Brill-Noether-type Theorems with a Movable Ramification Point

by

Rebecca C. Lehman

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Author . . . .
Department of Mathematics
April 23, 2007

Certified by . . . .
Jason M. Starr
Assistant Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Pavel Etingof
Chairman, Department Committee on Graduate Students

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Abstract

The classical Brill-Noether theorems count the dimension of the family of maps from a general curve of genus g to non-degenerate curves of degree d in projective space $\mathbb{P}^r$. These theorems can be extended to include ramification conditions at fixed general points. This thesis deals with the problem of imposing a ramification condition at an unspecified point. We solve the problem completely in dimension 1, prove a closed-form existence criterion and a finiteness result in dimension 2, and provide an existence test and bound the dimension of the family in the general case.

Thesis Supervisor: Jason M. Starr
Title: Assistant Professor of Mathematics

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It takes more than one village to raise a grad student. I am unable to name all the amazing friends and communities who have contributed to my experience writing this thesis, those whose names I know and those whose names I do not know. Let my silence not be misconstrued as ingratitude.

Blessed is the One Who grants wisdom to human beings.

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Chapter 1

Introduction

Algebraic curves have been natural objects of study for centuries. The classical founders of algebraic geometry conceived of curves as embedded in an ambient affine or projective space. With the invention of abstract curves, questions of representability in projective space became central to modern algebraic geometry. Up to projective equivalence, maps from a curve of genus g to a curve of degree d in $\mathbb{P}^r$ are given by linear series of degree d and dimension $(r+1)$, denoted as g^r_d's. So we ask the question, under what conditions do g^r_d's exist?

In their seminal paper on algebraic functions and their geometric applications ([2], 1879), Brill and Noether calculated the expected dimension ρ of the family of maps from a general curve of genus g to $\mathbb{P}^r$. However, they did not prove that the family has dimension at most ρ, or even that g^r_d's exist at all.

The existence half of the Brill-Noether theorem was first proved with twentieth-century rigor by Kleiman and Laksov ([19], 1972; [21], 1974) and independently by Kempf ([17], 1971). We shall discuss these proofs in detail in Section 3.

The non-existence theorem, and the upper bound on the dimension, were proved by Griffiths and Harris ([15], 1980) and refined by Eisenbud and Harris ([5], 1986, [4], 1986), by methods we shall discuss in Sections 2 and 4.

Griffiths, Harris and Eisenbud's proofs extend almost verbatim to the case when one imposes in addition the condition that the linear system must have a specified type of ramification at a general fixed point P of the curve. But this raises the more basic question of whether a g^r_d exists with the specified ramification at any point at all. This is the main question of this thesis:

Question 1.0-1. *Let X be a general curve of genus g, and let positive integers r, d and $(m_0 < \cdots < m_r)$ be given. Does there exist a g^r_d on X possessing vanishing sequence $(m_0, \cdots, m_r)$ at any point Q? If so, what is the dimension of the set of such pairs $(\mathcal{L}, Q)$? If the dimension is zero, how many are there?*

We shall proceed as follows:

In Chapter 2 we define the problem and some notation, and sketch a simple proof of the classical Brill-Noether Theorem due to Eisenbud and Harris. This beautiful proof includes both the existence and non-existence components, and generalizes automatically to the case of a fixed general ramification point, motivating what is to come. It also provides an opportunity to introduce some key notions of Schubert calculus and degeneration.

In Chapter 3 we present an enumerative proof of the existence half of the Brill-Noether Theorem, due to Kleiman, Laksov and Kempf, by means of the Porteous formula. This proof then motivates our proof of the following theorem in the case of moving ramification:

Theorem 1.0-2. *Let X be a general curve of genus g and let $r, d, m_0 \leq \cdots \leq m_r$ be nonnegative integers such that*

$$\rho(g, r, d, m_i) = g - \sum_{i=1}^{r} (m_i - i + g + r - d) + 1 \geq 0.$$

Then the class of the family of g^r_d's admitting a point Q with vanishing sequence m_i is given by an explicit formula (see 3.3-4) in terms of the Theta divisor on $\mathrm{Pic}^d(X)$ and the Schubert classes. If $\rho = 0$, then the class is Poincaré dual to a finite set of points, the number of which is a product of certain polynomial and factorial functions in g, r, d and the multiplicities m_i.

We then present some examples of this theorem and use it to derive simple existence criteria in case $r = 1$ and $r = 2$.

In Chapter 4 we present another proof of the non-existence half of the Brill-Noether Theorem, also due to Eisenbud and Harris, who define an appropriate notion of limit linear series on a flag curve consisting of a backbone of rational curves with g elliptic tails. They compute some inequalities on the possible ramification of any potential limit series on the rational curves, and derive a proof by contradiction. We then analyze the possible limit g^r_d's on the elliptic tails of the flag curve. The divisors in these linear series are all expressed as sums of certain torsion points, which we use to prove the following theorems:

Theorem 1.0-3 (Finiteness of Points). *Given nonnegative integers g, r, d, and an $(r+1)$-tuple $(m_0 \leq \cdots \leq m_r)$ such that $\rho(g, r, d, m_i) \leq 0$, then on a general curve of genus g, there are at most finitely many points Q such that X can be embedded as a curve of degree d in $\mathbb{P}^r$ such that the vanishing sequence at Q is $(m_0, \cdots, m_r)$.*

Moreover, by analyzing the possible limiting cases, we shall prove a weak bound on the dimension, but this bound will not in general be equal to ρ unless $r = 1$ or $r = 2$.

Theorem 1.0-4 (Weak General Bound). *If the expected dimension ρ is less than or equal to zero, then the actual dimension of the family of g^r_d's over a general curve of genus g with a ramification point of type $(m_0, \cdots, m_r)$ is bounded by $\rho+r-2$ if this number is nonnegative. Moreover, let $k+1$ be the size of the largest subset of the set of multiplicities $\{m_{i_0}, \cdots, m_{i_k}\} \subseteq \{m_0, \cdots, m_r\}$ whose pairwise differences all share a common factor. Then the dimension of $\mathcal{G}^r_d(m_0, \cdots, m_r)$ is bounded by $\rho + k - 1$.*

Chapter 2 is classical, and sets up the notation; it can be omitted or used for reference. Chapters 3 and 4 contain the main proofs and are essentially logically independent. Chapter 5 deals with open problems and future research directions, relying on both 3 and 4.

Chapter 2

Brill-Noether Theory

2.1 Definitions and Notation

We begin with a smooth, connected, projective curve C of genus g over the complex numbers $\mathbb{C}$.

Definition 2.1-1. *A linear system of degree d and dimension $r+1$, or g^r_d, on C, is an $(r+1)$-dimensional vector space of linearly equivalent divisors on C.*

It will be helpful to use both additive and multiplicative notation. Multiplicatively, a g^r_d can be given as a pair $(\mathcal{L}, V)$, where $\mathcal{L}$ is a line bundle on C and V is an $(r+1)$-dimensional subspace of $H^0(\mathcal{L})$. A basis of V will be denoted by $\sigma_0, \cdots, \sigma_r$. Additively, a g^r_d will be given as a vector space L of linearly equivalent divisors on C, with basis $D_0, \cdots, D_r$. If L is base-point-free, that is, if there is no point P contained in every divisor in L, then L determines a map ϕ_L of degree d from the curve C to projective space $\mathbb{P}^r$ up to projective equivalence. So a g^r_d can be given equivalently by the pair $(\mathcal{L}, V)$, by L, or by a base divisor B of degree $b \leq d$ and a map $\phi_{L-B} \colon C \to \mathbb{P}^r$ of degree $d-b$. By abuse of notation we shall use these notations interchangeably without further comment.

Definition 2.1-2. *Let $(\mathcal{L}, V)$ be a g^r_d on C, and let P be a point on C. An* order basis *for V at P is a basis $(\sigma_0, \cdots, \sigma_r)$ of V constructed as follows: Given $(\sigma_{j+1}, \cdots, \sigma_r)$, take σ_j to be a section linearly independent of $(\sigma_{j+1}, \cdots, \sigma_r)$ that van-*

ishes to the highest possible order at P.

In particular, given any basis $\tau_0, \cdots, \tau_r$, we can write

$$\sigma_i = \tau_i - (\sum_{j=0}^{i-1} c_j \tau_j)$$

for suitably chosen coefficients c_i.

Definition 2.1-3. *The* vanishing sequence *or* multiplicity sequence $(m_0, \cdots, m_r)$ *of a g^r_d $(\mathcal{L}, V)$ at a point P is given by the orders of vanishing $v_P(\sigma_i)$ of the elements of an order basis at P.*

Except in Chapter 3, we shall always order the vanishing sequence from least to greatest, as is customary.

Definition 2.1-4. *The* ramification sequence $(a_0, \cdots, a_r)$ *of $(\mathcal{L}, V)$ at P is given by*

$$a_i = m_i - i.$$

Note that the ramification sequence is also naturally ordered from least to greatest.

Definition 2.1-5. *The* weight *or* total weight *of $\mathcal{L}$ at P, is the sum*

$$w(\mathcal{L}, P) = \sum_{i=0}^{r} a_i.$$

It will be denoted $w(P)$ when $\mathcal{L}$ is understood.

Notation 2.1-6. *Let Pic^d_C be the Picard scheme of line bundles of degree d. Let W^r_d be the locus in Pic^d_C consisting of line bundles $\mathcal{L}$ with at least $r+1$ global sections, let $W^r_d(P, m_0, \cdots, m_r)$ be the locus of line bundles $\mathcal{L}$ with at least $r+1$ global sections vanishing to orders at least $m_0, \cdots, m_r$ at P, and let $W^r_d(m_0, \cdots, m_r)$ be the locus of line bundles $\mathcal{L}$ with at least $r+1$ global sections vanishing to orders at least $m_0, \cdots, m_r$ at* some *point Q.*

Notation 2.1-7. *Let $\mathcal{P}_d$ be a Poincaré sheaf on $\mathrm{Pic}^d_C \times C$, and let $\mathcal{E}$ be the push-forward of $\mathcal{P}_d$ to Pic^d.*

Notation 2.1-8. *Let* $\mathcal{G}^r_d$ *be the Grassmann bundle* $\mathbb{G}(r+1,\mathcal{E})$ *over* Pic^d_C *whose fiber over a point* $[\mathcal{L}]$ *is the set of* $(r+1)$*-dimensional subspaces of* $H^0(\mathcal{L})$. *Let* $\mathcal{G}^r_d(P,m_0,\cdots,m_r)$ *denote the locus in* $\mathcal{G}^r_d$ *consisting of pairs* $(\mathcal{L},V\subset H^0(\mathcal{L}))$ *such that* V *has a basis of sections vanishing to orders at least* $m_0,\cdots,m_r$ *at the given point* P, *and let* $\mathcal{G}^r_d(m_0,\cdots,m_r)$ *denote the subscheme of pairs* $(\mathcal{L},V)$ *such that* V *has a basis of sections vanishing to orders* $m_0,\cdots,m_r$ *at* some *point* Q *on* C.

We can now state the Brill-Noether problems:

Question 2.1-9 (Classical Brill-Noether). *For which triples of integers* (g,r,d) *does a general curve of genus* g *have a* g^r_d? *If such* g^r_d*'s exist, then what are the classes of* W^r_d *and* $\mathcal{G}^r_d$? *Do they have the expected dimensions? If they have dimension* 0, *how many distinct* g^r_d*'s exist?*

Question 2.1-10 (Brill-Noether with Fixed Ramification Point). *Given a triple of integers* (g,r,d) *and a ramification sequence* $(a_0,\cdots,a_r)$ *such that*

$$0\le a_0\le\ldots\le a_r\le d,$$

does a general curve of genus g *possess a* g^r_d *with ramification* $(a_0,\cdots,a_r)$ *at a fixed general point* P? *If so, what are the classes of* $W^r_d(P,a_0,\cdots,a_r)$ *and* $\mathcal{G}^r_d(P,a_0,\cdots,a_r)$? *Do they have the expected dimensions? If the dimension is* 0, *how many distinct points do they contain?*

Question 2.1-11 (Brill-Noether with Movable Ramification Point). *Given a triple of integers* (g,r,d) *and a ramification sequence* $(a_0,\cdots,a_r)$ *such that*

$$0\le a_0\le\ldots\le a_r\le d,$$

does a general curve of genus g *possess a* g^r_d *with ramification* $(a_0,\cdots,a_r)$ *at some point* Q? *If so, what are the loci* $W^r_d(a_0,\cdots,a_r)$ *and* $\mathcal{G}^r_d(a_0,\cdots,a_r)$? *In dimension* 0, *how many distinct points do they contain?*

2.2 The Brill-Noether Theorem

In modern language, Brill and Noether proved the following:

Theorem 2.2-1 (Brill, Noether, 1879). *The family W^r_d of g^r_d's on a curve of genus g has dimension at least*

$$\rho = g - (r+1)(g+r-d).$$

Proof: The Picard scheme Pic^d_C of line bundles of degree d on C has dimension g. Choose n sufficiently large that any line bundle of degree $d+n$ is nonspecial, i.e. has $n+d+1-g$ independent global sections.

The vector space of global sections $H^0(\mathcal{L})$ is the kernel of the map

$$H^0(\mathcal{L}(nP)) \to H^0(\mathcal{L}(nP)/\mathcal{L}).$$

We want it to have dimension at least $r+1$. The source $H^0(\mathcal{L}(nP))$ has dimension $n+d+1-g$, and the target $H^0(\mathcal{L}(nP)/\mathcal{L})$ has dimension n, so the locus where the kernel has dimension $(r+1)$ is cut out by $(r+1)(n-(n+d+1-g)+(r+1))$ or $(r+1)(g+r-d)$ equations. Hence the expected dimension is

$$\rho = g - (r+1)(g+r-d).$$

□

The complete answer to the classical Brill-Noether question is due to Kleiman, Laksov, and Kempf (existence) and Griffiths, Harris, and Eisenbud (non-existence).

Theorem 2.2-2 (Brill-Noether Theorem). *Let C be a general curve of genus g. Let $\rho(g,r,d)$ be the Brill-Noether number*

$$\rho = g - (r+1)(g+r-d).$$

If $0 \leq \rho \leq g$, then W^r_d is a non-empty subscheme of Pic^d_C of dimension ρ. If $\rho < 0$ then W^r_d is empty, and if $\rho \geq g$ then $W^r_d = \mathrm{Pic}^d_C$.

The classical Brill-Noether theorem was first proved by Griffiths and Harris [15] by specializing to nodal rational curves. In this section we shall provide a slightly simpler proof, due to Eisenbud and Harris [4], which immediately generalizes to the following:

Theorem 2.2-3 (Brill-Noether, Fixed Ramification Point). *Let P be a general point on a general curve C of genus g. Let $\rho(g,r,d,(a_i))$ be the* adjusted Brill-Noether number

$$\rho = g - \sum_{i=0}^{r}(a_i + g + r - d),$$

and let ρ_+ be the existence number

$$\rho_+ = g - \sum_{a_i+g+r-d\geq 0}(a_i + g + r - d).$$

If $\rho_+ < 0$, then the sets $W^r_d(a_i)$ and $\mathcal{G}^r_d(a_i)$ of g^r_d's with ramification sequence a_i at P are empty. If $\rho_+ \geq 0$, then $W^r_d(a_i)$ is non-empty and has dimension ρ_+, and $\mathcal{G}^r_d(a_i)$ is non-empty and has dimension ρ.

The key idea of the proof is to degenerate the general curve C of genus g to a g-cuspidal rational curve C_0. (See Fig. 2-1.)

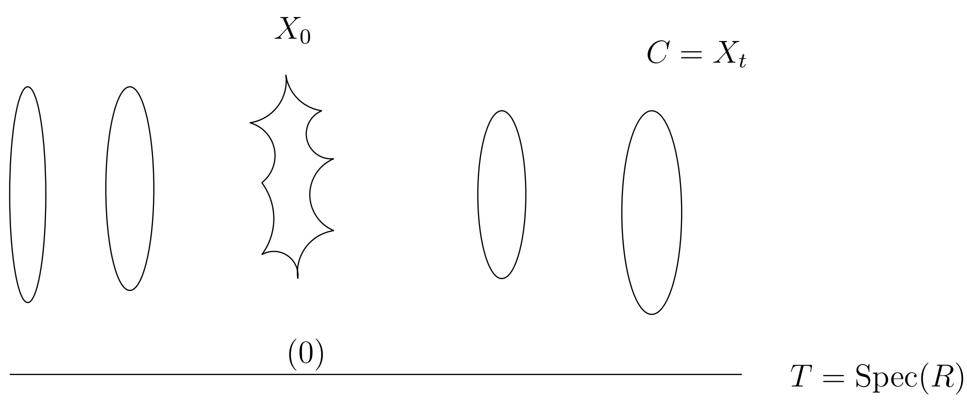

Figure 2-1: Degenerating to a g-cuspidal curve

By upper semicontinuity, every g^r_d on C specializes to a unique g^r_d on C_0 ([4], Prop. 5.5). Pulling this g^r_d back by the normalization map, we obtain a g^r_d on the rational normal curve of degree d, which has simple cuspidal ramification, of type

$(0, 2, 3, \cdots, r, r+1)$, on the g points that are the preimages of the g cusps. So it is enough to count the dimension of the family of projections of the rational normal curve of degree d that acquire g cusps at the specified g points.

The rational normal curve of degree d is embedded in $\mathbb{P}^d$. To obtain a projection to $\mathbb{P}^r$, we must project from a $(d-r-1)$-plane. The Grassmannian $\mathbb{G}(d-r-1, d)$ parametrizes $(d-r-1)$-planes in $\mathbb{P}^d$.

The ramification sequence of the g^r_d at the image of a point P is the sequence of dimensions in which the projection plane meets the flag of osculating spaces to the curve at P. In particular, a simple cusp, with vanishing sequence $(0, 2, 3, \cdots, r+1)$, and ramification sequence $(0, 1, 1, \cdots, 1)$, occurs when the projection plane meets the tangent line to the curve at P.

We need to calculate the locus in the Grassmannian $\mathbb{G}(d-r-1, d)$ of $(d-r-1)$-planes in $\mathbb{P}^d$ that meet g specified tangent lines to the curve C. The solution to this problem is given by Schubert calculus (see [20]).

Definition 2.2-4. *Given a flag*

$$F_0 \subset F_1 \subset \cdots \subset F_n$$

of subspaces of $\mathbb{P}^n$, *the Schubert cycle* $\Sigma_{(c_0, \cdots, c_{(n-r)})}(F)$ *is the locus in* $\mathbb{G}((n-r), n)$ *of* $(n-r)$*-planes that meet* $F_{(r-c_0)}$, *that meet* $F_{(r-c_1)}$ *in at least a line, and for each* i, *meet* $F_{(r-c_i)}$ *in at least an* i*-dimensional plane.*

Proposition 2.2-5. *Given a sequence of integers* $\mathbf{c} = (c_1, \cdots, c_{(n-r)})$ *and two different flags* F *and* G, *the Schubert cycles* $\Sigma_{\mathbf{c}}(F)$ *and* $\Sigma_{\mathbf{c}}(G)$ *are algebraically equivalent. The class of all cycles of form* $\Sigma_{\mathbf{c}}(F)$ *is denoted* $\sigma_{\mathbf{c}}$.

Proposition 2.2-6. *The Schubert class* $\sigma_{(c_1, \cdots, c_{(n-r)})}$ *has codimension* $\sum_{i=1}^{(n-r)}(c_i)$.

Definition 2.2-7. *The* special Schubert classes *are those of the form* $\sigma_a = \sigma_{(a,0,0,\cdots,0)}$.

Proposition 2.2-8. *The Chern class of the universal quotient on* $\mathcal{G}(n-r, n)$ *is* $1 + \sigma_1 + \cdots + \sigma_r$.

Multiplication of special Schubert classes is given by Pieri's formula:

Proposition 2.2-9 (Pieri's formula, [20] p.1073). *The product* $\sigma_{(m,0,0,\cdots,0)} \cdot \sigma_{(c_1,\cdots,c_{(n-r)})}$ *on the Grassmannian* $\mathbb{G}((n-r),n)$ *is given by the sum* $\sum \sigma_{(\alpha_1+c_1,\cdots,\alpha_m+c_m)}$*, where the sum is taken over all* m*-tuples* $(\alpha_0,\cdots,\alpha_m)$ *such that* $\alpha_0+\cdots+\alpha_{(n-r)} = m$*,* $\alpha_1+c_1 \le r$*, and* $\alpha_i + c_i \le c_{i-1}$ *for* $1 < i \le r$*.*

In our case, we need to compute the class of $(d-r-1)$-planes in $\mathbb{P}^d$ that meet g given tangent lines to the rational normal curve. Each tangent line imposes a Schubert condition of type σ_r, so the expected class of their intersection would be σ_r^g, which has dimension $(r+1)(d-r)-rg$, or $g-(r+1)(g+r-d)$. If we also impose a fixed vanishing sequence $(m_0,\cdots,m_r)$ at P, this imposes an additional condition of type $\sigma_{(c_1,\cdots,c_{n-r})}$ where c_i is the number of indices m_j greater than or equal to i. Hence the expected dimension of the intersection becomes

$$g-(r+1)(g+r-d)-\sum_i (m_i - i).$$

To prove the Brill-Noether theorem for C_0, it remains to show that these Schubert subschemes actually do intersect in the expected dimension, and that their intersection is nonzero:

Lemma 2.2-10 ("Dimensional Transversality", [4], Thm. 2.3). *Let* $p_1,\cdots,p_m$ *be distinct points on the rational normal curve* C *of degree* d*, and let* $\mathcal{F}(p_i)$ *be the flag of osculating spaces to* C *at* p_i*. If for each* i*,* τ_i *is any Schubert variety of* r*-planes defined in terms of the flag* $\mathcal{F}(p_i)$*, then the* τ_i *are dimensionally transverse; that is, every component of* $\cap_{i=1}^m \tau_i$ *has codimension equal to the expected codimension* $\sum_{i=1}^m \operatorname{codim} \tau_i$*. Dually, the Schubert varieties of linear series on* $\mathbb{P}^1$ *with defined vanishing sequences at* $p_1,\cdots,p_m$ *intersect in the expected codimension if the intersection is non-empty.*

Proof: First we prove that when the expected intersection class has negative dimension, then the interesection is in fact empty.

A linear series $(\mathcal{L},V)$ satisfies the condition $\mathbb{P}(V^\vee) \in \Sigma_{b^0,\cdots,b^r}(\mathcal{G}(p))$ if and only if for all i we have

$$\dim \mathbb{P}(V^\vee) \cap \mathcal{G}^{n+r+b_i-i}(p) \ge i.$$

In other words,

$$a_{r-i}(V) + r - i \geq a_{b_i+r-i}(H^0(\mathcal{L}, p)) + b_i + r - i.$$

for all i, which happens if and only if

$$a_{r-i}(V) \geq a_{b_i+r-i}(H^0(\mathcal{L}), p) + b_i,$$

for all i.

Summing over all p_i, we obtain the following lemma:

Lemma 2.2-11 (([4], Corollary 2.2)). *Let $p_1, \cdots, p_m$ be distinct points in C and b_i^j $(i = 0, \cdots, r, j = 1, \cdots, m)$ be Schubert indices. Then $\mathbb{P}(V^\vee) \in \cap_j \Sigma_{b_i}(\mathcal{G}(p_j))$ if and only if*

$$\sum_{j=1}^{m} w(p_j) \geq \sum_{i,j} (b_i^j)(b_j^i + a_{b_i^j+r-i}(H^0(\mathcal{L}), p_j)).$$

But we have the Plücker formula:

Lemma 2.2-12 (Plücker). *For any linear series V on a curve C of genus g, we have*

$$\sum_{p_j \in C} w(V, p_j) = (r+1)d + \binom{r+1}{2}(2g-2).$$

Hence we obtain the following corollary:

Corollary 2.2-13 ([4], Corollary 2.2). *Let $p_1, \cdots, p_m$ be distinct points on a curve C of genus g, and let b_i^j $(i = 0, \cdots, r, j = 1, \cdots, m)$ be Schubert indices. Then $\cap_j \Sigma_{b_i^j}(\mathcal{G}(p_j))$ is empty if*

$$\sum_{i,j} (b_i^j)(b_j^i + a_{b_i^j+r-i}(H^0(\mathcal{L}), p_j)) > (r+1)d + \binom{r+1}{2}(2g-2).$$

Setting $C = \mathbb{P}^1$, we have $g = 0$, so the Plücker number $(r+1)d + \binom{r+1}{2}(2g-2)$ becomes $(r+1)(d-r)$. Hence the intersection of the τ_j is empty whenever its expected dimension is negative.

To prove the lemma in general, suppose that the expected dimension is $k \geq 0$.

Then the dimension of each component must be at least k. We must prove that it is at most k. But we can then choose $k+1$ more points $p_{m+1}, \cdots, p_{m+k+1}$ on C, and for $j = m+1, \cdots, m+k+1$ we let τ_j be the hyperplane $\Sigma_1(\mathcal{G}(p_j))$, then the intersection $\cap_{j=1}^{m+k+1}\tau_j$ must be zero because its expected dimension is negative. Hence the original intersection $\cap_{j=1}^{m}\tau_j$ cannot have any components of dimension greater than k. □

Proof of Brill-Noether Theorem: Since the intersection is of the expected dimension, to prove the full theorem on C_0 we need only check that the class σ_r^g is nonzero. Use Pieri's formula to write σ_r^g as a sum of terms with nonnegative coefficients.

By induction on g, the leading term $\sigma_{r+1,r+1,\cdots}$, will always have a strictly positive coefficient. Hence the class is nonzero.

We can now use the curve C_0 to prove the same result for a general curve.

Consider a family of curves C_t specializing to the g-cuspidal curve C_0. Embed C_t by the canonical embedding in $\mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is the symmetric algebra over $\pi_*\omega_{C/T}$. Let I be the incidence corresponence

$$I = \{\Lambda \in \mathbb{G}(g-r-1, \mathcal{E}) : \Lambda \cap f(C) \geq 2g-2-d\}.$$

Then the fiber I_0 of I over $t = 0$ is of dimension ρ. The irreducible component of I containing I_0 has dimension $\rho + 1$, since it has I_0 as a divisor. Hence there is an open neighborhood of $t = 0$ on which the fibers are all of dimension ρ. Hence for a general curve C_t, the fiber $\mathcal{G}_d^r$ has dimension exactly ρ. □

Example 2.2-14. *Let $r = 1$, and $2d = g + 2$, so that $\rho = 0$. How many maps of degree d are there from the rational normal curve to $\mathbb{P}^1$ that factor through a curve with g cusps?*

By Pieri's formula, σ_1^g counts the number of paths from $(0, \cdots, 0)$ to $(2, 2, \cdots, 2)$ by paths of steps in which no coordinate increases beyond the previous value of that ahead of it. Let a be the number of 2's and b be the number of 1's and 2's among the coordinates. Then we are allowed steps that increase either a or b by 1, so long as b is greater than or equal to a. In other words, we want to count the number of monotonic paths from $(0, 0)$ to $(d-1, d-1)$ lying above the main diagonal.

This number is the $(d-1)^{st}$ Catalan number, and we can count it as follows: The total number of monotonic paths is $\binom{2(d-1)}{(d-1)}$ or $\frac{(2(d-1))!}{(d-1)!^2}$. Of these, we need to count those that lie above the diagonal. For any path P, let $E(P)$ denote the number of pairs of edges that lie below the diagonal. If $E(P) = 1$, then there is one pair of edges below the diagonal. If we swap the portion of P occurring before the vertical edge below the diagonal, with that occurring after it, then we obtain a well-defined path P' with $E(P) = 0$. Conversely, given any P' with $E(P') = 0$, we can recover the path P by exchanging the portions occurring before and after the first vertical segment that originates on the diagonal. Likewise, for any path P, we can construct a unique P' with $E(P') = E(P) - 1$, and conversely. So we see that the number of paths with $E(P) = 0$ is equal to the number of paths with $E(P) = n$ for $0 \leq n \leq d-1$, or $\frac{1}{d}$ of the total. Hence our number is $\frac{(2d-2)!}{(d-1)!d!}$ or $\frac{g!}{(\frac{g}{2})!(\frac{g}{2}+1)!}$.

Example 2.2-15. *Let $r = 2$, and $3d = 2g + 6$, so that $\rho = 0$. How many maps are there from the rational normal curve of degree d to a g-cuspidal plane curve?*

The answer is σ_2^g. As above, we need to count the number of paths from $(0, \cdots, 0)$ to $(3, \cdots, 3)$ by paths consisting of pairs of steps in which no coordinate increases beyond the previous value of that ahead of it. Let a be the number of 3's, b the number of 2's and 3's, and c the number of 1's, 2's and 3's. Then we need to count paths in 3-space from $(0, \cdots, 0)$ to $(d-2, d-2, d-2)$ by diagonals $(a, b, c) \mapsto (a, b+1, c+1)$, $(a, b, c) \mapsto (a+1, b, c+1)$ or $(a, b, c) \mapsto (a+1, b+1, c)$, such that at all times $c > b > a$. Change coordinates to a new bases $\mathbf{i}' = (1, 1, 0); \mathbf{j}' = (1, 0, 1); \mathbf{k}' = (0, 1, 1)$. In these coordinates, we want to count monotonic paths from $(0, 0, 0)$ to $(\frac{g}{3}, \frac{g}{3}, \frac{g}{3})$ such that $c' + b' - a' > c' + a' - b' > a' + b' - c'$, or $c' > b' > a'$. In other words, we need to count monotonic paths lying in the permitted region above the plane $z' = y'$ and to the right of the plane $y' = x'$.

The number of such paths is the 3-dimensional Catalan number $\frac{2(g!)}{(\frac{g}{3})!(\frac{g}{3}+1)!\frac{g}{3}+2)!}$, but the proof is much more involved.

In general, it is possible to compute the number of maps from the rational normal curve through the g-cuspidal curve to $\mathbb{P}^n$ explicitly as n-dimensional generalized Catalan numbers, but the proofs involve some messy combinatorics. Moreover, we still need to know that all the maps on the g-cuspidal rational curve deform to unique maps of an arbitrary plane curve; hypothetically there could be multiplicities. In the next chapter, we shall compute the number directly for a general curve of genus g.

Chapter 3

Existence Results

The idea behind these enumerative existence proofs is that the cycle class of an empty set must be zero. If we can compute the class of the locus of g^r_d's with a given property and show that it is nonzero, then such g^r_d's must exist. We do this by expressing the locus as the degeneracy locus of an appropriate map of vector bundles.

Throughout this section, we shall index all bases and matrix elements beginning with 0, and order all ramification sequences from greatest to least.

3.1 Brill-Noether Existence Without Ramification

We present a direct proof of the Brill-Noether existence theorem, first proved by Kleiman, Laksov and Kempf ([21] and [17]).

Theorem 3.1-1 (Brill-Noether Existence). *Let ρ be the Brill-Noether number*

$$\rho = g - (r+1)(g+r-d).$$

If $\rho \geq 0$, then any curve of genus g possesses a family of g^r_d's of dimension at least ρ.

Fix a point P, and choose an integer n sufficiently large that all line bundles of degree $d+n$ are non-special, that is, they have $h^0 = d+n+1-g$. We introduce vector bundles $\mathcal{E}$ and $\mathcal{F}$ on Pic^d_C whose fibers over the class of a line bundle

$\mathcal{L}$ are $\mathcal{E}_{[\mathcal{L}]} = H^0(\mathcal{L}(nP))$ and $\mathcal{F}_{[\mathcal{L}]} = H^0(\mathcal{L}(nP)/\mathcal{L})$. The vector bundle $\mathcal{E}$ can be realized as $\pi_{1*}\mathcal{P}_d(\mathrm{Pic}_C^d \times nP)$, where $\mathcal{P}_d$ is the Poincare line bundle on $\mathrm{Pic}_C^d \times C$, and π_1 is the projection to the first factor. The vector bundle $\mathcal{F}$ can be realized as $\pi_{1*}(\mathcal{P}_d(\mathrm{Pic}_C^d \times nP)/\mathcal{P}_d)$.

The locus W_d^r is the locus where the natural map $\mathcal{E} \to \mathcal{F}$ has kernel of rank at least $r+1$. We will use the Porteous formula ([12], Thm. 14.4) to compute the Chern class of this locus from the Chern classes of $\mathcal{E}$ and $\mathcal{F}$.

Since $\mathcal{F}_{[\mathcal{L}]}$ is always a skyscraper sheaf of rank n at P, its total Chern class is 1. We need to compute the Chern classes of $\mathcal{E}$.

Recall the cohomological structure of Pic_C^d and $\mathrm{Pic}_C^d \times C$.

Lemma 3.1-2 (Poincaré's Formula, [1] p. 320). *The algebraic cohomology classes of the Picard scheme Pic_C^d are generatedover $\mathbb{Q}$ by the theta divisor θ. The top class θ^g is Poincaré dual to a finite set of $g!$ points.*

Lemma 3.1-3 (Künneth Formula, [14], p.104). *The cohomology of the product $\mathrm{Pic}_C^d \times C$ is given by the Künneth decomposition:*

$$H^m(\mathrm{Pic}_C^d \times C) = \bigoplus_{p=0}^{2m} H^p(\mathrm{Pic}_C^d) \otimes H^{m-p}(C).$$

To calculate the Chern class of $\mathcal{E}$, we first calculate the Chern class and the Chern character of $\mathcal{P}_d(nP)$, and then use Grothendieck-Riemann-Roch.

Lemma 3.1-4. *The Chern class of $\mathcal{P}_d(nP)$ is $1 + (d+n)\zeta + \gamma$, where ζ is the pullback of the point class from C, and γ is the class of the intersection pairing on $H^1(C)$ and $H^1(\mathrm{Pic}_C^d)$.*

Proof: The sheaf $\mathcal{P}_d$ is a line bundle, so its zeroth Chern class is 1 and it has no higher Chern classes above the first. To compute $c_1(\mathcal{P}_d)$, use the Künneth decomposition: $c_1(\mathcal{P}_d(nP)) = c^{20} + c^{11} + c^{02}$. Since $\mathcal{P}_d(nP)$ is trivial on $\mathrm{Pic}_C^d \times \{P\}$, we have $c^{02} = 0$. Since $\mathcal{P}$ has degree $d+n$ on $\{[\mathcal{L}]\} \times C$, we have $c^{20} = (d+n)\zeta$, where ζ is the pullback of the point class on C. Finally, let $\delta_1, \cdots, \delta_{2g}$ and $\delta_1', \cdots, \delta_2 g'$ be the H^1 classes of C and $\mathrm{Pic}(C)$ respectively, such that $\sum_{i=1}^{g} \delta_i\delta_{g+i} = \zeta$ is the point class on C, and $\sum_{i=1}^{g} \delta_i' \delta_{g+i}' = \theta$ is the theta divisor on Pic_C^d. Then the diagonal class $c^{11}(\mathcal{P}_d)$ is the class of the intersection pairing on $H^1(C)$ and $H^1(\mathrm{Pic})$, namely $c^{11} = \gamma = \sum_i \delta_i \delta_{g+i}' - \delta_i' \delta_{g+i}$. Hence $c_1(\mathcal{P}_d) = (d+n)\zeta + \gamma$. □

Lemma 3.1-5. *The Chern class of $\mathcal{E}$ is $e^{-\theta}$.*

Proof: The Chern character of $\mathcal{P}_d(nP)$ is

$$ch(\mathcal{P}_d(nP)) = e^{c_1(\mathcal{P}_d(nP))} = \sum_{k \geq 0} \frac{((d+n)\zeta + \gamma)^k}{k!} = 1 + (d+n)\zeta + \gamma + \frac{1}{2}\gamma^2,$$

since all higher-order terms vanish. We need to compute γ^2 : it is

$$-2 \sum_{1 \leq i < j \leq g} \delta_i \delta_{g+i} \delta_j \delta_{g+j} = -2\theta\zeta.$$

To calculate $c(\mathcal{E})$, we apply Grothendieck-Riemann-Roch ([12], Thm. 15.2). The Todd class of the vertical tangent bundle is the pullback of the Todd class of the curve C, or $1 - \frac{1}{2}\omega_C$, or $1 + (1-g)\zeta$. Hence

$$ch(\mathcal{E}) = \pi_{1*}(\mathrm{Td}(T^v) ch(\mathcal{P}_d(nP))) = \pi_{1*}\left((1 + (1-g)\zeta)\left(1 + (d+n)\zeta + \gamma - \theta\zeta\right)\right).$$

The Gysin image π_{1*} takes the coefficient of ζ in the sum, which in our case is $(1-g) + (d+n) - \theta$, or $1 + d + n - g - \theta$. So

$$ch(\mathcal{E}) = 1 + d + n - g - \theta.$$

Hence $c_1(\mathcal{E}) = -\theta$, $c_2(\mathcal{E}) = \theta^2/2$, and in general

$$c_k(\mathcal{E}) = ((-1)^k \theta^k / k!).$$

□

We can now apply the Thom-Porteous Formula:

Theorem 3.1-6 (Thom-Porteous, [11], Thm. 14.4). *Let $\phi\colon \mathcal{E}^n \to \mathcal{F}^m$ be a map* *of vector bundles of ranks n and m respectively. The degeneracy locus on which the map ϕ has rank at most k is given by the $(n-k)\times(n-k)$ determinant*

$$|c_s(\mathcal{F}-\mathcal{E})|$$

where $s = m-k-i+j$. By convention, the negative Chern classes are taken to be zero in this expression.

Proof of Proof of Brill-Noether Existence: The class of our degeneracy locus W^r_d is

$$\det((c(\mathcal{E})^{-1})_{g-d+r-i+j})_{0\le i,j\le r},$$

assuming that $g-d+r>0$. (If $g-d+r\le 0$, then the expected codimension is zero; every line bundle has at least $r+1$ sections.) Since $c(\mathcal{E})=e^{-\theta}$, we obtain

$$\det(\theta^{g-d+r-i+j}/(g-d+r-i+j)!).$$

Clear denominators and factor out $\theta^{(r+1)(g+r-d)}$ to obtain

$$\frac{\theta^{(r+1)(g+r-d)}}{\prod_{i=0}^{r}(g-d+r-i+r)!}\begin{vmatrix}(g+r-d+1)\cdot(g+r-d+2)\cdots(g+r-d+r) & \dots & (g+r-d+r) & 1\\ \vdots & \ddots & \vdots & \vdots\\ (g-d)\cdot(g-d+1)\cdots(g-d+r) & \dots & (g-d+r) & 1\end{vmatrix}.$$

Each row of this matrix can be written in the form

$$\begin{vmatrix}a^r-\frac{r(r+1)}{2}a^{r-1}+\ldots\pm r! & a^{r-1}-\frac{(r-1)r}{2}+\ldots\pm(r-1)! & \dots & a-1 & 1\end{vmatrix}$$

where $a=g-d+(r-i)+(r+1)$. In general, the j^{th} column is a^{r-j} plus a linear combination of elements from the columns to the right. Hence by elementary column operations, the determinant reduces to the Vandermonde determinant

$$\frac{\theta^{(r+1)(g-d+r)}}{\prod_{i=0}^{r}(g-d+2r-i)!}\det((g-d+2r-i)^{r-j})_{0\le i\le r;\,0\le j\le r}.$$

We apply the Vandermonde determinant formula:

Lemma 3.1-7 (Vandermonde determinant, [13] 4.13). *Given constants* $a_0, \cdots, a_r$, *the determinant*

$$\det(a_i^{r-j})_{0\le i\le r; 0\le j\le r} = \frac{\prod_{i>j}(a_i - a_j)}{\prod_{0\le i\le r}(a_i + r)!}$$

So our determinant becomes

$$\theta^{(r+1)(g-d+r)} \prod_{i>j} \frac{(i-j)}{(g-d+2r-i)!}$$

Note that the product in the denominator $\prod_{i=0}^{r}(g-d+2r-i)!$ can be reordered as $\prod_{i=0}^{r}(g+r-d+i)$ by replacing i by $r-i$, and the numerator reduces to $\prod_{i=0}^{r} i!$. So the class $[W_d^r]$ finally becomes

$$\frac{\theta^{(r+1)(g-d+r)} \prod_{i=0}^{r} i!}{\prod_{i=0}^{r}(g-d+r+i)!}.$$

Since this number is always positive, the class $[W_d^r]$ is nonzero, so the locus W_d^r is non-empty. □

Note that these numbers agree with those calculated in section 2.2 for the g-cuspidal curve.

3.2 Existence with a Fixed Ramification Point

Theorem 3.2-1 (Brill-Noether Existence, Fixed Ramification Point). *Let* Q *be a fixed general point on a general curve* C *of genus* g, *and let* $m_0 < \cdots < m_r$ *be a vanishing sequence. Let* $\rho(g, r, d, (m_i))$ *be the* adjusted Brill-Noether number

$$\rho = g - \sum_{i=0}^{r}(m_i - i + g + r - d),$$

and let ρ_+ *be the* existence number

$$\rho_+ = g - \sum_{m_i - i + g + r - d \ge 0}(m_i - i + g + r - d).$$

If ρ_+ is nonnegative, then the locus $W^r_d(Q,(m_0,\cdots,m_r))$ of line bundles $\mathcal{L}$ with vanishing sequence $(m_0,\cdots,m_r)$ at the point Q is non-empty, and the locus $\mathcal{G}^r_d(Q,m_0,\cdots,m_r)$ of g^r_d's $(\mathcal{L},V^{r+1}\subset H^0(\mathcal{L}))$ with vanishing sequence $(m_0,\cdots,m_r)$ at Q has dimension at least ρ.

We first assume that for all m_i, the sum $m_i-i+g+r-d\geq 0$. Consider the maps of vector bundles $\mathcal{E}\to\mathcal{F}_i$, where $\mathcal{E}$ is as above, and $\mathcal{F}_i=\pi_*(\mathcal{P}_d(nP)/\mathcal{P}_d(-m_iQ))$. As in the proof of the ordinary Brill-Noether theorem, $\mathcal{E}$ has rank $d+n-g+1$ and $c(\mathcal{E})=e^{-\theta}$. Each $\mathcal{F}_i$ has rank $n+m_i$ and is filtered by trivial line bundles, so their Chern classes are trivial.

We are interested in the locus $W^r_d(m_i,Q)$ on Pic^d_C where the map from $\mathcal{E}$ to $\mathcal{F}_i$ has kernel of dimension $i+1$, and hence has rank $d+n-g-i$. The class of this locus is given by Fulton's generalization of Porteous' formula to filtered vector bundles, which we quote here in full generality for future reference:

Proposition 3.2-2 ([11], Thm. 10.1). *Suppose we are given partial flags of vector bundles*

$$A_1\subseteq\ldots\subseteq A_k$$

and

$$B_1\twoheadrightarrow\ldots\twoheadrightarrow B_k$$

on a scheme X, of ranks $a_1\leq\ldots\leq a_k$ and $b_1\geq\ldots\geq b_k$, and a morphism $h\colon A_k\to B_1$ of bundles. (Note that equalities are allowed in these bundles.)

Let $r_1,\ldots,r_k$ be nonnegative integers satisfying

$$0<a_1-r_1<\ldots<a_k-r_k,$$

$$b_1-r_1>\ldots>b_k-r_k>0,$$

Define $\Omega_r=\Omega_r(h)$ to be the subscheme defined by the conditions that the rank of the map from A_i to B_i is at most r_i for $1\leq i\leq k$. Let μ be the partition $(q_1^{n_1},\ldots,q_k^{n_k})$, where

$$q_i=b_i-r_i,$$

$$n_1 = a_1 - r_1,\ n_i = (a_i - r_i) - (a_{i-1} - r_{i-1})$$

for $2 \le i \le k$. *Let* $n = a_k - r_k$. *For* $1 \le i \le n$, *let*

$$\rho(i) = \min\{s \in [1,k] : i \le a_s - r_s = n_1 + \ldots + n_s\}.$$

If X *is purely* δ*-dimensional, then there is a class in* $A_{\delta-d(r)}(\Omega_r)$ *whose image in* $A_{\delta-d(r)}(X)$ *is* $P_r \cap [X]$, *where*

$$P_r = \det(c_{\mu_i - i + j}(B_{\rho(i)} - A_{\rho(i)}))_{1\le i,j\le n}.$$

We use this formula to prove the following:

Lemma 3.2-3. *The class* $[W^r_d(Q, m_0, \cdots, m_r)]$ *is given by*

$$\det\left(\frac{\theta^{m_i+g-d+j}}{(m_i+g-d+j)!}\right)_{0\le i,j\le r}.$$

Proof: We must transform the Fulton-Porteous formula into a form that applies to our problem.

First a trivial note: we can subtract 1 from all the indices, so they begin counting at 0 instead of 1. In the formulas for a_i, b_i, μ_i and $\rho(i)$ the number i appears only as a label. The only place the values of i and j are used is in c_{μ_i-i+j}). So subtracting 1 from i and j simultaneously changes nothing.

A more serious problem is that Fulton's formula is given for non-redundant conditions only, but does not require any particular number of conditions. We need to impose exactly $r+1$ conditions even if some of them are redundant. In order to apply the formula, therefore, we reduce to a set of non-redundant conditions and check that the formula is the same.

For convenience, we order the set of multiplicities in decreasing order:

$$m_0 > m_1 > \ldots > m_r.$$

In our case, the ranks of the two vector bundles are

$$a_i = \operatorname{rank} \mathcal{E} = d + n - g$$

for all i, and

$$b_i = \operatorname{rank} \mathcal{F}_i = n + m_i.$$

We want to impose the rank conditions $r_i = d + n - g - i$. Hence $a_i - r_i = i$, so the sequence of $a_i - r_i$ is strictly increasing. For all i, we have

$$n_i = i - (i - 1) = 1.$$

Suppose first that $m_i - m_{i+1} \geq 2$ for all i. Then since $b_i = n + m_i$ and

$$b_i - r_i = m_i + g - d + i,$$

the sequence $b_i - r_i$ is strictly decreasing, so Theorem 3.2-2 applies. Hence

$$\mu_i = m_i + i + g - d.$$

Otherwise, suppose that $m_k = m_{k+1} + 1$ for some k. There is redundancy in requiring all the multiplicity conditions. The condition that at most $k + 1$ basis elements vanish at the point Q to multiplicity at least m_{k+1} implies that at most k of them vanish to multiplicity at least m_k.

We can forget about $\mathcal{F}_k$ altogether and renumber the indices to omit it. Hence for $i > k$, we have $r_i = d + n - g - i - 1$, so $n_i = 1$ for all i except $i = k$, where $n_i = 2$. Hence q_k is to be repeated twice. Hence the sequence μ_i which counts the q_i with their multiplicities, is unchanged. We still have

$$\mu_i = m_i + i + g - d$$

for $0 \leq i \leq r$, whether this sequence is strictly decreasing or merely nonincreasing.

For all i, we have $c(B_i) = 1$ and $c(A_i) = e^{-\theta}$. So

$$c(B_{\rho(i)} - A_{\rho(i)}) = e^{\theta}.$$

Hence

$$c_{\mu_i - i + j}(B_{\rho(i)} - A_{\rho(i)}) = \frac{\theta^{m_i+g-d+j}}{(m_i + g - d + j)!}.$$

□

Now we can compute our determinant to complete the proof:

Proof of Brill-Noether Existence, Fixed Ramificatication Point, $g + r - d \geq 0$: We have

$$[W_d^r(Q, (m_0, \cdots, m_r))] = \det\left(\frac{\theta^{m_i+g-d+j}}{(m_i + g - d + j)!}\right) = \begin{vmatrix} \frac{\theta^{m_0+g-d}}{(m_0+g-d)!} & \frac{\theta^{m_0+g-d+1}}{(m_0+g-d+1)!} & \cdots & \frac{\theta^{m_0+g-d+r}}{(m_0+g-d+r)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\theta^{m_r+g-d}}{(m_r+g-d)!} & \frac{\theta^{m_r+g-d+1}}{(m_r+g-d+1)!} & \cdots & \frac{\theta^{m_r+g-d+r}}{(m_r+g-d+r)!} \end{vmatrix}.$$

The denominators in each row are increasing by 1. So when we factor out the powers of θ, we obtain a Vandermonde determinant of form

$$\theta^c \begin{vmatrix} \frac{1}{\alpha_0!} & \frac{1}{(\alpha_0+1)!} & \cdots & \frac{1}{(\alpha_0+r)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_r!} & \frac{1}{(\alpha_r+1)!} & \cdots & \frac{1}{(\alpha_r+r)!} \end{vmatrix} = \frac{\prod_{0 \leq i < j \leq r}(\alpha_i - \alpha_j)}{\prod_{i=0}^r (\alpha_i + r)!},$$

where$c = \sum_{i=0}^r (g + r - d + m_i - i)$ and $\alpha_i = m_i + g - d$.

In our case it evaluates to

$$\theta^c \frac{\prod_{0 \leq i < j \leq r}(m_i - m_j)}{\prod_{i=0}^r (m_i + g + r - d)!}.$$

Since the m_i are a strictly decreasing sequence, this determinant is always nonzero if the codimension $\sum_{i=0}^r (m_i - i + g + r - d)$ is less than g. □

Finally, we must consider the case when $g+r-d<0$, when every line bundle of degree d gives rise to g^r_d's.

Proof of Brill-Noether existence criterion, Fixed Ramification Point, $g+r-d<0$: Now suppose $g+r-d<0$. Then the condition for a g^r_d to exist is vacuous. Indeed, if $m_i-(r-i)+g+r-d<0$, then the condition for a g^r_d to have an $(i+1)$-dimensional family of sections that vanish to order m_i is vacuous. So it is sufficient to apply the Porteous formula to those conditions that are not vacuous. Hence the class $W^r_d(Q,m_0,\cdots,m_r)$ is

$$\theta^c \frac{\prod_{m_i+g+r-d>m_j+g+r-d\geq 0}(m_i-m_j)}{\prod_{m_i-(r-i)+g+r-d\geq 0}(m_i+g+r-d)!}$$

where $c=\sum_{m_i-(r-i)+g+r-d\geq 0}(m_i-(r-i)+g+r-d)$ This class is nonzero when

$$\sum_{m_i-i+g+r-d\geq 0}(m_i-i+g+r-d)\leq g,$$

or when ρ_+ is nonnegative.

But any class $[\mathcal{L}]\in \mathrm{Pic}^d_C$ gives rise to a whole family of g^r_d's when $g+r-d<0$. To calculate the actual dimension of the family of g^r_d's, we need to calculate the dimension of the class $\mathcal{G}^r_d(m_i)$ on the Grassmann bundle $\mathcal{G}^r_d=\mathbb{G}(r+1,\mathcal{E})$ of $(r+1)$-dimensional spaces of sections of $H^0(\mathcal{L}(nP))$.

Let π be the projection map from the Grassmann bundle $\mathcal{G}^r_d$ to Pic^d_C. The fibers of the universal subbundle $\mathcal{S}$ are our candidate g^r_d's. We still need to impose rank conditions such that the kernel of the map $\mathcal{S}\to\mathcal{F}_i$ should have rank $i+1$, so we set $r_i=r-i$. The rank b_i of $\mathcal{F}_i$ is still $n+m_i$, and the rank a_i of $\mathcal{S}$ is $r+1$. So when we apply the filtered Porteous formula again, we have

$$\mu_i=n+m_i+i-r.$$

We need to calculate the Chern classes of $\mathcal{S}$. Consider the exact sequence

$$0\to\mathcal{S}\to\pi^*\mathcal{E}\to\mathcal{Q}\to 0.$$

So

$$c(\mathcal{S}) \cdot c(\mathcal{Q}) = c(\pi^* \mathcal{E}).$$

Thus

$$c(\mathcal{S}) = c(\pi^* \mathcal{E}) \cdot c(\mathcal{Q})^{-1}.$$

The total Chern class $c(\mathcal{Q})$ of the universal quotient is $1 + \sigma_1 + \cdots + \sigma_k$, where k is the rank of the quotient $\mathcal{Q}$, in our case $n + d - r - g$. Hence

$$c(\mathcal{F}_i - \mathcal{S}) = e^\theta \cdot (1 + \cdots + \sigma_k).$$

Hence the class

$$[\mathcal{G}^r_d(Q, m_i)] = \det \left(c_{m_i-(r+1)+j} \left((1 + \cdots + \sigma_{d-r-g}) e^\theta \right) \right)_{0 \le i \le r;\, 0 \le j \le r}.$$

The codimension of this determinant is $\sum_{i=0}^{r} (m_i - (r - i))$. Since the dimension of $\mathbb{G}(r + 1, \mathcal{E})$ is $g + (r + 1)(n + d - r - g)$, the expected dimension is

$$g + (r + 1)(n + d - r - g) - \sum_{i=0}^{r'} (n + m_i - (r - i)) = \rho.$$

□

Remark 3.2-4. *Each row of the determinant contributes a factor of at least* $\theta^{(m_i-(r-i)+g+r-d)}$ *if this term is positive. So if* $\rho_+ < 0$*, then the power of* θ *in the determinant is greater than* g*. Hence the class* $[\mathcal{G}^r_d(m_i)]$ *is zero if* $\rho_+ < 0$*. Otherwise, it is a sum of nonnegative terms, which we can calculate explicitly using Pieri's Rule.*

3.3 Existence with a Movable Ramification Point

What happens if we allow the point Q to vary?

Once again, we first consider the case when $g + r - d > 0$.

Pull back the problem to $\mathrm{Pic}^d_C \times C \times C$, using a second copy of C to parametrize the moving point Q. Let Δ be the diagonal on $C \times C$. Pull back the Poincaré sheaf $\mathcal{P}_d$

to $\mathrm{Pic}_C^d \times C \times C$ by π_{12}^*. The fiber of the vector bundle $\pi_{12}^*(\mathcal{P}_d(nP))/(\pi_{12}^*\mathcal{P}_d)(-m_i\Delta)$ over a point Q of the second copy of C is just $\mathcal{P}_d(nP)/\mathcal{P}_d(-m_iQ)$. So we consider the map of vector bundles on $\mathrm{Pic} \times C$ given by

$$\mathcal{E} \to \mathcal{F}_i,$$

where

$$\mathcal{F}_i = \pi_{12*}(\pi_{12}^*(\mathcal{P}_d(nP))/(\pi_{12}^*\mathcal{P}_d)(-m_i\Delta)).$$

From now on we shall suppress the π_{12} for ease of notation.

As before, we have $c(\mathcal{E}) = e^{-\theta}$. But the targets $\mathcal{P}_d(nP)/\mathcal{P}_d(-m_i\Delta)$ are no longer trivial.

Lemma 3.3-1. *The total Chern class of $\mathcal{F}_i$ is*

$$1 + (d + (g-1)(m_i - 1))\zeta + m_i\gamma - m_i(m_i - 1)\zeta\theta.$$

Proof: We can filter $\mathcal{P}(nP)/\mathcal{P}(-m_i\Delta)$ with successive quotients of the form

$$\mathcal{P}(kP)/\mathcal{P}((k-1)P)$$

and of form

$$\mathcal{P}(-k\Delta)/\mathcal{P}(-(k+1)\Delta).$$

The former terms are trivial. The latter can be written as $\mathcal{P} \otimes \omega_C^{\otimes k}$. We know that $c(\mathcal{P}) = 1 + d\zeta + \gamma$ on $\mathrm{Pic}_C^d \times C$. Since the diagonal Δ is another degree 1 copy of C in $\mathrm{Pic}_C^d \times C \times C$, pulling back $\mathcal{P}_d$ to $\mathrm{Pic}_C^d C \times C$ and restricting to $\mathrm{Pic}_C^d \times \Delta$ gives the same Chern class $1 + d\zeta + \gamma$.

Since $c(\omega_C) = 1 + (2g-2)\zeta$, where ζ is the pullback of the point class from C, we get $c(\mathcal{P}_d \otimes \omega_C^{\otimes k}) = 1 + (d + 2k(g-1))\zeta + \gamma$. Hence the class $c(\mathcal{P}_d(nP)/\mathcal{P}_d(-m_i\Delta))$ is the product

$$\prod_{k=0}^{m_i-1}(1+(d+2k(g-1))\zeta+\gamma) = 1+m_i\left(d+(g-1)(m_i-1)\right)\zeta+m_i\gamma+\frac{(m_i-1)m_i}{2}\gamma^2.$$

All higher terms vanish because $\zeta^2 = \zeta\gamma = 0$.

Since $\gamma^2 = -2\zeta\theta$, we can rewrite this class as

$$1+m_i(d+(m_i-1)(g-1))\zeta+m_i\gamma-m_i(m_i-1)\zeta\theta.$$

□

We can now apply Fulton's filtered Porteous formula (3.2-2) on $\mathrm{Pic}^d_C \times C$, and then take the Gysin image on Pic^d_C. Again, $r_i = d+n-g-i$, and $a_i - r_i = i$, so the sequence of $a_i - r_i$ is strictly increasing and $n_i = 1$ for all i, so $\rho(i) = i$. We still have

$$b_i = n + m_i$$

and

$$b_i - r_i = m_i + g - d + i,$$

which is always positive and non-increasing. We obtain

$$W^r_d(m_i) = \det(c_{m_i+g-d+j}(B_{\rho(i)} - A_{\rho(i)}))_{0\le i\le r;\, 0\le j\le r}.$$

There is a slight complication in that the B_i are no longer equal to each other. In general,

$$c(B_i) = c(\mathcal{F}_i) = 1+m_i\left(d+(g-1)(m_i-1)\right)\zeta+m_i\gamma+\frac{(m_i-1)m_i}{2}\gamma^2.$$

However, if the multiplicity $m_k = m_{k+1}+1$ is a redundant condition, then we must renumber the $\mathcal{F}_i$ to omit it. The result is that $B_{\rho(k)}$ is really $\mathcal{F}_{k+1}$, not $\mathcal{F}_k$.

Set m_i' to be the greatest value $m_j \le m_i$ such that $m_{j+1} < m_j - 1$. Then we obtain the following:

Lemma 3.3-2. *The total Chern classes* $c(B_{\rho(i)} - A_{\rho(i)})$ *are given by*

$$c(B_{\rho(i)} - A_{\rho(i)}) = e^{\theta} \cdot \big(1 + m_i'(d + (m_i' - 1)(g-1))\zeta + m_i'\gamma - m_i'(m_i' - 1)\zeta\theta\big)$$

Thus the determinant becomes $\det(a_{ij})$, where

$$a_{ij} = \frac{\theta^{m_i+g-d+j}}{(m_i+g-d+j)!} + \zeta\theta^{m_i+g-d-1+j}\frac{m_i'(d+(m_i'-1)(g-1))}{(m_i+g-d-1+j)!}$$

$$-\zeta\theta^{m_i+g-d-1+j}\frac{(m_i'-1)m_i'}{(m_i+g-d-2+j)!} + \gamma\theta^{m_i+g-d-1+j}\frac{m_i'}{(m_i+g-d-1+j)!}$$

We can break up this matrix as a sum. Set

$$M_{ij} = \frac{\theta^{m_i+g-d+j}}{(m_i+g-d+j)!}.$$

This is the classical term that exists without the movable ramification point. All but one or two components of the product will be of this form. Set

$$N_{ij} = \zeta\theta^{m_i+g-d-2+j}\frac{(m_i')(d+(m_i'-1)(g-1))}{(m_i+g-d-2+j)!}.$$

This term comes from the ζ part of the canonical sheaf ω_C. It is always positive. Since it contains ζ, it is killed by multiplication with any other term containing ζ or γ. Set

$$L_{ij} = \zeta\theta^{m_i+g-d-1+j}\frac{(m_i'-1)m_i'}{(m_i+g-d-2+j)!}.$$

This term comes from the γ^2 in $c(\mathcal{F}_i)$, so it is subtracted. It contains ζ, so it is killed by any other term containing ζ or γ. Finally, set

$$G_{ij} = \gamma\theta^{m_i+g-d-1+j}\frac{m_i'}{(m_i+g-d-1+j)!}.$$

This term contains γ instead of ζ, it is killed by multiplication by anything containing ζ or θ^{g-1}. In order to contribute to the sum, it must be multiplied against another copy of itself. It will then contribute a negative number.

We want to evaluate $\det(M_{ij} + N_{ij} - L_{ij} + G_{ij})$. We need an elementary lemma on expanding determinants.

Lemma 3.3-3. *If $C = A + B$, then*

$$\det(C) = \sum_{S \subset \{1,2,\cdots,r+1\}} \det(D_{ij}(S)),$$

where $D_{ij}(S) = A_{ij}$ if $i \in S$, otherwise B_{ij}.

Proof: It follows immediately from expanding out the definition of the determinant,

$$\det(C) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{r+1} (A_{i\sigma(i)} + B_{i\sigma(i)}).$$

□

In our case, when we expand our determinant $\det(M_{ij} + N_{ij} - L_{ij} + G_{ij})$ all but three types of terms vanish and we are left with $X + Y + Z$, where the first term is

$$X = \sum_{k=1}^{r+1} \det(X_{ij}(k)),$$

where

$$X_{ij}(k) = \left\{ \begin{array}{ccc} M_{ij} & \text{if} & i \neq k \\ N_{kj} & \text{if} & i = k \end{array} \right\},$$

the second term is

$$Y = -\sum_{k=1}^{r+1} \det(Y_{ij}(k)),$$

where

$$Y_{ij}(k) \left\{ \begin{array}{ccc} M_{ij} & \text{if} & i \neq k \\ L_{kj} & \text{if} & i = k \end{array} \right\},$$

and the third term is

$$Z = \sum_{1 \leq k \leq l \leq r+1} \det(Z_{ij}(k,l)),$$

where

$$Z_{ij}(k,l) = \left\{ \begin{array}{cccccc} M_{ij} & \text{if} & i \neq k & \text{and} & i \neq l \\ G_{ij} & \text{if} & i = k & \text{or} & i = l \end{array} \right\}.$$

All the other possible combinations of M, N, L and G vanish because they contain ζ^2, $\zeta\gamma$, or else they fail to contain ζ, so their Gysin images vanish on Pic^d_C.

We expand each determinant separately. By pulling out the ζ and θ powers, we can write the first term as

$$X = \zeta\theta^c \sum_{k=0}^{r} \det(X'_{ij}(k)),$$

where $c = (r+1)(g-d+r) + \sum_{i=0}^{r}(m_i - i) - 1$ and

$$X'_{ij}(k) = \left\{ \begin{array}{ccc} \frac{1}{\alpha_{ij}!} & \text{if} & i \neq k \\ \frac{m'_k(d+(m'_k-1)(g-1))}{(\alpha_{ij}-1)!} & \text{if} & i = k \end{array} \right\},$$

where $\alpha_{ij} = (m_i + g - d + j)$. Expanding the Vandermonde determinant as before, we get

$$\zeta\theta^{(r+1)(g-d+r)+\sum_{i=0}^{r}(m_i-i)-1} \sum_{k=0}^{r} \Bigg[(m'_k)(d + (m'_k - 1)(g-1))(m_k + g + r - d)$$

$$\frac{\prod_{i>j,\, i\neq k,\, j\neq k}(m_i - m_j) \prod_{i\neq k} |m_i - m_k + 1|}{\prod_{i=0}^{r}(m_i + g + r - d)!} \Bigg].$$

Every summand in this sum is nonnegative, so the sum as a whole is nonnegative. The k^{th} term in this sum is zero if and only if $m_{k-1} = m_k + 1$. So the entire term is zero for a completely trivial ramification sequence $0, \cdots, g-1$. This makes sense, since it is not possible for the expected class of completely unramified points on the curve to be finite.

Likewise, pulling out the ζ and θ powers, we can write

$$Y = -\zeta\theta^c \sum_{k=0}^{r} \det(Y'_{ij}(k)),$$

where

$$Y'_{ij}(k) = \left\{ \begin{array}{ccc} \frac{1}{\alpha_{ij}!} & \text{if} & i \neq k \\ \frac{(m_k-1)m_k}{(\alpha_{ij}-2)!} & \text{if} & i = k \end{array} \right\}.$$

Expanding the Vandermonde determinant, we obtain

$$Y = -\zeta\theta^c \sum_{k=0}^{r} \Bigg[(m'_k - 1)m'_k(m_k + g - d)(m_k + g + r - d - 1)$$

$$\frac{\prod_{i>j,\, i\neq k,\, j\neq k}(m_i - m_j)\prod_{i<k}(m_i - m_k + 2)\prod_{i>k}(m_k - m_i - 2)}{\prod_{i=0}^{r}(m_i + g + r - d)!}\Bigg]$$

This term vanishes when $m_{k-1} = m_k + 2$ for some k. In particular, it does not contribute to the class of ordinary Weierstrass points.

Finally, we pull out the γ^2 and θ powers from Z to obtain

$$Z = -2\zeta\theta^c \sum_{0\leq k\leq l\leq r} \det(Z'_{ij}(k,l)),$$

where

$$Z'_{ij}(k) = \left\{ \begin{array}{ccccc} \frac{1}{\alpha_{ij}!} & \text{if } i \neq k & \text{and} & i \neq l & \\ \frac{m'_i}{(\alpha_{ij}-1)!} & \text{if} & i = k & \text{or} & i = l \end{array} \right\}.$$

Expanding the Vandermonde determinant, we obtain

$$Z = -2\zeta\theta^{(r+1)(g-d+r)+\sum_{i=0}^{r}(m_i-i)-1} \sum_{0\leq k<l\leq r} \Bigg[(m'_k)(m'_l)(m_k + g + r - d)(m_l + g + r - d)$$

$$\frac{|m_k - m_l|\prod_{i>j,\, i\neq k,\, i\neq l,\, j\neq k,\, j\neq l}(m_i - m_j)\prod_{i\neq k}|m_i - m_k + 1||m_i - m_l + 1|}{\prod_{i=1}^{r+1}(m_i + g + r - d)!}\Bigg]$$

Summing the three terms, and taking the Gysin image on Pic^d_C, we obtain the following

Theorem 3.3-4. *Let X be a general curve of genus g and let r, d, m_i be numbers such that $g + r - d \geq 0$ and*

$$\rho(g, r, d, m_i) \geq 0.$$

Then the class $[W^r_d(m_0, \cdots, m_r)]$ of the family of g^r_d's admitting a point Q with van-

ishing sequence m_i is given by

$$W^r_d(m_i) = \theta^c \sum_{k=0}^{r} \frac{(m'_k)(m_k+g+r-d)}{\prod_{i=0}^{r}(m_i+g+r-d)!}$$

$$\Bigg[(d+(m'_k-1)(g-1)) \prod_{i>j,\, i\neq k,\, j\neq k} (m_i-m_j) \prod_{i\neq k} |m_i-m_k+1|$$

$$-(m'_k-1)(m_k+g+r-d-1) \prod_{i>j,\, i\neq k,\, j\neq k} (m_i-m_j) \prod_{i<k}(m_i-m_k+2) \prod_{i>k}(m_k-m_i-2)$$

$$-\sum_{l\neq k}(m'_l)(m_l+g+r-d)|m_k-m_l| \prod_{i>j,\, i\neq k,\, i\neq l,\, j\neq k,\, j\neq l} (m_i-m_j) \prod_{i\neq k} |m_i-m_k+1||m_i-m_l+1|\Bigg],$$

where

$$c = \sum_{i=0}^{r}(m_i - i + g + d - r).$$

It is not immediately clear whether or not this sum is always positive, but we can calculate it in important examples.

Example 3.3-5. *A canonical curve has exactly*

$$(g-1)(g)(g+1)$$

ramification points.

Proof: A canonical curve has $d = 2g-2$, $r = g-1$, and the ramification must be at least $(g, g-2, g-3, \ldots, 1, 0)$. For any $k \neq 0, k \neq g$, there exists $i = k+1$ such that $m_i - m_k + 1 = 0$. For any $k \neq g$, there exists $i = k+2$ such that $m_i - m_k + 2 = 0$. When $k = g$, the coefficient m_g is zero. So it is sufficient to consider the first term

$$\theta^s \sum_{k=0}^{r} \frac{(m'_k)(m_k+g+r-d)}{\prod_{i=0}^{r}(m_i+g+r-d)!}(d+(m'_k-1)(g-1)) \prod_{i>j,\, i\neq k,\, j\neq k} (m_i-m_j) \prod_{i\neq k} |m_i-m_k+1|$$

for $k = 0$; $m'_k = g$. Note that the codimension is $s = g$.

We have

$$\theta^g \frac{(g)(g+g+(g-1)-(2g-2))}{\prod_{i=0}^{g-1}(m_i+g+(g-1)-(2g-2))!}((2g-2)+(g-1)(g-1)) \prod_{i>j\neq g}(m_i-m_j) \prod_{i\neq k}|g-m_i-1|$$

$$= \zeta\theta^g \frac{(g)(g+1)}{\prod_{i=0}^{g-2}(i+1)!(g+1)!}(g+1)(g-1) \prod_{g-1>i>j\geq 0}(i-j) \prod_{g-1>i>0}|g-i-1|$$

$$= \zeta\theta^g \frac{g(g+1)}{\prod_{i=0}^{g-1} i!(g+1)!}(g+1)(g-1)\prod_{i=0}^{g-1} i!(g-2)!]$$

Since $\theta^g = g!$, we have $\theta^g(g+1) = (g+1)!$. This cancels the $(g+1)!$ in the denominator. The products $\prod_{i=1}^{g-1} i!$ in the numerator and the denominator cancel with each other, leaving $g(g+1)(g-1)$. □

Example 3.3-6. *For any g, r, d such that $g + r - d \geq 0$, and*

$$g - (r+1)(g+r-d) \geq 0,$$

the expected class of g^r_d's possessing a point with the simplest possible ramification $(r+1, r-1, \cdots, 0)$ is positive.

Proof: The class is positive because the negative terms all contain factors of

$$m_k' \prod |m_i - m_k + 2|$$

and

$$m_k' \prod |m_i - m_k + 1||m_i - m_l + 1|,$$

so they vanish.

If the dimension of W^r_d on Pic^d_C is $\rho(g, r, d)$, the dimension of pairs $(\mathcal{L}, Q)$ on $\mathrm{Pic}^d_C \times C$ with $\mathcal{L} \in W^r_d$ is $\rho+1$. The ramification imposes one additional condition, so the expected dimension of pairs $(\mathcal{L}, Q)$ with simple ramification at Q is ρ. Since the coefficient of the class is positive, the locus is non-empty. □

Example 3.3-7. *If C is a general curve, the class of g^r_d's possessing a simple* *n-fold cusp, with multiplicity sequence*

$$(n+r-1, n+r-2, \cdots, n, 0),$$

is positive if the expected dimension is nonnegative.

Proof: For any $k \neq r, k \neq r+1$, there exists $i = k+1$ such that $m_i - m_k + 1 = 0$. For any $k \neq r-1, k \neq r, k \neq r+1$, there exists $i = k+2$ such that $m_i - m_k + 2 = 0$. When $k = r+1$, $m'_k = 0$, so these terms vanish automatically. When $k \neq r+1$, then $m'_k = n$. So there are two negative terms and one positive term,and the positive term dominates. □

This is D. Schubert's theorem [22] on the existence of n-fold cusps.

Example 3.3-8. *For $r = 1$ and $m_1 = 0$ (base point-free maps to $\mathbb{P}^1$), the dimension $\rho(g, r, d, m_0, 0)$ is nonnegative if and only if $d \geq \frac{g+m_0}{2}$, and the class $W^r_d(m_0, 0)$ is positive whenever $g > \rho(g, r, d) \neq 0$.*

Proof: The expected dimension is $g-2(g+1-d)-(m_0-1)+1 = -g+2d-m_0 \geq 0$ if and only if $2d \geq g+m_0$.

We have

$$W^1_d(m_0, 0) = \theta^{2(g-d+1)+m_0-2}\frac{(m_0)(m_0+g+1-d)}{(m_0+g+1-d)!(g+1-d)!}\Big[(d+(m_0-1)(g-1))|m_0-1|-$$

$$(m_0-1)(m_0+g-d)|m_0-2|-0\Big] = \theta^{2(g-d+1)+m_0-1}\frac{(m_0)(m_0+g+1-d)}{(m_0+g+1-d)!(g+1-d)!}$$

$$\Big[(d+(m_0-1)(g-1))(m_0-1)-(m_0-1)(m_0+g-d)(m_0-2)\Big]$$

$$= \theta^{2(g-d+1)+m_0-1}\frac{(m_0)(m_0+g+1-d)}{(m_0+g+1-d)!(g+1-d)!}(m_0-1)[(d+g-1)(m_0-1)-(m_0+g)(m_0-2)]$$

$$= \theta^{2(g-d+1)+m_0-1}\frac{(m_0)(m_0+g+1-d)}{(m_0+g+1-d)!(g+1-d)!}(m_0-1)(dm_0-m_0^2+g+1-d+m_0)$$

Since $d \geq m_0$, $dm_0 - m_0^2 \geq 0$. Since $2(g+1-d)$ must be between 0 and g we have

$g+1-d \geq 0$. Hence the total class is positive. $\square$

Example 3.3-9. *For $r=2, g=4, d=6, m_i=(0,3,5)$, we get $W_d^r=24$. We shall return to this example in Chapter 4.*

Example 3.3-10. *For $r=2, g=2k+1, d=2k+3$, and $m_i=(0,k+2,k+3)$ we have $\rho=0$, and $W_d^r=0$. This is not a surprise, since if we project away from a point other than Q, we obtain a g^1_{2k+2} with vanishing sequence $(0,k+2)$ at Q, which is not allowed because of Example 3.3-8.*

Example 3.3-11. *For $r=2$, when $g+r-d \geq 0$ and $\rho=0$, the class $W_d^r(t,s,0)$ is zero precisely in the case above, and positive in all other cases.*

Proof: If $\rho=0$, se can set

$$g=\frac{1}{2}(3d-s-t-2).$$

Assuming that $t \neq s+1$, the value of $W_d^2(t,s,0)$ is the postive factor

$$\frac{g!}{(t+g+2-d)!(s+g+2-d)(g+2-d)}$$

times

$$t(t+g+2-d)[(d+(g-1)(t-1))(t-1-s)-(t-1)(t+g+2-d-1)(t-2-s)-s(s+g+2-d)(t-s)]$$

$$+s(s+g+2-d)[(d+(g-1)(s-1))(t-s+1)-(s-1)(s+g+2-d-1)(t-s+2)-t(t+g+2-d)(t-s)].$$

It is this function which we shall attempt to minimize.

To show that this function is nonnegative, extend it to a function of real variables. We shall show that this function is strictly increasing in d for fixed s and t, and strictly increasing in $(t-s)$ for fixed g and d. Write $u=t-s$.

We compute the partial derivative with respect to d, obtaining

$$\frac{1}{2}\left((2su-2s^3u-3s^2u^2-su^3-3u+u^2+s^2u+su^2+u^3)+du(2s^2-2s+2su+2u^2-1-u)\right).$$

If $t > s > 1$, then we can show that this derivative is always positive. If $s > 1$, then we have

$$2su \geq 2s, \quad 3s^2u \geq s^2 + u + su, \quad asu^2 \geq 2u^2,$$

so the constant term is positive, and

$$2s^2 \geq 2s, 2su \geq 1, 2u^2 > u,$$

so the term containing d is positive.

If $s = 1$, then the constant term is bounded below by $-u^2$. But since $d \geq t - s$, the term containing du dominates.

We see that for fixed s and t, the value of $W_d^2(t, s, 0)$ is strictly increasing in d. So we need only consider the minimum value. When $g + r - d \geq 0$, we have $d \geq t + s - 2$, unless $s = 1$. If $s = 1$, we have the case of $(0, k+2, k+3)$. Otherwise, we may assume $d = t + s - 2$.

In this case,

$$t(t+g+2-d)\big[\,(d+(t-1)(g-1))\,(t-1-s)-(t-1)(t+g+2-d-1)(t-2-s)$$

$$-s(s+g+2-d)(t-s)\big] + s(s+g+2-d)\big[\,(d+(s-1)(g-1))\,(t+1-s)$$

$$-(s-1)(s+g+2-d-1)(t+2-s) - t(t+g+2-d)(t-s)\big]$$

$$= t^2\left[(t^2+ts-5t+3)(t-1-s)-(t-1)^2(t-2-s)-s^2(t-s)\right]$$

$$+s^2\left[(s^2+ts-5s+3)(t+1-s)-(s-1)^2(t+2-s)-t^2(t-s)\right]$$

$$= t^4s - 3t^3s^2 + 3t^2s^3 - s^4t - 2t^4 - 2t^4 + 2t^3s - 2ts^3 + 2s^4 + 3t^3 - 2t^2s + 2s^2t - 3s^3 - t^2 + s^2$$

$$= ts(t-s)^3 - 2(t^3-s^3)(t-s) + (t-s)(3t^2+ts+3s^2) - (t-s)(t+s)$$

$$= (t-s)[ts(t-s)^2 - 2(t-s)(t^2+ts+s^2) + 3t^2 + ts + 3s^2 - (t+s)] > 0.$$

□

Example 3.3-12. *Values of W_d^2 for $r = 2$, $\rho = 0$ and small values of g and d are included in Table 1.*

Finally, in the case when $g + r - d < 0$, we can do the same thing as in the case of a fixed point: work on $\mathbb{G}(r+1, \mathcal{E})$. Again, it is the universal subbundle $\mathcal{S}$ that parametrizes g_d^r candidates. Consider the exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0.$$

So

$$c(\mathcal{S}) \cdot c(\mathcal{Q}) = c(\pi^*\mathcal{E}).$$

So

$$c(\mathcal{S}) = c(\pi^*\mathcal{E}) \cdot c(\mathcal{Q})^{-1}.$$

The Chern class $c(\mathcal{Q})$ of the universal quotient is $1 + \cdots + \sigma_k$, where k is the rank of the quotient $\mathcal{Q}$. Hence

$$c(\mathcal{F}_i - \mathcal{S}) = e^\theta \cdot (1 + m_i'(d + (m_i' - 1)(g-1))\zeta + m_i'\gamma - m_i'(m_i' - 1)\zeta\theta) \cdot (1 + \cdots + \sigma_k).$$

If $\rho_+ \leq 0$, then every term in this determinant will contain higher powers of θ than θ^g, so it will have to vanish. If $\rho_+ \geq 0$, then in any given case it is possible to compute $\mathcal{G}_d^r(m_0, \cdots, m_r)$ explicitly, by Schubert calculus.

Table 3.1: Some Small Values of $W_d^2(0,s,t)$

d=g+2

g	d	s	t	$W_d^2(0,s,t)$	g	d	s	t	$W_d^2(0,s,t)$
t=s+1					t=s+2				
1	3	2	3	0	2	4	2	4	6
3	5	3	4	0	4	6	3	5	24
5	7	4	5	0	6	8	4	6	90
7	9	5	6	0	8	10	5	7	336
t=s+3					t=s+4				
3	5	2	5	24	4	6	2	6	60
5	7	3	6	120	6	8	3	7	360
7	9	4	7	504	8	10	4	8	1680
t=s+5					t=s+6				
5	7	2	7	120	6	8	2	8	210
7	9	3	8	840	8	10	3	9	1680

d=g+1

g	d	s	t	$W_d^2(0,s,t)$	g	d	s	t	$W_d^2(0,s,t)$
t=s+1					t=s+2				
4	5	2	3	24	3	4	1	3	24
6	7	3	4	240	5	6	2	4	240
8	9	4	5	1680	7	8	2	5	1680
t=s+3					t=s+4				
4	5	1	4	120	5	6	1	5	360
6	7	2	5	1080	7	8	2	6	3360
8	8	3	6	7056	9	10	3	7	22176
t=s+5					t=s+6				
6	7	1	6	840	7	8	1	7	1680
8	9	2	7	8400	9	10	2	8	18144
t=s+7					t=s+8				
8	9	1	8	3024	9	10	1	9	5040

Chapter 4

Finiteness and Non-Existence Results

Our goal in this section is to prove in as many cases as possible that the expected dimension of $W^r_d(m_i)$ is equal to the actual dimension. In particular, we will show that when ρ is sufficiently low, W^r_d and $\mathcal{G}^r_d$ are finite or even empty. As in Section 2, our method is to degenerate the curve, but this time to a semi-stable form. Instead of a cuspidal curve, we consider a reducible curve consisting of a tree of rational curves with g elliptic tails.

4.1 Limit Linear Series

Let $\pi\colon X \to T$ be a flat, proper map from a smooth variety X to the spectrum T of a discrete valuation ring $\mathcal{O}$ with parameter t, residue field $k(0)$, and function field $K(\eta)$. Suppose that the geometric generic fiber $X_{\overline{\eta}}$ is a smooth irreducible curve, whereas the special fiber X_0 is a reduced but reducible curve of compact type. Let $(\mathcal{L}, V)$ be a g^r_d on $X_{\overline{\eta}}$. We would like to define its limit on X_0. The solution is provided by Eisenbud and Harris' theory of limit linear series. We shall follow their method as presented in [5] and [3].

After a finite base change, we may assume that the sheaf $\mathcal{L}$ is defined on X_η. After blowing up if necessary, we may assume from now on that the ramification points of

$\mathcal{L}$ specialize to smooth points of X_0.

Since the total space X is smooth, $\mathcal{L}$ extends to a sheaf on X. That extension, however, is not unique: we can vary it by twisting by a divisor supported on X_0. If $\widetilde{\mathcal{L}}$ is an extension of $\mathcal{L}$ and D is any divisor of X supported on X_0, then $\widetilde{\mathcal{L}} \otimes \mathcal{O}_X(D)$ is another. Fortunately this is the only ambiguity: if $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$ are any two extensions of $\mathcal{L}$, then $\widetilde{\mathcal{L}} \otimes \widetilde{\mathcal{L}}'^{-1}$ is trivial away from X_0, so it must be the line bundle associated to some divisor D supported on X_0.

We would like to define "the limit" precisely. But there is no natural, canonical representative. The solution, in a sense, is to use them all.

The total degree of any extension $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ is d. So the sum of the degrees $\widetilde{\mathcal{L}}_Y$ over all components Y of X_0 is d. Since X_0 is of compact type and the intersection pairing on the components of X_0 is unimodular, there exists an extension $\mathcal{L}_Y$ of $\mathcal{L}$ whose degree is d on Y and 0 on all other components.

The pushforward $\pi_*\mathcal{L}_Y$ is a free $\mathcal{O}$-module, which restricts to an $\mathcal{O}$-lattice in the vector space $\pi_*\mathcal{L}_\eta$, defined up to multiplication by a power of the parameter t. Then $V \cap \pi_*\mathcal{L}_Y$ determines a free $\mathcal{O}$-module of rank $(r+1)$, up to a power of t. We set

$$V_Y := (V \cap \pi_*\mathcal{L}_Y) \otimes k(0);$$

it is an $(r+1)$-dimensional subspace of $H^0(\mathcal{L}_Y|_{X_0})$.

Since $\deg(\mathcal{L}_Y) = 0$ on all components of X_0 except Y, the sections of $\mathcal{L}_Y$ are determined by their restrictions to Y. So we can consider V_Y as a subspace of $H^0(\mathcal{L}_Y|_Y)$. We thus obtain a g^r_d $(\mathcal{L}_Y, V_Y)$ on Y.

Definition 4.1-1. *The g^r_d $(\mathcal{L}_Y, V_Y)$ will be called the Y-*aspect *of $\mathcal{L}$.*

Any one aspect determines all the others:

Proposition 4.1-2 ([5], p.349). *Let Y and Z be components of X_0 meeting at P. Let Y' and Z' be the connected components of $X_0 - P$ containing Y and Z respectively. Then $\mathcal{L}_Y(-dZ') = \mathcal{L}_Z$.*

Proof: Check intersection numbers: since X_0 is of compact type, $\mathcal{L}_Z$ and $\mathcal{L}_Y(-dZ')$ both have degree d on Z and zero everywhere else. □

We can now consider the sections and vanishing sequences of the aspects.

Definition 4.1-3. *Let $\sigma \in V$ be a section. Then the Y-aspect σ_Y of σ is the image of $t^n\sigma$ in V_Y, where n is the least power of t such that $t^n\sigma \in \pi_*\mathcal{L}_Y$.*

Proposition 4.1-4 (Adapted Bases, [3], Lemma 1.2; [5], Lemma 2.3). *Let Y and Z be two components that meet at P; let P' be any point on Y. Then there exists an order basis $(\sigma_0, \cdots, \sigma_r)$ of V_Y at P' such that for suitable integers n_i, the elements $t^{n_i}\sigma_i$ also form a basis of V_Z.*

Proof: By Gaussian elimination, reduce the matrix representing the inclusion $V_Z(-dZ') = V_Y \hookrightarrow V_Z$ to row-echelon form. This gives us a basis σ_i such that $t^{n_i}\sigma_i$ also form a basis for V_Z. This property is preserved when we replace σ_i with $\sigma_i + a\sigma_j$ to construct an order basis at P'. □

Proposition 4.1-5 (Compatibility Condition, [5], Prop. 2.5). *For any two components Y and Z of X meeting at a point P, and for any i, such that $0 \le i \le r$, we have*

$$m_i(V_Y, P) + m_{r-i}(V_Z, P) = d.$$

To prove this condition, we shall require some additional inequalities:

Lemma 4.1-6 ([5], Prop. 2.2). *Let P be the intersection of two components Y and Z of X_0. Then for any section $\sigma \in V$,*

$$v_P(\sigma_Y) + v_P(\sigma_Z) \ge d.$$

Proof: Let D be the closure in X of the divisor (σ) in $X - X_0$. Suppose that σ_Y vanishes along Z to order a and σ_Z vanishes along Y to order b. Then

$$v_P(\sigma_Y) = (Y \cdot (t^n\sigma))_P = a + (D \cdot Y)_P \ge a.$$

Likewise for Z,

$$v_P(\sigma_Z) = b + (D \cdot Z)_P \ge b.$$

Since σ_Y vanishes on Z, and $\mathcal{L}_Y$ has degree 0 on all components of Z' except Z, we have σ_Y must vanish on Z'. Hence $\sigma_Y \in \mathcal{L}_Y(-Z')$. By induction on a, we have $\sigma_Y \in \mathcal{L}_Y(-aZ')$. So $\sigma_Z = t^{d-a}\sigma_Y$. as sections of $\mathcal{L}_Z = \mathcal{L}_Y(-dZ')$.

Hence $t^{d-b}\sigma_Z = t^{(d-a)+(d-b)}\sigma_Y$ is a section of $\mathcal{L}_Z(-dY')$, which $\mathcal{L}_Y(-dX_0)$, and it does not vanish on Y. Since $t^d\sigma_Y$ is another one, and $\mathcal{L}_Y(-dX_0)$ is of degree zero, we must have $(d-a)+(d-b)=d$, so $a+b=d$.

So

$$v_P(\sigma_Y) + v_P(\sigma_Z) \geq d.$$

□

Proof of Compatibility Condition: The above lemma proves that

$$m_i(V_Y, P) + m_{r-i}(V_Z, P) \geq d.$$

We need to prove the other direction.

If a ramification point P_η of weight w on $\mathcal{L}_\eta$ specializes to a smooth point P', then we have $w(P_\eta) \geq w$. Hence by the Plücker formula, the sum over smooth points P'

$$\sum_{P'} w(P) \geq \sum w(P_\eta) = (r+1)d + \binom{r+1}{2}(2g-2).$$

We complete the proof by showing that

$$\sum_{P'} w(P) \leq (r+1)d + \binom{r+1}{2}(2g-2),$$

with equality if and only if the compatibility condition holds.

The proof is by induction on the number of components of X_0. If X_0 is smooth, there is nothing to prove. Let Y_1 be a smooth component of genus g', that meets only one other component Y_2 in a point P.

We have the inequality

$$w(\mathcal{L}_{Y_1}, P) \geq \sum_i (d - m_{r-i}(\mathcal{L}_{Y_2}, p) - i),$$

with equality if and only if $\mathcal{L}$ satisfies the compatibility condition at P.

We can bound the sum $w(\mathcal{L}_{Y_1}, P) + w(\mathcal{L}_{Y_2}, P)$ in two different ways.

$$w(\mathcal{L}_{Y_1}, P) + w(\mathcal{L}_{Y_2}, P) \geq \sum_i (m_i(p) - i) + \sum (d - m_i(p) - i) = (d - r)(r + 1)$$

The point P is the one point which is smooth on Y_1 and on $\overline{X_0 - Y_1}$ but not on X_0. So

$$w(\mathcal{L}_{Y_1}, P) + w(\mathcal{L}_{Y_2}, P) = \sum_{\text{smooth points of } Y_1} w(P') + \sum_{\text{smooth points of } X_0 - Y_1} w(P') - \sum_{\text{smooth points of } X_0} w(P')$$

$$= (r+1)d + \binom{r+1}{2}(2g'-2) + (r+1)d + \binom{r+1}{2}(2(g-g')-2) - (r+1)d - \binom{r+1}{2}(2g-2)$$
$$= (d - r)(r + 1)$$

by the induction hypothesis on Y_1 and $X_0 - Y_1$, each of which has fewer components than X_0.

Hence $w(\mathcal{L}_{Y_1}, P) + w(\mathcal{L}_{Y_2}, P) = (d - r)(r + 1)$ exactly, so $\mathcal{L}$ satisfies the compatibility condition as required. □

This construction motivates the following definition:

Definition 4.1-7. *A* limit linear series *is an association to each component Y of X_0 a g^r_d $(\mathcal{L}_Y, V_Y)$, a Y-aspect, satisfying the* Compatibility Condition*: For any two components Y and Z of X meeting at a point P, and for any i,*

$$0 \leq i \leq r,$$

we have

$$m_i(\mathcal{L}_Y, P) + m_{r-i}(\mathcal{L}_Z, P) = d.$$

Remark 4.1-8 (Warning). *Every linear series on X_η gives rise to a unique limit linear series on X_0, but the converse need not be true. Not every limit linear series on X_0 is the limit of a linear series. We shall see some examples of this in Section*

4.3.

We present a few fundamental equalities and inequalities about limit linear series.

Proposition 4.1-9. *Let Y and Z be irreducible components of X_0 meeting at P. Then*

$$w(V_Y, P) + w(V_Z, P) = (r+1)(d-r). \tag{1.1}$$

Proof: It follows immediately from the Compatibility Condition, by summing the weights. □

Proposition 4.1-10 ([3], Prop. 1.3). *Let X_0 be a reduced but reducible curve of compact type, let Y and Z be irreducible components of X_0 meeting at P, and let P' be another point of Y. Let $(\mathcal{L}, V)$ be a limit linear series on X_0. Then the multiplicities satisfy the inequality*

$$m_i(V_Y, P') \leq m_i(V_Z, P). \tag{1.1}$$

Proof:

$$m_i(V_Y, P') + m_{r-i}(V_Y, P) \leq d,$$

so

$$m_i(V_Y, P') \leq d - m_{r-i}(V_Y, P) = m_i(V_Z, P).$$

□

Proposition 4.1-11 ([3], Prop. 1.5). *Let Y be a rational component of X_0. Let P be the intersection between Y and a component of positive genus, or between Y and a chain of rational curves W_j^k terminating in a curve of positive genus. Then the aspect V_Y has at least a cusp at P.*

Proof: The proof is by induction on the length of the chain. If Y meets the curve W_1 at P, then $m_i(V_Y, P) = d - m_{r-i}(V_W, P)$ by 4.1-5. Suppose V_Y does not have a cusp at P. Then $m_i(V_Y, P) = (0, 1, \cdots)$, so $m_i(V_{W_1}, P) = (\cdots, d-1, d)$. By projecting away from P, we obtain a g_1^1 on W_1. Hence W_1 is a rational curve. Suppose W_i meets W_{i+1} at Q. Then by 1.1,

$$(\cdots, d-1, d) = m_i(V_{W_i}, P) \leq m_i(V_{W_{i+1}}, Q),$$

so W_{i+1} also has a g^1_1, so it is also rational This contradicts the assumption that the chain of W_i's ended in a curve of positive genus. $\square$

Corollary 4.1-12 ([3], Cor. 1.6). *Let Y be a rational component of X_0. Let Q be the intersection of Y with a chain of W^i_j's terminating in a curve of positive genus.*

- *Let P and P' be any two points of Y not equal to Q. Then there is at most one section of V vanishing only at P and P'.*
- *If Y meets another component Z at P, then $m_i(V_Y, P') < m_i(V_Z, P)$ for all but at most 1 value of i.*

Proof: Suppose there are 2 independent sections vanishing only at P and P'. They span a pencil in V that is totally ramified at P and P', and not at all elsewhere. But every limit linear series must have a cusp at Q by 4.1-11. Likewise, if

$$m_i(V_Y, P') = m_i(V_Z, P),$$

then

$$m_i(V_Y, P') = d - m_{r-i}(V_Y, P).$$

Choosing a compatible basis for P and P', there is a section σ_i that vanishes only at P and P'. This is impossible. $\square$

4.2 Proof of Classical Brill-Noether Non-Existence

We use the theory of limit linear series to present two more proofs of the classical Brill-Noether non-existence theorem, also due to Eisenbud and Harris ([5]), which will suggest a direction in which to proceed in the case when we impose ramification points.

Theorem 4.2-1 (Brill-Noether Non-Existence, Eisenbud-Harris 1986). *Let C be a general curve of genus g. Let $\rho(g, r, d)$ be the Brill-Noether number*

$$\rho = g - (r+1)(g + r - d).$$

If $\rho < 0$ then C admits no g^r_d's.

Proof: As in Section 2.2, we prove the theorem by deforming our curve to a special curve X_0. The dimension of the linear system on the special fiber is greater than or equal to the dimension on the generic fiber, so it's enough to prove the theorem for the special fiber. But instead of deforming to a cuspidal curve, we deform to a semi-stable form, namely a *flag curve*.

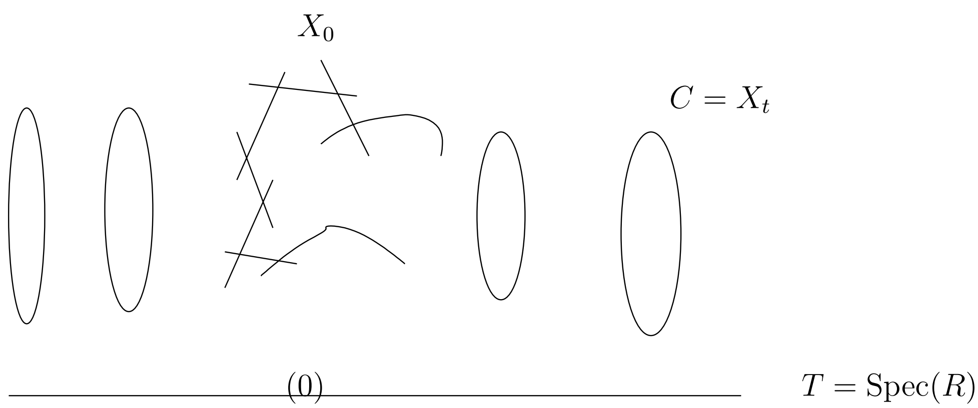

Figure 4-1: Degenerating to a flag curve

We start with a rational curve and instead of imposing g cusps, we attach g elliptic tails. We then blow up the attachment points, and continue to blow them up until no node is a limit of ramification points of $(\mathcal{L}, V)$. The resulting curve X_0 consists of a backbone of N rational curves Z_i. At g of these curves we attach a chain of rational curves W_j^k terminating in elliptic tails E_j.

Set $R_l = Z_l \cap Z_{l+1}$. Then for all i, we have $i \leq m_i(V_{Z_l}, R_l) \leq d - r + i$. So

$$(r+1)(d-r) \geq \sum_{i=0}^{r} m_i(V_{Z_N}, R_N) - m_l(V_{Z_2}, R_2)$$

since there are $r+1$ terms in the sum, each at most $d-r$.

We have

$$m_i(V_{Z_{l+1}}, R_{l+1}) \geq m_i(V_{Z_l}, R_l)$$

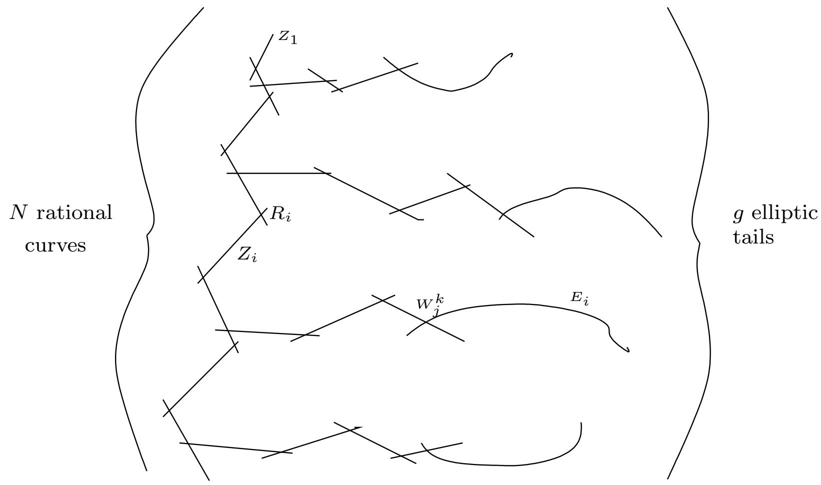

Figure 4-2: The flag curve X_0

by 1.1, and if Z_l meets one of the g tails, then for all but all but 1 value of i,

$$m_i(V_{Z_{l+1}}, R_{l+1}) > m_i(V_{Z_l}, R_l).$$

So for these Z_l,

$$\sum_{i=0}^{r} m_i(V_{Z_{l+1}}, R_{l+1}) - m_i(V_{Z_l}, R_l) \geq r. \tag{2.1}$$

In other words, each time the curve has an elliptic tail, the total weight drops by r.

Since there are g of these Z_l that meet the g tails, we have

$$\sum_{l=2}^{N} \sum_{i=0}^{r} m_i(V_{Z_{l+1}}, R_{l+1}) - m_i(V_{Z_l}, R_l) \geq rg.$$

Hence

$$(r+1)(d-r) \geq \sum_{i=0}^{r} m_i(V_N, R_N) - m_i(V_2, R_2) = \sum_{l=2}^{N} \sum_{i=0}^{r} m_i(V_{Z_{l+1}}, R_{l+1}) - m_i(V_{Z_l}, R_l) \geq rg.$$

□

We can also use limit linear series to prove a stronger version of the Brill-Noether theorem, for sufficiently general reducible curves of compact type.

Proposition 4.2-2 (Additivity For General Reducible Curves, [5], 4.5). *Let X be a curve of compact type whose components are general curves $X_1, \cdots, X_c$ of genus $g_1, \cdots, g_c$. Let $P_1, \cdots, P_s$ be a set of general points on $X_1, \cdots, X_c$,* and, let the nodes of X be general points on the components. *Then the dimension of the family of g^r_d's on X with specified multiplicities $m_i(P_j)$ is exactly equal to*

$$\rho = g - (r+1)(g+r-d) - \sum_{j=1}^{s}\sum_{i=0}^{r}(m_i(P_j) - i).$$

Proof: The proof has four steps. First we prove it for smooth rational and elliptic curves. Then we prove it for reducible curves whose components are rational and elliptic, by induction on the number of components. We then prove it for general smooth curves of any genus by degenerating to a flag curve, and finally for reducible curves by induction on the number of components.

If X is a smooth rational curve, then there is not much to prove. By the Plücker formula, the total ramification is $(d-r)(r+1)$. So if $\rho < 0$, if we are trying to impose total ramification of more than $(d-r)(r+1)$, then we have an immediate contradiction. If $\rho \geq 0$, then as in the proof of the Dimensional Transversality Lemma (2.2-10), we can try to impose an additional $\rho + 1$ ordinary ramification conditions. The Picard scheme of $\mathbb{P}^1$ is simply a point, so the Grassmann bundle $\mathcal{G}^r_d$ is simply an ordinary Grassmannian. Each additional ordinary ramification condition is simply a σ_1 class. If $\mathcal{G}^r_d(P_j, m_i)$ has dimension greater than ρ, the intersection must be non-zero. But we know that it is not possible to impose more than $(d-r)(r+1)$ ramification conditions, so we have a contradiction.

If $X = E$ is an elliptic curve, then we can degenerate it to a cuspidal rational curve. The dimension of the family of g^r_d's on $\mathbb{P}^1$ with an ordinary cusp of type $(0, 2, 3, \cdots, r+1)$ at a general point P, and vanishing orders $m_i(P_j)$ at general points

P_j, is

$$(d-r)(r+1)-\sum_{j=1}^{s}\sum_{i=1}^{r}(m_i(P_j)-i)-r=(d-1-r)(r+1)-\sum_{j=1}^{s}\sum_{i=1}^{r}(m_i(P_j)-i)$$
$$=1-(1+r-d)(r+1)-\sum_{j=1}^{s}\sum_{i=1}^{r}(m_i(P_j)-i)=\rho.$$

Now let X be a reducible union of rational and elliptic components; we do induction on the number of components. If X is the union of Y and Z meeting at a node P, then $g(Y)+g(Z)=g(X)$, and Y and Z both have fewer components than X. Assume that the fixed points $P_1,\cdots,P_n$ lie on Y, and $P_{n+1}\cdots,P_s$ lie on Z.

Given a multiplicity sequence $0\le m_0<m_1<\cdots<m_r\le d$, let

$$0\le d-m_r<\cdots<d-m_1\le d$$

be the complementary multiplicity sequence. By the induction hypothesis, the dimension of the set of linear series on Y with fixed multiplicities on $P_0,\cdots,P_n$ and multiplicities $(m_0,\cdots,m_r)$ at P, is

$$g(Y)-(r+1)(g(Y)+r-d)-\sum_{j=1}^{n}\sum i=0^r(m_i(P_j)-i)-\sum_{i=0}^{r}(m_i-i).$$

Likewise, the dimension of the set of Z-aspects with fixed multiplicities on $P_{n+1},\cdots,P_s$ and multiplicities $(d-m_r,\cdots,d-m_0)$ at P, is

$$g(Z)-(r+1)(g(Z)+r-d)-\sum_{j=n+1}^{s}\sum i=0^r(m_i(P_j)-i)-\sum_{i=0}^{r}(d-m_i-i).$$

The set of all limit linear series is equal to the set of pairs of a limit linear series on Y and one on Z, with complementary multiplicities. Hence its dimension is the sum

$$g-(r+1)(g+r-d)-\sum_{i=0}^{r}m_i-i.$$

If X is a general curve of genus g, we degenerate it to a flag curve, whose components are all rational and elliptic. The family of g^r_d's on the flag curve has dimension ρ by the previous argument. So by upper semicontinuity, so does the one on X.

Finally, if X is a union of general curves meeting at general points, we can do induction on the number of components, exactly as we did for unions of rational and elliptic components. □

Example 4.2-3 (Warning). *Note that this theorem fails for a reducible curve whose components are not joined at general points.*

For example, suppose that a curve C_4 of genus 4 is joined to an elliptic curve E at a point P where there is a g^2_5 with an ordinary cusp $(0, 2, 3)$. These exist; we saw in Chapter 3 that there are 24 of them. Then there is a g^2_5 on the composite curve; its C_4 aspect is the g^2_5 with the cusp at P, and its E-aspect has ramification $(2, 3, 5)$ at P. It has a basepoint of order 2 at P, plus an ordinary g^1_3 with simple ramification at P. However, a general curve of genus 5 has no g^2_5, since $\rho = 5 - 3(5 + 2 - 5) = -1$.

4.3 Non-Existence and Finiteness Conditions with Ramification

We can use a more refined version of the same methods to prove some finiteness and non-existence results, and obtain a bound on the dimension for the Brill-Noether problem with a movable ramification point.

Let $(g, r, d, m_0, \cdots, m_r)$ be positive integers. The moving-point Brill-Noether number is

$$\rho = 1 + g - (r + 1)(g + r - d) - \sum_{i=0}^{r} m_i - i.$$

Choose $(g, r, d, m_0, \cdots, m_r)$ such that $\rho \leq 0$. We wish to prove in as many cases as possible that there are at most finitely many g^r_d's with vanishing sequence (m_i).

Let X be a family of curves of genus g, specializing to the flag curve X_0. Let $(\mathcal{L}, V)$ be a g^r_d on the smooth fiber, possessing a ramification point with vanishing sequence $(m_0, \cdots, m_r)$. Assume that the ramification point specializes to a smooth point Q. If this is not the case, we can always blow up the nodes; the result will still be a flag curve with more rational components. Then we ask the question, what are the possible limit linear series on X_0? We begin with some basic inequalities.

Proposition 4.3-1. *The sum of the weights of the backbone curve Z_i at the nodes R_i and R_{i+1} where it meets other backbone curves is*

$$w(V_{Z_i}, R_i) + w(V_{Z_i}, R_{i+1}) \geq r(g-1)$$

Proof: By 2.1, we have

$$w(V_{Z_{l+1}}, R_{l+1}) - w(V_{Z_l}, R_l) \geq 0$$

for all Z_l and

$$w(V_{Z_{l+1}}, R_{l+1}) - w(V_{Z_l}, R_l) \geq r$$

for each Z_l that meets a chain curve. Hence the weight

$$w(V_{Z_i}, R_i) \geq rj$$

if there are j tails joined above Z_i. Likewise,

$$w(V_{Z_{l-1}}, R_l) - w(V_{Z_l}, R_{l+1}) \geq r,$$

so

$$w(V_{Z_i}, R_{i+1}) \geq rk$$

if there are k tails joined below Z_i. There are a total of $(g-1)$ tails joined above and below Z_i. Since by the Plücker formula, the total ramification on a rational curve is $(r+1)(d-r)$, and the rational curve Z_i can have ramification only at the nodes R_i, r_{i+1} and P_1, we have the weight $w(V_{Z_i}, P_1) \geq (r+1)(d-r) - r(g-1)$. □

Proposition 4.3-2. *Let P_0 be the intersection of a backbone curve Z_i with the j^{th} chain curve W_j^1. Then the weight of V_{Z_i} at P_0 is at most $(r+1)(d-r) - r(g-1)$.*

Proof: Since the total ramification on a rational curve is $(r+1)(d-r)$, and the

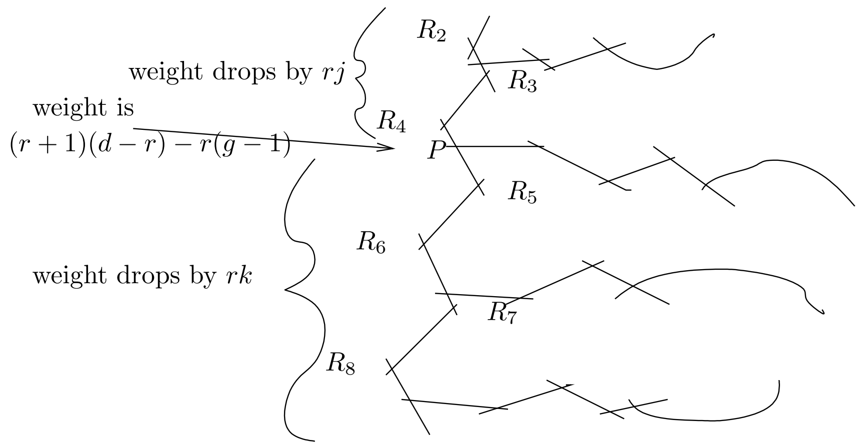

Figure 4-3: Where the Chain Attaches to the Backbone

weightes at R_i and R_{i+1} add up to $r(g-1)$, we have the weight

$$w(V_{Z_i}, P_0) \leq (r+1)(d-r) - r(g-1).$$

□

Proposition 4.3-3 (Minimum Weight at P). *Let P_k be the intersection of a chain curve W_j^k with the next chain curve W_j^{k+1} or with the elliptic tail E_j. Then the weight $w(V_{W_j^{k+1}}, P_k)$ or $w(V_{E_j}, P_k)$ is at least $r(g-1)$.*

Proof: For $0 < k \leq n$, let P_k be the intersection of W_j^k with W_j^{k+1}. The proof is by induction on k. Since $w(V_{Z_i}, P_0) \geq (r+1)(d-r) - r(g-1)$, and by the Compatibility Condition

$$w(V_{Z_i}, P) + w(V_{W_j^1}, P_1) = (r+1)(d-r),$$

we have

$$w(V_{W_j^1}, P_0) \geq r(g-1).$$

For the induction step, since the total ramification on a rational curve is $(r+1)(d-r)$ by the Plücker formula, we have

$$w(V_{W_j^k}, P_k) \geq (r+1)(d-r) - r(g-1).$$

Hence, by applying the Compatibility Condition,

$$w(V_{W_j^{k+1}}, P_k) \geq r(g-1).$$

□

Proposition 4.3-4. *If $\rho \leq 0$, the limit of the ramification point Q lies on one of the elliptic tails.*

Proof: The limit is a smooth point. So it can not be one of the nodes of a rational curve. But any smooth point on a rational curve has weight at most

$$(r+1)(d-r) - r(g-1).$$

In our case, however, if $\rho \leq 0$, then

$$w \geq (d-r)(r+1) - rg - 1 > (d-r)(r+1) - r(g-1).$$

□

Proposition 4.3-5 (Maximum Weight at P). *Let Q lie on the elliptic tail E_j. The vanishing sequence of V_{E_j} at its node P is at most $(d - m_{r-i})$, and*

$$w(V_{E_j}, P) \leq (r+1)d - \sum m_i - \frac{r(r+1)}{2}.$$

If ρ is the moving-point Brill-Noether number

$$\rho = 1 + g - (r+1)(g+r-d) - \sum_{i=0}^{r} m_i + \frac{r(r+1)}{2},$$

then

$$w(V_{E_j}, P) \leq r(g+1) + \rho + r - 1.$$

Proof: Since the sum

$$m_i(V_{E_j}, Q) + m_{r-i}(V_{E_j}, P) \leq d,$$

the vanishing sequence at P is at most $(d - m_i)$. Sum the multiplicities to get

$$w(V_{E_j}, P) \leq (r+1)d - \sum m_i - \frac{r(r+1)}{2}.$$

But we have

$$\rho = 1 - rg - r(r+1) + (r+1)d - \sum m_i + \frac{r(r+1)}{2} = 1 - rg + w(V_{E_j}, P).$$

So

$$w(V_{E_j}, P) \leq r(g+1) + \rho + r - 1.$$

□

Theorem 4.3-6 (Non-Existence For Sufficiently Low ρ). *Let ρ be the* Moving-Point Brill-Noether number

$$\rho = 1 + g - (r+1)(g+r-d) - \sum_{i=0}^{r}(m_i - i).$$

If $\rho < 1 - r$, then there is no g^r_d on a general curve of genus g with vanishing sequence (m_i) at any point Q.

Proof: Degenerate the curve to a flag curve, and consider the possible limits. By 4.3-4, the limiting position of Q must lie on an elliptic tail. Let P be the node where that limiting tail is attached to the rational components. By 4.3-3 and 4.3-5, any limit g^r_d $(\mathcal{L}, V)$ on the flag curve would have to satisfy

$$r(g-1) \leq w(V_{E_j}, P) \leq r(g-1) + \rho + r - 1.$$

Hence there is no such g^r_d on the flag curve, so there is no such g^r_d on the general smooth curve. □

Proposition 4.3-7 (Finiteness of Points for $\rho \leq 0$). *If $\rho \leq 0$, then there are at most finitely many points Q for which a g^r_d exists with multiplicities m_i at Q.*

Proof: As in the previous proof, the limit of any such point Q must lie on an elliptic tail. The flag curve only has finitely many elliptic tails, so it's enough to show that on any one tail E, there are only finitely many possible limiting points Q. We bound the weights at P;

$$r(g-1) \leq w(V_{E_j}, P) \leq r(g-1) + \rho + r - 1.$$

So the difference between the maximal and minimal possible weights is $\rho + r - 1$. Since $r \leq 0$, this is at most $r-1$. Therefore, since there are $r+1$ places in the multiplicity sequence and they differ by only $r-1$, there are at least two positions i and j where $m_{r-i}(P)$ is exactly the maximum value $d - m_i(Q)$ and $m_{r-j}(P)$ is exactly the maximum value $d - m_j(Q)$. Thus the linear system contains divisors $m_iQ + (d-m_i)P$ and $m_jQ + (d-m_j)P$.

So $Q - P$ must be $(m_i - m_j)$-torsion. Hence there are at most finitely many possible choices for Q. □

Remark 4.3-8. *Note that the finiteness of points implies the Brill-Noether non-existence theorem with a fixed general ramification point: if $\rho_{\text{ fixed } Q} < 0$ then we have $\rho_{\text{ moving } Q} \leq 0$. So there are only finitely many Q possessing a g^r_d with ramification $(m_0, \cdots, m_r)$. In particular, a general Q does not possess such a g^r_d.*

Theorem 4.3-9 (Finiteness and Non-Existence of Linear Systems for $r = 1$, $\rho \leq 0$). *If $r = 1$ and the expected dimension is $\rho(g, 1, d, m_0, m_1) = 0$, then a general curve of genus g possesses at most finitely many g^1_d's with a ramification point of type (m_0, m_1). If $\rho < 0$, then no such g^1_d's exist.*

Proof: We have seen that if we degenerate the curve to a flag curve, then the limiting ramification point Q must land on an elliptic component, and the aspect of the limit g^1_d on that elliptic component is $m_0Q + (d-m_0)P, m_1Q + (d-m_1)P$. But we need to count the complete limit linear series, not just their E-aspects.

We know that the only possible aspect on E is $m_0Q + (d-m_0)P, m_1Q + (d-m_1)P$. So by the Compatibility Condition, the Y-aspect must have ramification (m_0, m_1) at P. We calculate the dimension of the family of g^1_d's on $X_0 - E$ with a fixed ramification point of type (m_0, m_1). Since $X_0 - E$ consists of rational and elliptic curves, they are all general. Since there are at most three nodes on the rational components and only one on the elliptic components, the nodes are all general points (since there is an automorphism that replaces these nodes with any others), so $X_0 - E$ satisfies the Additivity Condition. Hence the dimension of possible g^r_d's on $X_0 - E$ with a fixed ramification point at P of type (m_0, m_1) is

$$\rho_{\text{fixed}}(g-1, 1, d, P, m_0, m_1) = (g-1) - 2(g-1+1-d) - m_0 - m_1 + 1$$

$$= g + 2(g+1-d) - 1 + 2 - m_0 - m_1 + 1 = \rho(g, 1, d, m_0, m_1).$$

So if $\rho = 0$ there are finitely many, and if $\rho < 0$ there are none. Since there are only finitely many possible choices for E and finitely many choices for $X_0 - E$, there are a total of finitely many possible limit linear series with this ramification, and therefore a total of finitely many possible g^r_d's on the general curve. □

When $r = 2$, we can not always prove non-existence for $\rho = -1$, but we can still prove finiteness when $\rho = 0$.

Theorem 4.3-10 (Finiteness of Linear Systems for $r = 2$, $\rho \leq 0$). *If $r = 2$ and the expected dimension is $\rho(g, 2, d, m_i) \leq 0$, then there are at most finitely many g^2_d's on a general curve of genus d that possess a ramification point with vanishing sequence (m_0, m_1, m_2).*

Proof: As before, we can degenerate the curve to the flag curve. The limiting position of the ramification point Q is a torsion point on an elliptic tail E relative to the node P. The difference between the minimum and maximum possible weights of the E-aspect at the node P is at most $r - 1 = 1$. So the linear system on E is generated by three divisors, at least two of which are linear combinations of P and Q exclusively.

If $(m_2 - m_1)$ and $(m_1 - m_0)$ are relatively prime, then the three divisors

$$m_0Q + (d - m_0)P,\ m_1Q + (d - m_1)P \text{ and } m_2Q + (d - m_2)P$$

can not all be linearly equivalent, since that would require that $Q = P$. So the linear system can only be of the form $m_0Q + (d - m_0)P$, $m_1Q + (d - m_1)P$ and $m_2Q + (d - m_2 - 1)P + R$, up to renumbering the m_i's. The point R is completely determined by the linear equivalence. So there are only finitely many such aspects on E. Since the ramification of the E aspect at P is $(d - m_0, d - m_1, d - m_2 - 1)$, the ramification on X_0 at P is $(m_0, m_1, m_2 + 1)$. We can compute the dimension of possible g^2_d's on the complement $X_0 - E$ with this ramification at the fixed point P :

$$(g-1)-3(g-1+2-d)-m_0-m_1-m_2-1+3 = g-3(g+2-d)+2-m_0-m_1+3 = \rho.$$

In case $(m_2 - m_1)$ and $(m_1 - m_0)$ have a common factor, then there is also the possibility that the E-aspect is just

$$m_0Q + (d - m_0)P,\ m_1Q + (d - m_1)P,\ m_2Q + (d - m_1)P.$$

In this case, the ramification of the E-aspect is $(d - m_0, d - m_1, d - m_2)$ at P, so the ramification of the $X_0 - E$-aspect is only (m_0, m_1, m_2). The dimension of the family of such limit linear series is 1.

Suppose that the general curve of genus g actually had a 1-parameter family of g^2_d's with ramification (m_0, m_1, m_2). Consider the class $[\Lambda]$ of this locus in the Grassmann bundle $\mathbb{G}(3, \mathcal{E})$. If it is actually a non-empty locus of dimension 1, then its class is $a\theta^{g-1}\sigma_{\text{top}} + b\theta^g\sigma_{\text{top}-1}$, for some nonnegative coefficients a and b. Then we should be able to intersect it with the codimension 1 class λ of linear series that are ramified at a fixed general point R. This class is of the form $c\theta + e\sigma_1$. Assume that the rank of $\mathcal{E}$ is at least 4, which we can force by choosing n sufficiently large. Then the coefficient e is nonzero, since the intersection with the fiber over any point of Pic^d_C is non-empty: if the line bundle $\mathcal{L}(nP)$ has a 4-dimensional family of sections, then we can certainly pick a 3-dimensional subfamily that vanish to orders at least $(0, 1, 3)$ at R. But the intersection of σ_1 with any class is positive. Hence $\lambda \cap \Lambda$ is positive.

Hence there must exist a non-empty set of g^2_d's with ramification (m_0, m_1, m_2) at Q and at least simple ramification at R. But what happens when we try to degenerate these g^2_d's to X_0? We can choose fixed points R whose limit is a fixed general point on E. But there is only one possibility for the E-aspect, and it can only be ramified at finitely many points. At a fixed general point R on E, there is no ramification. Hence we obtain a contradiction.

So there can be at most finitely many g^2_d's with a ramification point of type (m_0, m_1, m_2). □

Example 4.3-11. *Let* $r = 2$, $g = 4$, $d = 6$, *and* $m_i = (0, 3, 5)$. *Then the possible ramification sequences at* P *are* $(1, 3, 5)$, $(0, 3, 6)$, $(1, 2, 6)$ *and* $(1, 3, 6)$. *All of these caess except* $(1, 3, 6)$ *can occur on the flag curve, as they give rise to the three linear systems* $(5Q + P, 3Q + 3P, 5P + (Q - P))$, $(6Q, 3Q + 3P, 5P + (3Q - 2P))$, *and* $(6Q, 2Q+3P+(4Q-3P), 6P)$, *where* $P-Q$ *is torsion of order* 2, 3, *or* 5 *respectively. (The case* $(1, 3, 6)$ *is the the degenerate case* $Q = P$.*)*

Example 4.3-12. *If* $g = 4$, $d = 6$, *and the vanishing orders are* $(0, 3, 6)$, *then the expected dimension is* $\rho = -1$. *On the elliptic tail of the flag curve, we have the linear system* $(6P, 3P + 3Q, 6Q)$, *where* $P - Q$ *is* 3*-torsion. However, on a general curve of genus* 4 *there are no such* g^2_6*'s. If we project away from the point* Q, *we obtain a* g^1_3 *with ramification* $(0, 3)$. *This cannot happen on a curve of genus* 4, *by 3.3-8. It is not immediately clear what happens if* $g = 7$ *and* $d = 8$ *or for higher* g *and* d.

When $r = 3$, the situation becomes a bit more complicated and begins to resemble the general case.

Proposition 4.3-13 (Finiteness Condition for $r = 3$). *If* $r = 3$ *and the expected dimension is* $\rho(g, r, d, m_i) \leq 0$, *then the dimension of* $\mathcal{G}^r_d(m_0, \cdots, m_3)$ *is at most* 1. *If in addition, the differences* $m_i - m_j$ *are pairwise relatively prime, then* W^r_d *is finite.*

Proof: As before, we shall degenerate the curve to the flag curve X_0,and consider the possible vanishing sequences at the node P on E. As in the previous proofs, the vanishing sequence is bounded by $(d - m_i)$ and is allowed to differ from its maximum values by at most $r - 1 = 2$. We shall consider each possible ramification at P.

If all the pairwise differences among the multiplicities share a common factor, then the first possible E-aspect is simply

$$m_0Q + (d - m_0)P, m_1Q + (d - m_1)P, m_2Q + (d - m_2)P, m_3Q + (d - m_3)P.$$

In this case we have finitely many E-aspects and a 2-parameter family of possible $X_0 - E$-aspects. However, only finitely many of them can deform to the general curve of genus g because otherwise at least finitely many would have to have ramification at a general fixed point R, and in the limit there are only finitely many possible E-aspects and therefore only finitely many possible fixed ramification points on E.

If at least two of the pairwise differences share a common factor, then we could have an E-aspect of the form

$$m_0Q + (d - m_0P), m_1Q + (d - m_1)P, m_2Q + (d - m_2)P, m_3Q + (d - m_3 - 1)P + R$$

for some point R. We have finitely many E-aspects and a 1-parameter family of possible $X_0 - E$-aspects. Or we could have

$$m_0Q + (d - m_0)P, m_1Q + (d - m_1)P, m_2Q + (d - m_2)P, m_3Q + (d - m_3 - 2)P + R + S,$$

for some effective divisor $R + S$ of degree 2. In this case there is a 1-parameter family of possible E-aspects, since R can be chosen arbitrarily and then S is determined, but we are imposing a fixed point with vanishing sequence $(m_0, m_1, m_2, m_3 + 2)$ on Y, so there are only finitely many Y-aspects. So these cases contribute a 1 parameter family if the pairwise differences are not relatively prime.

Finally, if all the pairwise differences are relatively prime, then the only option is an E-aspect of the form

$$m_0Q + (d - m_0)P, m_1Q + (d - m_1)P, m_2Q + (d - m_2 - 1)P + R, m_3Q + (d - m_3 - 1)P + S.$$

There are finitely many possible such aspects. The corresponding Y-aspects have vanishing sequence $(m_0, m_1, m_2 + 1, m_3 + 1)$ at P, so there are finitely many of them as well. Hence if the pairwise differences are relatively prime, then there are only finitely many g_d^3's with the specified ramification type. □

If $r \geq 4$, then we never have all the pairwise differences relatively prime, since at least two of them are even. However, we can still prove a bound on the dimension.

Theorem 4.3-14 (Weak General Bound). *If the expected dimension ρ is less than or equal to zero, then the actual dimension of $\mathcal{G}_d^r(m_i)$ over a general curve of genus g is bounded by $\rho+r-2$ if this number is nonnegative. Moreover, let $k+1$ be the size of the largest subset of the set of multiplicities $\{m_{i_0}, \cdots, m_{i_k}\} \subseteq \{m_0, \cdots, m_r\}$ whose pairwise differences all share a common factor. Then the dimension of $\mathcal{G}_d^r(m_0, \cdots, m_r)$ is bounded by $\rho + k - 1$.*

Proof: As before, if we degenerate the curve to a flag curve. Since $\rho \leq 0$, we know that the limit of the ramification point Q on X_0 lies on one of the elliptic tails, and is in fact a torsion point. We have the upper and lower bounds

$$r(g-1) \leq w(V_E, P) \leq r(g-1) + \rho + r - 1.$$

The multiplicities of V_E at P are allowed to be equal to their maximum values at the $k+1$ places whose pairwise differences have a common factor. The multiplicities at the other $r-k$ places are required to drop by 1 because $Q \neq P$. So the difference between the actual lower and upper bounds on $w(V_E, P)$ is $\rho + k - 1$. If $\rho + k - 1 < 0$, then there are no possible g_d^r's. Assuming this difference is nonnegative, we can distribute it between E and $X_0 - E$.

Let t be any integer between 0 and $\rho + k - 1$. Then we can construct an E-aspect of the form

$$m_0Q + (d-m_0)P, \cdots, m_kQ + (d-m_k)P, m_{k+1}Q + (d - m_{k+1} - 1 - t)P + D_{k+1},$$

$$m_{k+2}Q + (d - m_{k+2} - 1)Q + D_{k+1}, \cdots, m_rQ + (d - m_r - 1)P + D_r,$$

where the D_i are effective divisors of degree d_i whose sum is $t + r - k$. There is a t-parameter family of such aspects. The corresponding $X_0 - E$-aspects must have multiplicity sequence

$$(m_0, \cdots, m_k, m_{k+1} + d_{k+1}, m_{k+2} + d_{k+2}, \cdots, m_r + d_r)$$

There is a $(\rho+k-1-t)$-parameter family of such X_0-E-aspects. Thus in every case, there is a $(\rho+k-1)$-parameter family of pairs of an E-aspect with a X_0-E-aspect.

However, in case $k=r$, if all the pairwise differences have a common factor, the bound is only $\rho+r-2$ if this is nonnegative. The reason is that if we subtract t from $w(V_E, P)$, we only gain a $(t-1)$-parameter family because one point is determined by the others, and it is not possible to have $m_0Q+(d-m_0)P, \cdots, m_rQ+(d-m_r)P$ on E and a $(\rho+r-1)$-dimensional family on X_0-E because the resulting g^r_d's would not be ramified at a general fixed point R on E. □

Chapter 5

Further Questions

Is the Weak Bound the best we can do, or does the generalized Brill-Noether conjecture hold? Can we find a criterion for when limit linear series smooth on a reducible curve to a generic curve?

What happens if we allow two ramification points? What if instead of a cusp we impose a node or other higher-order double point where two distinct points are identified? The Porteous method in these cases will be complicated by the fact that instead of a map to a single filtered vector bundle, we will have a map to a bundle with two distinct filtrations, with simultaneous degeneracy conditions on both. I would like to construct a generalized Porteous theorem covering this case. Moreover, some degenerate cases would arise from allowing the two points to coalesce, but they can be analyzed separately and subtracted off by excess intersection theory. Meanwhile, the limit linear series approach should break into cases, depending on whether the limits of the two points lie on the same component or different components. It should be possible to analyze these cases separately and obtain some weak bounds, but how weak are the bounds?

What divisors do we obtain on the moduli space of curves when $\rho = -1$? Harris, Mumford and Eisenbud ([16], [6]) used Brill-Noether divisors to prove that $\overline{\mathcal{M}_g}$ is of general type for $g \geq 23$. More recently, Farkas [9] proved that $\overline{\mathcal{M}_{22}}$ is of general type, by considering certain divisors on the moduli space obtained by imposing degeneracy conditions on line bundles. What divisors do we obtain on the moduli space $\overline{\mathcal{M}_g}$ by imposing ramification conditions such that $\rho = -1$? What are their slopes?

Farkas and Popa ([7] and [8]) disproved the slope conjecture and by constructing interesting divisors on the moduli space $\overline{\mathcal{M}_g}$ using Brill-Noether type conditions on rank-2 vector bundles. Can we impose similar conditions with ramification? Higher rank vector bundles correspond to maps to a Grassmannian instead of a projective space. We know the possible curve classes on Grassmannians, and their degrees are well behaved. We should be able to prove existence of ramified rank-2 vector bundles in some cases by the Porteous formulas on the moduli space of rank-2 vector bundles instead of the Picard scheme. Farkas, Popa and Teixidor (e.g. [10] [24] and [25]) have begun to develop a theory of limit linear series for suitably well behaved vector bundles. How far can this be extended? What divisors does it yield on the moduli space?

With such a proliferation of problems, the next century of Brill-Noether theory promises to be as fertile as the first.

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