpha^\vee_{i_{j_1}}(a) \in T_{>0}$ and by 1.3.8, ${g_{(1)}}^{-1}x_{i_{j_1}}(-a^{-1})g_{(1)} \in T_{>0}U^+$. Hence by 1.1.7, we have

$$\begin{aligned}\pi_T(u\dot{s_{i_{j_1}}}g_{(1)}) &= \pi_T\Big(u_1\alpha^\vee_{i_{j_1}}(a)y_{i_{j_1}}(a)g_{(1)}\big({g_{(1)}}^{-1}x_{i_{j_1}}(-a^{-1})g_{(1)}\big)\Big)\\ &\in T_{>0}\pi_T\big(U^+_{v^{-1}s_{i_{j_1}},>0}y_{i_{j_1}}(a)g_{(1)}\big)T_{>0}.\end{aligned}$$

Set

$$\begin{aligned}\mathbf{w}' &= (1, w^{-1}_{(j_1-1)}w_{(j_1)}, \ldots, w^{-1}_{(j_1-1)}w_{(n)}),\\ \mathbf{v}'_+ &= (1, s_{i_{j_1}}v_{(j_1)}, s_{i_{j_1}}v_{(j_1+1)}, \ldots, s_{i_{j_1}}v_{(n)}).\end{aligned}$$

Then $\mathbf{w}'$ is a reduced expression of $w^{-1}_{(j_1-1)}w_{(n)}$ and $\mathbf{v}'_+$ is a positive subexpression of $\mathbf{w}'$. For any $a \in \mathbf{R}_{>0}$, $y_{i_{j_1}}(a)g_{(1)} \in G_{\mathbf{v}'_+,\mathbf{w}',>0}$. Thus by induction hypothesis, for any $a \in \mathbf{R}_{>0}$, $\pi_T(U^+_{v^{-1}s_{i_{j_1}},>0}y_{i_{j_1}}(a)g_{(1)}) \subset T_{>0}$. Therefore, $\pi_T(ug) \in T_{>0}$. Part (a) is proved.

We have

$$\begin{aligned}
\pi_{U^+}(U^+_{v^{-1},>0}g) &= \pi_{U^+}(U^+_{v^{-1},>0}y\dot{s_{i_{j_1}}}g_{(1)}) = \pi_{U^+}(\pi_{U^+}\big(U^+_{v^{-1},>0}y)\dot{s_{i_{j_1}}}g_{(1)}\big)\\
&= \pi_{U^+}(U^+_{v^{-1},>0}\dot{s_{i_{j_1}}}g_{(1)}) = \bigcup_{a\in\mathbf{R}_{>0}} \pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0}x_{i_{j_1}}(a^{-1})\dot{s_{i_{j_1}}}g_{(1)}\big)\\
&= \bigcup_{a\in\mathbf{R}_{>0}} \pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0}\alpha^\vee_{i_{j_1}}(a^{-1})y_{i_{j_1}}(a^{-1})g_{(1)}f_1(a)\big)\\
&= \bigcup_{a\in\mathbf{R}_{>0}} \pi_{U^+}\Big(\pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0}\alpha^\vee_{i_{j_1}}(a^{-1})y_{i_{j_1}}(a^{-1})\big)g_{(1)}f_1(a)\Big)\\
&= \bigcup_{a\in\mathbf{R}_{>0}} \pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0}g_{(1)}f_1(a)\big)\\
&= \bigcup_{a\in\mathbf{R}_{>0}} \pi_{U^+}\Big(\pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0}g_{(1)}\big)f_1(a)\Big).
\end{aligned}$$

By induction hypothesis,

$$\pi_{U^+}(U^+_{v^{-1}s_{i_{j_1}},>0}g_{(1)}) = \{\pi_{U^+}\big(f_k(a_k)f_{k-1}(a_{k-1})\cdots f_2(a_2)\big) \mid a_2, a_3, \dots, a_k \in \mathbf{R}_{>0}\}.$$

Thus

$$\begin{aligned}
\pi_{U^+}(U^+_{v^{-1},>0}g) &= \bigcup_{a\in\mathbf{R}_{>0}} \pi_{U^+}\Big(\pi_{U^+}\big(U^+_{v^{-1}s_{i_{j_1}},>0}g_{(1)}\big)f_1(a)\Big)\\
&= \{\pi_{U^+}\big(f_k(a_k)f_{k-1}(a_{k-1})\cdots f_1(a_1)\big) \mid a_1, a_2, \dots, a_k \in \mathbf{R}_{>0}\}.
\end{aligned}$$

□

*Remark.* The referee pointed out to me that the assertion $t \in T_{>0}$ of 1.3.10(a) could also be proved using generalized minors.

**Lemma 1.3.11.** *Assume that $\alpha$ is a positive root and $u \in U_\alpha$, $u' \in U^+$ such that $u^n u' \in U^+_{\geqslant 0}$ for all $n \in \mathbf{N}$. Then $u = x_i(a)$ for some $i \in I$ and $a \in \mathbf{R}_{\geqslant 0}$.*

*Proof.* There exists $t \in T_{>0}$, such that $\alpha_i(t) = 2$ for all $i \in I$. Then $tut^{-1} = u^{\alpha(t)} = u^m$ for some $m \in \mathbf{N}$. By assumption, $t^n ut^{-n}u' \in U^+_{\geqslant 0}$ for all $n \in \mathbf{N}$. Thus $u\big(t^{-n}u't^n\big) = t^{-n}\big(t^n ut^{-n}u'\big)t^n \in U^+_{\geqslant 0}$. Moreover, it is easy to see that $\lim_{n\to\infty} t^{-n}u't^n = 1$. Since

$U^+_{\geqslant 0}$ is a closed subset of $U^+$, $\lim_{n\to\infty} ut^{-n}u't^n = u \in U^+_{\geqslant 0}$. Thus $u = x_i(a)$ for some $i \in I$ and $a \in \mathbf{R}_{\geqslant 0}$. $\square$

**Lemma 1.3.12.** *Assume that $w \in W$ and $i, j \in I$ such that $w^{-1}\alpha_i = \alpha_j$. Then there exists $c \in \mathbf{R}_{>0}$, such that $\dot{w}^{-1}x_i(a)\dot{w} = x_j(ca)$ for all $a \in \mathbf{R}$.*

*Proof.* There exist $c, c' \in \mathbf{R} - \{0\}$, such that $y_i(a)\dot{w} = \dot{w}y_j(c'a)$ and $x_i(a)\dot{w} = \dot{w}x_j(ca)$ for $a \in \mathbf{R}$. Since ${}^{\dot{w}}B^- \in \mathcal{B}_{\geqslant 0}$, we have ${}^{y_i(1)\dot{w}}B^+ = {}^{\dot{w}y_j(c')}B^+ \in \mathcal{B}_{\geqslant 0}$. By 3.6, $c' \geqslant 0$. Thus $c' > 0$. Moreover, since $w\alpha_j = \alpha_i > 0$, we have $ws_jw^{-1} = s_i$ and $l(ws_j) = l(s_iw) = l(w) + 1$. Hence, setting $w' = ws_j = s_iw$, we have $\dot{w}' = \dot{w}\dot{s_j} = \dot{s_i}\dot{w}$, that is $\dot{w}x_i(-1)y_i(1)x_i(-1) = x_j(-c)y_j(c')x_i(-c)\dot{w} = x_j(-1)y_j(1)x_j(-1)\dot{w}$. Therefore, $c = c'^{-1} > 0$. $\square$

**1.3.13. Proof of Proposition 1.3.7** If $v \in W^J$, then $v\alpha > 0$ for $\alpha \in \Phi^+_J$. So $\pi_{U^+_J}(\prod_{\alpha\in R(v)} U_\alpha) = \{1\}$. By 1.3.8, $f_m(a) \in T(\prod_{\alpha\in R(v_m)} U_\alpha) \cdot U_{v_m^{-1}\alpha_{i_{j_m}}}$ for all $m \in \{1, 2, \ldots, k\}$. Note that $v\alpha \in -\Phi^+$ for all $a \in R(v_m)$ and $vv_m^{-1}\alpha_{i_{j_m}} = v_{(j_m)}\alpha_{i_{j_m}} \in -\Phi^+$. So $f_m(a) \in T\prod_{\alpha\in R(v)} U_\alpha$ and $f_k(a_k)f_{k-1}(a_{k-1})\cdots f_1(a_1) \in T\prod_{\alpha\in R(v)} U_\alpha$. Hence by 1.3.10(b), $\pi_{U^+_J}(ug) = 1$ for all $u \in U^+_{v^{-1},>0}$. Therefore $\cap_{u\in U^+_{v^{-1},>0}} \big(\pi_{U^+_J}(ug)\big)^{-1} \cdot U^+_{w_0^J,\geqslant 0} = U^+_{w_0^J,\geqslant 0}$.

If $v \notin W^J$, then there exists $\alpha \in \Phi^+_J$ such that $v\alpha \in -\Phi^+_J$, that is, $v_m^{-1}\alpha_{i_{j_m}} \in \Phi^+_J$ for some $m \in \{1, 2, \ldots, k\}$. Set $k_0 = \max\{m \mid v_m^{-1}\alpha_{i_{j_m}} \in \Phi^+_J\}$. Then since $R(v_{k_0}) = \{v_m^{-1}\alpha_{i_{j_m}} \mid m > k_0\}$, we have that $v_{k_0}\alpha > 0$ for $\alpha \in \Phi^+_J$. Hence by 3.8, $\pi_{U^+_J}\big(f_{k_0}(a)\big) = \dot{v_{k_0}}^{-1}x_{i_{j_{k_0}}}(-a)\dot{v_{k_0}}$. If $u' \in \cap_{u\in U^+_{v^{-1},>0}} \big(\pi_{U^+_J}(ug)\big)^{-1} \cdot U^+_{w_0^J,\geqslant 0}$, then $\pi_{U^+_J}\big(f_k(a_k)f_{k-1}(a_{k-1})\cdots f_1(a_1)\big)u' \in U^+_{w_0^J,\geqslant 0}$ for all $a_1, a_2, \ldots, a_k \in \mathbf{R}_{>0}$. Since $U^+_{w_0^J,\geqslant 0}$ is a closed subset of $G$, $\pi_{U^+_J}\big(f_k(a_k)f_{k-1}(a_{k-1})\cdots f_1(a_1)\big)u' \in U^+_{w_0^J,\geqslant 0}$ for all $a_1, a_2, \ldots, a_k \in \mathbf{R}_{\geqslant 0}$. Now take $a_m = 0$ for $m \in \{1, 2, \ldots, k\} - \{k_0\}$, then $\pi_{U^+_J}\big(f_{k_0}(a)\big)u' \in U^+_{w_0^J,\geqslant 0}$ for all $a \in \mathbf{R}_{>0}$. Set $u_1 = \dot{v_{k_0}}^{-1}x_{i_{j_{k_0}}}(-1)\dot{v_{k_0}}$. Then $u_1^n u' \in U^+_{w_0^J,\geqslant 0}$ for all $n \in N$. Thus by 1.3.11, $v_{k_0}^{-1}\alpha_{i_{j_{k_0}}} = \alpha_{j'}$ for some $j' \in J$ and $u_1 \in U^+_{w_0^J,\geqslant 0}$. By 1.3.12, $u_1 = x_{j'}(-c)$ for some $c \in \mathbf{R}_{>0}$. That is a contradiction. The proposition is proved. $\square$

Let me recall that $L = P_J \cap Q_J$ (see 2.4). Now I will prove the main theorem.

**Theorem 1.3.14.** *For any $v, w, v', w' \in W^J$ such that $v \leqslant w, v' \leqslant w'$, set*

$$\tilde{Z}_{J,>0}^{v,w,v',w'} = \left\{ \left({}^{g}P_J, {}^{\psi(g')^{-1}}Q_J, gH_{P_J}lU_{Q_J}\psi(g')\right) \middle| \begin{matrix} g \in G_{\mathbf{v}_+,\mathbf{w},>0}, \quad g' \in G_{\mathbf{v}'_+,\mathbf{w}',>0} \\ \text{and } l \in L_{\geqslant 0} \end{matrix} \right\}.$$

*Then*

$$Z_{J,>0}^{v,w,v',w'} = \begin{cases} \tilde{Z}_{J,>0}^{v,w,v',w'}, & \text{if } v, w, v', w' \in W^J, v \leqslant w, v' \leqslant w'; \\ \varnothing, & \text{otherwise.} \end{cases}$$

*Proof.* Note that $\{(P, Q, \gamma) \in Z_J \mid P \in \mathcal{P}^J_{\geqslant 0}, \psi(Q) \in \mathcal{P}^J_{\geqslant 0}\}$ is a closed subset containing $Z_{J,>0}$. Hence it contains $Z_{J,\geqslant 0}$. Now fix $g \in G_{\mathbf{v}_+,\mathbf{w},>0}, g' \in G_{\mathbf{v}'_+,\mathbf{w}',>0}$ and $l \in L$. By 1.3.10 (a), for any $u \in U^+_{v^{-1},>0}$, $ug = at\pi_{U^+}(ug)$ for some $a \in U^-_{w,>0}$ and $t \in T_{>0}$. Similarly, for any $u' \in U^+_{v'^{-1},>0}$, $u'g' = a't'\pi_{U^+}(u'g')$ for some $a' \in U^-_{w',>0}$ and $t' \in T_{>0}$. Set $z = \left({}^{g}P_J, {}^{\psi(g')^{-1}}Q_J, gH_{P_J}lU_{Q_J}\psi(g')\right)$. We have

$$\begin{aligned} \left(u, \psi(u')^{-1}\right) \cdot z &= \left({}^{a}P_J, {}^{\psi(a')^{-1}}Q_J, at\pi_{U^+}(ug)H_{P_J}lU_{Q_J}\psi\left(\pi_{U^+}(u'g')\right)t'\psi(a')\right) \\ &= \left({}^{a}P_J, {}^{\psi(a')^{-1}}Q_J, aH_{P_J}t\pi_{U_J^+}(ug)l\psi\left(\pi_{U_J^+}(u'g')\right)t'U_{Q_J}\psi(a')\right). \end{aligned}$$

Then $\left(u, \psi(u')^{-1}\right) \cdot z \in Z^1_{J,>0}$ if and only if $t\pi_{U_J^+}(ug)l\psi\left(\pi_{U_J^+}(u'g')\right)t' \in L_{\geqslant 0}Z(L)$, that is,

$$\begin{aligned} l &\in \pi_{U_J^+}(ug)^{-1}L_{\geqslant 0}Z(L)\psi\left(\pi_{U_J^+}(u'g')\right)^{-1} \\ &= \left(\pi_{U_J^+}(ug)^{-1}U^+_{w_0^J,\geqslant 0}\right)T_{>0}Z(L)\psi\left(\pi_{U_J^+}(u'g')^{-1}U^+_{w_0^J,\geqslant 0}\right). \end{aligned}$$

So by 1.3.5, $z \in Z_{J,\geqslant 0}$ if and only if

$$\begin{aligned} l &\in \cap_{\substack{u \in U^+_{v^{-1},>0} \\ u' \in U^+_{v'^{-1},>0}}} \left(\pi_{U_J^+}(ug)^{-1}U^+_{w_0^J,\geqslant 0}\right)T_{>0}Z(L)\psi\left(\pi_{U_J^+}(u'g')^{-1}U^+_{w_0^J,\geqslant 0}\right) \\ &= \cap_{u \in U^+_{v^{-1},>0}}\left(\pi_{U_J^+}(ug)^{-1}U^+_{w_0^J,\geqslant 0}\right)T_{>0}Z(L)\psi\left(\cap_{u' \in U^+_{v'^{-1},>0}}\pi_{U_J^+}(u'g')^{-1}U^+_{w_0^J,\geqslant 0}\right). \end{aligned}$$

By 1.3.7, $z \in Z_{J,\geqslant 0}$ if and only if $v, v' \in W^J$ and $l \in L_{\geqslant 0}Z(L)$. The theorem is proved. $\square$

**1.3.15** It is known that $G_{\geqslant 0} = \bigsqcup_{w,w' \in W} U^-_{w,>0} T_{>0} U^+_{w',>0}$, where for any $w, w' \in W$, $U^-_{w,>0} T_{>0} U^+_{w',>0}$ is a semi-algebraic cell (see [L1, 2.11]) and is a connected component of $B^+ \dot{w} B^+ \cap B^- \dot{w}' B^-$ (see [FZ]). Moreover, Rietsch proved in [R2, 2.8] that $\mathcal{B}_{\geqslant 0} = \bigsqcup_{v \leqslant w} \mathcal{R}_{v,w,>0}$, where for any $v, w \in W$ such that $v \leqslant w$, $\mathcal{R}_{v,w,>0}$ is a semi-algebraic cell and is a connected component of $\mathcal{R}_{v,w}$.

The following result generalizes these facts.

**Corollary 1.3.16.** $\overline{G_{>0}} = \bigsqcup_{J \subset I} \bigsqcup_{\substack{v,w,v',w' \in W^J \\ v \leqslant w, v' \leqslant w'}} \bigsqcup_{y,y' \in W_J} Z^{v,w,v',w';y,y'}_{J,>0}$. *Moreover, for any* $v, w, v', w' \in W^J, y, y' \in W_J$ *with* $v \leqslant w$, $v' \leqslant w'$, $Z^{v,w,v',w';y,y'}_{J,>0}$ *is a connected component of* $Z^{v,w,v',w';y,y'}_J$ *and is a semi-algebraic cell which is isomorphic to* $\mathbf{R}^d_{>0}$, *where* $d = l(w) + l(w') + 2l(w^J_0) + \mid J \mid - l(v) - l(v') - l(y) - l(y')$.

*Proof.* $\mathcal{P}^J_{v,w,>0}$ (resp. $\mathcal{P}^J_{v',w',>0}$) is a connected component of $\mathcal{P}^J_{v,w}$ (resp. $\mathcal{P}^J_{v',w'}$) (see [L3]). Thus $\{(P, Q, \gamma) \in Z^{v,w,v',w';y,y'}_J \mid P \in \mathcal{P}^J_{v,w,>0}, \psi(Q) \in \mathcal{P}^J_{v',w',>0}\}$ is open and closed in $Z^{v,w,v',w';y,y'}_J$. To prove that $Z^{v,w,v',w';y,y'}_{J,>0}$ is a connected component of $Z^{v,w,v',w';y,y'}_J$, it is enough to prove that $Z^{v,w,v',w';y,y'}_{J,>0}$ is a connected component of $\{(P, Q, \gamma) \in Z^{v,w,v',w';y,y'}_J \mid P \in \mathcal{P}^J_{v,w,>0}, \psi(Q) \in \mathcal{P}^J_{v',w',>0}\}$.

Assume that $g \in G_{\mathbf{v}_+,\mathbf{w},>0}, g' \in G_{\mathbf{v}'_+,\mathbf{w}',>0}$ and $l \in L$. We have that $({}^g P_J)^{B^+}$ is the unique element $B \in \mathcal{R}_{v,w}$ that is contained in ${}^g P_J$(see 1.4). Therefore $({}^g P_J)^{B^+} ={}^g B^+$. Similarly, we have that $({}^g P_J)^{B^-} ={}^{g\dot{w}^J_0} B^+$, $({}^{\psi(g'^{-1})} Q_J)^{B^+} ={}^{\psi(g'^{-1})\dot{w}^J_0} B^-$ and $({}^{\psi(g'^{-1})} Q_J)^{B^-} ={}^{\psi(g')^{-1}} B^-$. Thus $\mathrm{pos}\Big(({}^g P_J)^{B^+}, {}^{gl\psi(g')}\big(({}^{\psi(g'^{-1})} Q_J)^{B^+}\big)\Big) = \mathrm{pos}(B^+, {}^{l\dot{w}^J_0} B^-)$ and $\mathrm{pos}\Big(({}^g P_J)^{B^-}, {}^{gl\psi(g')}\big(({}^{\psi(g'^{-1})} Q_J)^{B^-}\big)\Big) = \mathrm{pos}({}^{\dot{w}^J_0} B^+, {}^l B^-)$. Therefore we have that $({}^g P_J, {}^{\psi(g')^{-1}} Q_J, g H_{P_J} l U_{Q_J} \psi(g')) \in Z^{v,w,v',w';y,y'}_J$ if and only if $l \in B^+ \dot{y} \dot{w}_0 B^+ \dot{w}_0 \dot{w}^J_0 \cap \dot{w}^J_0 B^+ \dot{y}' \dot{w}_0 B^+ \dot{w}_0 = B^+ \dot{y} B^- \dot{w}^J_0 \cap \dot{w}^J_0 B^+ \dot{y}' B^-$.

Note that $L \cap B^+ \subset {}^{\dot{w}^J_0} B^-$. Thus for any $x \in W_J$, $(L \cap B^+) \dot{x} (L \cap B^+) \subset B^+ \dot{x} \dot{w}^J_0 B^- \dot{w}^J_0$. Therefore,

$$\begin{aligned} L \cap B^+ \dot{y} B^- \dot{w}^J_0 &= \bigsqcup_{x \in W_J} (L \cap B^+) \dot{x} (L \cap B^+) \cap B^+ \dot{y} B^- \dot{w}^J_0 \\ &= (L \cap B^+) \dot{y} \dot{w}^J_0 (L \cap B^+). \end{aligned}$$

Similarly, $L \cap \dot{w}_0^J B^+ \dot{y}' B^- = (L \cap B^-) \dot{w}_0^J \dot{y}' (L \cap B^-)$.

Then $\{(P, Q, \gamma) \in Z_J^{v,w,v',w';y,y'} \mid P \in \mathcal{P}_{v,w,>0}^J, \psi(Q) \in \mathcal{P}_{v',w',>0}^J\}$ is isomorphic to $G_{v,w,>0} \times G_{v',w',>0} \times \big((L \cap B^+)\dot{y}\dot{w}_0^J(L \cap B^+) \cap (L \cap B^-)\dot{w}_0^J\dot{y}'(L \cap B^-)\big)/Z(L)$. Note that $\big((L \cap B^+)\dot{y}\dot{w}_0^J(L \cap B^+) \cap (L \cap B^-)\dot{w}_0^J\dot{y}'(L \cap B^-)\big) \cap L_{\geqslant 0} = U^-_{yw_0^J,>0} T_{>0} U^+_{w_0^J y',>0}$. Therefore

$$\begin{aligned} Z_{J,>0}^{v,w,v',w';y,y'} &\cong G_{v,w,>0} \times G_{v',w',>0} \times U^-_{yw_0^J,>0} T_{>0} U^+_{w_0^J y',>0} / \big(Z(L) \cap T_{>0}\big) \\ &\cong \mathbf{R}_{>0}^{l(w)+l(w')+2l(w_0^J)+|J|-l(v)-l(v')-l(y)-l(y')}. \end{aligned}$$

By 1.3.15, we have that $U^-_{yw_0^J,>0} T_{>0} U^+_{w_0^J y',>0} / \big(Z(L) \cap T_{>0}\big)$ is a connected component of $\big((L \cap B^+)\dot{y}\dot{w}_0^J(L \cap B^+) \cap (L \cap B^-)\dot{w}_0^J\dot{y}'(L \cap B^-)\big)/Z(L)$. The corollary is proved. $\square$

# Chapter 2

# Total positivity of the flag varieties

In [MR], Marsh and Rietsch gave a parametrization of the totally nonnegative part of the flag varieties. They proved the result using the theory of generalized Chamber Ansatz. In this chapter, I will give a new proof of this result using the inductive method in [R2]. In the appendix, I will also give an elementary proof of the symmetry of the totally positive part of the flag varieties, which was first proved by Lusztig in [L1] using the canonical basis.

## 2.1 Introduction

We keep the notations of section 1.1. and 1.3.6. Marsh and Rietsch proved the following theorem about the totally nonnegative part of the flag varieties in [MR, 11.3].

**Theorem.** *We have* $\mathcal{R}_{w,w'}^{>0} \subset \mathcal{R}_{\mathbf{w}_+,\mathbf{w}'}$. *Moreover, the isomorphism* $G_{\mathbf{w}_+,\mathbf{w}'} \xrightarrow{\sim} \mathcal{R}_{\mathbf{w}_+,\mathbf{w}'}$ *restricts to an isomorphism of real semi-algebraic varieties* $G_{\mathbf{w}_+,\mathbf{w}'}^{>0} \xrightarrow{\sim} \mathcal{R}_{w,w'}^{>0}$.

Note that Marsh and Rietsch's proof of the theorem relies on a generalization of Berenstein and Zelevinsky's Chamber Ansatz. Although the theory of Chamber Ansatz is more elementary than the theory of canonical basis, it is quite complicated. In section 2.2 and section 2.3, we will give a new proof of this theorem without using Chamber Ansatz and we will see that the theorem is a consequence of the inductive method in [R2]. The inductive method is based on the theory of canonical basis and thus is a non-elementary statement. However, once we have got the inductive method, we can easily prove the theorem.

## 2.2 Some technical tools

In the sequel, we will write $g \cdot B$ instead of ${}^gB = gBg^{-1}$.

**2.2.1** Let $w, v \in W$ such that $l(wv) = l(w) + l(v)$. Then we can define a morphism $\phi_{w,v} : B^+wv \cdot B^+ \to B^+w \cdot B^+$ which sends $B \in B^+wv \cdot B^+$ to the unique element $B' \in B^+w \cdot B^+$ that satisfies $\mathrm{pos}(B', B) = v$. Now I will prove an elementary property of the morphism.

**Lemma 2.2.2.** *Let $u, v, w \in W$ with $l(uvw) = l(u) + l(v) + l(w)$. Then $l(uv) = l(u) + l(v)$ and $l(vw) = l(v) + l(w)$. For any $B \in B^+vw \cdot B^+$ and $g \in B^+uB^+$, we have $g \cdot B \in B^+uvw \cdot B^+$ and $g \cdot \phi_{v,w}(B) = \phi_{uv,w}(g \cdot B)$.*

*Proof.* The fact that $g \cdot B \in B^+uvw \cdot B^+$ follows from the properties of the Bruhat decomposition. Moreover, $\mathrm{pos}\big(g \cdot \phi_{v,w}(B), g \cdot B\big) = \mathrm{pos}\big(\phi_{v,w}(B), B\big) = w$, $\mathrm{pos}(B^+, g \cdot B^+) = u$, $\mathrm{pos}\big(g \cdot B^+, g \cdot \phi_{v,w}(B)\big) = \mathrm{pos}\big(B^+, \phi_{v,w}(B)\big) = v$. Since $l(uv) = l(u) + l(v)$, $\mathrm{pos}\big(B^+, g \cdot \phi_{v,w}(B)\big) = uv$. By the definition of $\phi_{uv,w}$, we have $g \cdot \phi_{v,w}(B) = \phi_{uv,w}(g \cdot B)$. □

The following property was first proved by Marsh and Rietsch and is also needed in our proof.

**Proposition 2.2.3.** *Let $v \leqslant w$ in $W$ and suppose $\alpha$ is a simple root such that $u^{-1}\alpha > 0$ for all $v \leqslant u \leqslant w$. Then for all $x \in U_\alpha$ and $B \in \mathcal{R}_{v,w}$, we have $x \cdot B = B$.*

The original proof of this fact uses the theory of Chamber Ansatz. I will reprove the fact here without using the theory of Chamber Ansatz.

Note that $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$ is a dense subset of $\mathcal{R}_{v,w}$. Therefore, it is enough to show that for all $x \in U_\alpha$ and $B \in \mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$, we have $x \cdot B = B$. Hence, it suffices to prove the following lemma.

**Lemma 2.2.4.** *We define $\mathbf{N} \cdot R = \{n\alpha \mid n \in \mathbf{N}, \alpha \in R\}$ and $\mathbf{N} \cdot R^+ = \{n\alpha \mid n \in \mathbf{N}, \alpha \in R^+\}$. Let $v \leqslant w$ in $W$ and suppose $\alpha$ is a simple root such that $u^{-1}\alpha > 0$ for all $v \leqslant u \leqslant w$. Let $\mathbf{w} = (w_{(0)}, w_{(1)}, \ldots, w_{(n)})$ be a reduced expression of $w$ and $\mathbf{v}_+ = (v_{(0)}, v_{(1)}, \ldots, v_{(n)})$ the positive subexpression of $v$. Let $R(\mathbf{v}_+, \mathbf{w})$ be the subset of $\mathbf{N} \cdot R$ consisting of the elements of the form $v^{-1}(n_0\alpha - \sum_{j \in J^{\circ}_{\mathbf{v}_+}} n_j v_{(j-1)}\alpha_{i_j})$ such that $n_0 \in \mathbf{N}, n_j \in \mathbf{N} \cup \{0\}$ and for any $k$, $v_{(k)}^{-1}(n_0\alpha - \sum_{j \leqslant k, j \in J^{\circ}_{\mathbf{v}_+}} n_j v_{(j-1)}\alpha_{i_j}) \in \mathbf{N} \cdot R$. Then we have $R(\mathbf{v}_+, \mathbf{w}) \subset \mathbf{N} \cdot R^+$ and $g^{-1}xg \in \prod_{\beta \in R(\mathbf{v}_+, \mathbf{w}) \cap R} U_\beta$ for all $g \in G_{\mathbf{v}_+, \mathbf{w}}$ and $x \in U_\alpha$.*

*Proof.* Write $v'$ for $v_{(n-1)}$ and $w'$ for $w_{(n-1)}$. Set $\mathbf{v}'_+ = (v_{(0)}, v_{(1)}, \ldots, v_{(n-1)})$, $\mathbf{w}' = (w_{(0)}, w_{(1)}, \ldots, w_{(n-1)})$. Then $\mathbf{v}'_+$ is the positive expression of $v'$ in $\mathbf{w}'$. Now assume that $w_{(j)} = w_{(j-1)}s_{i_j}$ for all $j = 1, 2, \ldots, n$. I will show that $m\alpha_{i_n} \notin R(\mathbf{v}'_+, \mathbf{w}')$ for any $m \in \mathbf{N}$.

Suppose this is not true. Then $m\alpha_{i_n} = v_{(n-1)}^{-1}(n_0\alpha - \sum_{j \in J^{\circ}_{\mathbf{v}_+}, j \neq n} n_j v_{(j-1)}\alpha_{i_j})$ for some $m, n_0 \in \mathbf{N}, n_j \in \mathbf{N} \bigcup \{0\}$. Since $\mathbf{v}_+$ is a positive expression, $v_{(j-1)} < v_{(j-1)}s_{i_j}$ for all $j$. Thus $v_{(j-1)}\alpha_{i_j} \in R^+$ for all $j$. We have $n_0\alpha - \sum_{j \in J^{\circ}_{\mathbf{v}_+}, j \neq n} n_j v_{(j-1)}\alpha_{i_j} = mv_{(n-1)}\alpha_{i_n}$. Therefore, $n_0\alpha \geqslant mv_{(n-1)}\alpha_{i_n}$. Since $\alpha$ is a simple root and $v_{(n-1)}\alpha_{i_n}$ is a positive root, we must have $\alpha = v_{(n-1)}\alpha_{i_n}$. However, Marsh and Rietsch proved that $\alpha \neq v_{(n-1)}\alpha_{i_n}$ (see [MR, 11.8]). That is a contradiction. The statement is proved.

Now I will prove the lemma by induction on $l(w)$.

Note that if $u' \in W$ satisfies $v' \leqslant u' \leqslant w'$, then $u'^{-1}\alpha > 0$ (see [MR, 11.8]). Thus by induction hypothesis, $R(\mathbf{v}'_+, \mathbf{w}') \subset \mathbf{N} \cdot R^+$ and $g'^{-1}xg' \in \prod_{\beta \in R(\mathbf{v}'_+, \mathbf{w}') \cap R} U_\beta$ for all $g' \in G_{\mathbf{v}'_+, \mathbf{w}'}$ and $x \in U_\alpha$.

If $n \in J^+_{\mathbf{v}_+}$, then $R(\mathbf{v}_+, \mathbf{w}) = s_{i_n} \cdot R(\mathbf{v}'_+, \mathbf{w}')$. Since $R(\mathbf{v}'_+, \mathbf{w}') \subset \mathbf{N} \cdot R^+$ and $m\alpha_{i_n} \notin R(\mathbf{v}'_+, \mathbf{w}')$ for all $m \in \mathbf{N}$, we have $R(\mathbf{v}_+, \mathbf{w}) \subset \mathbf{N} \cdot R^+$. Note that $G_{\mathbf{v}_+, \mathbf{w}} = G_{\mathbf{v}'_+, \mathbf{w}'} \dot{s_{i_n}}$. So for any $g \in G_{\mathbf{v}_+, \mathbf{w}}$, we have

$$g^{-1}xg \in \dot{s_{i_n}}^{-1}\Big(\prod_{\beta \in R(\mathbf{v}'_+, \mathbf{w}') \cap R} U_\beta\Big)\dot{s_{i_n}} = \prod_{\beta \in R(\mathbf{v}'_+, \mathbf{w}') \cap R} U_{s_{i_n}\beta} = \prod_{\beta \in R(\mathbf{v}_+, \mathbf{w}) \cap R} U_\beta.$$

If $n \in J^{\circ}_{\mathbf{v}_+}$, then $v = v'$ and for any $g \in G_{\mathbf{v}_+, \mathbf{w}}$, there exists $g' \in G_{\mathbf{v}'_+, \mathbf{w}'}$ and $x' \in U_{-\alpha_{i_n}}$ such that $g = g'x'$. Then $g^{-1}xg \in x'^{-1}\big(\prod_{\beta \in R(\mathbf{v}'_+, \mathbf{w}') \cap R} U_\beta\big)x' =$

$\prod_{\beta\in R(\mathbf{v}'_+,\mathbf{w}')\cap R} x'^{-1}U_\beta x'$. Note that $\beta \neq \pm\alpha_{i_n}$ for any $\beta \in R(\mathbf{v}'_+,\mathbf{w}') \cap R^+$. Then by [S, 8.2.3], we have $u_1^{-1}U_\beta u_1 \subset \prod_{\gamma=m\beta-l\alpha_{i_n}\in R,\text{ for some } m\in\mathbf{N},l\in\mathbf{N}\bigcup\{0\}} U_\gamma$. It is easy to see that $\{\gamma\in R | \gamma = m\beta - l\alpha_{i_n} \text{ for some } \beta\in R(\mathbf{v}'_+,\mathbf{w}'), m\in\mathbf{N}, l\in\mathbf{N}\bigcup\{0\}\} = R(\mathbf{v}_+,\mathbf{w})$. So we have $g^{-1}xg \in \prod_{\beta\in R(\mathbf{v}_+,\mathbf{w})\cap R} U_\beta$. Moreover, since $R(\mathbf{v}'_+,\mathbf{w}') \subset \mathbf{N}\cdot R^+$ and $m\alpha_{i_n} \notin R(\mathbf{v}'_+,\mathbf{w}')$ for all $m\in\mathbf{N}$, we have $R(\mathbf{v}_+,\mathbf{w}) \subset \mathbf{N}\cdot R^+$. The lemma is proved. □

**2.2.5** Define the morphism $\pi_{U^-} : U^-B^+ \to U^-$ by $\pi_{U^-}(ub) = u$ for $u \in U^-$ and $b\in B^+$. Then it is easy to see that for any $u\in U^-, x\in U^-B^+, b\in B^+$, we have $\pi_{U^-}(uxb) = u\pi_{U^-}(x)$.

We will recall the following properties of totally nonnegative part of the flag varieties.

(1) Suppose $v, w, w' \in W$, such that $l(w) = l(wv)+l(v)$ and $l(w') = l(w'v)+l(v)$. If $wv \leqslant w'v$, then $w\leqslant w'$ and $\phi_{w'v,v} : \mathcal{R}_{w,w'} \to \mathcal{R}_{wv,w'v}$ is an isomorphism and its restriction to $\mathcal{R}^{>0}_{w,w'}$ is a bijection between $\mathcal{R}^{>0}_{w,w'}$ and $\mathcal{R}^{>0}_{wv,w'v}$ (see [R2, 2.3]).

(2) Let $w\in W$, $y\in U^+(w^{-1})$ and $B \in B^-w\cdot B^+$, then $B\in\mathcal{B}_{\geqslant 0}$ if and only if $y\cdot B\in\mathcal{B}_{\geqslant 0}$. In this case, we have $y\cdot B \in U^-\cdot B^+$. (see [R2, 2.2 & 2.6]).

(3) Let $w, w'\in W$ and $i\in I$, such that $w < ws_i$ and $w\leqslant w's_i < w'$. Then for any $y\in U^+(w^{-1})$, the map $\psi_y : \mathcal{R}^{>0}_{w,w's_i}\times\mathbf{R}_{>0} \to \mathcal{R}^{>0}_{w,w'}$ defined by $\psi_y(g\cdot B^+, a) = y^{-1}\pi_{U^-}(yg)y_i(a)\cdot B^+$ is a bijection (see [R2, 2.7]).

By property (2), we have $yg\in U^-B^+$. Thus $\pi_{U^-}(yg)$ is defined. It is easy to see that $\psi_y(B,a)$ does not depend on the choice of $g\in G$ such that $g\cdot B^+ = B$. Therefore $\psi_y$ is well-defined.

Note that properties (1) and (3) allows us to study the totally nonnegative part of the flag varieties in an inductive way. We will make use of these properties freely.

## 2.3 Proof of the theorem

Since $g\mapsto g\cdot B^+$ gives rise to an isomorphism $G_{\mathbf{w}_+,\mathbf{w}'} \xrightarrow{\sim} \mathcal{R}_{\mathbf{w}_+,\mathbf{w}'}$, it suffices to show that $\mathcal{R}^{>0}_{\mathbf{w}_+,\mathbf{w}'} = \{g\cdot B^+ \mid g\in G^{>0}_{\mathbf{w}_+,\mathbf{w}'}\}$. Note that $1\in J^\circ_{\mathbf{w}_+}$ if and only if

$w \leqslant w'_{(0)}w'$. Moreover, we have $\mathcal{R}_{1,1}^{>0} = \{B^+\}$. Therefore, it is enough to prove the following theorem.

**Theorem 2.3.1.** *Let $w \leqslant w' \in W$ and $s$ a simple reflection with $sw' < w'$. Then we have*

$$\mathcal{R}_{w,w'}^{>0} = \begin{cases} U^-(s) \cdot \mathcal{R}_{w,sw'}^{>0}, & \text{if } w \leqslant sw'; \\ \dot{s} \cdot \mathcal{R}_{sw,sw'}^{>0}, & \text{otherwise} . \end{cases}$$

*Proof.* It is easy to see that $\mathcal{R}_{1,s}^{>0} = U^-(s) \cdot B^+$ and $\mathcal{R}_{s,s}^{>0} = \dot{s} \cdot B^+$. Therefore the theorem holds if $l(w') = 1$. Now assume that $l(w') > 1$. I will prove the theorem by induction on $l(w')$.

Fixing a reduced expression $\mathbf{w}' = (w'_{(0)}, w'_{(1)}, \dots, w'_{(n)})$ of $w'$ such that $w'_{(1)} = s$. Then there exists a unique positive subexpression $\mathbf{w}_+ = (w_{(0)}, w_{(1)}, \dots, w_{(n)})$ of $w$ in $\mathbf{w}'$. Set $s' = {w'_{(n-1)}}^{-1} w'$. Since $n \geqslant 2$, we have $sw's' < w's' < w'$. We need to check the following two cases.

Case 1: $ws' < w$.

In this case, $n \in J^+_{\mathbf{w}_+}$. Hence $ws' \leqslant w's'$ and $\phi_{w's',s'} : \mathcal{R}_{w,w'} \to \mathcal{R}_{ws',w's'}$ is an isomorphism and its restriction to $\mathcal{R}_{w,w'}^{>0}$ is a bijection between $\mathcal{R}_{w,w'}^{>0}$ and $\mathcal{R}_{ws',w's'}^{>0}$.

If $w \leqslant sw'$, then $1 \in J^\circ_{\mathbf{w}_+}$ and $ws' \leqslant sw's'$. For any $B \in \mathcal{R}_{w,w'}^{>0}$, $\phi_{w's',s'}(B) \in \mathcal{R}_{ws',w's'}^{>0}$. By induction hypothesis, we have $\phi_{w's',s'}(B) \in U^-(s) \cdot \mathcal{R}_{ws',sw's'}^{>0}$. Since $\mathcal{R}_{ws',sw's'}^{>0} = \phi_{sw's',s'}(\mathcal{R}_{w,sw'}^{>0})$, there exist $u \in U^-(s)$ and $B' \in \mathcal{R}_{w,sw'}^{>0}$ such that $\phi_{w's',s'}(B) = u \cdot \phi_{sw's',s'}(B') = \phi_{w's',s'}(u \cdot B')$. Since $B$ and $u \cdot B'$ are contained in $\mathcal{R}_{w,w'}$, we have $B = u \cdot B'$. On the other hand, $U^-(s) \cdot \mathcal{R}_{w,sw'}^{>0} \subset \mathcal{B}_{\geqslant 0} \cap \mathcal{R}_{w,w'} = \mathcal{R}_{w,w'}^{>0}$. Thus the theorem holds for $\mathcal{R}_{w,w'}^{>0}$.

If $w \nleqslant sw'$, then $1 \in J^+_{\mathbf{w}_+}$. Hence $ws' \nleqslant sw's'$ and $sws' < sw \leqslant sw'$. For any $B \in \mathcal{R}_{sw,sw'}^{>0}$, $\text{pos}(B^+, \dot{s} \cdot B^+) = s$ and $\text{pos}(\dot{s} \cdot B^+, \dot{s} \cdot B) = \text{pos}(B^+, B) = sw'$. Thus $\text{pos}(B^+, \dot{s} \cdot B) = w'$. Moreover, $\text{pos}(B^-, \dot{s} \cdot B^-) = \text{pos}(\dot{w}_0 \cdot B^+, \dot{s}\dot{w}_0 \cdot B^+) = w_0 s w_0$ and $\text{pos}(\dot{s} \cdot B^-, \dot{s} \cdot B) = \text{pos}(B^-, B) = w_0 sw$. Thus $\text{pos}(B^-, \dot{s} \cdot B) = w_0 sw$ or $w_0 w$. Therefore, $\dot{s} \cdot B \in \mathcal{R}_{sw,w'}$ or $\mathcal{R}_{w,w'}$. Note that $\phi_{w's',s'}(\mathcal{R}_{sw,w'}) = \mathcal{R}_{sws',w's'}$. Since $\phi_{w's',s'}(\dot{s} \cdot B) = \dot{s} \cdot \phi_{sw's',s'}(B) \in \dot{s} \cdot \mathcal{R}_{sws',sw's'}^{>0}$. By induction hypothesis, $\phi_{w's',s'}(\dot{s} \cdot B) \in \mathcal{R}_{ws',w's'}^{>0}$. Therefore $\dot{s} \cdot B \notin \mathcal{R}_{sw,w'}$ and we have $\dot{s} \cdot B \in \mathcal{R}_{w,w'}^{>0}$.

On the other hand, for any $B \in \mathcal{R}^{>0}_{w,w'}$, $\phi_{w's',s'}(B) \in \mathcal{R}^{>0}_{ws',w's'}$. By induction hypothesis $\phi_{w's',s'}(B) \in \dot{s} \cdot \mathcal{R}^{>0}_{sws',sw's'} = \dot{s} \cdot \phi_{sw's',s'}(\mathcal{R}^{>0}_{sw,sw'})$. Therefore, there exists $B' \in \mathcal{R}^{>0}_{sw,sw'}$, such that $\phi_{w's',s'}(B) = \dot{s} \cdot \phi_{sw's',s'}(B') = \phi_{w's',s'}(\dot{s} \cdot B')$. Since $B$ and $\dot{s} \cdot B'$ are contained in $\mathcal{R}_{w,w'}$, we have $B = \dot{s} \cdot B'$. Thus the theorem holds for $\mathcal{R}^{>0}_{w,w'}$.

Case 2: $w < ws'$.

In this case, $n \in J^{\circ}_{\mathbf{W}_+}$ and $w < w's'$. We assume that $s = s_i$ and $s' = s_j$.

If $w \leqslant sw'$, then $1 \in J^{\circ}_{\mathbf{W}_+}$ and $w \leqslant sw's'$. Take $y \in U^+(w^{-1})$. For any $B \in \mathcal{R}^{>0}_{w,w'}$, there exists $B' \in \mathcal{R}^{>0}_{w,w's'}$ and $a \in \mathbf{R}_{>0}$ such that $B = \psi_y(B', a)$. By induction hypothesis, $B' = ug \cdot B^+$ for some $u \in U^-(s)$ and $g \cdot B^+ \in \mathcal{R}^{>0}_{w,sw's'}$. Then $\psi_y(B', a) = y^{-1}\pi_{U^-}(yug)y_i(a) \cdot B^+$. By [L1, 2.11], $yu = u_1 y_1 t$ for some $u_1 \in U^-(s), y_1 \in U^+(w^{-1})$ and $t \in T_{>0}$. Then $\pi_{U^-}(yug) = \pi_{U^-}(u_1 y_1 t g) = u_1 \pi_{U^-}(y_1 t g)$ and $y^{-1}\pi_{U^-}(yug)y_i(a) \cdot B^+ = ut^{-1}y_1^{-1}\pi_{U^-}(y_1 t g)y_i(a) \cdot B^+$. Since $y_1^{-1}\pi_{U^-}(y_1 t g)y_i(a) \cdot B^+ \in \mathcal{R}^{>0}_{w,sw'}$ and $T_{>0} \cdot \mathcal{R}^{>0}_{w,sw'} \subset \mathcal{R}^{>0}_{w,sw'}$, we have $B \in U^-(s) \cdot \mathcal{R}^{>0}_{w,sw'}$. On the other hand, $U^-(s) \cdot \mathcal{R}^{>0}_{w,sw'} \subset \mathcal{B}_{\geqslant 0} \cap \mathcal{R}_{w,w'} = \mathcal{R}^{>0}_{w,w'}$. Thus the theorem holds for $\mathcal{R}^{>0}_{w,w'}$.

If $w \nleqslant sw'$, then $1 \in J^{+}_{\mathbf{W}_+}$. Hence $sw < w$ and $sw \leqslant sw's' < sw'$. For any $B \in \mathcal{R}^{>0}_{sw,sw'}$, we have seen that $\dot{s} \cdot B \in \mathcal{R}_{sw,w'}$ or $\mathcal{R}_{w,w'}$. Choosing $y \in U^+(w^{-1}s)$, then there exists $B' = g \cdot B^+ \in \mathcal{R}^{>0}_{sw,sw's'}$ and $a \in \mathbf{R}_{>0}$, such that $B = \psi_y(B', a)$. So we have $\mathrm{pos}(\dot{s} \cdot B', \dot{s} \cdot B) = \mathrm{pos}(B', B) = \mathrm{pos}(g \cdot B^+, y^{-1}\pi_{U^-}(yg)y_j(a) \cdot B^+)$. Note that $g^{-1}y^{-1}\pi_{U^-}(yg)y_j(a) \subset g^{-1}y^{-1}ygB^+y_j(a) \subset B^+s'B^+$. Thus $\mathrm{pos}(\dot{s} \cdot B', \dot{s} \cdot B) = s'$. Moreover, we have $\mathrm{pos}(B^-, \dot{s} \cdot B') = w_0 w$ since $\dot{s} \cdot B' \in \mathcal{R}^{>0}_{w,w's'}$ by induction hypothesis. Therefore, $\mathrm{pos}(B^-, \dot{s} \cdot B) = w$ or $ws'$. If $\mathrm{pos}(B^-, \dot{s} \cdot B) \neq w$, then $\mathrm{pos}(B^-, \dot{s} \cdot B) = ws'$ and $\mathrm{pos}(B^-, \dot{s} \cdot B) = sw$. However, $sw < w < ws'$. That is a contradiction and we must have $\mathrm{pos}(B^-, \dot{s} \cdot B) = w$ and $\dot{s} \cdot B \in \mathcal{R}_{w,w'}$. Now for any $a \in \mathbf{R}_{>0}$, we have $x_i(a)\dot{s} = y_i(a^{-1})\alpha_i^\vee(a)x_i(-a^{-1})$. Then $yx_i(a)\dot{s} \cdot B = yy_i(a^{-1})\alpha_i^\vee(a)x_i(-a^{-1}) \cdot B$. By 2.2.3, $x_i(-a^{-1}) \cdot B = B \in \mathcal{B}_{\geqslant 0}$. So $yx_i(a)\dot{s} \cdot B = yy_i(a^{-1})\alpha_i^\vee(a) \cdot B \in \mathcal{B}_{\geqslant 0}$. Note that $yx_i(a) \in U^+(w^{-1})$. Then we have $\dot{s} \cdot B \in \mathcal{B}_{\geqslant 0}$. Hence $\dot{s} \cdot B \in \mathcal{R}^{>0}_{w,w'}$.

On the other hand, for any $B \in \mathcal{R}^{>0}_{w,w'}$ and $y \in U^+(w^{-1})$, there exists $B' \in \mathcal{R}^{>0}_{w,w's'}$ and $a \in \mathbf{R}_{>0}$, such that $B = \psi_y(B', a)$. By induction hypothesis, $B' = \dot{s}g \cdot B^+$ for some $g \in G$ with $g \cdot B^+ \in \mathcal{R}^{>0}_{sw,sw's'}$. We have that $y = y_1 x_i(b)$ for some $y_1 \in U^+(w^{-1}s)$ and $b \in \mathbf{R}_{>0}$. Then $y\dot{s}g = y_1 x_i(b)\dot{s}g = y_1 y_i(b^{-1})\alpha_i^\vee(b)x_i(-b^{-1})g$. By 2.2.3,

$x_i(-b^{-1})g \cdot B^+ = g \cdot B^+$. So $x_i(-b^{-1})g \in gB^+$. We have that $y_1 y_i(b^{-1}) = uy_2t$ for some $u \in U^-(s), y_2 \in U^+(w^{-1})$ and $t \in T_{>0}$. Therefore,

$$\begin{aligned} B &= y^{-1}\pi_{U^-}(y\dot{s}g)y_j(a) \cdot B^+ = y^{-1}\pi_{U^-}(uy_2t\alpha_i^\vee(b)g)y_j(a) \cdot B^+ \\ &= x_i(-b)y_i(b^{-1})t^{-1}y_2^{-1}\pi_{U^-}(y_2t\alpha_i^\vee(b)g)y_j(a) \cdot B^+ \\ &= \dot{s}x_i(b)\alpha_i^\vee(b^{-1})t^{-1}y_2^{-1}\pi_{U^-}(y_2t\alpha_i^\vee(b)g)y_j(a) \cdot B^+. \end{aligned}$$

We have $t\alpha_i^\vee(b)g \cdot B^+ \in \mathcal{R}^{>0}_{sw,sw't}$ and $y_2^{-1}\pi_{U^-}(y_2t\alpha_i^\vee(b)g)y_j(a) \cdot B^+ \in \mathcal{R}^{>0}_{sw,sw'}$. Hence, $B \in \dot{s}x_i(b)\alpha_i^\vee(b^{-1})t^{-1} \cdot \mathcal{R}^{>0}_{sw,sw'} = \dot{s} \cdot \mathcal{R}^{>0}_{sw,sw'}$. The theorem is proved. □

## 2.4 Appendix

We will give a new proof of the following property.

**Proposition.** *We have that $U^+_{>0} \cdot B^- = U^-_{>0} \cdot B^+$.*

*Proof.* At first, I will prove the following statement:

if $s_iw > w$ for $i \in I$, then $U^+(w^{-1}s_i)\dot{s}_i\dot{w} \cdot B^+ \subset U^-(s_i)U^+(w^{-1})\dot{w} \cdot B^+$.

For any $u \in U^+(w^{-1}s_i)$, we have $u = u_1x_i(a)$ for some $u_1 \in U^+(w^{-1})$ and $a \in \mathbf{R}_{>0}$. Then $u\dot{s}_i\dot{w} = u_1x_i(a)\dot{s}_i\dot{w} = u_1y_i(a^{-1})\alpha^\vee(a)x_i(-a^{-1})\dot{w}$. We have $u_1y_i(a^{-1}) \in U^-(s_i)U^+(w^{-1})T$ (see [L, 2.11]). Since $s_iw > w$, we have $w^{-1}\alpha_i > 0$ and $\dot{w}^{-1}Tx_i(-a^{-1})\dot{w} \subset TU_{w^{-1}\alpha_i} \subset B^+$. Thus $u\dot{s}_i\dot{w} \cdot B^+ \in U^-(s_i)U^+(w^{-1})\dot{w} \cdot B^+$. The statement is proved.

As a consequence of the statement, we can see easily that $U^+(w^{-1})\dot{w} \cdot B^+ \subset U^-(w) \cdot B^+$. In particular, $U^+_{>0} \cdot B^- = U^+_{>0}\dot{w}_0 \cdot B^+ \subset U^-_{>0} \cdot B^+$. Similarly, $U^-_{>0} \cdot B^+ \subset U^+_{>0} \cdot B^-$. The proposition is proved. □ □

# Chapter 3

# Unipotent variety in the group compactification

The unipotent variety of a reductive algebraic group $G$ plays an important role in the representation theory. In this paper, we will consider the closure $\bar{\mathcal{U}}$ of the unipotent variety in the De Concini-Procesi compactification $\bar{G}$ of a quasi-simple, adjoint group $G$. We will prove that $\bar{\mathcal{U}} - \mathcal{U}$ is a union of some $G$-stable pieces introduced by Lusztig in [L8]. This was first conjectured by Lusztig. We will also give an explicit description of $\bar{\mathcal{U}}$. It turns out that similar results hold for the closure of any Steinberg fiber in $\bar{G}$.

## 3.0. Introduction

A connected simple algebraic group $G$ has a "wonderful" compactification $\bar{G}$, introduced by De Concini and Procesi. The variety $\bar{G}$ is a smooth, projective variety with $G \times G$ action on it. The $G \times G$-orbits of $\bar{G}$ are indexed by the subsets of the simple roots.

The group $G$ acts diagonally on $\bar{G}$. Lusztig introduced a partition of $\bar{G}$ into finitely many $G$-stable pieces. The $G$-orbits on each piece are in one-to-one correspondence to the conjugacy classes of a certain reductive group. Based on the partition, he developed the theory of "Parabolic Character Sheaves" on $\bar{G}$.

In this chapter, we study the closure $\bar{\mathcal{U}}$ of the unipotent variety $\mathcal{U}$ of $G$ in $\bar{G}$, partially based on the previous work of [Spr3]. The main result is that the boundary of the closure is a union of some $G$-stable pieces. (see Theorem 4.3.)

The unipotent variety plays an important role in the representation theory. One would expect that $\bar{\mathcal{U}}$, the subvariety of $\bar{G}$, which is analogous to the subvariety $\mathcal{U}$ of $G$, also plays an important role in the theory of "Parabolic Character Sheaves". Our result is a step toward this direction.

The arrangement of this chapter is as follows. In section 3.1, we briefly recall some results on the $B \times B$-orbits of $\bar{G}$ (where $B$ is a Borel subgroup of $G$) and results on $\bar{\mathcal{U}}$, which were proved by Springer in [Spr2] and [Spr3]. In section 3.2, we first recall the definition of the $G$-stable pieces and then in 3.2.6, we show that any $G$-stable piece is the minimal $G$-stable subset of $\bar{G}$ that contains a particular $B \times B$-orbit. In the remaining part of section 3.2, we establish some basic facts about the Coxeter elements, which will be used in section 3.4 to prove our main theorem. In section 3.3, we show case-by-case that certain $G$-stable pieces are contained in $\bar{\mathcal{U}}$. Hence a lower bound of $\bar{\mathcal{U}}$ is established.

A naive thought about $\bar{\mathcal{U}}$ is that the boundary of the "unipotent elements" are "nilpotent cone". In fact, it is true. A precise statement is given and proved in 4.3. Thus we obtain an upper bound of $\bar{\mathcal{U}}$. We also show in 4.3 that the lower bound is actually equal to the upper bound. Therefore, our main theorem is proved. In section 4, we also consider the closure of arbitrary Steinberg fiber of $G$ in $\bar{G}$. (An example of Steinberg fiber is $\mathcal{U}$.) The results are similar. In the end of section 4, we calculate the number of points of $\bar{\mathcal{U}}$ over a finite field. The formula bears some resemblance to the formula for $\bar{G}$.

## 3.1 Preliminaries

**3.1.1** We keep the notation of 1.1.1-1.1.4. In this chapter, we assume that $G$ is a simple group.

For $J \subset I$, let $P_J^- \supset B^-$ be the opposite of $P_J$. Set $L_J = P_J \cap P_J^-$. Then $L_J$ is a Levi subgroup of $P_J$ and $P_J^-$. Let $Z_J$ be the center of $L_J$ and $G_J = L_J/Z_J$ be its adjoint group. We denote by $\pi_{P_J}$ (resp. $\pi_{P_J^-}$) the projection of $P_J$ (resp. $P_J^-$) onto $G_J$.

Let $\bar{G}$ be the wonderful compactification of $G$ ([DP] deals with the case $k = \mathbf{C}$. The generalization to arbitrary $k$ was given in [Str]). It is an irreducible, projective smooth $G \times G$-variety. The $G \times G$-orbits $Z_J$ of $\bar{G}$ are indexed by the subsets $J$ of $I$. Moreover, $Z_J = (G \times G) \times_{P_J^- \times P_J} G_J$, where $P_J^- \times P_J$ acts on the right on $G \times G$ and on the left on $G_J$ by $(q, p) \cdot z = \pi_{P_J^-}(q) z \pi_{P_J}(p)^{-1}$. Let $h_J$ be the image of $(1, 1, 1)$ in $Z_J$.

We will identify $Z_I$ with $G$ and the $G \times G$-action on it is given by $(g, h) \cdot x = gxh^{-1}$.

For any subvariety $X$ of $\bar{G}$, we denote by $\bar{X}$ the closure of $X$ in $\bar{G}$.

**3.1.2** For any closed subgroup $H$ of $G$, we denote by $H_{diag}$ the image of the diagonal embedding of $H$ in $G \times G$.

We will simply write $U$ for $U^+$. For $J \subset I$, set $U_J = U \cap L_J$ and $U_J^- = U^- \cap L_J$.

For parabolic subgroups $P$ and $Q$, define

$$P^Q = (P \cap Q) U_P.$$

It is easy to see that for $J, K \subset I$ and $u \in {}^J W^K$, $P_J^{({}^{\dot{u}} P_K)} = P_{J \cap \mathrm{Ad}(u) K}$.

Let $\mathcal{U}$ be the unipotent variety of $G$. Then $\mathcal{U}$ is stable under the action of $G_{diag}$ and $U$ is stable under the action of $U \times U$ and $T_{diag}$. Moreover, $\mathcal{U} = G_{diag} \cdot U$. Similarly, $\bar{\mathcal{U}} = G_{diag} \cdot \bar{U}$ (see [Spr3, 1.4]).

**3.1.3** Now consider the $B \times B$-orbits on $\bar{G}$. We use the same notation as in [Spr2]. For any $J \subset I$, $u, v \in W$, set $[J, u, v] = (B \times B)(\dot{u}, \dot{v}) \cdot h_J$. It is easy to see that $[J, u, v] = [J, x, vz^{-1}]$, where $u = xz$ with $x \in W^J$ and $z \in W_J$. Moreover, $\bar{G} = \bigsqcup_{J \subset I} \bigsqcup_{x \in W^J, w \in W} [J, x, w]$. Springer proved the following result in [Spr2, 2.4].

**Theorem 3.1.4.** *Let $x \in W^J$, $x' \in W^K$, $w, w' \in W$. Then $[K, x', w']$ is contained in $\overline{[J, x, w]}$ if and only if $K \subset J$ and there exists $u \in W_K, v \in W_J \cap W^K$ with $xvu^{-1} \leqslant x'$, $w'u \leqslant wv$ and $l(wv) = l(w) + l(v)$.*

As a consequence of the theorem, we have the following properties which will be used later.

(1) For any $K \subset J$, $w \in W^J$ and $v \in W_J$, $[K, wv, v] \subset \overline{[J, w, 1]}$.

(2) For any $J \subset I$, $w, w' \in W^J$ with $w \leqslant w'$, then $[J, w', 1] \subset \overline{[J, w, 1]}$.

**3.1.5** In this subsection, we recall some results of [Spr3].

Let $\epsilon$ be an indeterminate. Put $o = k[[\epsilon]]$ and $K = k((\epsilon))$. An $o$-valued point of a $k$-variety $Z$ is a $k$-morphism $\gamma : \mathrm{Spec}(o) \to Z$. We write $Z(o)$ for the set of all $o$-valued points of $Z$. Similarly, we write $Z(K)$ for the set of all $K$-valued points of $Z$. For $\gamma \in Z(o)$, we have that $\gamma(0) \in Z$, where 0 is the closed point of $\mathrm{Spec}(o)$.

By the valuative criterion of completeness (see [EGA, ChII, 7.3.8, 7.3.9]), for the complete $k$-variety $\bar{G}$, the inclusion $o \hookrightarrow K$ induces a bijective from $\bar{G}(o)$ onto $\bar{G}(K)$. Therefore, any $\gamma \in \bar{G}(K)$ defines a point $\gamma(0) \in \bar{G}$. In particular, any $\gamma \in U(K)$ defines a point $\gamma(0) \in \bar{G}$. Here we regard $U(K)$ as a subset of $\bar{G}(K)$ in the natural way.

We have that $x \in \bar{U}$ if and only if there exists $\gamma \in U(K)$ such that $\gamma(0) = x$ (see [Spr3, 2.2]).

Let $Y$ be the cocharacter group of $T$. An element $\lambda \in Y$ defines a point in $T(k[\epsilon, \epsilon^{-1}])$, hence a point $p_\lambda$ of $T(K)$. Let $H \subset G(o)$ be the subgroup consisting of elements $\gamma$ with $\gamma(0) \in B$. Then for $\gamma \in U(K)$, there exists $\gamma_1, \gamma_2 \in H$, $w \in W$ and $\lambda \in Y$, such that $\gamma = \gamma_1 \dot{w} p_\lambda \gamma_2$. Moreover, $w$ and $\lambda$ are uniquely determined by $\gamma$ (see [Spr3, 2.6]). In this case, we will call $(w, \lambda)$ admissible. Springer showed that $(w, \lambda - w^{-1}\lambda)$ is admissible for any dominant regular coweight $\lambda$ (see [Spr3, 3.1]).

For $\lambda \in Y$ and $x \in W$ with $x^{-1} \cdot \lambda$ dominant, we have that $p_\lambda(0) = (\dot{x}, \dot{x}) \cdot h_{I(x^{-1}\lambda)}$, where $I(x^{-1}\lambda)$ is the set of simple roots orthogonal to $x^{-1}\lambda$ (see [Spr3, 2.5]). If moreover, $(w, \lambda)$ is admissible, then there exists some $t \in T$ such that $(U \times U)(\dot{w}\dot{x}t, \dot{x}) \cdot h_{I(x^{-1}\lambda)} \subset \bar{U}$.

## 3.2 the partition of $Z_J$

**3.2.1** We will follow the set-up of [L8, 8.18].

For $J, J' \subset I$ and $y \in {}^{J'}W^J$ with $\mathrm{Ad}(y)J = J'$, define

$$\tilde{Z}_J^y = \{(P, P', \gamma) \mid P \in \mathcal{P}^J, P' \in \mathcal{P}^{J'}, \gamma = U_{P'} g U_P, \mathrm{pos}(P', {}^g P) = y\}$$

with the $G \times G$ action given by $(g_1, g_2) \cdot (P, Q, \gamma) = ({}^{g_1}P, {}^{g_2}P', g_2 \gamma g_1^{-1})$.

To $z = (P, P', \gamma) \in \tilde{Z}_J^y$, we associate a sequence $(J_k, J'_k, u_k, y_k, P_k, P'_k, \gamma_k)_{k \geqslant 0}$ with $J_k, J'_k \subset I$, $u_k \in W$, $y_k \in {}^{J'_k}W^{J_k}, \mathrm{Ad}(y_k)J_k = J'_k$, $P_k \in \mathcal{P}_{J_k}, P'_k \in \mathcal{P}_{J'_k}, \gamma_k = U_{P'_k} g U_{P_k}$ for some $g \in G$ satisfies $\mathrm{pos}(P'_k, {}^g P_k) = u_k$. The sequence is defined as follows.

$$P_0 = P, P'_0 = P', \gamma_0 = \gamma, J_0 = J, J'_0 = J', u_0 = \mathrm{pos}(P'_0, P_0), y_0 = y.$$

Assume that $k \geqslant 1$, that $P_m, P'_m, \gamma_m, J_m, J'_m, u_m, y_m$ are already defined for $m < k$ and that $u_m = \mathrm{pos}(P'_m, P_m), P_m \in \mathcal{P}_{J_m}, P'_m \in \mathcal{P}_{J'_m}$ for $m < k$. Let

$$J_k = J_{k-1} \cap \mathrm{Ad}(y_{k-1}^{-1} u_{k-1}) J_{k-1}, J'_k = J_{k-1} \cap \mathrm{Ad}(u_{k-1}^{-1} y_{k-1}) J_{k-1},$$

$$P_k = g_{k-1}^{-1} ({}^{g_{k-1}}P_{k-1})^{(P'_{k-1}{}^{P_{k-1}})} g_{k-1} \in \mathcal{P}_{J_k}, P'_k = P_{k-1}^{P'_{k-1}} \in \mathcal{P}_{J'_k}$$

where

$$g_{k-1} \in \gamma_{k-1} \text{ is such that } {}^{g_{k-1}}P_{k-1} \text{ contains some Levi of } P_{k-1} \cap P'_{k-1},$$

$$u_k = \mathrm{pos}(P'_k, P_k), y_k = u_{k-1}^{-1} y_{k-1}, \gamma_k = U_{P'_k} g_{k-1} U_{P_k}.$$

It is known that the sequence is well defined. Moreover, for sufficient large $n$, we have that $J_n = J'_n = J_{n+1} = J'_{n+1} = \cdots$ and $u_n = u_{n+1} = \cdots = 1$. Now we set $\beta(z) = u_0 u_1 \cdots u_n$, $n \gg 0$. Then we have that $\beta(z) \in {}^{J'}W$. By [L8, 8.18] and [L7, 2.5], the sequence $(J_k, J'_k, u_k, y_k)_{k \geqslant 0}$ is uniquely determined by $(J, \beta(z), y)$.

The map $w \mapsto yw^{-1}$ is a bijection between $W^J$ and ${}^{J'}W$. For $w \in W^J$, set

$$\tilde{Z}^y_{J,w} = \{z \in \tilde{Z}^y_J \mid \beta(z) = yw^{-1}\}.$$

Then $(\tilde{Z}^y_{J,w})_{w \in W^J}$ is a partition of $\tilde{Z}^y_J$ into locally closed $G$-stable subvarieties. For $w \in W^J$, let $(J_k, J'_k, u_k, y_k)_{k \geqslant 0}$ be the sequence uniquely determined by $(J, yw^{-1}, y)$. Then $(P, P', \gamma) \mapsto (P_1, P'_1, \gamma_1)$ define a $G$-equivariant map $\vartheta : \tilde{Z}^y_{J,w} \to \tilde{Z}^{y_1}_{J_1, u_0^{-1} w}$.

**3.2.2** Let $J \subset I$. Set $\tilde{Z}_J = \tilde{Z}^{w_0 w_0^J}_J$ and $J^* = \mathrm{Ad}(w_0 w_0^J) J$. For $w \in W^J$, set $w_J = w_0 w_0^J w^{-1}$. The map $w \mapsto w_J$ is a bijection between $W^J$ and ${}^{J^*}W$. For any $w \in W^J$, let

$$\tilde{Z}_{J,w} = \{z \in \tilde{Z}_J \mid \beta(z) = w_J\}.$$

Then $(\tilde{Z}_{J,w})_{w \in W^J}$ is a partition of $\tilde{Z}_J$ into locally closed $G$-stable subvarieties. Let $(J_k, J'_k, u_k, y_k)_{k \geqslant 0}$ be the sequence determined by $(J, w_J, w_0 w_0^J)$ (see 3.2.1). Assume that $J_n = J'_n = J_{n+1} = J'_{n+1} = \cdots$ and $u_n = u_{n+1} = \cdots = 1$. Set $v_0 = w_J$ and $v_k = u_{k-1}^{-1} v_{k-1}$ for $k \in \mathbf{N}$. By [L8, 8.18] and [L7, 2.3], we have $u_k \in {}^{J'_k}W^{J_k}$ and $u_{k+1} \in W_{J_k}$ for all $k \geqslant 0$. Hence $v_{k+1} \in W_{J_k}$ for all $k \geqslant 0$. Moreover, it is easy to see by induction on $k$ that $y_k = v_k w$. In particular, $w = y_n \in {}^{J_n}W^{J_n}$, $Ad(w) J_n = J_n$ and $\dot{w}$ normalizes $B \cap L_{J_n}$. We have the following result.

**Lemma 3.2.3.** *Keep the notation of 3.2.2. Let* $z = (P_J, {}^{\dot{w}_J^{-1}} P_{J^*}, \dot{w}_J^{-1} U_{P_{J^*}} \dot{w}_J \dot{w} b U_{P_J})$, *where* $b \in {}^{\dot{w}^{n-1} \dot{v}_n^{-1}} (U_{P_{J'_n}} \cap U_{J_{n-1}}) {}^{\dot{w}^{n-2} \dot{v}_{n-1}^{-1}} (U_{P_{J'_{n-1}}} \cap U_{J_{n-2}}) \cdots {}^{\dot{v}_1^{-1}} (U_{P_{J'_1}} \cap U_{J_0}) T$ *or* $b \in B$. *Then* $z \in \tilde{Z}_{J,w}$.

*Proof.* For any $k$, set $P_k = P_{J_k}, P'_k = {}^{\dot{v}_k^{-1}} P_{J'_k}$. Then

$$P_k \cap P'_k = P_{J_k} \cap {}^{\dot{v}_{k+1}^{-1} \dot{u}_k^{-1}} P_{J'_k} = {}^{\dot{v}_{k+1}^{-1}} (P_{J_k} \cap {}^{\dot{u}_k^{-1}} P_{J'_k}).$$

Note that $u_k^{-1} \in {}^{J_k} W^{J'_k}$. Then $L_{J_k} \cap {}^{\dot{u}_k^{-1}} L_{J'_k} = L_{J_k \cap Ad(\dot{u}_k^{-1}) J'_k} = L_{J'_{k+1}}$. Thus

${}^{\dot v_{k+1}^{-1}}L_{J'_{k+1}} = {}^{\dot v_{k+1}^{-1}}(L_{J_k} \cap {}^{\dot u_k^{-1}} L_{J'_k})$ is a Levi factor of $P_k \cap P'_k$. Moreover, we have

$$\begin{aligned} P_k^{P'_k} &= P_{J_k}^{({}^{\dot v_k^{-1}} P_{J'_k})} = {}^{\dot v_{k+1}^{-1}}(P_{J_k}^{({}^{\dot u_k^{-1}} P_{J'_k})}) = {}^{\dot v_{k+1}^{-1}} P_{J_k \cap Ad(\dot u_k^{-1})J'_k} = {}^{\dot v_{k+1}^{-1}} P_{J'_{k+1}} \\ P_k'^{P_k} &= {}^{\dot v_k^{-1}}(P_{J'_k}^{({}^{\dot v_k} P_{J_k})}) = {}^{\dot v_k^{-1}}(P_{J'_k}^{({}^{\dot u_k} P_{J_k})}) = {}^{\dot v_k^{-1}} P_{J'_k \cap Ad(\dot u_k)J_k} \\ &= {}^{\dot v_k^{-1}} P_{Ad(\dot y_k)(J_k \cap Ad(\dot y_k^{-1}\dot u_k)J_k)} = {}^{\dot v_k^{-1}} P_{Ad(\dot y_k)J_{k+1}} \end{aligned}$$

If $b \in B$, then set $g_k = \dot w b$, $\gamma_k = U_{P'_k} g_k U_{P_k}$ and $z_k = (P_k, P'_k, \gamma_k)$ for all $k$. In this case, ${}^{\dot v_{k+1}^{-1}}L_{J'_{k+1}} = {}^{\dot w \dot y_{k+1}^{-1}} L_{J'_{k+1}} = {}^{\dot w} L_{J_{k+1}} \subset {}^{\dot w} P_k = {}^{g_k} P_k$. Thus ${}^{g_k}P_k$ contains some Levi of $P_k \cap P'_k$. Moreover,

$$\begin{aligned} g_k^{-1}({}^{g_k}P_k)^{({}^{\dot v_k^{-1}} P_{Ad(\dot y_k)J_{k+1}})} g_k &= P_k^{({}^{b^{-1}\dot w^{-1}\dot v_k^{-1}} P_{Ad(\dot y_k)J_{k+1}})} = {}^{b^{-1}}(P_k^{{}^{\dot y_k^{-1}} P_{Ad(\dot y_k)J_{k+1}}}) \\ &= {}^{b^{-1}} P_{J_k \cap Ad(\dot y_k^{-1})Ad(\dot y_k)J_{k+1}} = {}^{b^{-1}} P_{J_{k+1}} = P_{J_{k+1}}. \end{aligned}$$

Therefore, $\vartheta(z_k) = z_{k+1}$.

If $b = (\dot w^{n-1}\dot v_n^{-1} b_n \dot v_n \dot w^{-n+1})(\dot w^{n-2}\dot v_{n-1}^{-1} b_{n-1}\dot v_{n-1}\dot w^{-n+2}) \cdots (\dot v_1^{-1} b_1 \dot v_1)(\dot w^n t \dot w^{-n})$, where $b_j \in U_{P_{J'_j}} \cap U_{J_{j-1}}$ for $1 \leqslant j \leqslant n$ and $t \in T$, then set

$$a_k = (\dot w^{n-k}\dot v_n^{-1} b_n \dot v_n \dot w^{-k})(\dot w^{n-k-1}\dot v_{n-1}^{-1} b_{n-1}\dot v_{n-1}\dot w^{-n+k+1}) \cdots (\dot v_k^{-1} b_k \dot v_k)(\dot w^{n+1-k} t \dot w^{-n-1+k}).$$

In this case, set $g_k = \dot w a_{k+1}$, $\gamma_k = U_{P'_k} g_k U_{P_k}$ and $z_k = (P_k, P'_k, \gamma_k)$.

For $j \geqslant 0$, $J_{j+1} = J_j \cap \mathrm{Ad}(\dot y_{j+1}^{-1})J_j$ and $v_{j+1} \in W_{J_j}$. Thus ${}^{\dot w}L_{J_{j+1}} = {}^{\dot v_{j+1}^{-1}\dot y_{j+1}} L_{J_{j+1}} \subset {}^{\dot v_{j+1}^{-1}} L_{J_j} = L_{J_j}$. Then ${}^{\dot w^j \dot v_{k+j+1}^{-1}}U_{J_{k+j}} \subset {}^{\dot w^j} L_{J_{k+j}} \subset L_{J_k}$. So $a_{k+1} \in P_k$. Thus ${}^{g_k}P_k = {}^{\dot w} P_k$ contains some Levi of $P_{J_k} \bigcap {}^{\dot v_k^{-1}} P_{J'_k}$. Moreover,

$$g_k^{-1}({}^{g_k}P_k)^{({}^{\dot v_k^{-1}} P_{Ad(\dot y_k)J_{k+1}})} g_k = {}^{a_{k+1}^{-1}} P_{J_{k+1}}.$$

Thus $\vartheta(z_k) = (Q, Q', \gamma')$, where $Q = {}^{a_{k+1}^{-1}} P_{J_{k+1}}$, $Q' = {}^{\dot v_{k+1}^{-1}} P_{J'_{k+1}}$ and $\gamma' = U_{Q'} g_k U_Q$. Note that ${}^{\dot v_{k+1}^{-1}}U_{P_{J'_{k+1}}} \subset Q'$ and $T \subset Q'$. Moreover, for $j \geqslant 1$, ${}^{\dot w^j \dot v_{k+j+1}^{-1}}U_{J_{k+j}} \subset {}^{\dot w^j} L_{J_{k+j}} \subset {}^{\dot w} L_{J_{k+1}} = {}^{\dot v_{k+1}^{-1}\dot y_{k+1}} L_{J_{k+1}} = {}^{\dot v_{k+1}^{-1}} L_{J'_{k+1}} \subset Q'$. Thus $a_{k+1} \in Q'$. Hence, $z_{k+1} = (a_{k+1}, a_{k+1}) \cdot \vartheta(z_k)$.

In both cases, $\vartheta(z_k)$ is in the same $G$ orbit as $z_{k+1}$. Thus

$$\beta(z) = \beta(z_0) = u_1\beta(z_1) = \cdots = u_1u_2\cdots u_n = w_J.$$

□

*Remark.* 1. From the proof of the case where $b \in B$, we can see that

$$\vartheta^n(P_J, {}^{\dot{w}_J^{-1}} P_{J^*}, \dot{w}_J^{-1}U_{P_{J^*}}\dot{w}_J\dot{w}bU_{P_J}) = (P_{J_n}, P_{J_n}, U_{P_{J_n}}\dot{w}bU_{P_{J_n}}).$$

This result will be used to establish a relation between the $G$-stable pieces and the $B \times B$-orbits.

2. The fact that $(P_J, {}^{\dot{w}_J^{-1}} P_{J^*}, \dot{w}_J^{-1}U_{P_{J^*}}\dot{w}_J\dot{w}bU_{P_J})$ is contained in $\tilde{Z}_{J,w}$ for any $b \in {}^{\dot{w}^{n-1}\dot{v}_n^{-1}}(U_{P_{J'_n}} \cap U_{J_{n-1}})^{\dot{w}^{n-2}\dot{v}_{n-1}^{-1}}(U_{P_{J'_{n-1}}} \cap U_{J_{n-2}}) \cdots {}^{\dot{v}_1^{-1}}(U_{P_{J'_1}} \cap U_{J_0})T$ plays an important role in section 3.3. We will discuss about it in more detail in 3.3.1.

**3.2.4** Let $(J_n, J'_n, u_n, y_n)_{n \geqslant 0}$ be the sequence that is determined by $w_J$ and $w_0w_0^J$. Assume that $J_n = J'_n = J_{n+1} = J'_{n+1} = \cdots$ and $u_n = u_{n+1} = \cdots = 1$. Then $z \mapsto \vartheta^n(z)$ is a $G$-equivariant morphism from $\tilde{Z}_{J,w}$ to $\tilde{Z}^w_{J_n,1}$ and induces a bijection from the set of $G$-orbits on $\tilde{Z}_{J,w}$ to the set of $G$-orbits on $\tilde{Z}^w_{J_n,1}$.

Set $\tilde{L}_{J,w} = L_{J_n}$ and $\tilde{C}_{J,w} = \dot{w}\tilde{L}_{J,w}$. Let $N_G(\tilde{L}_{J,w})$ be the normalizer of $\tilde{L}_{J,w}$ in $G$. Then $\tilde{C}_{J,w}$ is a connected component of $N_G(\tilde{L}_{J,w})$ and $\tilde{Z}^w_{J_n,1}$ is a fibre bundle over $\mathcal{P}^{J_n}$ with fibres isomorphic to $\tilde{C}_{J,w}$. There is a natural bijection between $\tilde{C}_{J,w}$ and $F = \{z = (P_{J_n}, P_{J_n}, \gamma_n) \mid z \in \tilde{Z}^w_{J_n,1}\}$ under which the action of $\tilde{L}_{J,w}$ on $\tilde{C}_{J,w}$ by conjugation corresponds to the action of $P_{J_n}/U_{P_{J_n}}$ on $F$ by conjugation. Therefore, we obtain a canonical bijection the set of $G$-stable subvarieties of $\tilde{Z}_{J,w}$ and the set of $\tilde{L}_{J,w}$-stable subvarieties of $\tilde{C}_{J,w}$ (see [L8, 8.21]). Moreover, a $G$-stable subvariety of $\tilde{Z}_{J,w}$ is closed if and only if the corresponding $\tilde{L}_{J,w}$-stable subvariety of $\tilde{C}_{J,w}$ is closed. By the remark 1 of 3.2.3, for any $b \in B \cap \tilde{L}_{J,w}$, the $G$-orbit that contains $(P_J, {}^{\dot{w}_J^{-1}} P_{J^*}, \dot{w}b)$ corresponds to the $\tilde{L}_{J,w}$-orbit that contains $\dot{w}b$ via the bijection.

**3.2.5** Since $G$ is adjoint, the center of $P/U_P$ is connected for any parabolic subgroup $P$. Let $H_P$ be the inverse image of the (connected) center of $P/U_P$ under $P \to P/U_P$. We can regard $H_P/U_P$ as a single torus $\Delta_J$ independent of $P$. Now $\Delta_J$ acts (freely) on $\tilde{Z}_J$ by $\delta : (P, P', \gamma) \mapsto (P, P', \gamma z)$ where $z \in H_P$ represents $\delta \in \Delta_J$. The action of $G$ on $\tilde{Z}_J$ commutes with the action of $\Delta_J$ and induces an action of $G$ on $\Delta_J \setminus \tilde{Z}_J$. There exists a $G$-equivariant isomorphism from $Z_J$ to $\Delta_J \setminus \tilde{Z}_J$ which sends $(g_1, g_2) \cdot h_J$ to $({}^{g_2}P_J, {}^{g_1}P_J^-, U_{{}^{g_1}P_J^-} g_1 g_2^{-1} H_{{}^{g_2}P_J})$. We will identify $Z_J$ with $\Delta_J \setminus \tilde{Z}_J$.

It is easy to see that $\Delta_J(\tilde{Z}_{J,w}) = \tilde{Z}_{J,w}$. Set $Z_{J,w} = \Delta_J \setminus \tilde{Z}_{J,w}$. Then

$$Z_J = \bigsqcup_{w \in W^J} Z_{J,w}.$$

Moreover, we may identify $\Delta_J$ with a closed subgroup of the center of $\tilde{L}_{J,w}$. Set $L_{J,w} = \tilde{L}_{J,w}/\Delta_J$ and $C_{J,w} = \tilde{C}_{J,w}/\Delta_J$. Thus we obtain a bijection between the set of $G$-stable subvarieties of $Z_{J,w}$ and the set of $L_{J,w}$-stable subvarieties of $C_{J,w}$ (see [L8, 11.19]). Moreover, a $G$-stable subvariety of $Z_{J,w}$ is closed if and only if the corresponding $L_{J,w}$-stable subvariety of $C_{J,w}$ is closed and for any $b \in B \cap \tilde{L}_{J,w}$, the $G$-orbit that contains $(P_J, {}^{\dot{w}_J^{-1}}P_{J^*}, \dot{w}b)$ corresponds to the $L_{J,w}$-orbit that contains $\dot{w}b\Delta_J$ via the bijection.

**Proposition 3.2.6.** *For any $w \in W^J$, $Z_{J,w} = G_{diag} \cdot [J, w, 1]$.*

*Proof.* By 3.2.3, $(\dot{w}, b) \cdot h_J \in Z_{J,w}$ for all $b \in B$. Since $Z_{J,w}$ is $G$-stable, $G_{diag}[J, w.1] \subset Z_{J,w}$.

For any $z \in Z_{J,w}$, let $C$ be the $L_{J,w}$-stable subvariety corresponding to $G_{diag} \cdot z$ and let $c$ be an element in $\tilde{C}_{J,w}$ such that $c\Delta_J \in C$. By 3.2.2, $\dot{w}$ normalizes $B \cap \tilde{L}_{J,w}$. Thus $c$ is $\tilde{L}_{J,w}$-conjugate to an element of $\dot{w}(B \cap \tilde{L}_{J,w})$. Therefore, $z$ is $G$-conjugate to $(\dot{w}, b) \cdot h_J$ for some $b \in B \cap \tilde{L}_{J,w}$. The proposition is proved. □

**Proposition 3.2.7.** *For any $w \in W^J$, $\overline{Z_{J,w}} = \overline{G_{diag}(\dot{w}T, 1) \cdot h_J}$.*

*Proof.* Since $(\dot{w}T, 1) \cdot h_J \subset Z_{J,w}$ and $\overline{Z_{J,w}}$ is a $G$-stable closed variety, we have that $\overline{G_{diag}(\dot{w}T, 1) \cdot h_J} \subset \overline{Z_{J,w}}$.

Set $X = \{(\dot{w}t, u) \cdot h_J \mid t \in T, u \in U\}$. For any $u \in {}^{\dot{w}}U_J$ and $t \in T$, we have that $Ad(\dot{w}t)^{-1}u \in U_J$ and $u \in {}^{\dot{w}}U_J \subset U$. Consider the map $\phi :^{\dot{w}} U_J \times T \to X$ defined by $\phi(u,t) = (u,u)(\dot{w}t, 1) \cdot h_J = (\dot{w}t, (\dot{w}t)^{-1}u\dot{w}tu^{-1}) \cdot h_J$, for $u \in {}^{\dot{w}}U_J, t \in T$.

It is easy to see that there is an open subset $T'$ of $T$, such that the restriction of $\phi$ to ${}^{\dot{w}}U_J \times T'$ is injective. Note that $\dim(X) = \dim(T) + \dim(U/U_{P_J}) = \dim(T) + \dim(U_J) = \dim({}^{\dot{w}}U_J \times T)$. Then the image of $\phi$ is dense in $X$. The proposition is proved. □

**3.2.8** For $w \in W$, recall that $\text{supp}(w)$ is the set of simple roots whose associated simple reflections occur in a reduced expression of $w$. An element $w \in W$ is called a Coxeter element if it is a product of the simple reflections, in some order, or in other words, $|\text{supp}(w)| = l(w) = |I|$. We have the following properties.

**Proposition 3.2.9.** *Fix $i \in I$. Then all the Coxeter elements are conjugate under elements of $W_{I-\{i\}}$.*

*Proof.* Let $c, c'$ be Coxeter elements. We say that $c'$ can be obtained from $c$ via a cyclic shift if $c = s_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression and $c' = s_{i_1}cs_{i_1}$. It is known that for any Coxeter elements $c, c'$, there exists a finite sequences of Coxeter elements $c = c_0, c_1, \ldots, c_m = c'$ such that $c_{k+1}$ can be obtained from $c_k$ via a cyclic shift (see [Bo, p. 116, Prop. 1]).

Now assume that $c = s_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression of a Coxeter element. If $i_1 \neq i$, then $s_{i_1}cs_{i_1}$ and $c$ are conjugated by $s_{i_1} \in W_{I-\{i\}}$. If $i_1 = i$, then $s_{i_1}cs_{i_1} = s_{i_2}s_{i_3}\cdots s_{i_n}c(s_{i_2}s_{i_3}\cdots s_{i_n})^{-1}$. Therefore, if a Coxeter element can be obtained from another Coxeter element via a cyclic shift, then they are conjugated by elements of $W_{I-\{i\}}$. The proposition is proved. □

*Remark.* The proof of [loc. cit] also can be used to prove this proposition.

**Proposition 3.2.10.** *Let $J \subset I$ and $w \in W^J$ with* $\text{supp}(w) = I$*. Then there exist a Coxeter element $w'$, such that $w' \in W^J$ and $w' \leqslant w$.*

*Proof.* We prove the statement by induction on $l(w)$.

Let $i \in I$ with $s_i w < w$. Then $s_i w \in W^J$. If $\text{supp}(s_i w) = I$, then the statement holds by induction hypothesis on $s_i w$. Now assume that $\text{supp}(s_i w) = I - \{i\}$. By induction, there exists a Coxeter element $w'$ of $W_{I-\{i\}}$, such that $w' \in W^{J-\{i\}}$ and $w' \leqslant s_i w$. Then $s_i w'$ is a Coxeter element of $w$ and $s_i w' \leqslant w$.

Since $w' \in W_{I-\{i\}}$, $(w')^{-1}\alpha_i$ is either $\alpha_i$ or a non-simple positive root. We also have that $w'$ is a Coxeter element of $W_{I-\{i\}}$. Thus if $(w')^{-1}\alpha_i = \alpha_i$, then $< \alpha_i, \alpha_j^\vee >= 0$ for all $j \neq i$. It contradicts the assumption that $G$ is simple. Hence $(w')^{-1}\alpha_i$ is a non-simple positive root. Note that if $s_i w' \notin W^J$, then $s_i w' = w' s_j$ for some $j \in J$, that is, $(w')^{-1}\alpha_i = \alpha_j$. Therefore, $s_i w' \in W^J$. The proposition is proved. □

**Corollary 3.2.11.** *Let $i \in I$, $J = I - \{i\}$ and $w$ be a Coxeter element of $W$ with $w \in W^J$. Then $\bigsqcup_{K \subset J} \bigsqcup_{w' \in W^K, \text{supp}(w') = I} Z_{K,w'} \subset \overline{Z_{J,w}}$.*

*Proof.* By 3.1.4, $[K, wv, v] \subset \overline{[J, w, 1]}$ for $K \subset J$ and $v \in W_J$. Since $\overline{Z_{J,w}}$ is $G$-stable, $(\dot{v}^{-1}\dot{w}\dot{v}T, 1) \cdot h_K \subset \overline{Z_{J,w}}$. By 3.2.9, $(\dot{w}'T, 1) \cdot h_K \subset \overline{Z_{J,w}}$ for all Coxeter element $w'$. By 3.2.7, $Z_{K,w'} \subset \overline{Z_{J,w}}$ for all Coxeter element $w'$ with $w' \in W^K$. For any $u \in W^K$ with $\text{supp}(u) = I$, there exists a Coxeter element $w'$, such that $w' \in W^K$ and $w' \leqslant u$. Thus by 3.1.4, we have that $[K, u, 1] \subset \overline{Z_{J,w}}$. By 3.2.6, $Z_{K,u} \subset \overline{Z_{J,w}}$. The corollary is proved. □

*Remark.* In 3.4.4, we will show that the equality holds.

## 3.3 Some combinatorial results

**3.3.1** Fix $i \in I$. Define subsets $I_k$ of $I$ for all $k \in \mathbf{N}$ in the following way. Set $I_1 = \{i\}$. Assume that $I_k$ is already defined. Set

$$I_{k+1} = \{\alpha_j \mid j \in I - \cup_{l=1}^k I_l, < \alpha_j^\vee, \alpha_m > \neq 0 \text{ for some } m \in I_k\}.$$

It is easy to see that if $j_1, j_2 \in I_k$ with $j_1 \neq j_2$, then $< \alpha_{j_1}, \alpha_{j_2}^\vee >= 0$. Thus $s_{I_k} = \prod_{j \in I_k} s_j$ is well-defined. For sufficiently large $n$, we have $I_n = I_{n+1} = \cdots = \varnothing$ and $s_{I_n} = s_{I_{n+1}} = \cdots = 1$. Now set $w_k = s_{I_n} s_{I_{n-1}} \cdots s_{I_k}$ for $k \in \mathbf{N}$. We will write

$w^J$ for $w_1$. Set $J_{-1} = I$ and $J_0 = J = I - \{i\}$. Then $w^J$ is a Coxeter element and $w^J \in W^J$. Let $(J_n, J'_n, u_n, y_n)$ be the sequence determined by $w^J$ and $w_0 w_0^J$. Then we can show by induction that for $k \geqslant 0$, $J_k = J_{k-1} - I_{k+1}$, $u_k = w_0^{J_{k-1}} w_0^{J_k} s_{I_{k+1}} w_0^{J_{k+1}} w_0^{J_k}$, $y_k = w_0^{J_{k-1}} w_0^{J_k} s_{I_k} s_{I_{k-1}} \cdots s_{I_1}$ and $J'_k = Ad(y_k) J_k$. In particular, $J_n = \varnothing$. Thus $\tilde{L}_{J,w^J} = T$ and $\tilde{C}_{J,w^J} = \dot{w}^J T$. Since $w$ is a Coxeter element, the homomorphism $T \to T$ sending $t \in T$ to $(\dot{w}^J)^{-1} t \dot{w}^J t^{-1}$ is surjective. Thus $\tilde{L}_{J,w^J}$ acts transitively on $\tilde{C}_{J,w^J}$. By 3.2.5, $G$ acts transitively on $Z_{J,w^J}$.

For $k \in \mathbf{N}$, we set $v_k = w_0^{J_{k-1}} w_0^{J_k} w_{k+1}^{-1}$. Then it is easy to see that ${}^{\dot{v}_k^{-1}}(U_{P_{J'_k}} \cap U_{J_{k-1}}) = {}^{w_{k+1}}(U_{P^-_{J_k}} \cap U^-_{J_{k-1}})$. Therefore by 3.2.3, $(\dot{w}^J b, 1) \cdot h_J \in Z_{J,w^J}$ for all $b \in {}^{w^{n-1} w_{n+1}}(U_{P^-_{J_n}} \cap U^-_{J_{n-1}}) {}^{w^{n-2} w_n}(U_{P^-_{J_{n-1}}} \cap U^-_{J_{n-2}}) \cdots {}^{w_2}(U_{P^-_{J_1}} \cap U^-_{J_0}) T$.

In the remaining part of 3.3.1, we will keep the notations of $J$, $J_k$, $w^J$ and $w_k$ as above. Let $X$ be a subset of $\bar{G}$ such that for any admissible pair $(w, \lambda)$ and $x \in W$ with $x^{-1}\lambda$ is dominant, there exist $t \in T$, such that $G_{diag}(U \times U)(\dot{w}\dot{x}t, \dot{x}) \cdot h_{I(x^{-1}\lambda)} \subset X$. (An example is $\bar{\mathcal{U}}$.) We will show that $Z_{J,w^J} \subset X$ for all $i \in I$. The proof is based on case-by-case checking.

*Remark.* The outline of the case-by-case checking is as follows.

For $\lambda \in Y$, we write $\lambda \geqslant 0$ if $\lambda \in \sum_{l \in I} \mathbf{R}_{\geqslant 0} \alpha_l^\vee$.

We start with the fundamental coweight $\omega_i^\vee$. Find $x \in W$ that satisfies the conditions (1) $x\omega_i^\vee \geqslant 0$ and (2) for $l \in I$, either $(s_l - 1)x\omega_i^\vee \geqslant 0$ or $s_l x \omega_i^\vee \ngeqslant 0$. Such $x$ always exists, as we will see by case-by-case checking. The elements $x\omega_i^\vee$ that we obtain in this way are not unique, in general. Fortunately, there always exists some $x \in W$ that satisfies the conditions (1) and (2) and allows us to do the procedures that we will discuss below.

In the rest of the remark, we fix such $x$. Since $x\omega_i^\vee \in Y$, there exists $n \in \mathbf{N}$, such that $nx\omega_i^\vee$ is contained in the coroot lattice. Set $\lambda = nx\omega_i^\vee$. Now we can find $v \in W$ such that $(v, \lambda)$ is admissible. (In practice, we find $v \in W$ with $l(v) = |\text{supp}(v)|$ and $-v\lambda \geqslant 0$. Then we can use lemma 3.3.2 to check that if $(v, \lambda)$ is admissible.) By the assumption on $X$, $G_{diag}(U \times U)(\dot{v}\dot{x}t, \dot{x}) \cdot h_J \subset X$ for some $t \in T$.

In some cases, $x^{-1}vx = w_J$. Since $w_J$ is a Coxeter element, $(\dot{w}_J T, 1) \cdot h_J = T_{diag}(\dot{w}_J t, 1) \cdot h_J \subset X$. By 3.2.7, $Z_{J,w_J} \subset X$.

In other cases, the situation is more complicated. We need to choose some $u \in U$, such that $(u\dot{v}\dot{x}t, \dot{x}) \cdot h_J \in Z_{J,w_J}$. This is the most difficult part of the case-by-case checking. The lemma 3.3.3 and lemma 3.2.3 will be used to overcome the difficulties.

Throughout this section, we will use the same labelling of Dynkin diagram as in [Bo]. For $a, b \in I$, we denote by $s_{[a,b]}$ the element $s_b s_{b-1} \cdots s_a$ of the Weyl group $W$ and $\dot{s}_{[a,b]} = \dot{s}_b \dot{s}_{b-1} \cdots \dot{s}_a$. (If $b < a$, then $s_{[a,b]} = 1$ and $\dot{s}_{[a,b]} = 1$.)

**Lemma 3.3.2.** *Let $x = s_{i_1} s_{i_2} \cdots s_{i_n}$ with $|\mathrm{supp}(x)| = n$. Then $(1 - x^{-1})\omega_k^\vee = 0$ if $k \notin \{i_1, i_2, \ldots, i_n\}$ and $(1 - x^{-1})\omega_{i_j}^\vee = s_{i_n} s_{i_{n-1}} \cdots s_{i_{j+1}} \alpha_{i_j}^\vee$. Thus $(x, \lambda)$ is admissible for all $\lambda \in \sum_{j=1}^n \mathbf{N} s_{i_n} s_{i_{n-1}} \cdots s_{i_{j+1}} \alpha_{i_j}^\vee$.*

The lemma is a direct consequence of [Bo, p. 226, Ex. 22a].

**Lemma 3.3.3.** *Let $w, x, y_1, y_2 \in W$ and $t \in T$. Assume that $y_1 = s_{i_1} s_{i_2} \cdots s_{i_l}$, $y_2 = s_{i_{l+1}} s_{i_{l+2}} \cdots s_{i_{l+k}}$ with $k + l = |\mathrm{supp}(y_1 y_2)|$. If moreover, $< \alpha_{i_{l_1}}^\vee, \alpha_{i_{l_2}} >= 0$ for all $1 \leqslant l_1 < l_2 \leqslant l$ and $(1 - y_1 y_2) x \omega_i^\vee, (1 - y_1) w \omega_i^\vee \in \sum_{j=1}^k \mathbf{R}_{>0} \alpha_{i_j}^\vee$, then there exists $u \in U_{-w^{-1}\alpha_{i_{l+1}}} U_{-w^{-1}\alpha_{i_{l+2}}} \cdots U_{-w^{-1}\alpha_{i_{l+k}}}$ such that $(\dot{x}^{-1}\dot{w}ut, 1) \cdot h_J \in G_{diag}(U \times U)(\dot{w}t, \dot{y}_1 \dot{y}_2 \dot{x}) \cdot h_J$.*

*Proof.* We have that $(1 - y_1 y_2) x \omega_i^\vee = \sum_{j=1}^{k+l} (1 - s_{i_j}) s_{i_{j+1}} \cdots s_{i_{l+k}} x \omega_i^\vee$. Note that $i_1, i_2, \ldots, i_{k+l}$ are distinct and $(1 - s_{i_j}) s_{i_{j+1}} \cdots s_{i_{l+k}} x \omega_i^\vee \in \mathbf{R} \alpha_{i_j}^\vee$ for all $j$. Hence $(1 - s_{i_j}) s_{i_{j+1}} \cdots s_{i_{l+k}} x \omega_i^\vee \in \mathbf{R}_{>0} \alpha_{i_j}^\vee$ for all $j$, i. e., $< s_{i_{j+1}} \cdots s_{i_k} x \omega_i^\vee, \alpha_{i_j} >\in \mathbf{R}_{>0}$. Therefore $\dot{x}^{-1} \dot{s}_{i_{l+k}}^{-1} \cdots \dot{s}_{i_{j+1}}^{-1} U_{\alpha_{i_j}} \dot{s}_{i_{j+1}} \cdots \dot{s}_{i_{l+k}} \dot{x} \subset U_{P_J}$. Similarly, we have that $\dot{w}^{-1} U_{-\alpha_{i_j}} \dot{w} \in U_{P_J^-}$ for $j \leqslant l$.

There exists $u_j \in U_{\alpha_{i_j}}$ and $u_j' \in U_{-\alpha_{i_j}}$ such that $u_j \dot{s}_{i_j} u_j = u_j'$. Note that $u_1' u_2' \cdots u_{l+k-1}' \in L_{I - \{i_{l+k}\}}$, $u_{l+k} \in U_{P_{I - \{i_{l+k}\}}}$ and $\dot{x}^{-1} u_{l+k} \dot{x} \subset U_{P_J}$. Thus

$$
\begin{aligned}
u_1' u_2' \cdots u_{l+k}' \dot{x} = u_1' u_2' \cdots u_{l+k-1}' u_{l+k} \dot{s}_{i_k} u_{l+k} \dot{x} &\in U_{P_{I - \{i_k\}}} u_1' u_2' \cdots u_{l+k-1}' \dot{s}_{i_k} \dot{x} U_{P_J} \\
&\subset U u_1' u_2' \cdots u_{l+k-1}' \dot{s}_{i_k} \dot{x} U_{P_J}.
\end{aligned}
$$

We can show in the same way that $u_1' u_2' \cdots u_{l+k}' \dot{x} \in U \dot{y}_1 \dot{y}_2 \dot{x} U_{P_J}$. Therefore, $(\dot{w}t, u_1' u_2' \cdots u_{l+k}' \dot{x}) \cdot h_J \in (U \times U)(\dot{w}t, \dot{y}_1 \dot{y}_2 \dot{x}) \cdot h_J$. Set $u = \dot{w}^{-1} u_{l+1}' u_{l+2}' \cdots u_{l+k}' \dot{w}$

and $u' = t^{-1}\dot{w}^{-1}(u_1'u_2'\cdots u_l')^{-1}\dot{w}t \in U_{P_J^-}$. Then

$$(\dot{x}^{-1}\dot{w}ut, 1)\cdot h_J = (\dot{x}^{-1}\dot{w}utu', 1)\cdot h_J = \big(\dot{x}^{-1}(u_1'u_2'\cdots u_{l+k}')^{-1}\dot{w}t, 1\big)\cdot h_J$$
$$\in G_{diag}(U\times U)(\dot{w}t, \dot{y}_1\dot{y}_2\dot{x})\cdot h_J.$$

□

**3.3.4** In 3.3.4 to 3.3.7, we assume that $G$ is $PGL_n(k)$. Without loss of generality, we assume that $i \leqslant n/2$. In this case, $w^J = s_{[i+1,n-1]}s_{[1,i]}^{-1}$. For any $a \in \mathbf{R}$, we denote by $[a]$ the maximal integer that is less than or equal to $a$.

For $1 \leqslant j \leqslant i$, set $a_j = [(j-1)n/i]$. For convenience, we will set $a_{i+1} = n-1$. Note that for $j \leqslant i-1$, $a_{j+1} - a_j = [jn/i] - [(j-1)n/i] \geqslant [n/i] \geqslant 2$. Therefore, we have that $0 = a_1 < a_1 + 1 < a_2 < a_2 + 1 < \cdots < a_i < a_i + 1 \leqslant a_{i+1} = n-1$. Now set $b_0 = 0$. For $k \in \{1, 2, \dots, n-1\} - \{a_2, a_3, \dots, a_i\} - \{a_2+1, a_3+1, \dots, a_i+1\}$, set $b_k = i$. For $j \in \{2, 3, \dots, i\}$, set $b_{a_j} = (j-1)n - ia_k$ and $b_{a_j+1} = i - b_{a_j}$.

Now set $v = s_{[a_1+1, a_2-\delta_{b_{a_2},0}]}s_{[a_2+1, a_3-\delta_{b_{a_3},0}]}\cdots s_{[a_i+1, a_{i+1}-\delta_{b_{a_{i+1}},0}]}$, where $\delta_{a,b}$ is the Kronecker delta. Set $v_j = s_{[a_j+1, a_{j+1}]}s_{[a_{j+1}+1, a_{j+2}]}\cdots s_{[a_i+1, a_{i+1}]}$ for $1 \leqslant j \leqslant i$. Set $\lambda = \sum_{j=1}^{i}\sum_{k=1}^{a_{j+1}-a_j} b_{a_j+k}(s_{[a_j+1, a_j+k-1]}v_{j+1})^{-1}\alpha_{a_j+k}^\vee$. It is easy to see that for $1 \leqslant a \leqslant b \leqslant n-1$ and $1 \leqslant k \leqslant n-1$,

$$s_{[a,b]}\alpha_k^\vee = \begin{cases} \sum_{l=a-1}^{b}\alpha_l^\vee, & \text{if } k = a-1; \\ -\sum_{l=a}^{b}\alpha_l^\vee, & \text{if } k = a; \\ \alpha_{k-1}^\vee, & \text{if } a < k \leqslant b; \\ \alpha_b^\vee + \alpha_{b+1}^\vee, & \text{if } k = b+1; \\ \alpha_k^\vee, & \text{otherwise}. \end{cases}$$

If $b_{a_j+k} \neq 0$, then $(s_{[a_j+1, a_j+k-1]}s_{[a_{j+1}+1, a_{j+2}-\delta_{b_{a_{j+2}},0}]}\cdots s_{[a_i+1, a_{i+1}-\delta_{b_{a_{i+1}},0}]})^{-1}\alpha_{a_j+k}^\vee = (s_{[a_j+1, a_j+k-1]}v_{j+1})^{-1}\alpha_{a_j+k}^\vee$. By 3.2, $(v, \lambda)$ is admissible.

We have that

$$
\begin{aligned}
\lambda &= \sum_{j=1}^{i} \sum_{k=1}^{a_{j+1}-a_j-1} b_{a_j+k} v_{j+1}^{-1} s_{[a_j+1,a_j+k-1]}^{-1} \alpha_{a_j+k}^\vee + \sum_{j=1}^{i} b_{a_{j+1}} v_{j+1}^{-1} s_{[a_j+1,a_{j+1}-1]}^{-1} \alpha_{a_{j+1}}^\vee \\
&= \sum_{j=1}^{i} \sum_{k=1}^{a_{j+1}-a_j-1} \sum_{l=1}^{k} b_{a_j+k} \alpha_{a_j+l}^\vee + \sum_{j=1}^{i-1} b_{a_{j+1}} \sum_{l=1}^{a_{j+1}-a_j+1} \alpha_{a_j+l}^\vee + b_{a_{i+1}} \sum_{l=1}^{a_{i+1}-a_i} \alpha_{a_i+l}^\vee \\
&= \sum_{j=1}^{i} \sum_{k=1}^{a_{j+1}-a_j} \sum_{l=1}^{k} b_{a_j+k} \alpha_{a_j+l}^\vee + \sum_{j=1}^{i-1} b_{a_{j+1}} \alpha_{a_{j+1}+1}^\vee \\
&= \sum_{j=1}^{i} \sum_{l=1}^{a_{j+1}-a_j} \sum_{k=l}^{a_{j+1}-a_j} b_{a_j+k} \alpha_{a_j+l}^\vee + \sum_{j=1}^{i-1} b_{a_{j+1}} \alpha_{a_{j+1}+1}^\vee \\
&= \sum_{j=1}^{i} \sum_{l=2}^{a_{j+1}-a_j} \big((a_{j+1}-a_j-l)i + b_{a_{j+1}}\big) \alpha_{a_j+l}^\vee + \big((a_2-1)i + b_{a_2}\big)\alpha_1^\vee \\
&\quad + \sum_{j=2}^{i} \big(b_{a_j} + (a_{j+1}-a_j-2)i + b_{a_{j+1}} + b_{a_j+1}\big) \alpha_{a_j+1}^\vee \\
&= \sum_{j=1}^{i} \sum_{l=1}^{a_{j+1}-a_j} \big((a_{j+1}-a_j-l)i + b_{a_{j+1}}\big) \alpha_{a_j+l}^\vee = nx\omega_i^\vee.
\end{aligned}
$$

Note that $a_j \geqslant j$ for $j \geqslant 2$. Set $x_i = 1$ and $x_j = s_{[j+1,a_{j+1}]} s_{[j+2,a_{j+2}]} \cdots s_{[i,a_i]}$ for $1 \leqslant j \leqslant i-1$. If $j = 1$, we will simply write $x$ for $x_1$.

**Lemma 3.3.5.** *For $1 \leqslant j \leqslant i$, we have that*

$$
\begin{aligned}
nx_j\omega_i^\vee = \sum_{l=1}^{j-1} l(n-i)\alpha_l^\vee + \sum_{l=j}^{a_{j+1}} \big(jn - il\big)\alpha_l^\vee \\
+ \sum_{k=j+1}^{i} \sum_{l=1}^{a_{k+1}-a_k} \big((a_{k+1}-a_k-l)i + b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee.
\end{aligned}
$$

*In particular, $nx\omega_i^\vee = \sum_{j=1}^{i} \sum_{l=1}^{a_{j+1}-a_j} \big((a_{j+1}-a_j-l)i + b_{a_{j+1}}\big)\alpha_{a_j+l}^\vee$.*

*Proof.* We will prove the lemma by induction on $j$. Note that $n\omega_i^\vee = \sum_{l=1}^{i-1} l(n-i)\alpha_l^\vee + \sum_{l=i}^{n-1} i(n-l)\alpha_l^\vee$. Thus the lemma holds for $j = i$.

Note that $jn - i(a_j + l) = jn - ia_{j+1} + i(a_{j+1} - a_j - l) = b_{a_{j+1}} + i(a_{j+1} - a_j - l)$. Assume that the lemma holds for $j$. Then

$$nx_{j-1}\omega_i^\vee = s_{[j,a_j]}\Big(\sum_{l=1}^{j-1} l(n-i)\alpha_l^\vee + \sum_{l=j}^{a_{j+1}}(jn-il)\alpha_1^\vee + \sum_{k=j+1}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i+b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee\Big)$$

$$= \sum_{l=1}^{j-2} l(n-i)\alpha_l^\vee + (j-1)(n-i)\sum_{l=j-1}^{a_j}\alpha_l^\vee - j(n-i)\sum_{l=j}^{a_j}\alpha_l^\vee + \sum_{l=j+1}^{a_j}(jn-il)\alpha_{l-1}^\vee$$

$$+ \big(jn-i(a_j+1)\big)(\alpha_{a_j}^\vee + \alpha_{a_{j+1}}^\vee) + \sum_{l=a_j+2}^{a_{j+1}}(jn-il)\alpha_l^\vee + \sum_{k=j+1}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i+b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee$$

$$= \sum_{l=1}^{j-2} l(n-i)\alpha_l^\vee + (j-1)(n-i)\sum_{l=j-1}^{a_j}\alpha_l^\vee - j(n-i)\sum_{l=j}^{a_j}\alpha_l^\vee + \sum_{l=j+1}^{a_j}(jn-il)\alpha_{l-1}^\vee$$

$$+ \big(jn-i(a_j+1)\big)\alpha_{a_j}^\vee + \sum_{k=j}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i+b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee$$

$$= \sum_{l=1}^{j-2} l(n-i)\alpha_l^\vee + (j-1)(n-i)\alpha_{j-1}^\vee + \sum_{l=j}^{a_j}\big((j-1)(n-i)-j(n-i)+jn-i(l+1)\big)\alpha_l^\vee$$

$$+ \sum_{k=j}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i+b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee$$

$$= \sum_{l=1}^{j-2} l(n-i)\alpha_l^\vee + \sum_{l=j-1}^{a_j}\big((j-1)n-il\big)\alpha_l^\vee + \sum_{k=j}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i+b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee.$$

Thus the lemma holds for $j$. □

**Lemma 3.3.6.** *We have that $x^{-1}v_1x = w^J$.*

*Proof.* If $a_j \geqslant j+1$, then $s_{[j+1,a_{j+1}]}^{-1}s_{[a_j+1,a_{j+1}]} = s_{[j+1,a_j]}^{-1}$. If $j \geqslant 2$ and $a_j < j+1$, then $j=2$, $a_j = 2$ and $s_{[3,a_3]}^{-1}s_{[a_2+1,a_3]} = 1 = s_{[3,a_2]}^{-1}$. In conclusion, $s_{[j+1,a_{j+1}]}^{-1}s_{[a_j+1,a_{j+1}]} = s_{[j+1,a_j]}^{-1}$ for $j \geqslant 2$. Moreover, $s_{[2,a_2]}^{-1}s_{[a_1+1,a_2]} = s_1$. Thus

$$s_{[2,a_2]}^{-1}v_1s_{[2,a_2]} = s_{[2,a_2]}^{-1}s_{[a_1+1,a_2]}v_2s_{[2,a_2]} = s_1v_2s_{[2,a_2]} = v_2s_1s_{[2,a_2]} = v_2s_{[3,a_2]}s_1s_2$$

$$s_{[j+1,a_{j+1}]}^{-1}v_js_{[j+1,a_j]}s_{[1,j]}^{-1}s_{[j+1,a_{j+1}]} = s_{[j+1,a_{j+1}]}^{-1}s_{[a_j+1,a_{j+1}]}v_{j+1}s_{[j+1,a_j]}s_{[1,j]}^{-1}s_{[j+1,a_{j+1}]}$$

$$= s_{[j+1,a_j]}^{-1}v_{j+1}s_{[j+1,a_j]}s_{[1,j]}^{-1}s_{[j+1,a_{j+1}]} = v_{j+1}s_{[1,j]}^{-1}s_{[j+2,a_{j+1}]}s_{j+1} = v_{j+1}s_{[j+2,a_{j+1}]}s_{[1,j+1]}^{-1}.$$

Thus, we can prove by induction on $j$ that $x^{-1}v_1x = x_j^{-1}v_js_{[j+1,a_j]}s_{[1,j]}^{-1}x_j$ for $1 \leqslant j \leqslant i$. In particular, $x^{-1}v_1x = s_{[i+1,n-1]}s_{[1,i]}^{-1}$. The lemma is proved. □

**3.3.7** By 3.3.4 and 3.3.5, there exists $t \in T$, such that $(U \times U)(\dot{v}\dot{x}t, \dot{x}) \cdot h_J \subset X$. Consider $K = \{a_j \mid b_{a_j} = 0\}$. Then for any $j, j' \in K$ with $j \neq j'$, we have that $|j - j'| \geqslant 2$ and $< \alpha_j^\vee, \alpha_{j'} >= 0$. Set $y = \prod_{j \in K} s_j$. Then $y$ is well-defined. Note that $(1-y)yx\omega_i^\vee, (1-y)vx\omega_i^\vee \in \sum_{j \in K} \mathbf{R}_{>0}\alpha_j^\vee$. By 3.3.3, $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$. Therefore, $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$. By 3.3.6, $x^{-1}yvx = x^{-1}v_1x = w^J$. Therefore, $Z_{J,w^J} \cap X \neq \varnothing$. By 3.3.1, $G$ acts transitively on $Z_{J,w^J}$. Therefore $Z_{J,w^J} \subset X$.

**3.3.8** We assume that $G$ is of type $C_n$ and set

$$\epsilon = \begin{cases} 1, & \text{if } 2 \mid i; \\ 0, & \text{otherwise.} \end{cases}$$

Now set $v = s_{n-i+1}s_{n-i+3}\cdots s_{n-\epsilon}$, $x_1 = s^{-1}_{[n-i,n-1]}s^{-1}_{[n-i-1,n-2]}\cdots s^{-1}_{[1,i]}$ and $x_2 = s^{-1}_{[n+\epsilon-1,n]}s^{-1}_{[n+\epsilon-3,n]}\cdots s^{-1}_{[n-i+2,n]}$. Set $\lambda = \alpha^\vee_{n-i+1} + \alpha^\vee_{n-i+3} + \cdots + \alpha^\vee_{n-\epsilon}$. Then we have that $(v, \lambda)$ is admissible.

Now set $\lambda' = \sum_{j \in I} \min(i,j)\alpha_j^\vee \in \mathbf{N}\omega_i^\vee$. Set $x_{1,j} = s^{-1}_{[j-i+1,j]}s^{-1}_{[j-i,j-1]}\cdots s^{-1}_{[1,i]}$ for $i-1 \leqslant j \leqslant n-1$, s. Then we can show by induction that $x_{1,j}\lambda' = \sum_{k=1}^{i} k\alpha^\vee_{j-i+1+k} + i\sum_{l=j+2}^{n}\alpha_l^\vee$. In particular, $x_1\omega_i^\vee = \sum_{k=1}^{i} k\alpha^\vee_{n-i+k}$.

For $0 \leqslant j \leqslant (i+\epsilon-1)/2$, set $x_{2,j} = s^{-1}_{[n-i+2j,n]}s^{-1}_{[n-i+2j-2,n]}\cdots s^{-1}_{[n-i+2,n]}$. Then we can show by induction that $x_{2,j}x_1\lambda' = \sum_{k=0}^{j-1}\alpha^\vee_{n-i+1+2k} + \sum_{l=1}^{i-2j} l\alpha^\vee_{n-i+2j+l}$. In particular, we have that $x_2x_1\lambda' = \lambda$. Therefore, there exists $t \in T$, such that $(U,U)(\dot{v}\dot{x}_2\dot{x}_1t, \dot{x}_2\dot{x}_1) \cdot h_J \subset X$.

Now set $y_1 = s_{n+\epsilon-1}s_{n+\epsilon-3}\cdots s_{n-i}$ and $y_2 = s_{[1,n-i-1]}$. For $1 \leqslant j \leqslant n-i-1$, set $\beta_k = -(vx_2x_1)^{-1}\alpha_k = -\alpha_{k+i}$. Thus by 3.3, there exists $u \in U_{\beta_1}U_{\beta_2}\cdots U_{\beta_{n-i}}$, such that $(\dot{x}_1^{-1}\dot{x}_2^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}_2\dot{x}_1ut, 1) \cdot h_J \in X$.

For $0 \leqslant j \leqslant (i+\epsilon-1)/2$, set

$$v_{2,j} = s_{[1,n-i]}(s_{n-i+2}s_{n-i+4}\cdots s_{n-i+2j})(s_{n-i+1}s_{n-i+3}\cdots s_{n-i+2j-1})s^{-1}_{[n-i+2j+1,n]}.$$

It is easy to see that $s_{[n-i+2j,n]}v_{2,j}s^{-1}_{[n-i+2j,n]} = v_{2,j-1}$. Therefore, we can show by induction that $x_2^{-1}y_1y_2vx_2 = x_{2,j}^{-1}v_{2,j}x_{2,j}$ for $0 \leqslant j \leqslant (i+\epsilon-1)/2$. In particular,

$x_2^{-1}y_1y_2vx_2 = s_{[1,n-i]}s_{[n-i+1,n]}^{-1}$.

For $i-1 \leqslant j \leqslant n-1$, set $v_{1,j} = s_{[1,j-i+1]}s_{[j+2,n]}s_{[j-i+2,j+1]}^{-1}$. Then we have that $s_{[j-i+1,j]}v_{1,j}s_{[j-i+1,j]}^{-1} = v_{1,j-1}$. Therefore, we can show by induction that $x_1^{-1}s_{[1,n-i]}s_{[n-i+1,n]}^{-1}x_1 = x_{1,j}^{-1}v_{1,j}x_{1,j}$ for $i-1 \leqslant j \leqslant n-1$. In particular, $x_2^{-1}y_1y_2vx_2 = s_{[i+1,n]}s_{[1,i]}^{-1} = w^J$.

Moreover, $w_{n-i-k+1}^{-1}w^{-n+i+k+1}\beta_k = w_{n-i-k+1}^{-1}(-\alpha_{n-1}) = -\sum_{l=n-k}^{n}\alpha_l$. Since $n-k \in J_{n-i-k-1} - J_{n-i-k}$, $U_{\beta_k} \subset^{\dot{w}^{n-i-k-1}\dot{w}_{n-i-k+1}} (U_{P^-_{J_{n-i-k}}} \cap U^-_{J_{n-i-k-1}})$. By 3.3.1, we have that $(\dot{x}_1^{-1}\dot{x}_2^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}_2\dot{x}_1ut, 1)\cdot h_J \in Z_{J,w^J}$. Therefore, $Z_{J,w^J} \subset X$.

For type $B_n$, we have the similar results.

**3.3.9** In 3.3.9 and 3.3.10, we assume that $G$ is of type $D_n$. In 3.3.9, assume that $i \leqslant n-2$.

If $2 \mid i$, then set $v = s_{n-i}s_{n-i+2}\cdots s_{n-2}$, $\lambda = \alpha_{n-i}^\vee + \alpha_{n-i+2}^\vee + \cdots + \alpha_{n-2}^\vee$ and $x = (s_{[n-1,n]}^{-1}s_{[n-3,n]}^{-1}\cdots s_{[n-i+1,n]}^{-1})(s_{[n-i-1,n-2]}^{-1}s_{[n-i-2,n-3]}^{-1}\cdots s_{[1,i]}^{-1})$.

If $2 \nmid i$, then set $v = (s_{n-i}s_{n-i+2}\cdots s_{n-1})s_n$, $\lambda = \alpha_{n-i}^\vee + \alpha_{n-i+2}^\vee + \cdots + \alpha_{n-3}^\vee + 1/2(\alpha_{n-1}^\vee + \alpha_n^\vee)$ and $x = (s_{[n-2,n]}^{-1}s_{[n-4,n]}^{-1}\cdots s_{[n-i+1,n]}^{-1})(s_{[n-i-1,n-2]}^{-1}s_{[n-i-2,n-3]}^{-1}\cdots s_{[1,i]}^{-1})$.

By the similar calculation to what we did for type $C_{n-1}$, we have that in both cases $(v,\lambda)$ is admissible and $x^{-1}\lambda = \omega_i^\vee$. Moreover, by the similar argument to what we did for type $C_{n-1}$, we can show that $Z_{J,w^J} \subset X$.

**3.3.10** Assume that $i = n$. Set

$$\epsilon = \begin{cases} 1, & \text{if } 2 \mid [n/2]; \\ 0, & \text{otherwise.} \end{cases}$$

If $2 \nmid n$, set $v = s_{n+\epsilon-1}(s_1s_3\cdots s_{n-2})s_{n-\epsilon}$, $x = s_{n+\epsilon-1}(s_{[n-3,n]}^{-1}s_{[n-5,n]}^{-1}\cdots s_{[2,n]}^{-1})s_{n-1}$ and $\lambda = \frac{3}{2}\alpha_{n-\epsilon}^\vee + \frac{1}{2}\alpha_{a+\epsilon-1}^\vee + \sum_{j=0}^{(n-3)/2}\alpha_{2j+1}^\vee$. Then $\lambda = 2x\omega_n^\vee$ and $(v,\lambda)$ is admissible. Set $y = s_2s_4\cdots s_{n-3}$. Then $(\dot{v}\dot{x}t, \dot{y}^{-1}\dot{x})\cdot h_J \in X$ for some $t \in T$. By 3.3, $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1)\cdot h_J \in X$. Since $x^{-1}yvx = s_{n-1}s_{[1,n-2]}^{-1}s_n = w^J$, $Z_{J,w^J} \subset X$.

If $2 \mid n$, set $v = (s_1 s_3 \cdots s_{n-3}) s_{n-\epsilon}$, $\lambda = \alpha^\vee_{n-\epsilon} + \sum_{j=0}^{n/2-2} \alpha^\vee_{1+2j}$ and

$$x = \begin{cases} s_2 s_4, & \text{if } n = 4; \\ s_{n-2} s_{n+\epsilon-1} (s^{-1}_{[n-4,n]} s^{-1}_{[n-6,n]} \cdots s^{-1}_{[2,n]}) s_{n-1}, & \text{otherwise.} \end{cases}$$

Then $\lambda = 2x\omega^\vee_n$ and $(v, \lambda)$ is admissible. Therefore, there exists $t \in T$, such that $(U, U)(\dot{v}\dot{x}t, \dot{x}) \cdot h_J \subset X$. Set $y_1 = s_2 s_4 \cdots s_{n-2}$, $y_2 = s_{n+\epsilon-1}$ and $\beta = -(vx)^{-1}\alpha_{n+\epsilon-1} = -\alpha_{n/2}$. By 3.3.3, there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$.

It is easy to see that $x^{-1} y_1 y_2 v x = s_{n-1} s^{-1}_{[1,n-2]} s_n = w^J$ and

$$w_2^{-1}\beta = \begin{cases} -\sum_{l=1}^{3} \alpha_l, & \text{if } n = 4; \\ -\sum_{l=n/2-1}^{n-2} \alpha_l, & \text{otherwise.} \end{cases}$$

Note that $J_0 = I - \{n\}$ and $J_1 = I - \{n-2, n\}$. Thus $U_\beta \subset^{w_2} (U_{P^-_{J_1}} \cap U^-_{J_0})$. By 3.3.1, $Z_{J, w^J} \subset X$.

Similarly, $Z_{I-\{i-1\}, s_n s^{-1}_{[1,n-2]} s_{n-1}} \subset X$.

**3.3.11. Type $G_2$** Set $v = s_i$, $x = w^J$ and $\lambda = \alpha^\vee_i = x\omega^\vee_i$. Then $(v, \lambda)$ is admissible. Set $y = s_{3-i}$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1} y v x = w^J$. Therefore, $Z_{J, w^J} \subset X$.

**3.3.12. Type $F_4$** If $i = 1$, then set $v = s_2$, $x = s_1 s_4 w^2$ and $\lambda = \alpha^\vee_2 = x\omega^\vee_1$. Thus $(v, \lambda)$ is admissible. Set $y_1 = s_1 s_3$, $y_2 = s_4$ and $\beta = -(vx)^{-1}\alpha_4 = -(\alpha_2 + \alpha_3)$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1} y_1 y_2 v x = w^J$ and $w_2^{-1}\beta = -(\alpha_2 + 2\alpha_3 + \alpha_4)$. By 3.3.1, $Z_{J, w^J} \subset X$.

If $i = 2$, then set $v = s_1 s_3$, $x = s_2 w^2$ and $\lambda = \alpha^\vee_1 + \alpha^\vee_3 = x\omega^\vee_2$. Thus $(v, \lambda)$ is admissible. Set $y = s_2 s_4$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1} y v x = w^J$. Thus $Z_{J, w^J} \subset X$.

If $i = 3$, then set $v = s_2 s_4$, $x = s_3 w^2$ and $\lambda = 2\alpha^\vee_2 + \alpha^\vee_4 = x\omega^\vee_3$. Thus $(v, \lambda)$ is admissible. Set $y = s_1 s_3$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1} y v x = w^J$. Thus $Z_{J, w^J} \subset X$.

If $i = 4$, then set $v = s_3$, $x = s_1 s_4 w^2$ and $\lambda = \alpha_3^\vee = x\omega_1^\vee$. Thus $(v, \lambda)$ is admissible. Set $y_1 = s_2 s_4$, $y_2 = s_1$ and $\beta = -(vx)^{-1}\alpha_1 = -(\alpha_2 + 2\alpha_3)$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$ and $w_2^{-1}\beta = -(\alpha_1 + 2\alpha_2 + 2\alpha_3)$. By 3.3.1, $Z_{J,w^J} \subset X$.

**3.3.13. Type** $E_6$ If $i = 1$, then set $v = s_1 s_5 s_3 s_6$, $x = s_1 s_4 s_3 s_1 s_6 w^J$ and $\lambda = \alpha_1^\vee + 2\alpha_3^\vee + \alpha_5^\vee + 2\alpha_6^\vee = 3x\omega_1^\vee$. Thus $(v, \lambda)$ is admissible. Set $y_1 = s_4$, $y_2 = s_2$ and $\beta = -(vx)^{-1}\alpha_2 = -(\alpha_3 + \alpha_4 + \alpha_5)$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$ and $w_2^{-1}\beta = -(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$. By 3.3.1, $Z_{J,w^J} \subset X$.

Similarly, $Z_{I-\{6\},s_2s_1s_3s_4s_5s_6} \subset X$.

If $i = 2$, then set $v = s_4$, $x = s_2 s_3 s_5 s_4 s_2 w^J$ and $\lambda = \alpha_4^\vee = x\omega_1^\vee$. Thus $(v, \lambda)$ is admissible. Set $y_1 = s_2 s_3 s_5$, $y_2 = s_1 s_6$, $\beta_1 = -(vx)^{-1}\alpha_1 = -(\alpha_4 + \alpha_5)$ and $\beta_2 = -(vx)^{-1}\alpha_6 = -(\alpha_3 + \alpha_4)$. Then there exists $u \in U_{\beta_1}U_{\beta_2}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\sum_{l=3}^{6}\alpha_l$ and $w_2^{-1}\beta_2 = -(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)$. By 3.3.1, $Z_{J,w^J} \subset X$.

If $i = 3$, then set $v = s_3 s_6 s_1 s_4 s_5$, $x = s_2 s_3 s_4 s_1 s_3 w^J$ and $\lambda = 2\alpha_1^\vee + \alpha_3^\vee + 3\alpha_4^\vee + 5\alpha_5^\vee + \alpha_6^\vee = 3x\omega_3^\vee$. Thus $(v, \lambda)$ is admissible. Set $y = s_2$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

Similarly, $Z_{I-\{5\},s_2s_1s_3s_4s_6s_5} \subset X$.

If $i = 4$, then set $v = s_2 s_3 s_5$, $x = s_4(w^J)^2$ and $\lambda = \alpha_2^\vee + \alpha_3^\vee + 5\alpha_5^\vee = x\omega_3^\vee$. Thus $(v, \lambda)$ is admissible. Set $y = s_1 s_4 s_6$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

**3.3.14. Type** $E_7$ If $i = 1$, then set $v = s_4$, $x = s_3 s_1 s_2 s_5 s_4 s_3 s_1 s_7 (w^J)^2$ and $\lambda = \alpha_4^\vee = x\omega_1^\vee$. Thus $(v, \lambda)$ is admissible. Set $y_1 = s_3 s_2 s_5$ , $y_2 = s_1 s_6 s_7$, $\beta_1 = -(vx)^{-1}\alpha_1 = -\sum_{l=3}^{6}\alpha_l$, $\beta_2 = -(vx)^{-1}\alpha_6 = -(\alpha_4 + \alpha_5)$ and $\beta_3 = -(vx)^{-1}\alpha_7 = -(\alpha_2 + \alpha_3 + \alpha_4)$. Then there exists $u \in U_{\beta_3}U_{\beta_2}U_{\beta_1}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\alpha_4 - \sum_{l=2}^{7}\alpha_l$, $w_2^{-1}\beta_2 = -\sum_{l=2}^{6}\alpha_l$ and $w_3^{-1}(w^J)^{-1}\beta_3 = -(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)$. By 3.3.1, $Z_{J,w^J} \subset X$.

If $i=2$, then set $v=s_2s_3s_5s_7$, $x=s_4s_2s_7(w^J)^3$ and $\lambda=\alpha_2^\vee+2\alpha_3^\vee+\alpha_5^\vee+\alpha_7^\vee=2x\omega_2^\vee$. Thus $(v,\lambda)$ is admissible. Set $y=s_1s_4s_6$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t,1)\cdot h_J\in X$ for some $t\in T$. Note that $x^{-1}yvx=w^J$. Thus $Z_{J,w^J}\subset X$.

If $i=3$, then set $v=s_2s_3s_5$, $x=s_1s_4s_3s_7(w^J)^3$ and $\lambda=\alpha_2^\vee+\alpha_3^\vee+\alpha_5^\vee=x\omega_3^\vee$. Thus $(v,\lambda)$ is admissible. Set $y_1=s_1s_4s_6$, $y_2=s_7$ and $\beta=-(vx)^{-1}\alpha_7=-(\alpha_4+\alpha_5)$. Then there exists $u\in U_{\beta_3}U_{\beta_2}U_{\beta_1}$ and $t\in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut,1)\cdot h_J\in X$. Note that $x^{-1}y_1y_2vx=w^J$ and $w_2^{-1}\beta=-(\alpha_2+\alpha_4+\alpha_5+\alpha_6)$. By 3.3.1, $Z_{J,w^J}\subset X$.

If $i=4$, then set $v=s_1s_4s_6$, $x=s_2s_3s_5s_4(w^J)^3$ and $\lambda=\alpha_1^\vee+2\alpha_4^\vee+\alpha_6^\vee=x\omega_4^\vee$. Thus $(v,\lambda)$ is admissible. Set $y=s_2s_3s_5s_7$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t,1)\cdot h_J\in X$ for some $t\in T$. Note that $x^{-1}yvx=w^J$. Thus $Z_{J,w^J}\subset X$.

If $i=5$, then set $v=s_2s_3s_5s_7$, $x=s_4s_6s_5(w^J)^3$ and $\lambda=\alpha_2^\vee+2\alpha_3^\vee+3\alpha_5^\vee+\alpha_7^\vee=2x\omega_5^\vee$. Thus $(v,\lambda)$ is admissible. Set $y=s_1s_4s_6$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t,1)\cdot h_J\in X$ for some $t\in T$. Note that $x^{-1}yvx=w^J$. Thus $Z_{J,w^J}\subset X$.

If $i=6$, then set $v=s_4s_6$, $x=s_1s_5s_7s_6(w^J)^3$ and $\lambda=\alpha_4^\vee+\alpha_6^\vee=x\omega_6^\vee$. Thus $(v,\lambda)$ is admissible. Set $y_1=s_2s_3s_5s_7$, $y_2=s_1$ and $\beta=-(vx)^{-1}\alpha_1=-(\alpha_3+\alpha_4+\alpha_5)$. Then there exists $u\in U_\beta$ and $t\in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut,1)\cdot h_J\in X$. Note that $x^{-1}y_1y_2vx=w^J$ and $w_2^{-1}\beta=-\alpha_4-\sum_{l=1}^5\alpha_l$. By 3.3.1, $Z_{J,w^J}\subset X$.

If $i=7$, then set $v=s_2s_5s_7$, $x=s_6s_7s_4s_5s_6s_7s_1(w^J)^2$ and $\lambda=\alpha_2^\vee+\alpha_5^\vee+\alpha_7^\vee=2x\omega_7^\vee$. Thus $(v,\lambda)$ is admissible. Set $y_1=s_4s_6$, $y_2=s_3s_1$, $\beta_1=-(vx)^{-1}\alpha_3=-(\alpha_3+\alpha_4+\alpha_5)$ and $\beta_2=-(vx)^{-1}\alpha_1=-(\alpha_2+\alpha_4+\alpha_5+\alpha_6)$. Then there exists $u\in U_{\beta_3}U_{\beta_1}$ and $t\in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut,1)\cdot h_J\in X$. Note that $x^{-1}y_1y_2vx=w^J$, $w_2^{-1}\beta_1=-\alpha_4-\sum_{l=1}^6\alpha_l$, $w_3^{-1}(w^J)^{-1}\beta_2=-\alpha_4-\sum_{l=1}^5\alpha_l$. By 3.3.1, $Z_{J,w^J}\subset X$.

**3.3.15. Type** $E_8$ If $i=1$, then set $v=s_4s_6$, $x=s_3s_1s_2s_5s_4s_3s_1s_8(w^J)^5$ and $\lambda=\alpha_4^\vee+\alpha_6^\vee=x\omega_1^\vee$. Thus $(v,\lambda)$ is admissible. Set $y_1=s_2s_3s_5s_7$, $y_2=s_1s_8$, $\beta_1=-(vx)^{-1}\alpha_1=-\alpha_4-\sum_{l=2}^6\alpha_l$ and $\beta_2=-(vx)^{-1}\alpha_8=-\sum_{l=3}^7\alpha_l$. Then there exists $u\in U_{\beta_2}U_{\beta_1}$ and $t\in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut,1)\cdot h_J\in X$. Note that $x^{-1}y_1y_2vx=w^J$, $w_2^{-1}\beta_1=-\alpha_4-\alpha_5-\sum_{l=2}^7\alpha_l$ and $w_2^{-1}\beta_2=-\alpha_4-\sum_{l=2}^8\alpha_l$. By 3.3.1, $Z_{J,w^J}\subset X$.

If $i=2$, then set $v=s_2s_3s_5s_7$, $x=s_4s_2s_7s_8(w^J)^6$ and $\lambda=\alpha_2^\vee+\alpha_3^\vee+\alpha_5^\vee+\alpha_7^\vee=x\omega_2^\vee$. Thus $(v,\lambda)$ is admissible. Set $y=s_1s_4s_6s_8$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t,1)\cdot h_J\in X$ for some $t\in T$.

Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 3$, then set $v = s_2s_3s_5s_7$, $x = s_1s_4s_3s_7s_8(w^J)^6$ and $\lambda = \alpha_2^\vee + \alpha_3^\vee + 2\alpha_5^\vee + \alpha_7^\vee = x\omega_3^\vee$. Thus $(v, \lambda)$ is admissible. Set $y = s_1s_4s_6s_8$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 4$, then set $v = s_1s_4s_6s_8$, $x = s_2s_5s_3s_4s_8(w^J)^6$ and $\lambda = \alpha_1^\vee + 3\alpha_4^\vee + 2\alpha_6^\vee + \alpha_8^\vee = x\omega_4^\vee$. Thus $(v, \lambda)$ is admissible. Set $y = s_2s_3s_5s_7$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 5$, then set $v = s_2s_3s_5s_7$, $x = s_4s_6s_5(w^J)^6$ and $\lambda = \alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_5^\vee + \alpha_7^\vee = x\omega_5^\vee$. Thus $(v, \lambda)$ is admissible. Set $y = s_1s_4s_6s_8$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 6$, then set $v = s_1s_4s_6$, $x = s_1s_5s_7s_6(w^J)^6$ and $\lambda = \alpha_1^\vee + 2\alpha_4^\vee + \alpha_6^\vee = x\omega_6^\vee$. Thus $(v, \lambda)$ is admissible. Set $y_1 = s_2s_3s_5s_7$, $y_2 = s_8$ and $\beta = -(vx)^{-1}\alpha_8$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$ and $w_2^{-1}\beta = -\alpha_4 - \sum_{l=1}^{5} \alpha_l$. By 3.3.1, $Z_{J,w^J} \subset X$.

If $i = 7$, then set $v = s_2s_3s_5$, $x = s_6s_7s_8s_4s_5s_6s_7(w^J)^5$ and $\lambda = \alpha_2^\vee + \alpha_3^\vee + \alpha_5^\vee = x\omega_7^\vee$. Thus $(v, \lambda)$ is admissible. Set $y_1 = s_1s_4s_6$, $y_2 = s_7s_8$, $\beta_1 = -(vx)^{-1}\alpha_7 = -(\alpha_3 + \alpha_4 + \alpha_5)$ and $\beta_2 = -(vx)^{-1}\alpha_8 = -(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)$. Then there exists $u \in U_{\beta_2}U_{\beta_1}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\alpha_4 - \sum_{l=1}^{6} \alpha_l$ and $w_3^{-1}(w^J)^{-1}\beta_2 = -\alpha_4 - \sum_{l=1}^{5} \alpha_l$. By 3.3.1, $Z_{J,w^J} \subset X$.

If $i = 8$, then set $v = s_4$, $x = s_1s_5s_6s_7s_8(w^J)^5$ and $\lambda = \alpha_4^\vee = x\omega_8^\vee$. Thus $(v, \lambda)$ is admissible. Set $y_1 = s_5s_2s_3$, $y_2 = s_1s_6s_7s_8$, $\beta_1 = -(vx)^{-1}\alpha_1 = -\alpha_4 - \sum_{l=2}^{7} \alpha_l$, $\beta_2 = -(vx)^{-1}\alpha_6 = -(\alpha_3 + \alpha_4 + \alpha_5)$, $\beta_3 = -(vx)^{-1}\alpha_7 = w^J\beta_2$ and $\beta_4 = -(vx)^{-1}\alpha_8 = (w^J)^2\beta_2$. Then there exists $u \in U_{\beta_4}U_{\beta_3}U_{\beta_2}U_{\beta_1}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\sum_{l=3}^{6} \alpha_l - \sum_{l=1}^{7} \alpha_l$, $w_2^{-1}\beta_2 = -\alpha_4 - \sum_{l=1}^{7} \alpha_l$, $w_3^{-1}(w^J)^{-1}\beta_3 = -\alpha_4 - \sum_{l=1}^{6} \alpha_l$ and $w_4^{-1}(w^J)^{-2}\beta_4 = -\alpha_4 - \sum_{l=1}^{5} \alpha_l$. By 3.3.1, $Z_{J,w^J} \subset X$.

## 3.4 The explicit description of $\bar{\mathcal{U}}$

**3.4.1** We assume that $G^1$ is a disconnected algebraic group such that its identity component $G^0$ is reductive. Following [St, 9], an element $g \in G^1$ is called quasi-semisimple if $gBg^{-1} = B, gTg^{-1} = T$ for some Borel subgroup $B$ of $G^0$ and some maximal torus $T$ of $B$. We have the following properties.

(a) *if $g$ is semisimple, then it is quasi-semisimple.* See [St, 7.5, 7.6].

(b) *$g$ is quasi-semisimple if and only if the $G^0$-conjugacy class of $g$ is closed in $G^1$.* See [Spa, 1.15].

(c) *Let $g \in G^1$ is a quasi-semisimple element and $T_1$ be a maximal torus of $Z_{G^0}(g)^0$, where $Z_{G^0}(g)^0$ is the identity component of $\{x \in G^0 \mid xg = gx\}$. Then any quasi-semisimple element in $gG^0$ is $G^0$-conjugate to some element of $gT_1$.* See [L8, 1.14].

**3.4.2** Let $\rho_i : G \to GL(V_i)$ be the irreducible representation of $G$ with lowest weight $-\omega_i$ and $\bar{\rho}_i : \bar{G} \to P\big(\mathrm{End}(V_i)\big)$ be the morphism induced from $\rho_i$. Let $\mathcal{N}$ be the subvariety of $\bar{G}$ consisting of elements such that for all $i \in I$, the images under $\bar{\rho}_i$ are represented by nilpotent endomorphisms of $V_i$. We have the following result.

**Theorem 3.4.3.** *We have that*

$$\bar{\mathcal{U}} - \mathcal{U} = \mathcal{N} = \bigsqcup_{J \subsetneq I} \bigsqcup_{w \in W^J, \mathrm{supp}(w)=I} Z_{J,w}.$$

*Proof.* By 3.2.11 and the results in section 3.3, we have that

$$\bigsqcup_{J \subsetneq I} \bigsqcup_{w \in W^J, \mathrm{supp}(w)=I} Z_{J,w} \subset \bar{\mathcal{U}} - \mathcal{U}.$$

For $i \in I$, let $X_i$ be the subvariety of $P\big(\mathrm{End}(V_i)\big)$ consisting of the elements that can be represented by unipotent or nilpotent endomorphisms of $V_i$. Then $X_i$ is closed in $P(\mathrm{End}(V_i))$. Thus, $\bar{\rho}_i(z) \in X_i$ for $z \in \bar{\mathcal{U}}$. Moreover, since $G$ is simple, for any $g \in \bar{G}$, $\bar{\rho}_i(g)$ is represented by an automorphism of $V_i$ if and only if $g \in G$. Thus if $z \in \bar{\mathcal{U}} - \mathcal{U}$, then $\bar{\rho}_i(z)$ is represented by an nilpotent endomorphism of $V_i$. Therefore

$\bar{\mathcal{U}} - \mathcal{U} \subset \mathcal{N}$.

Assume that $w \in W^J$ with $\mathrm{supp}(w) \neq I$ and $\mathcal{N} \cap Z_{J,w} \neq \varnothing$. Let $C$ be the closed $L_{J,w}$-stable subvariety that corresponds to $\mathcal{N} \cap Z_{J,w}$. We have seen that $\dot{w}$ is a quasi-semisimple element of $N_G(L_{J,w})$. Moreover, there exists a maximal torus $T_1$ in $Z_{L_{J,w}}(w)^0$ such that $T_1 \subset T$. Since $C$ is an $L_{J,w}$-stable nonempty closed subvariety of $C_{J,w}$, $\dot{w}t \in C$ for some $t \in T_1$. Set $z = (\dot{w}t, 1) \cdot h_J$. Then $z \in \mathcal{N}$.

Since $\mathrm{supp}(w) \neq I$, there exists $i \in I$ with $i \notin \mathrm{supp}(w)$. Then $-w\omega_i = -\omega_i$. Let $v$ be a lowest weight vector in $V_i$. Assume that $\bar{\rho}_i(z)$ is represented by an endomorphism $A$ of $V$. Then $Av \in k^* v$. Thus $z \notin \mathcal{N}$. That is a contradiction. Therefore $\mathcal{N} \subset \bigsqcup_{J \subsetneqq I} \bigsqcup_{w \in W^J, \mathrm{supp}(w) = I} Z_{J,w}$. The theorem is proved. □

**Corollary 3.4.4.** *Let $i \in I$ and $J = I - \{i\}$ and $w$ be a Coxeter element of $W$ with $w \in W^J$. Then $\overline{Z_{J,w}} = \bigsqcup_{K \subset J} \bigsqcup_{w' \in W^K, \mathrm{supp}(w') = I} Z_{K,w'}$.*

*Proof.* Note that $Z_{J,w} \subset \bar{\mathcal{U}} \cap (\bigsqcup_{K \subset J} Z_K)$. Since $\bar{\mathcal{U}}$ and $\bigsqcup_{K \subset J} Z_K$ are closed, $\overline{Z_{J,w}} \subset \bar{\mathcal{U}} \cap (\bigsqcup_{K \subset J} Z_K) = \bigsqcup_{K \subset J} \bigsqcup_{w' \in W^K, \mathrm{supp}(w') = I} Z_{K,w'}$. Therefore by 3.2.11,

$$\overline{Z_{J,w}} = \bigsqcup_{K \subset J} \bigsqcup_{w' \in W^K, \mathrm{supp}(w') = I} Z_{K,w'}.$$

□

**3.4.5** Let $\sigma : G \to T/W$ be the morphism which sends $g \in G$ to the $W$-orbit in $T$ that contains an element in the $G$-conjugacy class of the semisimple part $g_s$. The map $\sigma$ is called Steinberg map. The fibers of $\sigma$ are called Steinberg fibers. The unipotent variety is an example of Steinberg fiber. Some other interesting examples are the regular semisimple conjugacy classes of $G$.

Let $F$ be a fiber of $\sigma$. It is known that $F$ is a union of finitely many $G$-conjugacy classes. Let $t$ be a representative of $\sigma(F)$ in $T$, then $F = G_{diag} \cdot tU$ and $\bar{F} = G_{diag} \cdot t\bar{U}$ (see [Spr3, 1.4]). It is easy to see that $t(\bar{U} - U) \subset \mathcal{N}$. Thus $\bar{F} - F = G_{diag} \cdot t(\bar{U} - U) \subset \mathcal{N}$. Therefore, if $(w, \lambda)$ is admissible and $x^{-1} \cdot \lambda$ dominant, then there exists some $t' \in T$ such that $(U \times U)(\dot{w}\dot{x}t', \dot{x}) \cdot h_{I(x^{-1}\lambda)} \subset t\bar{U}$. Thus by 3.2.11 and the results in section 3.3, $\bigsqcup_{J\subsetneqq I}\bigsqcup_{w\in W^J,\mathrm{supp}(w)=I} Z_{J,w}\subset \bar{F}-F$. Therefore, we have

$$\bar{F}-F=\mathcal{N}=\bigsqcup_{J\subsetneqq I}\bigsqcup_{w\in W^J,\mathrm{supp}(w)=I} Z_{J,w}.$$

Thus $\bar{F}-F$ is independent of the choice of the Steinberg fiber $F$. As a consequence, in general, $\bar{F}$ contains infinitely many $G$-orbits (answering a question that Springer asked in [Spr3]).

**3.4.6** For any variety $X$ that is defined over the finite field $\mathbf{F}_q$, we write $|X|_q$ for the number of $\mathbf{F}_q$-rational points in $X$.

If $G$ is defined and split over the finite field $\mathbf{F}_q$, then for any $w\in W^J$, $|\tilde{Z}_{J,w}|_q=|G|_q q^{-l(w)}$ (see [L8, 8.20]). Thus

$$|Z_{J,w}|_q=|G|_q q^{-l(w)}(q-1)^{-|I-J|}=(\sum_{w\in W} q^{l(w)})(q-1)^{|J|}q^{l(w_0 w)}.$$

Set $L(w)=\{i\in I\mid ws_i<w\}$. Then $w\in W^J$ if and only if $J\subset L(w_0w)$. Moreover, if $w\neq 1$, then $L(w_0w)\neq I$. Therefore

$$\begin{aligned}|\bar{\mathcal{U}}-\mathcal{U}|_q&=\sum_{J\neq I}\sum_{w\in W^J,\mathrm{supp}(w)=I}|Z_{J,w}|_q=(\sum_{w\in W}q^{l(w)})\sum_{J\neq I}\sum_{w\in W^J,\mathrm{supp}(w)=I}(q-1)^{|J|}q^{l(w_0w)}\\&=(\sum_{w\in W}q^{l(w)})\sum_{\mathrm{supp}(w)=I}\sum_{J\subset L(w_0w)}q^{l(w_0w)}(q-1)^{|J|}\\&=(\sum_{w\in W}q^{l(w)})\sum_{\mathrm{supp}(w)=I}q^{l(w_0w)+|L(w_0w)|}.\end{aligned}$$

*Remark.* Note that $|\bar{G}|_q=\sum_{w\in W}q^{l(w)}\sum_{w\in W}q^{l(w_0w)+|L(w_0w)|}$ (see [DP, 7.7]). Our formula for $|\bar{\mathcal{U}}-\mathcal{U}|_q$ bears some resemblance to the formula for $|\bar{G}|_q$.

# Chapter 4

# The $G$-stable pieces of the wonderful compactification

Let $G$ be a connected, simple algebraic group over an algebraically closed field. There is a partition of the wonderful compactification $\bar{G}$ of $G$ into finite many $G$-stable pieces, which were introduced by Lusztig. In this chapter, we will investigate the closure of any $G$-stable piece in $\bar{G}$. We will show that the closure is a disjoint union of some $G$-stable pieces, which was first conjectured by Lusztig. We will also prove the existence of cellular decomposition if the closure contains finitely many $G$-orbits.

## 4.0. Introduction

An adjoint semi-simple group $G$ has a "wonderful" compactification $\bar{G}$, introduced by De Concini and Procesi in [DP]. The variety $\bar{G}$ is a smooth variety with $G \times G$ action. Denote by $G_{diag}$, the image of the diagonal embedding of $G$ into $G \times G$. The $G_{diag}$ orbits of $\bar{G}$ were studied by Lusztig in [L8]. He introduced a partition of $\bar{G}$ into finitely many $G$-stable pieces. The $G$ orbits on each piece can be described explicitly. Based on the partition, he established the theory of "parabolic character sheaves" on $\bar{G}$.

The main results of this chapter concern the closure of the $G$-stable pieces. The closure of each piece is a union of some other pieces and if the closure contains finitely many $G$-orbits, then it admits a cellular decomposition.

We now review the content of this chapter in more detail.

In section 4.1, we recall the definition of $G$-stable pieces in [L8] and establish some basic results. The pieces are indexed by the pairs $\mathcal{I} = \{(J, w)\}$, where $J$ is a subset of the simple roots and $w$ is an element of the Weyl group $W$, which has minimal length in the coset $wW_J$. One interesting result is that any $G$-stable piece is the minimal $G$-stable subset that contains a particular $B \times B$-orbit, where $B$ is the Borel subgroup. The closure of any $B \times B$-orbit in $\bar{G}$ was studied by Springer in [Spr2]. Based on his result and the relations between $G$-stable pieces and $B \times B$-orbits, we are able to investigate the closure of the $G$-stable pieces.

In section 4.2, we recall the definition of the "wonderful" compactification and introduce "compactification through the fibres", a technique tool that will be used to prove the existence of cellular decomposition. In section 4.3, we describe a partial order on $\mathcal{I}$, which is the partial order that corresponds to the closure relation of the $G$-stable piece, as we will see in section 4.4. In section 4.4, we also discuss the closure of any $G$-stable piece that appears in [L7].

In section 4.5, we discuss the existence of cellular decomposition. Each piece does not have a cellular decomposition. However, a union of certain pieces has a cellular decomposition. (This is motivated by Springer in [Spr2], in which he showed that a union of certain $B \times B$-orbits is isomorphic to an affine space.) In fact, if the closure contains finitely many $G$-orbits, then it has a cellular decomposition.

The methods work for arbitrary connected component of a disconnected algebraic group with identity component $G$. The results for that component is just a "twisted" version of the results for $G$ itself.

## 4.1 The $G$-stable pieces

**4.1.1** We keep the notation of 1.1.1-1.1.4.

In the sequel, we assume that $G$ is adjoint. Let $\hat{G}$ be a possibly disconnected reductive algebraic group over an algebraically closed field with identity component

$G$. Let $G^1$ be a fixed connected component of $\hat{G}$. There exists an isomorphism $\delta : W \to W$ such that $\delta(I) = I$ and ${}^gP \in \mathcal{P}^{\delta(J)}$ for $g \in G^1$ and $P \in \mathcal{P}^J$. There exists $g_0 \in G^1$ such that $g_0$ normalizes $T$ and $B$. Moreover, $g_0$ can be chosen in such a way that ${}^{g_0}L_J = L_{\delta(J)}$ for $J \subset I$. We will fix such $g_0$ in the rest of this chapter.

In particular, if $G^1 = G$, then $\delta = id$, where $id$ is the identity map. In this case, we will choose $g_0$ to be the unit element 1 of $G$.

**4.1.2** We will follow the set-up of [L8].

Let $J, J' \subset I$ and $y \in {}^{J'}W^J$ be such that $\mathrm{Ad}(y)\delta(J) = J'$. For $P \in \mathcal{P}^J$, $P' \in \mathcal{P}^{J'}$, define $A_y(P, P') = \{g \in G^1 \mid \mathrm{pos}(P', {}^gP) = y\}$. Define

$$\tilde{Z}_{J,y,\delta} = \{(P, P', \gamma) \mid P \in \mathcal{P}^J, P' \in \mathcal{P}^{J'}, \gamma \in U_{P'} \backslash A_y(P, P')/U_P\}$$

with $G \times G$ action defined by $(g_1, g_2)(P, P', \gamma) = ({}^{g_2}P, {}^{g_1}P', g_1 \gamma g_2^{-1})$.

By [L8, 8.9], $A_y(P, P')$ is a single $P', P$ double coset. Thus $G \times G$ acts transitively on $Z_{J,y,\delta}$.

Let $z = (P, P', \gamma) \in \tilde{Z}_{J,y,\delta}$. Then there exists $g \in \gamma$ such that ${}^gP$ contains some Levi of $P \cap P'$. Now set $P_1 = g^{-1}({}^gP)^{(P'^P)}g$, $P'_1 = P^{P'}$. Define $\alpha(P, P', \gamma) = (P_1, P'_1, U_{P'_1} g U_{P_1})$. By [L8, 8.11], The map $\alpha$ doesn't depend on the choice of $g$.

To $z = (P, P', \gamma) \in \tilde{Z}_{J,y,\delta}$, we associate a sequence $(J_n, J'_n, u_n, y_n, P_n, P'_n, \gamma_n)_{n \geqslant 0}$ with $J_n, J'_n \subset I$, $u_n \in W$, $y_n \in {}^{J'_n}W^{\delta(J_n)}$, $\mathrm{Ad}(y_n)\delta(J_n) = J'_n$, $P_n \in \mathcal{P}^{J_n}$, $P'_n \in \mathcal{P}^{J'_n}$, $\gamma_n = U_{P'_n} g U_{P_n}$ for some $g \in G$ satisfies $\mathrm{pos}(P'_n, {}^gP_n) = y_n$. The sequence is defined as follows.

$$P_0 = P, P'_0 = P', \gamma_0 = \gamma, J_0 = J, J'_0 = J', u_0 = \mathrm{pos}(P'_0, P_0), y_0 = y.$$

Assume that $n \geqslant 1$, that $P_m, P'_m, \gamma_m, J_m, J'_m, u_m, y_m$ are already defined for $m < n$ and that $u_m = \mathrm{pos}(P'_m, P_m), P_m \in \mathcal{P}^{J_m}, P'_m \in \mathcal{P}^{J'_m}$ for $m < n$. Let

$$J_n = J_{n-1} \cap \delta^{-1}\mathrm{Ad}(y_{n-1}^{-1}u_{n-1})J_{n-1}, J'_n = J_{n-1} \cap \mathrm{Ad}(u_{n-1}^{-1}y_{n-1})\delta(J_{n-1}),$$

$$(P_n, P'_n, \gamma_n) = \alpha(P_{n-1}, P'_{n-1}, \gamma_{n-1}) \in \tilde{Z}_{J_n,y_n,\delta}(\text{see [L8, 8.10]}),$$

$$u_n = \text{pos}(P'_n, P_n), y_n = u_{n-1}^{-1} y_{n-1}, \gamma_n = U_{P'_n} g_{n-1} U_{P_n}.$$

It is known that the sequence is well defined. Moreover, for sufficient large $n$, we have that $J_n = J'_n = J_{n+1} = J'_{n+1} = \cdots = J_\infty$, $u_n = u_{n+1} = \cdots = 1$, $y_n = y_{n+1} = \cdots = y_\infty$, $P_n = P_{n+1} = \cdots = P_\infty$, $P'_n = P'_{n+1} = \cdots = P'_\infty$ and $\gamma_n = \gamma_{n+1} = \cdots = \gamma_\infty$. Now we set $\beta(z) = y_\infty$. Then we have that $\beta(z) \in W^{\delta(J)}$. By [L8, 8.18] and [L7, 2.5], the sequence $(J_n, J'_n, u_n, y_n)_{n \geqslant 0}$ is uniquely determined by $\beta(z)$ and $y$.

For $w \in W^{\delta(J)}$, set

$$\tilde{Z}^w_{J,y,\delta} = \{z \in \tilde{Z}_{J,y,\delta} \mid \beta(z) = w\}.$$

Then $(\tilde{Z}^w_{J,y,\delta})_{w \in W^{\delta(J)}}$ is a partition of $\tilde{Z}_{J,y,\delta}$ into locally closed $G$-stable subvarieties. For $w \in W^{\delta(J)}$, let $(J_n, J'_n, u_n, y_n)_{n \geqslant 0}$ be the sequence determined by $w$ and $y$. The restriction of the map $\alpha$ on $\tilde{Z}^w_{J,y,\delta}$ is a $G$-equivariant morphism from $\tilde{Z}^w_{J,y,\delta}$ onto $\tilde{Z}^w_{J_1,y_1,\delta}$. We also denote this morphism by $\alpha$. It is known that $\alpha$ induces a bijection from the set of $G$-orbits on $\tilde{Z}^w_{J,y,\delta}$ to the set of $G$-orbits on $\tilde{Z}^w_{J_1,y_1,\delta}$.

For sufficiently large $n$, $\vartheta = \alpha^n : \tilde{Z}^w_{J,y,\delta} \to \tilde{Z}^w_{J_\infty,w,\delta}$ is independent of the choice of $n$ and is a $G$-equivariant morphism. Moreover, $\vartheta$ induces a bijection from the set of $G$-orbits on $\tilde{Z}^w_{J,y,\delta}$ to the set of $G$-orbits on $\tilde{Z}^w_{J_\infty,w,\delta}$.

**4.1.3** In the rest of this section, we will fix $J, y, \delta, w$ and $J_\infty$. First, we will give an explicit description of $J_\infty$ in terms of $J, \delta$ and $w$.

**Lemma 4.1.4.** *Keep the notion of 4.1.2. Then*

$$J_\infty = \max\{K \subset J \mid \text{Ad}(w)\delta(K) = K\}.$$

*Proof.* Set $v = y_1 w^{-1}$. By [H2, 2.2], $v \in W_J$. Now $J_1 = J \cap \delta^{-1}\text{Ad}(y_1^{-1})J$. Thus $\Phi_{\delta(J_1)} \subset \text{Ad}(y_1^{-1})\Phi_J = \text{Ad}(w^{-1})\text{Ad}(v^{-1})\Phi_J = \text{Ad}(w^{-1})\Phi_J$.

Let $i \in J$. Assume that $\alpha_{\delta(i)} \in \text{Ad}(y_1^{-1})\Phi_J$. Then $\alpha_{\delta(i)} = \text{Ad}(y_1^{-1})\alpha = \text{Ad}(y^{-1})\text{Ad}(u_0)\alpha$

for some $\alpha \in \Phi_J$. Then $\alpha_{\mathrm{Ad}(y)\delta(i)} = \mathrm{Ad}(u_0)\alpha$. Note that $\alpha_{\mathrm{Ad}(y)\delta(i)}$ is a simple root and $u_0 \in W^J$. Then $\alpha = \alpha_j$ for some $j \in J$. Hence $i = \delta^{-1}\mathrm{Ad}(y_1^{-1})j$. Therefore, $i \in J \cap \delta^{-1}\mathrm{Ad}(y_1^{-1})J = J_1$. So

$$J_1 = \max\{K \subset J \mid \Phi_{\delta(K)} \subset \mathrm{Ad}(w^{-1})\Phi_J\}.$$

Set $J'_\infty = \max\{K \subset J \mid \mathrm{Ad}(w)\delta(K) = K\}$. Then $J'_\infty \subset J$. Moreover, $\Phi_{\delta(J'_\infty)} = \mathrm{Ad}(w^{-1})\Phi_{J'_\infty} \subset \mathrm{Ad}(w^{-1})\Phi_J$. Thus $J'_\infty \subset J_1$. We can show by induction that $J'_\infty \subset J_n$ for all $n$. Thus $J'_\infty \subset J_\infty$. By the definition, $J_\infty = J_\infty \cap \delta^{-1}\mathrm{Ad}(w^{-1})J_\infty$. Thus $\mathrm{Ad}(w)\delta(J_\infty) = J_\infty$. So $J_\infty = J'_\infty$. The lemma is proved. □

**4.1.5** Now set $\tilde{h}_{J,y,\delta} = (P_J, {}^{\dot{y}^{-1}}P_{J'}, U_{\dot{y}^{-1}P_{J'}} g_0 U_{P_J}) \in Z_{J,y,\delta}$. For $w \in W^{\delta(J)}$ and $v \in W$, set $\widetilde{[J,w,v]}_{y,\delta} = (B \times B)(\dot{w},\dot{v}) \cdot \tilde{h}_{J,y,\delta}$. Then we have the following result.

**Lemma 4.1.6.** *Keep the notion of 4.1.2. Let $g \in P_{J_1}$. Set $z = (\dot{w}, g) \cdot \tilde{h}_{J,y,\delta}$ and $z' = (\dot{w}, g) \cdot \tilde{h}_{J_1,y_1,\delta}$. Then $\alpha(z) = z'$.*

*Proof.* Set $P = P_J$, $P' = {}^{\dot{w}\dot{y}^{-1}}P_{J'}$, $g_1 = \dot{w}g_0 g$ and $v = y_1 w^{-1}$. Then $v \in W_J$.

By the proof of [H2, 2.3], ${}^{\dot{v}^{-1}}L_{J'_1}$ is a Levi factor of $P \cap P'$ and $P^{P'} = {}^{\dot{v}^{-1}}P_{J'_1}$, $(P')^P = {}^{\dot{w}\dot{y}^{-1}}P_{\mathrm{Ad}(y)\delta(J_1)}$. Moreover,

$${}^{\dot{v}^{-1}}L_{J'_1} = {}^{\dot{w}\dot{y}_1^{-1}}L_{J'_1} = {}^{\dot{w}}L_{\delta(J_1)} = {}^{\dot{w}g_0}L_{J_1} \subset {}^{\dot{w}g_0}P_J = {}^{\dot{w}g_0 g}P_J.$$

So ${}^{g_1}P$ contains some Levi of $P \cap P'$. We have that

$$\begin{aligned} g_1^{-1}({}^{g_1}P)^{({}^{\dot{w}\dot{y}^{-1}}P_{\mathrm{Ad}(y_1)\delta(J_1)})}g_1 &= {}^{g^{-1}}P^{({}^{g_0^{-1}\dot{y}^{-1}}P_{\mathrm{Ad}(y)\delta(J_1)})} = {}^{g^{-1}}P_{J\cap\delta^{-1}\mathrm{Ad}(y^{-1})\mathrm{Ad}(y)\delta(J_1)} \\ &= {}^{g^{-1}}P_{J_1} = P_{J_1}. \end{aligned}$$

Thus $\alpha(z) = z'$. The lemma is proved. □

**Proposition 4.1.7.** *We have that*

$$\tilde{Z}^w_{J,y,\delta} = G_{diag} \cdot \widetilde{[J,w,1]}_{y,\delta} = G_{diag} \cdot (P_J, {}^{\dot{w}\dot{y}^{-1}}P_{J'}, U_{\dot{w}\dot{y}^{-1}P_{J'}} \dot{w} g_0 (B \cap L_{J_\infty}) U_{P_J}).$$

*Proof.* It is easy to see that $\tilde{Z}^w_{J_\infty,w,\delta} = G_{diag}(\dot{w}, L_{J_\infty}) \cdot \tilde{h}_{J_\infty,w,\delta}$. Thus for any $b \in B$, $\alpha^n\big((\dot{w}, b) \cdot \tilde{h}_{J,y,\delta}\big) \in \tilde{Z}^w_{J_\infty,w,\delta}$ for sufficiently large $n$. Therefore, $G_{diag}(\dot{w}, B) \cdot \tilde{h}_{J,y,\delta} \subset \tilde{Z}^w_{J,y,\delta}$.

Note that $\dot{w}g_0$ normalizes $(L_{J_\infty})$ and $(L_{J_\infty}) \cap B$. Thus $\dot{w}g_0L_{J_\infty} = \{l\dot{w}g_0bl^{-1} \mid l \in L_{J_\infty}, b \in L_{J_\infty} \cap B\}$. Hence any element in $\tilde{Z}^w_{J_\infty,w,\delta}$ is $G$-conjugate to $(\dot{w}, l) \cdot \tilde{h}_{J_\infty,w,\delta}$ for some $l \in L_{J_\infty} \cap B$. Now let $z \in \tilde{Z}^w_{J,y,\delta}$. Then $\vartheta(z)$ is $G$-conjugate to $(\dot{w}, l) \cdot \tilde{h}_{J_\infty,w,\delta}$ for some $l \in L_{J_\infty} \cap B$. Set $z' = (\dot{w}, l) \cdot \tilde{h}_{J,y,\delta} \in \tilde{Z}^w_{J,y,\delta}$. Then $\vartheta(z')$ lies in the same $G$-orbit as $\vartheta(z)$. Since $\vartheta$ induces a bijection from the set of $G$-orbits on $\tilde{Z}^w_{J,y,\delta}$ to the set of $G$-orbits on $\tilde{Z}^w_{J_\infty,w,\delta}$. Thus $z$ is $G$-conjugate to $z'$. So $\tilde{Z}^w_{J,y,\delta} = G_{diag}(\dot{w}, B \cap L_{J_\infty})) \cdot \tilde{h}_{J,y,\delta}$. The proposition is proved. □

**4.1.8** In [L8, 8.20], Lusztig showed that $\tilde{Z}^w_{J,y,\delta}$ is an iterated affine space bundle over a fibre bundle over $\mathcal{P}^{J_\infty}$ with fibres isomorphic to $L_{J_\infty}$. In 4.1.10, we will prove a similar (but more explicit) result, which will be used to establish the cellular decomposition. Before doing that, some simple results about fibre products will be discussed.

**Lemma 4.1.9.** *Let $X$ be a variety with $G$-action and $Y$ be a subvariety such that $X = G \cdot Y$. Let $H$ be a closed subgroup of $G$ such that if $gy = y'$ for some $y, y' \in Y$, then $g \in H$. Consider the morphism $\pi : G \times (H \cdot Y) \to X$ sending $(g, y)$ to $g \cdot y$ for $g \in G$ and $y \in H \cdot Y$. Then the morphism is invariant under the $H$-action given by $h(g, y) = (gh^{-1}, h \cdot y)$. Denote by $G \times_H (H \cdot Y)$ the quotient, then $\pi$ induces an isomorphism $G \times_H (H \cdot Y) \xrightarrow{\sim} X$.*

*Proof.* Let $x \in X$. Then there exists $g \in G$ and $y \in Y$ such that $x = g \cdot y$. The subvariety $\pi^{-1}(x)$ of $G \times (H \cdot Y)$ is stable under the action of $H$. Moreover, if $(g_1, h_1y_1), (g_2, h_2y_2) \in \pi^{-1}(x)$ with $h_1, h_2 \in H$ and $y_1, y_2 \in Y$, then $(h_1^{-1}g_1^{-1}g_2h_2) \cdot y_2 = y_1$. Thus $h_1^{-1}g_1^{-1}g_2h_2 \in H$. So $g_2 \in g_1H$. So $H$ acts simply transitively on $\pi^{-1}(x)$ for each $x \in X$. Hence, $\pi$ induces an isomorphism $G \times_H (H \cdot Y) \xrightarrow{\sim} X$. □

**Proposition 4.1.10.** *For $a \in W$, set $U_a = U \cap {}^{\dot{a}}U^-$. Set $\tilde{L}^w_{J,y,d} = (L_{J_\infty}, L_{J_\infty})(\dot{w}, 1) \cdot \tilde{h}_{J,y,\delta}$. Then we have the following results.*

*(1) $\tilde{Z}^w_{J,y,\delta}$ is isomorphic to $G \times_{P_{J_\infty}} \big((P_{J_\infty}) \cdot \tilde{L}^w_{J,y,\delta}\big)$.*

*(2)* $(P_{J_\infty})\cdot \tilde{L}^w_{J,y,\delta} = (B\times B)\cdot \tilde{L}^w_{J,y,\delta} \cong (U\cap {}^{\dot{w}_0^{J_\infty}\dot{w}\dot{y}^{-1}\dot{w}_0^{J'}}U^-)\times \tilde{L}^w_{J,y,\delta}$, *where* $\tilde{L}^w_{J,y,\delta}$ *is isomorphic to* $L_{J_\infty}$*.*

*(3)* $G_{diag}(\dot{w}T,1)\cdot \tilde{h}_{J,y,\delta}$ *is dense in* $\tilde{Z}^w_{J,y,\delta}$*.*

*Proof.* It is easy to see that $\tilde{L}^w_{J,y,\delta} = (\dot{w}, L_{J_\infty})\cdot \tilde{h}_{J,y,\delta}$ is isomorphic to $L_{J_\infty}$. By 4.1.7, $\tilde{Z}^w_{J,y,\delta} = G_{diag}\cdot \tilde{L}^w_{J,y,\delta}$. Consider the $G$-equivariant map $p : \tilde{Z}^w_{J_\infty,w,\delta} \to \mathcal{P}^{J_\infty}$ defined by $p(P,P,\gamma) = P$ for $(P,P,\gamma)\in \tilde{Z}^w_{J_\infty,w,\delta}$. For $l\in L_{J_\infty}$ and $g\in G$, if $p\circ\vartheta\big((g,g)(\dot{w},l)\cdot \tilde{h}_{J,y,\delta}\big) = P_{J_\infty}$, then $g\in P_{J_\infty}$. Thus $(P_{J_\infty})_{diag}\cdot \tilde{L}^w_{J,y,\delta} = (p\circ\vartheta)^{-1}(P_{J_\infty})$.

Assume that $(g,g)(\dot{w},l_1)\cdot \tilde{h}^w_{J,y,\delta} = (\dot{w},l_2)\cdot \tilde{h}^w_{J,y,\delta}$ for $g\in G$ and $l_1,l_2\in L_{J_\infty}$. Then ${}^gP_{J_\infty} = p\circ\vartheta\big((g,g)(\dot{w},l_1)\cdot \tilde{h}^w_{J,y,\delta}\big) = p\circ\vartheta\big((\dot{w},l_2)\cdot \tilde{h}^w_{J,y,\delta}\big) = P_{J_\infty}$. So $g\in P_{J_\infty}$. Part (1) is proved.

We have that $(B\times B)\cdot \tilde{L}^w_{J,y,\delta} = (B)_{diag}(1,B)\cdot \tilde{L}^w_{J,y,\delta}$ and $p\circ\vartheta\big((1,B)\cdot \tilde{L}^w_{J,y,\delta}\big) = P_{J_\infty}$. Thus $(B\times B)\cdot \tilde{L}^w_{J,y,\delta} \subset (B)_{diag}(P_{J_\infty})_{diag}\tilde{L}^w_{J,y,\delta} = (P_{J_\infty})_{diag}\tilde{L}^w_{J,y,\delta}$. On the other hand, $(P_{J_\infty})_{diag}\tilde{L}^w_{J,y,\delta} \subset (P_{J_\infty},P_{J_\infty})\cdot \tilde{L}^w_{J,y,\delta} = (B\times B)\cdot \tilde{L}^w_{J,y,\delta}$. Hence $(P_{J_\infty})\cdot \tilde{L}^w_{J,y,\delta} = (B\times B)\cdot \tilde{L}^w_{J,y,\delta}$. Now consider $\pi : (U\cap {}^{\dot{w}_0^{J_\infty}\dot{w}\dot{y}^{-1}\dot{w}_0^{J'}}U^-)\times \tilde{L}^w_{J,y,\delta} \to (B\times B)\cdot \tilde{L}^w_{J,y,\delta}$ defined by $\pi(u,l) = (u,1)l$ for $u\in U\cap {}^{\dot{w}_0^{J_\infty}\dot{w}\dot{y}^{-1}\dot{w}_0^{J'}}U^-$ and $l\in \tilde{L}^w_{J,y,\delta}$.

Note that $(1,BL_{J_\infty})\cdot \tilde{h}_{J,y,\delta} = (1,U_{P_J}L_{J_\infty}U_J)\cdot \tilde{h}_{J,y,\delta} = (U_{\delta(J)},L_{J_\infty})\cdot \tilde{h}_{J,y,\delta}$. Since $w\in W^{\delta(J)}$, $B\dot{w}U_{\delta(J)} = B\dot{w} = U_{P_{J_\infty}}L_{J_\infty}\dot{w}$. Hence $(B\dot{w},BL_{J_\infty})\cdot \tilde{h}_{J,y,\delta} = (U_{P_{J_\infty}}\dot{w}L_{\delta(J_\infty)},L_{J_\infty})\cdot \tilde{h}_{J,y,\delta} = (U_{P_{J_\infty}}\dot{w},L_{J_\infty})\cdot \tilde{h}_{J,y,\delta}$. Since $wy^{-1}\in W^{J'}$ and $\mathrm{Ad}(yw^{-1})J_\infty \subset J'$, then

$$\begin{aligned} U_{P_{J_\infty}} &= (U_{P_{J_\infty}}\cap {}^{\dot{w}\dot{y}^{-1}\dot{w}_0^{J'}}U^-)(U_{P_{J_\infty}}\cap {}^{\dot{w}\dot{y}^{-1}}U_{P_{J'}}) \\ &= (U\cap {}^{\dot{w}_0^{J_\infty}\dot{w}\dot{y}^{-1}\dot{w}_0^{J'}}U^-)(U_{P_{J_\infty}}\cap {}^{\dot{w}\dot{y}^{-1}}U_{P_{J'}}). \end{aligned}$$

Therefore, $(B\dot{w},BL_{J_\infty})\cdot \tilde{h}_{J,y,\delta} = \big((U\cap {}^{\dot{w}_0^{J_\infty}\dot{w}\dot{y}^{-1}\dot{w}_0^{J'}}U^-)\dot{w},L_{J_\infty}\big)\cdot \tilde{h}_{J,y,\delta}$. So $\pi$ is surjective.

Let $u\in U\cap {}^{\dot{w}_0^{J_\infty}\dot{w}\dot{y}^{-1}\dot{w}_0^{J'}}U^-$ and $l_1,l_2\in L_{J_\infty}$. Assume that $(u,1)(\dot{w},l_1)\cdot \tilde{h}_{J,y,\delta} = (\dot{w},l_2)\cdot \tilde{h}_{J,y,\delta}$. Note that the isotropy subgroup of $G\times G$ at the point $(\dot{w},1)\cdot \tilde{h}_{J,y,\delta}$ is $\{(U_{\dot{w}\dot{y}^{-1}P_{J'}}l', U_{P_J}g_0^{-1}\dot{w}^{-1}l'\dot{w}g_0) \mid l'\in {}^{\dot{w}}L_{\delta(J)}\}$. Thus $u\in U_{\dot{w}\dot{y}^{-1}P_{J'}}l'$ and $l_2^{-1}l_1\in U_{P_J}g_0^{-1}\dot{w}^{-1}l'\dot{w}g_0$ for some $l'\in {}^{\dot{w}}L_{\delta(J)}$. Then $l'\in L_{J_\infty}$ and $u=1$. Thus $\pi$ is injective.

In fact, we can show that the bijective morphism $\pi$ is an isomorphism. The verification is omitted.

Part (3) can be proved in the same way as in [H2, 2.7]. $\square$

**4.1.11** For $P \in \mathcal{P}^J$, let $H_P$ be the inverse image of the connected center of $P/U_P$ under $P \to P/U_P$. We can regard $H_P/U_P$ as a single torus $\Delta_J$ independent of $P$. Then $\Delta_J$ acts (freely) in the natural way on $\tilde{Z}_{J,y,\delta}$ and the action commutes with the action of $G$. Moreover, each piece $\tilde{Z}^w_{J,y,\delta}$ is $\Delta_J$-stable.

Define

$$\begin{aligned} Z_{J,y,\delta} &= \{(P, P', \gamma) \mid P \in \mathcal{P}^J, P' \in \mathcal{P}^{J'}, \gamma \in H_{P'} \backslash A_y(P, P')/U_P\} \\ &= \{(P, P', \gamma) \mid P \in \mathcal{P}^J, P' \in \mathcal{P}^{J'}, \gamma \in U_{P'} \backslash A_y(P, P')/H_P\} \end{aligned}$$

with $G \times G$ action defined by $(g_1, g_2)(P, P', \gamma) = ({}^{g_2}P, {}^{g_1}P', g_1 \gamma g_2^{-1})$.

Then $Z_{J,y,\delta}$ and $\Delta_J \backslash \tilde{Z}_{J,y,\delta}$ can be identified in the natural way as varieties with $G$-action. Set $Z^w_{J,y,\delta} = \Delta_J \backslash \tilde{Z}^w_{J,y,\delta}$. Then

$$Z_{J,y,\delta} = \sqcup_{w \in W^{\delta(J)}} Z^w_{J,y,\delta}.$$

Set $h_{J,y,\delta} = (P_J, {}^{\dot{y}^{-1}}P_{J'}, H_{\dot{y}^{-1}P_{J'}} g_0 U_{P_J}) \in Z_{J,y,\delta}$, $L^w_{J,y,\delta} = (L_{J_\infty}, L_{J_\infty})(\dot{w}, 1) \cdot h_{J,y,\delta}$. For $w \in W^{\delta(J)}$ and $v \in W$, set $[J, w, v]_{y,\delta} = (B \times B)(\dot{w}, \dot{v}) \cdot h_{J,y,\delta}$. Then as a consequence of 4.1.7 and 4.1.10, we have the following result.

**Proposition 4.1.12.** *For $w \in W^{\delta(J)}$, we have that*

*(1) $Z^w_{J,y,\delta} = G_{diag} \cdot [J, w, 1]_{y,\delta}$.*

*(2) $Z^w_{J,y,\delta}$ is isomorphic to $G \times_{P_{J_\infty}} \big((P_{J_\infty}) \cdot L^w_{J,y,\delta}\big)$.*

*(3) $(P_{J_\infty}) \cdot L^w_{J,y,\delta} = (B \times B) \cdot L^w_{J,y,\delta} \cong (U \cap {}^{\dot{w}_0^{J_\infty} \dot{w} \dot{y}^{-1} \dot{w}_0^{J'}} U^-) \times L^w_{J,y,\delta}$, where $L^w_{J,y,\delta}$ is isomorphic to $L_{J_\infty}/Z(L_J)$.*

*(4) $G_{diag}(\dot{w}T, 1) \cdot h_{J,y,\delta}$ is dense in $Z^w_{J,y,\delta}$.*

## 4.2 Compactification through the fibres

**4.2.1** For any connected, semi-simple algebraic group of adjoint type, De Concini and Procesi introduced its wonderful compactification $\bar{G}$(see [DP]). It is an irreducible, projective smooth $G \times G$-variety. The $G \times G$-orbits $Z_J$ of $\bar{G}$ are indexed by the subsets $J$ of $I$. Moreover, $Z_J = (G \times G) \times_{P_J^- \times P_J} G_J$, where $P_J^- \times P_J$ acts on the right on $G \times G$ and on the left on $G_J$ by $(q,p) \cdot z = \pi_{P_J^-}(q) z \pi_{P_J}(p)$. Denote by $h_J$ the image of $(1,1,1)$ in $Z_J$. We will identify $Z_J$ with $Z_{J,w_0 w_0^J, id}$ and $h_J$ with $h_{J,w_0 w_0^J, id}$, where $id$ is the identity map on $I$ (see [H2, 2.5]).

Let us consider the $B \times B$-orbits on $\bar{G}$. For any $J \subset I$, $u \in W^J$ and $v \in W$, set $[J,u,v] = (B \times B)(\dot{u}, \dot{v}) \cdot h_J$. Then $\bar{G} = \bigsqcup_{J \subset I} \bigsqcup_{x \in W^J, w \in W} [J,x,w]$. The following result is due to Springer (see [Spr2, 2.4]).

**Theorem.** *Let $x \in W^J$, $x' \in W^K$, $w, w' \in W$. Then $[K,x',w']$ is contained in the closure of $[J,x,w]$ if and only if $K \subset J$ and there exists $u \in W_K, v \in W_J \cap W^K$ with $xvu^{-1} \leqslant x'$, $w'u \leqslant wv$ and $l(wv) = l(w) + l(v)$. In particular, for $J \subset I$ and $w \in W^J$, the closure of $[J,w,1]$ in $\bar{G}$ is $\sqcup_{K \subset J} \sqcup_{x \in W^K, u \in W_J,}$ and $_{x \geqslant wu}$ $[K,x,u]$.*

**4.2.2** We have defined $Z_{J,y,\delta}$ in 4.1.11. As we have seen, $Z_{J,y,\delta}$ is a locally trivial fibre bundle over $\mathcal{P}^J \times \mathcal{P}^{J'}$ with fibres isomorphic to $L_J/Z(L_J)$. Note that $L_J/Z(L_J)$ is a connected, semi-simple algebraic group of adjoint type. Thus we can define the wonderful compactification $\overline{L_J/Z(L_J)}$ of $L_J/Z(L_J)$. In this section, we will define $\underline{Z_{J,y,\delta}}$, which is a locally trivial fibre bundle over $\mathcal{P}^J \times \mathcal{P}^{J'}$ with fibres isomorphic to $\overline{L_J/Z(L_J)}$.

**4.2.3** We keep the notation of 4.1.2. Fix $g \in A_y(P,P')$. Then $A_y(P,P')g^{-1} = P' U_{{}^g P}$ (see [L8, 8.9]). Now define $L_{P,P',g} = {}^g P \cap P' / H_{{}^g P \cap P'}$ and $\Phi_g : H_{P'} \backslash A_y(P,P')/H_P \to L_{P,P',g}$ defined by $H_{P'} \backslash A_y(P,P')/H_P \xrightarrow{\cdot g^{-1}} H_{P'} \backslash A_y(P,P') g^{-1}/H_{{}^g P} \xleftarrow{i} L_{P,P',g}$, where $i$ is the obvious isomorphism. The $P \times P'$ action on $H_{P'} \backslash A_y(P,P')/H_P$ induces a $P \times P'$ action on $L_{P,P',g}$. Now for $g, g' \in A_y(P,P')$, set $\Phi_{g,g'} = \Phi_{g'} \Phi_g^{-1} : L_{P,P',g} \xrightarrow{\sim} L_{P,P',g'}$. Then $\Phi_{g,g'}$ is compatible with the $P \times P'$ action. Moreover, $(L_{P,P',g}, \Phi_{g,g'})$ forms an inverse system and

$$H_{P'}\backslash A_y(P,P')/H_P = \lim_{\leftarrow} L_{P,P',g}.$$

Note that $L_{P,P',g}$ is a semi-simple group of adjoint type. Then we can define the De Concini-Procesi compactification $\overline{L_{P,P',g}}$ of $L_{P,P',g}$. The $P \times P'$ action on $L_{P,P',g}$ can be extended in the unique way to a $P \times P'$ action on $\overline{L_{P,P',g}}$. The isomorphism $\Phi_{g,g'} : L_{P,P',g} \xrightarrow{\sim} L_{P,P',g'}$ can be extended in the unique way to an isomorphism from $\overline{L_{P,P',g}}$ onto $\overline{L_{P,P',g'}}$. We will also denote this isomorphism by $\Phi_{g,g'}$. It is easy to see that this isomorphism is compatible with the $P \times P'$ action. Now $(\overline{L_{P,P',g}}, \Phi_{g,g'})$ forms an inverse system. Define

$$\overline{H_{P'}\backslash A_y(P,P')/H_P} = \lim_{\leftarrow} \overline{L_{P,P',g}}.$$

We also obtain a $P \times P'$ action on $\overline{H_{P'}\backslash A_y(P,P')/H_P}$. Thus we can identify $\overline{H_{P'}\backslash A_y(P,P')/H_P}$ with $\overline{L_{P,P',g}}g$ as varieties with $P \times P'$ action.

*Remark.* $\overline{H_{P'}\backslash A_y(P,P')/H_P}$ is isomorphic to $\overline{L_{P,P',g}}$ as a variety. However, we are also concerned with the $P' \times P$ action. In this case, $\overline{H_{P'}\backslash A_y(P,P')/H_P}$ is regarded as $\overline{L_{P,P',g}}g$ with "twisted" $P' \times P$ action.

**4.2.4** In this section, we will consider a special case, namely, $P = P' = G^0$. In this case, $A_y(P,P') = G^1$ and we will identify $H_G\backslash A_y(G,G)/H_G$ with $G^1$.

Let $\mathcal{V}_G$ be the projective variety whose points are the $\dim(G)$-dimensional Lie subalgebras of $\mathrm{Lie}(G \times G)$. The $\hat{G} \times \hat{G}$ action on $\mathrm{Lie}(G \times G)$ which is defined by $(g_1, g_2) \cdot (a, b) = (\mathrm{Ad}(g_2)a, \mathrm{Ad}(g_1)b)$ for $g_1, g_2 \in \hat{G}$ and $a, b \in \mathrm{Lie}(G)$ induces a $\hat{G} \times \hat{G}$ action on $\mathcal{V}_G$. To each $g \in \hat{G}$, we associate a $\dim(G)$-dimensional subspace $V_g = \{(a, \mathrm{Ad}(g)a) \mid a \in \mathrm{Lie}(G)\}$ of $\mathrm{Lie}(G \times G)$. Then $V_{g_1 g g_2^{-1}} = (g_1, g_2) \cdot V_g$ for $g_1, g, g_2 \in \hat{G}$ and $g \mapsto V_g$ is an embedding $G^1 \subset \mathcal{V}_G$. We denote the image by $i(G^1)$.

If $G^1 = G$, then the closure of $i(G)$ in $\mathcal{V}_G$ is $\bar{G}$ (see [DP]). Note that $V_{gg_0} = (1, g_0^{-1})V_g$ for all $g \in G$. Thus $i(G^1) = (1, g_0^{-1})i(G)$. Hence the closure of $i(G^1)$ in $\mathcal{V}_G$ is $(1, g_0^{-1})\bar{G}$, which is just $\bar{G}^1$ defined above.

*Remark.* In [L8, 12.3], Lusztig defined the compactification of $G^1$ to be the closure of

$i(G^1)$ in $\mathcal{V}_G$. As we have seen, our definition coincides with his definition.

**4.2.5** In [L8, 12.3], Lusztig showed that $\bar{G^1} = \sqcup_{J \subset I} Z_{J, w_0 w_0^{\delta(J)}, \delta}$, where the base point $h_{J, w_0 w_0^{\delta(J)}, \delta} = (P_J, P^-_{\delta(J)}, H_{P^-_{\delta(J)}} g_0 H_{P_J})$ is identified with the $\dim(G)$-dimensional subalgebra $\{(lu, g_0 l g_0^{-1} u') \mid l \in L_J, u \in U_{P_J}, u' \in U_{P^-_{\delta(J)}}\}$ of Lie$(G \times G)$. We will simply write $h_{J, w_0 w_0^{\delta(J)}, \delta}$ as $h_{J,\delta}$, $[J, w, v]_{w_0 w_0^{\delta(J)}, \delta}$ as $[J, w, v]_\delta$ and $Z^w_{J, w_0 w_0^{\delta(J)}, \delta}$ as $Z^w_{J,\delta}$. If $G^1 = G$, then $h_{J,id} = h_J$ and $[J, w, v]_{id} = [J, w, v]$.

Note that $h_J$ corresponds to the $\dim(G)$-dimensional subalgebra $\{(lu, lu') \mid l \in L_J, u \in U_{P_J}, u' \in U_{P^-_J}\}$ of Lie$(G \times G)$. Thus $h_{J,\delta} = (1, g_0^{-1}) h_{\delta(J)}$. Hence $[J, w, v]_\delta = (B \times B)(\dot{w}, \dot{v}) \cdot h_{J,\delta} = (B \times B)(\dot{w}, \dot{v})(1, g_0^{-1}) \cdot h_J = (1, g_0^{-1})(B \times B)(\dot{w}, \dot{\delta(v)}) \cdot h_{\delta(J)} = (1, g_0^{-1})[\delta(J), w, \delta(v)]$. Thus we have the following result.

**Proposition.** *Let $J \subset I$ and $w \in W^{\delta(J)}$. Then the closure of $[J, w, 1]_\delta$ in $\bar{G^1}$ is $\sqcup_{K \subset J} \sqcup_{x \in W^{\delta(K)}, u \in W_J, \text{ and } x \geqslant w\delta(u)} [K, x, u]_\delta$.*

**4.2.6** Define

$$\underline{Z_{J,y,\delta}} = \{(P, P', \gamma) \mid P \in \mathcal{P}^J, P' \in \mathcal{P}^{J'}, \gamma \in \overline{H_{P'} \backslash A_y(P, P') / H_P}\}$$

with $G \times G$ action defined by $(g_1, g_2)(P, P', \gamma) = ({}^{g_2}P, {}^{g_1}P', g_1 \gamma g_2^{-1})$.

Set $P = P_J$ and $P' = {}^{\dot{y}^{-1}} P_{J'}$. Then $\overline{A_y(P, P')}$ can be identified with $\overline{L_{P,P',g_0}} g_0$ as varieties with $P' \times P$ action. Moreover, we have a canonical isomorphism between $\overline{L_{P,P',g_0}}$ and $\overline{L_{\delta(J)}}$. For $K \subset J$, I will identify $h_{\delta(K)} g_0$ with the corresponding element in $\overline{A_y(P, P')}$.

Then the $G \times G$-orbits in $\underline{Z_{J,y,\delta}}$ are one-to-one correspondence with the subsets of $J'$, i. e.,

$$\underline{Z_{J,y,\delta}} = \sqcup_{K \subset J} (G \times G) \cdot (P, P', h_{\delta(K)} g_0).$$

Set $y_K = y w_0^{\delta(J)} w_0^{\delta(K)}$. Note that $U_{P_J}(L_J \cap U_{P_K}) = U_{P_K}$ and

$$\begin{aligned} U_{P'}({}^{\dot{y}^{-1}} L_{J'} \cap U_{P^-_{\delta(K)}}) &= ({}^{\dot{y}_K^{-1}} U_{\dot{y}_K P'})^{\dot{y}_K^{-1}} ({}^{\dot{y}_K \dot{y}^{-1}} L_{J'} \cap {}^{\dot{y}_K} U_{P^-_{\delta(K)}}) \\ &= {}^{\dot{y}_K^{-1}} (U_{P_{J'}}(L_{J'} \cap U_{P_{\mathrm{Ad}(y_K)\delta(K)}})) = {}^{\dot{y}_K^{-1}} U_{P_{\mathrm{Ad}(y_K)\delta(K)}}. \end{aligned}$$

The isotropic subgroup of $G \times G$ at $(P, P', h_{\delta(K)} g_0)$ is $\{(l_1 u_1, g_0^{-1} l_2 g_0 u_2) \mid l_1, l_2 \in L_{\delta(K)}, l_1 l_2^{-1} \in Z(L_{\delta(K)}), u_1 \in U_{\dot{y}_K^{-1} P_{\mathrm{Ad}(y_K)\delta(K)}}, u_2 \in U_{P_K}\}$. Now set $Q = P_K, Q' = \dot{y}_K^{-1} P_{\mathrm{Ad}(y_K)\delta(K)}$ and $\gamma = H_{Q'} g_0 H_Q$. Then $\mathrm{pos}(Q', {}^{g_0}Q) = y_K$ and $(Q, Q', \gamma) \in Z_{K, y_K, \delta}$. The isotropic subgroup of $G \times G$ at $(P, P', h_{\delta(K)} g_0)$ is the same as the isotropic subgroup of $G \times G$ at $(Q, Q', \gamma) \in Z_{K, y_K, \delta}$. Thus we can identify $(P, P', h_{\delta(K)} g_0)$ with $(Q, Q', \gamma)$ and $(G \times G) \cdot (P, P', h_{\delta(K)} g_0)$ with $Z_{K, y_K, \delta}$ as varieties with $G \times G$ action. In other words,

$$\underline{Z_{J,y,\delta}} = \sqcup_{K \subset J} Z_{K, y w_0^{\delta(J)} w_0^{\delta(K)}, \delta}.$$

## 4.3 Partial order on $\mathcal{I}_\delta$

In this section, we will only consider subvarieties of $G$ and for any subvariety $X$ of $G$, we denote by $\bar{X}$ the closure of $X$ in $G$.

**4.3.1** Let $y, w \in W$. Then $y \leqslant w$ if and only if for any reduced expression $w = s_1 s_2 \cdots s_q$, there exists a subsequence $i_1 < i_2 < \cdots < i_r$ of $1, 2, \ldots, q$ such that $y = s_{i_1} s_{i_2} \cdots s_{i_r}$. (see [L4, 2.4])

The following assertion follows from the above property.

(1) If $l(wu) = l(w) + l(u)$, then for any $w_1 \leqslant w$ and $u_1 \leqslant u$, $w_1 u_1 \leqslant wu$.

(2) Let $u, v \in W$ and $i \in I$. Assume that $s_i v < v$, then $u \leqslant v \Leftrightarrow su \leqslant v$.

(3) Let $u, v \in W$ and $i \in I$. Assume that $u < s_i u$, then $u \leqslant v \Leftrightarrow u \leqslant s_i v$.

The assertion (1) follows directly from the above property. The proofs of assertions (2) and (3) can be found in [L4, 2.5].

**4.3.2** It is known that $G = \sqcup_{w \in W} B\dot{w}B$ and for $w, w' \in W$, $B\dot{w}B \subset \overline{B\dot{w}'B}$ if and only if $w \leqslant w'$. Moreover,

$$\overline{B\dot{s}_i B\dot{w}B} = \begin{cases} \overline{B\dot{w}B}, & \text{if } s_i w < w; \\ \overline{B\dot{s}_i\dot{w}B}, & \text{if } s_i w > w. \end{cases}$$

Similarly, $G = \sqcup_{w \in W} B\dot{w}B^-$ and for $w, w' \in W$, $B\dot{w}B^- \subset \overline{B\dot{w}'B^-}$ if and only if

$w \geqslant w'$. Moreover,

$$\overline{B\dot{s}_i B\dot{w}B^-} = \begin{cases} \overline{B\dot{w}B^-}, & \text{if } s_i w > w; \\ \overline{B\dot{s}_i\dot{w}B^-}, & \text{if } s_i w < w. \end{cases}$$

**Lemma 4.3.3.** *Let $u, w \in W$. Then*

*(1) The subset $\{vw \mid v \leqslant u\}$ of $W$ contains a unique minimal element $y$. Moreover, $l(y) = l(w) - l(yw^{-1})$ and $\overline{B\dot{u}B\dot{w}B^-} = \overline{B\dot{y}B^-}$.*

*(2) The subset $\{vw \mid v \leqslant u\}$ of $W$ contains a unique maximal element $y'$. Moreover, $l(y') = l(w) + l(y'w^{-1})$ and $\overline{B\dot{u}B\dot{w}B} = \overline{B\dot{y}'B}$.*

*Proof.* We will only prove part (1). Part (2) can be proved in the same way.

For any $v \leqslant u$, $B\dot{v} \subset \overline{B\dot{u}B}$. Thus $\overline{B\dot{v}\dot{w}B^-} \subset \overline{B\dot{u}B}\dot{w}B^- \subset \overline{B\dot{u}B\dot{w}B^-}$. On the other hand, $\overline{B\dot{u}B\dot{w}B^-}$ is an irreducible, closed, $B \times B^-$-stable subvariety of $G$. Thus there exists $y \in W$, such that $\overline{B\dot{u}B\dot{w}B^-} = \overline{B\dot{y}B^-}$. Since $B\dot{v}\dot{w}B^- \subset \overline{B\dot{y}B^-}$, we have that $vw \geqslant y$. Now it suffices to prove that $y = vw$ for some $v \leqslant u$ with $l(vw) = l(w) - l(v)$.

We argue by induction on $l(u)$. If $l(u) = 0$, then $u = 1$ and statement is clear. Assume now that $l(u) > 0$. Then there exists $i \in I$, such that $s_i u < u$. We denote $s_i u$ by $u'$. Now

$$\overline{B\dot{u}B\dot{w}B^-} = \overline{B\dot{s}_i B\dot{u}'B\dot{w}B^-} = \overline{B\dot{s}_i\overline{B\dot{u}'B\dot{w}B^-}}.$$

By induction hypothesis, there exists $v' \leqslant u'$, such that $l(v'w) = l(w) - l(v')$ and $\overline{B\dot{u}'B\dot{w}B^-} = \overline{B\dot{v}'\dot{w}B^-}$. Thus

$$\overline{B\dot{s}_i\overline{B\dot{u}'B\dot{w}B^-}} = \overline{B\dot{s}_i\overline{B\dot{v}'\dot{w}B^-}} = \overline{B\dot{s}_i B\dot{v}'\dot{w}B^-} = \begin{cases} \overline{B\dot{v}'\dot{w}B^-}, & \text{if } s_i v'w > v'w; \\ \overline{B\dot{s}_i\dot{v}'\dot{w}B^-}, & \text{if } s_i v'w < v'w. \end{cases}$$

Note that $s_i u < u$ and $v' \leqslant s_i u < u$. Thus $s_i v' \leqslant u$. Moreover, if $s_i v'w < v'w$, then $l(s_i v'w) = l(v'w) - 1 = l(w) - l(v') - 1$. Thus we have that $l(s_i v') = l(v) + 1$ and $l(s_i v'w) = l(w) - l(s_i v')$. Therefore, the statement holds for $u$. □

**Corollary 4.3.4.** *Let $u, w, w' \in W$ with $w' \leqslant w$. Then*

*(1) There exists $v \leqslant u$, such that $vw' \leqslant uw$.*

*(2) There exists $v' \leqslant u$, such that $uw' \leqslant v'w$.*

*Proof.* Let $v \leqslant u$ be the element of $W$ such that $vw'$ is the unique minimal element in $\{v'w' \mid v' \leqslant u\}$. Then $\overline{B\dot{u}B\dot{w}'B^-} = \overline{B\dot{v}\dot{w}'B^-}$. Since $w' \leqslant w$, we have that $B\dot{w}B^- \subset \overline{B\dot{w}'B^-}$. Thus

$$B\dot{u}\dot{w}B^- \subset B\dot{u}B\dot{w}B^- \subset B\dot{u}\overline{B\dot{w}'B^-} \subset \overline{B\dot{u}B\dot{w}'B^-} = \overline{B\dot{v}\dot{w}'B^-}.$$

So $uw \geqslant vw'$. Thus Part (1) is proved. Part (2) can be proved in the same way. □

**4.3.5** We will recall some known results about $W^J$.

(1) If $w \in W^J$ and $i \in I$, then there are three possibilities.

(a) $s_i w > w$ and $s_i w \in W^J$;

(b) $s_i w > w$ and $s_i w = w s_j$ for some $j \in J$;

(c) $s_i w < w$ in which case $s_i w \in W^J$.

(2) If $w \in W^J$, $v \in W_J$ and $K \subset J$, then $v \in W^K$ if and only if $wv \in W^K$.

(3) If $w \in {}^{J'}W^J$ and $u \in W_{J'}$, then $uw \in W^J$ if and only if $u \in W^K$, where $K = J' \cap \mathrm{Ad}(w)J$.

**Lemma 4.3.6.** *Let $w \in {}^{J'}W^J$, $u \in W_{J'}$ and $K = J' \cap \mathrm{Ad}(w)J$, then $uw = vwu'$ for some $v \in W_{J'} \cap W^K$ and $u' \in W_{\mathrm{Ad}(w^{-1})K}$.*

*Proof.* We argue by induction on $l(u)$. If $u = 1$, then the statement is clear. Now assume that $u = s_i u_1$ for some $i \in J'$ and $l(u_1) < l(u)$. Then by induction hypothesis, $u_1 w = v_1 w u'_1$ for some $v_1 \in W_{J'} \cap W^K$ and $u'_1 \in W_{\mathrm{Ad}(w^{-1})K}$.

If $s_i v_1 w \in W^J$, then the statement holds for $u$. Now assume that $s_i v_1 w \notin W^J$. Then $s_i v_1 w > v_1 w$. Hence $s_i v_1 > v_1$. Moreover, $s_i v_1 \notin W^K$. Thus $s_i v_1 = v_1 s_k$ for some $k \in K$. Note that $s_k w = w s_l$ for some $l \in \mathrm{Ad}(w^{-1})K$. Thus the statement holds for $u$. The lemma is proved. □

**4.3.7** Let $J \subset I$ and $w, w' \in W$ with $l(w) = l(w')$. We say that $w'$ can be obtained from $w$ via a $(J,\delta)$-cyclic shift if $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression and either (1) $i_1 \in J$ and $w' = s_{i_1} w s_{\delta(i_1)}$ or (2) $i_n \in \delta(J)$ and $w' = s_{\delta^{-1}(i_n)} w s_{i_n}$. We say that $w$ and $w'$ are equivalent in J if there exists a finite sequences of elements $w = w_0, w_1, \ldots, w_m = w'$ such that $w_{k+1}$ can be obtained from $w_k$ via a $(J,\delta)$-cyclic shift. (We then write $w \sim_{J,\delta} w'$.)

**4.3.8** Let $(J, w) \in \mathcal{I}_\delta$. For $x \in W$, we say that $x \geqslant_{J,\delta} w$ if $x \geqslant w'$ for some $w' \sim_{J,\delta} w$. It is easy to see that $x \geqslant w \Rightarrow x \geqslant_{J,\delta} w \Rightarrow l(x) \geqslant l(w)$.

Now for $(J_1, w_1), (J_2, w_2) \in \mathcal{I}_\delta$, we say that $(J_1, w_1) \leqslant_\delta (J_2, w_2)$ if $J_1 \subset J_2$ and $w_1 \geqslant_{J,\delta} w_2$. In the end of this section, we will show that $\leqslant$ is a partial order on $\mathcal{I}_\delta$. (The definition of partial order can be found in 4.3.12). Before doing that, we will investigate some properties of $\geqslant_{J,\delta}$.

**Proposition 4.3.9.** *Let $(J, w) \in \mathcal{I}_\delta$. Then for any $u \in W_J$ and $v \in W_{\delta(J)}$ with $\delta(u) \leqslant v$, we have that $u^{-1} w v \geqslant_{J,\delta} w$.*

*Proof.* I will prove the proposition by induction on $|J|$. Assume that the statement holds for all $J' \subset I$ with $|J'| < |J|$. Then I will prove that the statement holds for $J$ by induction on $l(v)$.

Set $w = w'y$ with $w' \in W_J$ and $y \in {}^J W^{\delta(J)}$. Set $K = \delta(J) \cap \mathrm{Ad}(y^{-1})J$, $v = v_1 v_2$ with $v_1 \in W_K, v_2 \in {}^K W$ and $u = u_1 u_2$ with $\delta(u_1) \leqslant v_1$, $\delta(u_2) \leqslant v_2$ and $l(u) = l(u_1) + l(u_2)$. There are two cases.

Case 1. $u_2 = v_2 = 1$.

In this case, $u \in w_K$, $v \in W_{\delta(K)}$ and $w \in W^{\delta(K)}$. If $|K| < |J|$, then by induction hypothesis we have that $u^{-1} w v \geqslant_{K,\delta} w$. Thus $u^{-1} w v \geqslant_{J,\delta} w$. If $K = J$, then since $w = w'y \in W^{\delta(J)}$, we have that $w' = 1$. Thus $u^{-1} w v \geqslant w$. The statement is proved in this case.

Case 2. $v_2 \neq 1$.

In this case, $l(v_1) < l(v)$. By induction hypothesis, there exists $w_1 \sim_{J,\delta} w$, such that $w_1 \leqslant u_1^{-1} w v_1$. Let $u_3 \leqslant u_2$ be the element in $W$ such that $u_3^{-1} w_1$ is the unique minimal element in $\{(u')^{-1} w_1 \mid u' \leqslant u_2\}$. Then $l(u_3^{-1} w_1) = l(w_1) - l(u_3)$ and $u_3^{-1} w_1 \leqslant$

$u_2^{-1}u_1^{-1}wv_1 = u^{-1}wv_1$. By 4.3.6, $u^{-1}w = ab$ for some $a \in W^{\delta(J)}$ and $b \in W_K$. Thus $l(u^{-1}wv_1v_2) = l(abv_1v_2) = l(a) + l(bv_1v_2) = l(a) + l(bv_1) + l(v_2) = l(abv_1) + l(v_2) = l(u^{-1}wv_1) + l(v_2)$. By 4.3.1, $u_3^{-1}w_1\delta(u_3) \leqslant u^{-1}wv$. Now assume that $u_3 = s_{i_1}s_{i_2}\cdots s_{i_k}$ and $u_3^{-1}w_1 = s_{j_1}s_{j_2}\cdots s_{j_l}$ are reduced expressions. Now for $m = 1, 2, \dots, k+1$, set $x_m = (s_{i_m}s_{i_{m+1}}\cdots s_{i_k})(s_{j_1}s_{j_2}\cdots s_{j_l})(s_{\delta(i_1)}s_{\delta(i_2)}\cdots s_{\delta(i_{m-1})})$. Then $l(x_m) \leqslant k + l = l(w_1)$ for all $m$. On the other hand, for any $m$, there exists $x \in W_J$, such that $x_m = x^{-1}w\delta(x)$. Note that $w \in W^J$, we have that $l(x^{-1}w\delta(x)) \geqslant l(w\delta(x)) - l(x^{-1}) = l(w) = l(w_1)$ for all $x \in W_J$. Therefore, $l(x_m) = l(w_1)$ and $x_m \sim_{J,\delta} w_1$ for all $m$. In particular, $u_3^{-1}w_1u_3 = x_{k+1} \sim_{J,\delta} w_1$. The statement is proved in this case. □

*Remark.* 1. As a consequence of the proposition, $w' \geqslant_{J,\delta} w$ if and only if $w' \geqslant x^{-1}w\delta(x)$ for some $x \in W_J$.

2. We can see from the proof that $u^{-1}wv \geqslant x^{-1}w\delta(x)$ for some $x \leqslant u$. This result will be used in the proof of 4.5.2.

**Lemma 4.3.10.** *Let $J \subset I$, $w \in W^J$, $u \in W$ with $l(uw) = l(u) + l(w)$. Assume that $uw = xv$ with $x \in W^J$ and $v \in W_J$. Then for any $v' \leqslant v$, there exists $u' \leqslant u$, such that $u'w = xv'$.*

*Proof.* We argue by induction on $l(u)$. If $l(u) = 0$, then $u = 1$ and statement is clear. Assume now that $l(u) > 0$. Then there exists $i \in I$, such that $s_iu < u$. We denote $s_iu$ by $u_1$. Let $u_1w = x_1v_1$ with $x_1 \in W^J$ and $v_1 \in W_J$. Then $s_ix_1 > x_1$.

If $s_ix_1 \in W^J$, then the lemma holds by induction hypothesis. If $s_ix_1 \notin W^J$, then there exists $j \in J$, such that $s_ix_1 = x_1s_j$. In this case, $s_jv_1 > v_1$. Let $v' \leqslant s_jv_1$. If $v' \leqslant v_1$, then the lemma holds by induction hypothesis. If $v' \nleqslant v_1$, then $v' = s_jv_1'$ for some $v_1' \leqslant v_1$. By induction hypothesis, there exists $u_1' \leqslant u_1$, such that $u_1'w = x_1v_1'$. Thus $s_iu_1'w = x_1s_jv_1'$. The lemma holds in this case. □

**Lemma 4.3.11.** *Fix $J \subset I$ and $w \in W^{\delta(J)}$. For any $K \subset J$, $w' \in W^{\delta(K)}$ with $w' \geqslant_{J,\delta} w$, there exists $x \in W^{\delta(K)}$, $u \in W_J$ and $u_1 \in W_K$, such that $x \geqslant w\delta(u)$ and $w' = u_1^{-1}u^{-1}x\delta(u_1)$.*

*Proof.* Since $w' \geqslant_{J,\delta} w$, there exists $v_1 \in W_J$, such that $w' \geqslant v_1^{-1}w\delta(v_1)$. By 4.3.4, there exists $v' \leqslant v_1$, such that $v'w' \geqslant w\delta(v_1) \geqslant w\delta(v')$. Let $v$ be a minimal element in the set $\{v \in W_J \mid vw' \geqslant w\delta(v)\}$. Then $l(vw') = l(v) + l(w')$. Now assume that $vw' = x\delta(v')$ for some $x \in W^{\delta(K)}$ and $v' \in W_K$. Then there exists $v'_1 \leqslant v'$, such that $x \geqslant w\delta(v)\delta(v'_1)^{-1}$. By 4.3.10, $x\delta(v'_1) = v_2 w'$ for some $v_2 \leqslant v$. Since $l(x\delta(v'_1)) = l(x) + l(v'_1)$, $v_2 w' = x\delta(v'_1) \geqslant w\delta(v) \geqslant w\delta(v_2)$. Therefore, $v_2 = v$ and $v'_1 = v'$. So $x \geqslant w\delta(v)\delta(v')^{-1}$. Now set $u = v(v')^{-1}$ and $u_1 = v'$. Then $w' = v^{-1}x\delta(v') = u_1^{-1}u^{-1}x\delta(u_1)$. □

**4.3.12** A relation $\leqslant$ is a partial order on a set $S$ if it has:

1. Reflexivity: $a \leqslant a$ for all $a \in S$.
2. Antisymmetry: $a \leqslant b$ and $b \leqslant a$ implies $a = b$.
3. Transitivity: $a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$.

**Proposition 4.3.13.** *The relation $\leqslant_\delta$ on the set $\mathcal{I}_\delta$ is a partial order.*

*Proof.* Reflexivity is clear from the definition.

For $(J_1, w_1), (J_2, w_2) \in \mathcal{I}_\delta$ with $(J_1, w_1) \leqslant_\delta (J_2, w_2)$ and $(J_2, w_2) \leqslant_\delta (J_1, w_1)$, we have that $J_1 = J_2$ and $l(w_1) = l(w_2)$. Since $w_1 \geqslant w'_2$ for some $w'_2 \sim_{J,\delta} w_2$ and $l(w_1) = l(w_2) = l(w'_2)$, $w_1 = w'_2 \in W^{\delta(J_2)}$. Hence $w_1 = w'_2 = w_2$. Therefore $(J_1, w_1) = (J_2, w_2)$. Antisymmetry is proved.

Let $(J_1, w_1), (J_2, w_2)$ and $(J_3, w_3) \in \mathcal{I}_\delta$. Assume that $(J_1, w_1) \leqslant_\delta (J_2, w_2)$ and $(J_2, w_2) \leqslant_\delta (J_3, w_3)$. Then $J_1 \subset J_2 \subset J_3$. Moreover, there exists $x \in W^{\delta(J_2)}$, $u \in W_{J_3}$ and $u_1 \in W_{J_2}$, such that $x \geqslant w_3\delta(u)$ and $w_2 = u_1^{-1}u^{-1}x\delta(u_1)$. Since $w_1 \geqslant_{J_2,\delta} w_2$, there exists $u_2 \in W_{J_2}$, such that $w_1 \geqslant u_2^{-1}u^{-1}x\delta(u_2)$. Note that $l(x\delta(u_2)) = l(x) + l(u_2)$ and $x \geqslant w_3\delta(u)$. Thus $x\delta(u_2) \geqslant w_3\delta(uu_2)$. By 4.3.4, there exists $v \leqslant uu_2$, such that $w_1 \geqslant v^{-1}w_3\delta(uu_2)$. By 4.3.7, $w_1 \geqslant_{J_3,\delta} w_3$. Transitivity is proved. □

## 4.4 The closure of any $G$-stable piece

**4.4.1** We have that

$$G^1 = \sqcup_{w \in W} B\dot{w}U^-\dot{w}_0^{\delta(J)}g_0 = \sqcup_{w \in W} B\dot{w}\dot{w}_0^{\delta(J)}U_{P^-_{\delta(J)}}U_{\delta(J)}g_0.$$

Moreover, $B\dot{w}U^- = \sqcup_{b\in U_J^-\cap {}^{\dot{w}^{-1}}U^-} B\dot{w}U_{P_J^-}b$. Thus

$$\begin{aligned} B\dot{w}U_{P_J^-}U_J = B\dot{w}\dot{w}_0^J U^- \dot{w}_0^J &= \sqcup_{b\in U_J^-\cap {}^{(\dot{w}\dot{w}_0^J)^{-1}}U^-} B\dot{w}\dot{w}_0^J U_{P_J^-} b\dot{w}_0^J \\ &= \sqcup_{b\in U_J\cap {}^{\dot{w}^{-1}}U^-} B\dot{w}U_{P_J^-}b. \end{aligned}$$

Note that if $w = w'u$ with $w' \in W^J$ and $u \in W_J$, then

$$U_J \cap {}^{\dot{w}^{-1}}U^- = {}^{\dot{u}^{-1}}({}^{\dot{u}}U_J \cap {}^{(\dot{w}')^{-1}}U^-) = {}^{\dot{u}^{-1}}({}^{\dot{u}}U_J \cap U_J^-) = U_J \cap {}^{\dot{u}^{-1}}U_J^-.$$

**Lemma 4.4.2.** *Let $(J, w) \in \mathcal{I}_\delta$. For any $u \in W$ and $b \in B$, there exists $v \leqslant u$, such that $\dot{u}b\dot{w} \in B\dot{v}\dot{w}U_{P^-_{\delta(J)}}U_{\delta(J)}$.*

*Proof.* We will prove the statement by induction on $l(u)$.

If $u = 1$, then the statement holds. If $u = s_i u_1$ with $l(u_1) = l(u) - 1$, then by induction hypothesis, there exists $v_1 \leqslant u_1$, such that $\dot{u}_1 b\dot{w} \in b'\dot{v}_1\dot{w}U_{P^-_{\delta(J)}}U_{\delta(J)}$ for some $b' \in B$. Write $b' = b_1 b_2$, where $b_1 \in U_{P_{\{i\}}}$ and $b_2 \in U_{\{i\}}$. Then $\dot{s}_i b'\dot{v}_1\dot{w} = (\dot{s}_i b_1 \dot{s}_i^{-1})\dot{s}_i b_2 \dot{v}_1 \dot{w}$ with $\dot{s}_i b_1 \dot{s}_i^{-1} \in B$.

If $(\dot{v}_1\dot{w})^{-1} b_2 \dot{v}_1 \dot{w} \in U_{P^-_{\delta(J)}}U_{\delta(J)}$, then $\dot{s}_i b_2 \dot{v}_1 \dot{w} \in \dot{s}_i \dot{v}_1 \dot{w} U_{P^-_{\delta(J)}}U_{\delta(J)}$. Otherwise, $b_2 \neq 1$ and $(\dot{v}_1\dot{w})^{-1}U^-_{\{i\}}\dot{v}_1\dot{w} \subset U_{P^-_{\delta(J)}}U_{\delta(J)}$. Note that $\dot{s}_i b_2 \in BU^-_{\{i\}}$. Thus $\dot{s}_i b_2 \dot{v}_1 \dot{w} \in B\dot{v}_1\dot{w}U_{P^-_{\delta(J)}}U_{\delta(J)}$. The statement holds in both cases. □

**4.4.3** Let $z \in (G, 1) \cdot h_{J,\delta}$. Then $z$ can be written as $z = (b\dot{w}\dot{u}b', 1) \cdot h_{J,\delta}$ with $b \in B$, $w \in W^{\delta(J)}$, $u \in W_{\delta(J)}$ and $b' \in U_{\delta(J)} \cap {}^{\dot{u}^{-1}}U^-_{\delta(J)}$. Moreover, $w, u, b'$ are uniquely determined by $z$.

Set $J_0 = J$. To $z \in (G, 1) \cdot h_{J,\delta}$, we associate a sequence $(J_i, w_i, v_i, v'_i, c_i, z_i)_{i \geqslant 1}$ with $J_i \subset J$, $w_i \in W^{\delta(J)}$, $v_i \in W_{J_{i-1}} \cap {}^{J_i}W$, $v'_i \in W_{J_i}$, $c_i \in U_{\delta(J_{i-1})} \cap {}^{\dot{\delta}(v_i^{-1})}U^-_{\delta(J_{i-1})}$ and $z_i \in (B\dot{w}_i\dot{\delta}(v'_i)U_{\delta(J)}\dot{\delta}(v_i)c_i, 1) \cdot h_{J,\delta}$ and in the same $G_{diag}$-orbit as $z$. The sequence is defined as follows.

Assume that $z \in (B\dot{w}\dot{\delta}(u)U_{\delta(J)}, 1) \cdot h_{J,\delta}$ with $w \in W^{\delta(J)}$ and $u \in W_{\delta(J)}$. Then set $z_1 = z$, $J_1 = J$, $w_1 = w$, $v_1 = 1$, $v'_1 = u$ and $c_1 = 1$.

Assume that $k \geqslant 1$, that $J_k, w_k, v_k, v'_k, c_k, z_k$ are already defined and that $J_k \subset$

$J_{k-1}$, $w_k \in W^{\delta(J)}$, $W_{J_{k-1}} w_k \subset W^{\delta(J)} W_{\delta(J_k)}$, $v_k \in W_{J_{k-1}} \cap {}^{J_k}W$, $v'_k \in W_{J_k}$, $c_k \in U_{\delta(J_{k-1})} \cap {}^{\dot{\delta}(v_k^{-1})} U^-_{\delta(J_{k-1})}$ and $z_k \in (B\dot{w}_k \dot{\delta}(v'_k) U_{\delta(J)} \dot{\delta}(v_k) c_k, 1) \cdot h_{J,\delta}$.

Set $z_{k+1} = (g_0^{-1} \dot{\delta}(v_k) c_k g_0, g_0^{-1} \dot{\delta}(v_k) c_k g_0) z_k$. Then $z_{k+1} \in (G, 1) \cdot h_{J,\delta}$. Moreover, by 4.4.2, there exists $x_k \leqslant v_k$, such that $z_{k+1} \in (B\dot{x}_k w_k \dot{\delta}(v'_k) U_{\delta(J)}, 1) \cdot h_{J,\delta}$. Let $y_{k+1}$ be the unique element of the minimal length in $W_{J_k} x_k w_k \delta(v'_k) W_{\delta(J)}$. Set $J_{k+1} = J_k \cap \delta^{-1} \mathrm{Ad}(y_{k+1}^{-1}) J_k$. Since $W_{J_{k-1}} w_k \subset W^{\delta(J)} W_{\delta(J_k)}$, then $x_k w_k \delta(v'_k) = w_{k+1} \delta(v'_{k+1} v_{k+1})$ for some $w_{k+1} \in W^{\delta(J)}$, $v'_{k+1} \in W_{J_{k+1}}$ and $v_{k+1} \in W_{J_k} \cap {}^{J_{k+1}}W$. Note that $W_{J_k} w_{k+1} \subset W^{\delta(J)} W_{\delta(J) \cap \mathrm{Ad}(y_{k+1}^{-1}) J_k}$. On the other hand, $W_{J_k} w_{k+1} \subset W_{J_{k-1}} w_k W_{\delta(J_k)} \subset W^{\delta(J)} W_{\delta(J_k)}$. Thus $W_{J_k} w_{k+1} \subset (W^{\delta(J)} W_{\delta(J) \cap \mathrm{Ad}(y_{k+1}^{-1}) J_k}) \cap (W^{\delta(J)} W_{\delta(J_k)}) = W^{\delta(J)} W_{\delta(J_{k+1})}$. Moreover $z_{k+1} \in (B\dot{w}_{k+1} \dot{\delta}(v'_{k+1}) U_{\delta(J)} \dot{\delta}(v_{k+1}) c_{k+1}, 1) \cdot h_{J,\delta}$ for a unique $c_{k+1} \in U_{J_k} \cap {}^{\dot{\delta}(v_{k+1}^{-1})} U^-_{J_k}$.

This completes the inductive definition. Moreover, for sufficient large $n$, we have that $J_n = J_{n+1} = \cdots$, $w_n = w_{n+1} = \cdots$, $v'_n = v'_{n+1} = \cdots$ and $v_n = v_{n+1} = \cdots = 1$.

**4.4.4** Let $K \subset J$, $y \in {}^K W^{\delta(K)}$ and $K = \mathrm{Ad}(y) \delta(K)$. Then for any $u \in W_K$, we have that $(B\dot{y} \dot{\delta}(u) U_{\delta(K)}, 1) \cdot h_{J,\delta} \subset G_{diag}(\dot{y} L_{\delta(K)}, B) \cdot h_{J,\delta} = G_{diag}(\dot{y} L_{\delta(K)}, U_{P_K}) \cdot h_{J,\delta}$. Note that for any $l \in L_K$, there exists $l' \in L_K$, such that $l' \dot{y} g_0 l (l')^{-1} \in \dot{y} g_0 (L_K \cap B)$. Thus $(L_K)_{diag}(\dot{y}(L_{\delta(K)} \cap B), U_{P_K}) \cdot h_{J,\delta} = (\dot{y} L_{\delta(K)}, U_{P_K}) \cdot h_{J,\delta}$. Hence $(B\dot{y} \dot{\delta}(u) U_{\delta(K)}, 1) \cdot h_{J,\delta} \subset G_{diag}(\dot{y}(L_{\delta(K)} \cap B), U_{P_K}) \cdot h_{J,\delta} = G_{diag}(\dot{y}, B) \cdot h_{J,\delta} = Z^y_{J,\delta}$.

Now for any $z \in (G, 1) \cdot h_{J,\delta}$, let $(z_i, J_i, w_i, v_i, v'_i, c_i)_{i \geqslant 1}$ be the sequence associated to $z$. Assume that $J_n = J_{n+1} = \cdots$, $w_n = w_{n+1} = \cdots$, $v'_n = v'_{n+1} = \cdots$ and $v_n = v_{n+1} = \cdots = 1$. Then we have showed that $z_n \in Z^{w_n}_{J,\delta}$. Thus $z \in Z^{w_n}_{J,\delta}$.

Note that for any $z \in Z_{J,\delta}$, $z$ is in the same $G$-orbit as an element of the form $(G, 1) \cdot h_{J,\delta}$. Therefore, given $z \in Z_{J,\delta}$, our procedure determines the $G$-stable piece $Z^w_{J,\delta}$ that contains $z$.

Now we are able to describe the closure of $Z^w_{J,\delta}$. In 4.5, we will only consider subvarieties of $\bar{G}^1$ and for any subvariety $X$ of $\bar{G}^1$, we denote by $\bar{X}$ the closure of $X$ in $\bar{G}^1$.

**Theorem 4.4.5.** *For any $(J, w) \in \mathcal{I}_\delta$, we have that*

$$\overline{Z^w_{J,\delta}} = \sqcup_{(K,w') \leqslant_\delta (J,w)} Z^{w'}_{K,\delta}.$$

*Proof.* Define $\pi' : G \times \overline{[I,1,1]_\delta} \to \bar{G}^1$ by $\pi(g,z) = (g,g) \cdot z$. The morphism is invariant under the $B$-action defined by $b(g,z) = (gb^{-1}, \pi'(b,z))$. Denote by $G \times_B \overline{[I,1,1]_\delta}$ the quotient, we obtain a morphism $\pi : G \times_B \overline{[I,1,1]_\delta} \to \bar{G}^1$. Because $G/B$ is projective, $\pi$ is proper and hence surjective.

Note that $\overline{[J,w,1]_\delta} = \sqcup_{K \subset J} \sqcup_{x \in W^{\delta(K)}, u \in W_J, \text{ and } x \geqslant w\delta(u)} [K,x,u]_\delta$. Since $Z^w_{J,\delta} = \pi(G \times_B [J,w,1]_\delta)$, we have that

$$\overline{Z^w_{J,\delta}} = \sqcup_{K \subset J} \cup_{x \in W^{\delta(K)}, u \in W_J, \text{ and } x \geqslant w\delta(u)} G_{diag} \cdot [K,x,u]_\delta.$$

For any $z \in [K,x,u]_\delta$ with $x \in W^{\delta(K)}$, $u \in W_J$ and $x \geqslant w\delta(u)$, we have that $z \in (B\dot{x}, B\dot{u}) \cdot h_{K,\delta} = G_{diag}(\dot{u}^{-1}B\dot{x}, 1) \cdot h_{K,\delta} \subset \sqcup_{v \leqslant u^{-1}} G_{diag}(B\dot{v}\dot{x}U_{\delta(K)}, 1) \cdot h_{K,\delta}$.

Fix $v \leqslant u^{-1}$ and $z' \in (B\dot{v}\dot{x}U_{\delta(K)}, 1) \cdot h_{K,\delta}$. Let $(z_i, J_i, w_i, v_i, v'_i, c_i)_{i \geqslant 1}$ be the sequence associated to $z'$. Then for any $i$, there exists $x_i \leqslant v_i$, such that $x_i w_i \delta(v'_i) = w_{i+1}\delta(v'_{i+1}v_{i+1})$. Assume that $J_n = J_{n+1} = \cdots$, $w_n = w_{n+1} = \cdots$, $v'_n = v'_{n+1} = \cdots$ and $v_n = v_{n+1} = \cdots = 1$. Set $x_\infty = x_n x_{n-1} \cdots x_2$ and $v_\infty = v'_n(v_n v_{n-1} \cdots v_2)$. Note that $x_1 = v_1 = 1$. Then $x_\infty v x = x_\infty w_1 \delta(v'_1) = w_n \delta(v_\infty)$. Since $v'_n \in W_{J_{n+1}}$ and $v_i \in W_{J_i} \cap {}^{J_{i+1}}W$, we have that $l(v_\infty) = l(v'_n) + l(v_n) + l(v_{n-1}) + \cdots + l(v_2)$. Thus $x_\infty \leqslant v_\infty$. By 4.4.4, $z' \in Z_{K, x_\infty v x \delta(v_\infty^{-1})}$. Note that $v^{-1} \leqslant u$ and $l(wu) = l(w) + l(u)$. Thus $w\delta(v^{-1}) \leqslant w\delta(u) \leqslant x$. Similarly, $w\delta(v^{-1}x_\infty^{-1}) \leqslant x\delta(v_\infty^{-1})$. By 4.3.4, there exist $v' \leqslant v^{-1}x_\infty^{-1}$, such that $(v')^{-1}w\delta(v^{-1}x_\infty^{-1}) \leqslant x_\infty v x \delta(v_\infty^{-1})$. Thus by 4.3.9, $x_\infty v x \delta(v_\infty^{-1}) \geqslant_{J,\delta} w$.

Now for any $K \subset J$ and $w' \in W^{\delta(K)}$ with $w' \geqslant_{J,\delta} w$, there exists $x \in W^{\delta(K)}$, $u \in W_J$ and $u_1 \in W_K$, such that $x \geqslant w\delta(u)$ and $w' = u_1^{-1}u^{-1}x\delta(u_1)$. Since $[K,x,u]_\delta \subset \overline{[J,w,1]_\delta}$. We have that $(\dot{x}T, \dot{u}) \cdot h_{K,\delta} \subset \overline{Z^w_{J,\delta}}$. Therefore $(\dot{u}^{-1}\dot{x}T, 1) \cdot h_{K,\delta} \subset \overline{Z^w_{J,\delta}}$. Note that $u^{-1}x = u_1 w' \delta(u_1)^{-1}$. Then $(\dot{u}_1 \dot{w}' \dot{\delta}(u_1)^{-1}T, 1) \cdot h_{K,\delta} \subset \overline{Z^w_{J,\delta}}$. Thus

$$(\dot{u}_1^{-1}, \dot{u}_1^{-1})(\dot{u}_1 \dot{w}' \dot{\delta}(u_1)^{-1}T, 1) \cdot h_{K,\delta} = (\dot{w}'T, 1) \cdot h_{K,\delta} \subset \overline{Z^w_{J,\delta}}.$$

By 4.1.12, $Z^{w'}_{K,\delta} \subset \overline{Z^{w}_{J,\delta}}$. The theorem is proved. □

Our method also works in another situation.

**Proposition 4.4.6.** *The closure of $Z^w_{J,1,\delta}$ in $Z_{J,1,\delta}$ is $\sqcup_{w' \in W^{\delta(J)}, w \geqslant_J w'} Z^{w'}_{J,1,\delta}$.*

*Proof.* In the proof, we will only consider subvarieties of $Z_{J,1,\delta}$ and for any subvariety $X$ of $Z_{J,1,\delta}$, we denote by $\bar{X}$ its closure in $Z_{J,1,\delta}$.

Note that the morphism $\pi : Z_{J,1,\delta} \to \mathcal{P}^J$ defined by $\pi(P,Q,\gamma) = P$ for $(P,Q,\gamma) \in Z_{J,1,\delta}$ is a locally trivial fibration with isomorphic fibers. Moreover, $i : \pi^{-1}(P_J) \to G^1/H_{P_J}$ defined by $i(P,Q,\gamma) = \gamma$ for $(P,Q,\gamma) \in \pi^{-1}(P_J)$ is an isomorphism. Now $[J,w,1]_{1,\delta} \subset \pi^{-1}(P_J)$ and $i([J,w,1]_{1,\delta}) = B\dot{w}Bg_0/H_{P_J}$. Thus $\overline{[J,w,1]_{1,\delta}} = \sqcup_{w' \leqslant w}[J,w',1]_{1,\delta}$. For any $w' \in W^J$ with $w \geqslant_{J,\delta} w'$, there exists $u \in W_J$, such that $w \geqslant u^{-1}w'\delta(u)$. Thus $(\dot{u}^{-1}\dot{w}'\dot{\delta}(u)T,1)\cdot h_{J,1,\delta} \subset \overline{[J,w,1]_{1,\delta}}$. Hence $G_{diag}(\dot{u}^{-1}\dot{w}'\dot{\delta}(u)T,1)\cdot h_{J,1,\delta} = G_{diag}(\dot{w}'T,1)\cdot h_{J,1,\delta} \subset \overline{Z^w_{J,1,\delta}}$. So $Z^{w'}_{J,1,\delta} \subset \overline{Z^w_{J,1,\delta}}$.

On the other hand, for any $z \in [J,w',1]_{1,\delta}$, by the similar argument as we did in 4.4.3 and 4.4.4, there exists $u \leqslant v \in W_J$, such that $uw'\delta(v^{-1}) \in W^{\delta(J)}$ and $z \in Z^{uw'v^{-1}}_{J,1,\delta}$. If moreover, $w' \leqslant w$, then $w \geqslant u^{-1}(uw'\delta(v^{-1}))\delta(v)$. Thus $w \geqslant_{J,\delta} uw'v^{-1}$. Therefore $z \in \sqcup_{w' \in W^{\delta(J)}, w \geqslant_{J,\delta} w'} Z^{w'}_{J,1,\delta}$. The proposition is proved. □

## 4.5 The cellular decomposition

**4.5.1** A finite partition of a variety $X$ into subsets is said to be an $\alpha$-partition if the subsets in the partition can be indexed $X_1, X_2, \ldots, X_n$ in such a way that $X_1 \cup X_2 \cup \cdots \cup X_i$ is closed in $X$ for $i = 1, 2, \ldots, n$. We say that a variety has a cellular decomposition if it admits an $\alpha$-partition into subvarieties which are affine spaces. It is easy to see that if a variety $X$ admits an $\alpha$-partition into subvarieties and each subvariety has a cellular decomposition, then $X$ has a cellular decomposition.

**Lemma 4.5.2.** *Let $(J,w) \in \mathcal{I}_\delta$, $K \subset J$ and $w' \in W$ with $\mathrm{Ad}(w')\delta(K) = K$. If $w'v \geqslant_{J,\delta} w$ for some $v \in W_{\delta(K)}$, then $w' \geqslant_{J,\delta} w$.*

*Proof.* Fix $w'$ and $(J,w)$. It suffices to prove the following statement:

Let $u \in W_J$ and $v \in W_{\delta(K)}$. If $w'v \geqslant u^{-1}w\delta(u)$, then $w' \geqslant_{J,\delta} w$.

We argue by induction on $l(u)$. Assume that the statement holds for all $u' < u$. Then I will prove that the statement holds for $u$ by induction on $l(v)$. If $l(v) = 0$, then $v = 1$ and the statement holds in this case. Now assume that $l(v) > 0$.

Set $u = u_1 u_2$ with $u_1 \in W^K$ and $u_2 \in W_K$. If $u_2 = 1$, then $u \in W^K$ and $w\delta(u) \in W^{\delta(K)}$. By 4.3.4, there exists $u' \leqslant u$, such that $u'w'v \geqslant w\delta(u)$. Assume that $v = v's_k$ for $v' < v$ and $k \in \delta(K)$. Then $w\delta(u) < w\delta(u)s_k$. By 4.3.1, $w\delta(u) \leqslant u'w'v'$. By 4.3.4, there exists $u_1' \leqslant u' \leqslant u$, such that $w'v' \geqslant (u_1')^{-1}w\delta(u)$. By the remark of 4.3.9, $w'v' \geqslant (u_2')^{-1}w\delta(u_2')$ for some $u_2' \leqslant u_1'$. Thus by induction hypothesis, $w' \geqslant_{J,\delta} w$.

If $u_2 \neq 1$. Then $l(u_1) < l(u)$. By 4.3.4, there exists $u_3 \leqslant u_2$ and $u_4 \leqslant u_2^{-1}$, such that $u_3 w' v \delta(u_4) \geqslant u_1^{-1} w\delta(u_1)$. Note that $u_3 w' v u_4 = w'((w')^{-1}u_3 w')v\delta(u_4) \in w'W_{\delta(K)}$. By induction hypothesis on $l(u_1)$, $w' \geqslant_{J,\delta} w$. □

**4.5.3** Let $J \subset I$. For $w \in W$, set

$$I_1(J, w, \delta) = \max\{K \subset J \mid w \in W^{\delta(K)}\},$$
$$I_2(J, w, \delta) = \max\{K \subset J \mid \mathrm{Ad}(w)\Phi_{\delta(K)} = \Phi_K\}.$$

Now let $(J, w) \in \mathcal{I}_\delta$. Set

$$W_\delta(J, w) = \{u \in W \mid u \geqslant_{J,\delta} w, I_2(J, u, \delta) \subset I_1(J, u, \delta)\}.$$

For any $u \in W_\delta(J, w)$, set

$$\begin{aligned} X_u^{(J,w,\delta)} &= \sqcup_{K \subset I_1(J,u,\delta)} \sqcup_{v \in W_{\delta(I_2(J,u,\delta))} \cap W^{\delta(K)}} Z_{K,\delta}^{uv} \\ &= \sqcup_{v \in W_{\delta(I_2(J,u,\delta))}} \sqcup_{K \subset I_1(J,uv,\delta)} Z_{K,\delta}^{uv}. \end{aligned}$$

For $w' \geqslant_{J,\delta} w$, we have that $w' = uv$ for some $u \in W^{\delta(I_2(J,w'))}$ and $v \in W_{\delta(I_2(J,w'))}$. Then $I_2(J, u, \delta) = I_2(J, w', \delta) \subset I_1(J, u, \delta)$. By 4.5.2, $u \geqslant_{J,\delta} w$. Thus $u \in W_\delta(J, w)$ and $\sqcup_{K \subset I_1(J,w',\delta)} Z_{K,\delta}^{w'} \subset X_u^{(J,w,\delta)}$.

For $u_1, u_2 \in W(J,w)$ and $v_1 \in W_{\delta(I_2(J,u_1))}, v_2 \in W_{\delta(I_2(J,u_2))}$ with $u_1v_1 = u_2v_2$, we have that $I_2(J,u_1,\delta) = I_2(J,u_1v_1,\delta) = I_2(J,u_2v_2,\delta) = I_2(J,u_2,\delta)$. Note that $u_1, u_2 \in W^{\delta(I_2(J,u_1))}$. Thus $u_1 = u_2$ and $v_1 = v_2$.

Therefore $\overline{Z^w_{J,\delta}} = \sqcup_{u \in W_\delta(J,w)} X_u^{(J,w,\delta)}$.

**Lemma 4.5.4.** *Let $(J,w) \in \mathcal{I}_\delta$. Set $I_2 = I_2(J,w,\delta)$. For $K \subset J$, we have that*

$$\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (L_{I_2})_{diag}(\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K,\delta} = (L_{I_2}, L_{I_2})(\dot{w}, 1) \cdot h_{K,\delta}.$$

*Proof.* At first, we will prove the case when $K \subset I_2$. In this case, set $g_1 = g_0\dot{w}$. Then $g_1 L_{\delta(I_2)} g_1^{-1} = L_{\delta(I_2)}$ and $g_1(L_{\delta(I_2)} \cap B) g_1^{-1} = L_{\delta(I_2)} \cap B$. Now consider $\overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})} g_1$ (a variety that is isomorphic to $\overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})}$, but with "twisted" $L_{\delta(I_2)} \times L_{\delta(I_2)}$ action, see 4.2.3). We have that

$$\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (L_{\delta(I_2)})_{diag}(\dot{v}, B \cap L_{\delta(I_2)}) \cdot (h_{\delta(K)} g_1) = (L_{\delta(I_2)}, L_{\delta(I_2)}) \cdot (h_{\delta(K)} g_1).$$

(In the case when $g_1^n \in L_{\delta(I_2)}$ for some $n \in \mathbf{N}$, $L_{\delta(I_2)} g_1$ is a connected component of the group generated by $L_{\delta(I_2)}$ and $g_1$. In this case, the left hand side is the union of some $L_{\delta(I_2)}$-stable pieces and the equality follows from [L8, 12.3]. The general case can be shown in the same way.)

Therefore

$$\begin{aligned} &\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, g_0)(L_{I_2})_{diag}(\dot{w}, g_0^{-1})(\dot{v}, B \cap L_{\delta(I_2)}) \cdot h_{\delta(K)} \\ &= \sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (1, g_1)(L_{\delta(I_2)})_{diag}(1, g_1^{-1})(\dot{v}, B \cap L_{\delta(I_2)}) \cdot h_{\delta(K)} \\ &= (L_{\delta(I_2)}, L_{\delta(I_2)})(\dot{w}, 1) \cdot h_{\delta(K)}. \end{aligned}$$

Note that $h_{K,\delta} = h_{\delta(K)} g_0$. Then $\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, 1)(L_{I_2})_{diag}(\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K,\delta} = (L_{\delta(I_2)}, L_{I_2}) \cdot h_{K,\delta}$. Hence $\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (L_{I_2})_{diag}(\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K,\delta} = (L_{I_2}, L_{I_2}) \cdot h_{K,\delta}$.

In the general case, Consider $\pi : (L_{\delta(I_2)}, L_{I_2}) \cdot h_{K,\delta} \to \overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})} g_0$ defined by $\pi\big((l_1, l_2) h_{K,\delta}\big) = (l_1, l_2) \cdot (h_{\delta(K) \cap \delta(I_2)} g_0)$ for $l_1 \in L_{\delta(I_2)}, l_2 \in L_{I_2}$. Here $h_{\delta(K) \cap \delta(I_2)}$ on the right side is the base point in $\overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})}$ that corresponds to $\delta(K) \cap \delta(I_2)$.

It is easy to see that the morphism is well-defined. Now define the $T$-action on $(L_{\delta(I_2)}, L_{I_2}) \cdot h_{K,\delta}$ by $t \cdot \big((l_1, l_2)h_{K,\delta}\big) = (tl_1, l_2)h_{K,\delta}$ for $t \in T$ and $l_1 \in L_{\delta(I_2)}, l_2 \in L_{I_2}$. Then $T$ acts transitively on $\pi^{-1}(a)$ for any $a \in (L_{\delta(I_2)}, L_{I_2}) \cdot (h_{\delta(K)\cap\delta(I_2)} g_0)$. Now

$$
\begin{aligned}
&\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} \pi\big((\dot{w}^{-1}, 1)(L_{I_2})_{diag} \cdot (\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K,\delta}\big) \\
&= \sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, 1)(L_{I_2})_{diag}(\dot{w}\dot{v}, B \cap L_{I_2}) \cdot (h_{\delta(K)\cap\delta(I_2)} g_0) \\
&= (L_{\delta(I_2)}, L_{I_2}) \cdot (h_{\delta(K)\cap\delta(I_2)} g_0).
\end{aligned}
$$

Moreover $\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, 1)(L_{I_2})_{diag} \cdot (\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K,\delta}$ is stable under $T$-action. Thus $\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, 1)(L_{I_2})_{diag} \cdot (\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K,\delta} = (L_{\delta(I_2)}, L_{I_2}) \cdot h_{K,\delta}$. The lemma is proved. □

**Lemma 4.5.5.** *Let $(J, w) \in \mathcal{I}_\delta$ and $u \in W_\delta(J, w)$. Set $I_1 = I_1(J, u, \delta)$, $I_2 = I_2(J, u, \delta)$ and $L_u^{(J,w,\delta)} = \sqcup_{K \subset I_1} (L_{I_2}, L_{I_2})(\dot{u}, 1) \cdot h_{K,\delta}$. Then we have that*

*(1) $L_u^{(J,w,\delta)}$ is a fibre bundle over $\overline{L_{I_2}/Z(L_{I_2})}$ with fibres isomorphic to an affine space of dimension $|I_1| - |I_2|$.*

*(2) $X_u^{(J,w,\delta)} = G_{diag} \cdot L_u^{(J,w,\delta)}$ is isomorphic to $G \times_{P_{I_2}} \big((P_{I_2})_{diag} \cdot L_u^{(J,w,\delta)}\big)$.*

*(3) $(P_{I_2})_{diag} \cdot L_u^{(J,w,\delta)} = (B \times B) \cdot L_u^{(J,w,\delta)} \cong (U \cap {}^{\dot{w}_0^{I_2} \dot{u} \dot{w}_0} U^-) \times L_u^{(J,w,\delta)}$.*

*Proof.* For part (1), note that $L_u^{(J,u,\delta)} = \sqcup_{K \subset I_1} (\dot{u}, 1)(L_{\delta(I_2)}, L_{I_2}) \cdot h_{K,\delta} = (\dot{u} L_{\delta(I_2)}, L_{I_2}) \cdot \big(\sqcup_{K \subset I_1} (T, 1) h_{K,\delta}\big)$ is a variety. Consider $\pi' : \sqcup_{K \subset I_1} (L_{\delta(I_2)}, L_{I_2}) \cdot h_{K,\delta} \to \overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})} g_0$ defined by $\pi'\big((l_1, l_2)h_{K,\delta}\big) = (l_1, l_2) \cdot (h_{\delta(K)\cap\delta(I_2)} g_0)$ for $l_1 \in L_{\delta(I_2)}, l_2 \in L_{I_2}$. It is easy to see that $\pi'$ is well defined and is a locally trivial fibration with fibers isomorphic to an affine space of dimension $|I_1| - |I_2|$.

Let $v \in W_{\delta(I_2)}$. For $K \subset J$, if $\mathrm{Ad}(uv)\delta(K) = K$, then $\mathrm{Ad}(u)\Phi_{\delta(K)} = \mathrm{Ad}(uv^{-1}u^{-1})\Phi_K$. Since $uv^{-1}u^{-1} \in W_{I_2}$, we have that $\mathrm{Ad}(u)\Phi_{\delta(K)} \subset \Phi_{K \cup I_2}$. Thus $\mathrm{Ad}(u)\Phi_{\delta(K \cup I_2)} \subset \Phi_{K \cup I_2}$. By the maximal property of $I_2$, $K \cup I_2 \subset I_2$. Thus $I_2(J, uv, \delta) \subset I_2$. Therefore,

$$
\begin{aligned}
G_{diag} \cdot L_u^{(J,w,\delta)} &= G_{diag}\big(\sqcup_{K\subset I_1}(L_{I_2}, L_{I_2})(\dot{u},1)\cdot h_{K,\delta}\big)\\
&= G_{diag}\big(\sqcup_{K\subset I_1}\sqcup_{v\in W_{\delta(I_2)}\cap W^{\delta(K)}}(L_{I_2})_{diag}(\dot{u}\dot{v}, B\cap L_{I_2(J,u,\delta)})\cdot h_{K,\delta}\big)\\
&= \sqcup_{K\subset I_1}\sqcup_{v\in W_{\delta(I_2)}\cap W^{\delta(K)}} G_{diag}(\dot{u}\dot{v}, B\cap L_{I_2})\cdot h_{K,\delta}\\
&= \sqcup_{K\subset I_1}\sqcup_{v\in W_{\delta(I_2)}} Z_{J,\delta}^{uv} = X_u^{(J,w,\delta)}.
\end{aligned}
$$

Assume that $(g,g)a = b$ for some $g \in G$ and $a, b \in L_u^{(J,u,\delta)}$. Then $a, b$ are in the same $G$ orbit. Note that any element in $L_u^{(J,u,\delta)}$ is conjugate by $L_{I_2}$ to an element of the form $(\dot{u}\dot{v}, l)h_{K,\delta}$ with $v \in W_{\delta(I_2)}$, $K \subset I_1(J,uv,\delta)$ and $l \in L_{I_2}\cap B$. Moreover, $(\dot{u}\dot{v}, L_{I_2}\cap B)\cdot h_{K,\delta} \subset Z_{K,\delta}^{uv}$. Thus if $v_1 \neq v_2$ or $K_1 \neq K_2$, then for any $l, l' \in L_{I_2}\cap B$, $(\dot{u}\dot{v}_1, l)\cdot h_{K_1,\delta}$ and $(\dot{u}\dot{v}_2, l')\cdot h_{K_2,\delta}$ are not in the same $G$ orbit. Thus $(g,g)(\dot{u}\dot{v}, l_1)\cdot h_{K,\delta} = (\dot{u}\dot{v}, l_2)\cdot h_{K,\delta}$ for some $v \in W_{\delta(I_2)}$, $K \subset I_1(J,uv,\delta)$ and $l_1, l_2 \in L_{I_2}\cap B$. By 4.1.12, $g \in P_{I_2(K,uv,\delta)}$. Since $I_2(K,uv,\delta) \subset I_2(J,uv,\delta) \subset I_2$, we have that $g \in P_{I_2}$. By 4.1.9, $X_u^{(J,w,\delta)} \cong G \times_{P_{I_2}} \big((P_{I_2})_{diag}\cdot L_u^{(J,w,\delta)}\big)$. Part (2) is proved.

For part (3), it is easy to see that $(P_{I_2(J,u,\delta)})_{diag}\cdot L_u^{(J,w,\delta)} \subset (B\times B)\cdot L_u^{(J,w,\delta)}$. On the other hand,

$$
\begin{aligned}
(B\times B)\cdot L_u^{(J,w,\delta)} &= (U_{P_{I_2}}, U_{P_{I_2}})(L_{I_2})_{diag}\big(\sqcup_{v\in W_{\delta(I_2)}}\sqcup_{K\subset I_2(J,uv,\delta)}(\dot{u}\dot{v}, B)\cdot h_{K,\delta}\big)\\
&= (L_{I_2})_{diag}(U_{P_{I_2}}, U_{P_{I_2}})\big(\sqcup_{v\in W_{\delta(I_2)}}\sqcup_{K\subset I_2(J,uv,\delta)}(\dot{u}\dot{v}, B)\cdot h_{K,\delta}\big).
\end{aligned}
$$

By 4.1.12, $(U_{P_{I_2}}, U_{P_{I_2}})(\dot{u}\dot{v}, B)\cdot h_{K,\delta} = (B\times B)(\dot{u}\dot{v},1)\cdot h_{K,\delta} \subset (P_{I_2(K,uv,\delta)})_{diag}\cdot (L_{I_2(K,uv,\delta)}, L_{I_2(K,uv,\delta)})(\dot{u}\dot{v},1)h_{K,\delta}$. We have showed that $I_2(K,uv,\delta) \subset I_2$. Hence $(U_{P_{I_2}}, U_{P_{I_2}})(\dot{u}\dot{v}, B)\cdot h_{K,\delta} \subset (P_{I_2(J,u,\delta)})_{diag}\cdot L_u^{(J,w,\delta)}$. Therefore, $(P_{I_2})_{diag}\cdot L_u^{(J,w,\delta)} = (B\times B)\cdot L_u^{(J,w,\delta)}$.

Consider the morphism $\pi : (U \cap {}^{\dot{w}_0^{I_2}\dot{u}\dot{w}_0}U^-) \times L_u^{(J,w,\delta)} \to (B\times B)\cdot L_u^{(J,w,\delta)}$ defined by $\pi(b,l) = (b,1)\cdot l$ for $b \in U \cap {}^{\dot{w}_0^{I_2}\dot{u}\dot{w}_0}U^-$ and $l \in L_u^{(J,w,\delta)}$. By the similar argument as we did in 4.1.10, we can show that $\pi$ is an isomorphism. □

**Corollary 4.5.6.** *We keep the notation of 4.5.5. If moreover, $I_2 = \varnothing$, then $X_u^{(J,w,\delta)}$ admits a cellular decomposition.*

*Proof.* If $I_2 = \varnothing$, then $L_u^{(J,w,\delta)}$ is an affine space. Thus $X_u^{(J,w,\delta)}$ is a fibre bundle over $\mathcal{P}^{I_2}$ with fibres isomorphic to an affine space of fixed dimension. Thus $X_u^{(J,w,\delta)}$ admits a cellular decomposition. □

**4.5.7** For $w_1, w_2 \in W_\delta(J,w)$, we say that $w_2 \leqslant' w_1$ if there exists $w_1 = x_0, x_1, \cdots, x_n = w_2$, $v_i \in \delta(I_2(J, x_{i+1}, \delta))$ for all $i$, such that $x_{i+1}v_i \geqslant_{I_1(J,x_i,\delta),\delta} x_i$. By 4.5, $\overline{X_{u_1}^{(J,w,\delta)}} \cap X_{u_2}^{(J,w,\delta)} = \varnothing$ if $u_2 \nleqslant' u_1$. hence if $\leqslant'$ is a partial order on $W_\delta(J,w)$, then $\overline{Z_{J,\delta}^w} = \sqcup_{u \in W_\delta(J,w)} X_u^{(J,w,\delta)}$ is an $\alpha$-partition. We will show that $\leqslant'$ is a partial order if $\overline{Z_{J,\delta}^w}$ contains finitely many $G$-orbits.

**Lemma 4.5.8.** *Let $J \subset I$, $u \in W$, $w \in W^J$ and $v \in W_J$. Assume that $uwv = w'v'$ for some $w' \in W^J$ and $v' \in W_J$. If $l(uwv) = l(wv) - l(u)$, then $w' \leqslant w$. If moreover, $w' = w$, then* $\mathrm{Ad}(w^{-1})\mathrm{supp}(u) \subset J$.

*Proof.* If $u = s_i$ for some $i \in J$ and $l(s_j wv) = l(wv) - 1$, then either $s_i w < w$ and $s_i w \in W^J$ or $s_i w = w s_j$ for some $j \in J$. It is easy to check that the statement holds in both cases.

The general case can be proved by induction on $l(u)$. □

**Lemma 4.5.9.** *If $w_1, w_2 \in W_\delta(J,w)$ with $w_1 \leqslant' w_2$ and $w_2 \leqslant' w_1$ and $I_2(J, w_1, \delta) = I_2(J, w_2, \delta) = \varnothing$, then $w_1 = w_2$.*

*Proof.* We will prove the case: if $w_1 \geqslant_{I_1(J,w_2,\delta),\delta} w_2$, $w_2 \geqslant_{I_1(J,w_1,\delta),\delta} w_1$ and $I_2(J, w_1, \delta) = \varnothing$, then $w_1 = w_2$. The general case can be proved in the similar way.

We argue by induction on $|J|$. Since $l(w_1) \geqslant l(w_2)$ and $l(w_2) \geqslant l(w_1)$, we have that $l(w_1) = l(w_2)$. Thus $w_1 = u_2^{-1} w_2 \delta(u_2)$ and $w_2 = u_1^{-1} w_1 \delta(u_1)$ for some $u_1 \in W_{I_1(J,w_1,\delta)}$ and $u_2 \in W_{I_1(J,w_2,\delta)}$. By induction hypothesis, it suffices to prove the case when $J = \mathrm{supp}(u_1) \cup \mathrm{supp}(u_2)$.

We have that $w_1 = w_1' \delta(v_1)$ and $w_2 = w_2' \delta(v_2)$ for some $w_1', w_2' \in W^{\delta(J)}$ and $v_1, v_2 \in W_J$. Note that $w_1' \delta(v_1) = u_2^{-1} w_2' \delta(v_2 u_2)$ and $l(u_2^{-1} w_2' \delta(v_2 u_2)) = l(w_2' \delta(v_2 u_2)) - l(u_2)$. By 4.5.8, $w_1' \leqslant w_2'$. Similarly $w_2' \leqslant w_1'$. Therefore $w_1' = w_2'$. By 4.5.8, $\mathrm{Ad}(w_2')^{-1}\mathrm{supp}(u_2) \subset \delta(J)$ and $\mathrm{Ad}(w_1')^{-1}\mathrm{supp}(u_1) \subset \delta(J)$. Therefore $\mathrm{Ad}(w_1')^{-1} J \subset$

$\delta(J)$. Hence $\mathrm{Ad}(w_1)^{-1}\Phi_J = \Phi_{\delta(J)}$. Since $I_2(J, w_1, \delta) = \varnothing$, we have that $J = \varnothing$. Therefore $w_1 \geqslant w_2$ and $w_2 \geqslant w_1$. Thus $w_1 = w_2$. The case is proved. $\square$

As a summary, we have the following result.

**Theorem 4.5.10.** *If $\overline{Z^w_{J,\delta}}$ contains only finitely many $G$-orbits, then it has a cellular decomposition.*

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