

PROBABILISTIC ANALYSIS OF
ROUTE DEVIATION BUS LINES

by

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des Travaux Publics de l'Etat
(1981)

Submitted to the Department of
Civil Engineering
in Partial Fulfillment of the
Requirement of the Degree of

MASTER OF SCIENCE IN TRANSPORTATION

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1982

[i.e. June 1982]

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ABSTRACT

Route Deviation Bus Lines (RDBLs) constitute a hybrid between traditional fixed-route bus lines and demand responsive, dial-a-ride systems; they are viewed as a promising way to combine responsive transportation services with high productivity.

We investigate the performance of Route Deviation Bus Lines through development of some simplified probabilistic models. The dependence of performance on such parameters as the demand intensity at checkpoints, the magnitude and distribution of headways and the number of checkpoints are explored.

This study begins by describing the problem and then identifies the different issues in the probabilistic analysis of RDBLs with emphasis on the two generic processes of bus systems: the arrival process of passengers, and the service process. We derive closed form expressions for the one call box case (Chapter III) and describe the way of solving the two call boxes cases (Chapter IV) and then the n call boxes case (Chapter V). Throughout this work we assume homogeneous Poisson arrivals at the call boxes and independent and identically distributed headways between buses. Our analysis clearly demonstrates that relaxation of these assumptions would have a major negative impact on the mathematical tractability of our models. Several interesting questions for further research are also identified.

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ACKNOWLEDGEMENTS

I wish to express my deep gratitude to Professor Amedeo R. Odoni, whose patience, help, and many suggestions made this thesis possible.

I would like to thank the Ecole Nationale des Travaux Publics de l'Etat in Lyon and the Ministère des Relations Extérieures of the French Government who made possible my stay in the United States.

I would also like to extend mes amitiés to Eric, Georges, and Margie.

I am forever indebted to my parents for their constant encouragement and support in completing this thesis.

Finally, Anne deserves special feelings for her patient bearing with my absence during this past year.

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CHAPTER I

STATEMENT OF THE PROBLEM

I.1 Introduction

The goal of this thesis is to investigate the performance of Route-Deviation Bus Lines (RDBLs) through development of some simplified probabilistic models. This can be seen as the beginning of the evaluation of tools for planning the operation of Route-Deviation Bus Lines.

RDBLs, some examples of which have already been implemented in Europe, are viewed as a promising way to combine the flexibility of demand responsive transportation services with the higher productivity of traditional fixed route bus lines. In addition to that characteristic, the implementation of such systems stimulates the development of new technologies such as specially designed mini-buses, call-box and checkpoint hardware, communications equipment and system control software. As route-derivation bus line technologies are intended to satisfy the service needs of suburban locations and of middle- and small-size towns, the potential market for these technologies would seem to be very large.

So far the literature on this subject has been very limited. Eric F. Peyrard [1] has recently reviewed the

characteristics of existing systems in the U.S. and in Europe and has completed an important research effort on one of RDBL-based area-wide systems in Saumur, France.

However, to the extent of the author's knowledge, no systematic analysis of the performance of such systems has been performed yet. This was one of the motivations for the the work reported herein.

I.2 Background

Route-deviation bus lines constitute a hybrid between traditional fixed-route bus lines and demand responsive, dial-a-ride systems. A schematic representation of a RDBL is provided in Figure I.1. The line is designed to operate as a fixed-route service between stations A and B most of the time. However, one or more "deviation checkpoints" (or simply, "checkpoints") have been established at some distance from the regular route (see points C_1 and C_2 in Figure I.1). These checkpoints are visited by a bus on the route only in response to a demand which is made known to the bus operator through some communications device. The term "route-deviation" derives from the fact that a bus must deviate from its regular route in order to serve the checkpoints.

RDBLs are currently often mentioned by transportation planners as one possible way of retaining some of the flexibility of demand responsive transportation systems

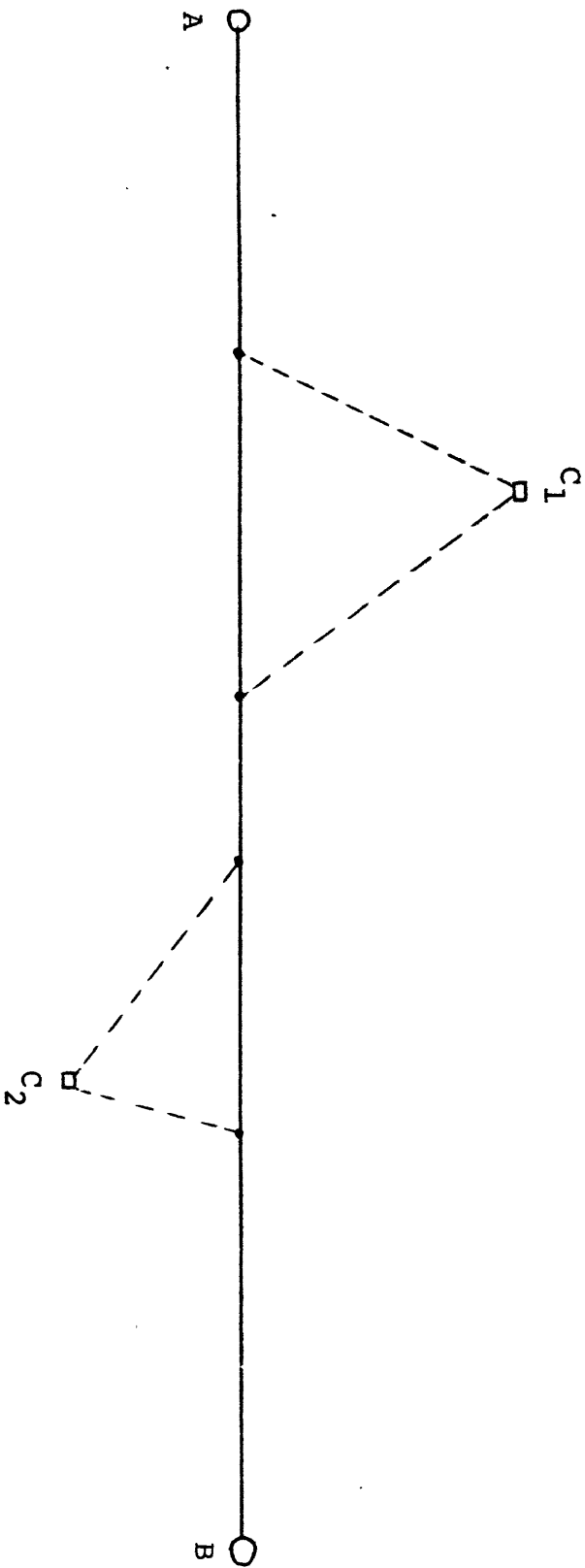


Figure I.1 A Simple Route Deviation Bus Line

while at the same time achieving the higher productivity and lower costs normally associated with fixed-route services. RDBLs are most appropriate for adoption in areas with demand of intermediate density, such as in suburban locations. In denser urban environments, RDBLs could also be used to serve specific locations generating infrequent demands by special-needs passengers. For example, checkpoint C_1 and C_2 in Figure I.1 might be the locations of nursing homes or of clusters of housing for the elderly. In such a context, it may not be worthwhile to design the bus route so as to always pass from C_1 and C_2 , but due to the special needs of the infrequent passengers from C_1 and C_2 , it may be appropriate to retain the flexibility of making a detour to these checkpoints, as needed.

Peyrard, in the first part of his thesis, reviewed different hybrid services which can alleviate what he called the dial-a-ride syndrome (low productivity, escalating costs, mediocre performance), then analyzed their main features and selected route deviation systems as the most promising ones. This selection has been made on the basis of the evaluation of several existing systems which seem promising in terms of technologies, operating costs and patronage. The selected system has then been evaluated through the case study of Saumur.

To date, RDBLs in the United States have developed on a more or less ad hoc basis, primarily as "evolutions" of dial-a-ride systems. A set of RDBLs, for example, is

operating today in Natick, MA, and serves as a feeder to the bus lines that leave Natick Common for downtown Boston. Each route-deviation bus line at Natick (there is one line for each of North, East, West and South Natick) has designated stops which are visited only upon request (made by telephone to a dispatcher). These optional stops ("checkpoints") have been located so that the great majority of the population is within a short walking distance of a checkpoint.

By contrast, West Europeans seem to be approaching RDBLs in a more systematic way. In France, RDBL-based area-wide systems have been operating over the last few years in several middle-size towns around the country. The Saumur system now in operation for nearly 4 years was planned such that each call box has a service area of 250 meters of radius.

"At any time, there is only one bus heading in the right direction on each line, this bus, of course, is expected to pick up the passenger requesting service. In a very special case (extreme delay) the dispatcher will assign the next bus to a call box request but this is very unlikely to happen." According to Peyrard the Saumur call box system appears to be efficient, well planned and satisfying to users on both technical and social aspects.

These area-wide systems are apparently sufficiently successful to prompt the Ministry of Transportation in France to consider their implementation at many other

locations. This has also stimulated development of new call-box technology to facilitate the operation of the systems.

In West Germany, an even more ambitious integrated bus system, having a large RDBL component, is currently being planned for the city of Hannover. The design of the software and hardware for the system is the responsibility of a consortium of three companies (MBB, Dornier and Rufbus) all of which have been involved in earlier West German efforts in this field (at Friedrichshagen, Wunstorf and West Berlin).

Despite the widespread current interest in RDBLs there are no published materials on techniques for planning for the efficient operation of such bus lines. Besides Peyrard's work the only research we can cite is earlier works based on simulation in order to compare various systems (Montgomery [2], Englisher and Sobel [3]). These investigations provided some insights on how carefully planned checkpoint RDBLs can become an important aspect of an overall public transportation system. This thesis will hopefully begin to fill the need for more systematic study of RDBLs. More precisely we will attempt to study how the performance of a route deviation bus line is affected by such characteristics as the number of deviation checkpoints along a route, the intensity of the demand at these checkpoints, the headways between buses, the location of checkpoints, etc.

I.3 Methodology of the Study

As pointed out in the previous paragraph we would like to model the operation of a route-deviation bus line.

The service provided by this system faces two kinds of uncertainties: first, as any other "more classical" transportation system, it must deal with the variabilities which characterize the operation (travel speed, traffic congestion, headways, fluctuations in demand,...). The second type of uncertainty which is specific to RDBLs is introduced by the time of occurrence and the number of demands at the checkpoints.

In view of these uncertainties it is clear that any model of a route deviation bus line must be probabilistic in nature; furthermore, the principal motivation for setting up a RDBL is the fact that "with high probability", the number of route deviations will not be large, meaning that RDBLs involve probabilistic concepts by definition.

Thus, our major concerns will be to study the probabilistic behavior of a RDBL; this behavior will be evaluated through the derivation (if possible) of the probability distribution for the time length at a typical bus run. The knowledge of this probability distribution as a function of the demand at checkpoints and of the operational characteristics of the fixed route bus line (associated with the call boxes) may allow us to estimate many of the parameters which might be of interest in planning for

RDBLs.

Before any attempt is made to derive some operational implications of RDBLs such as the interface of RDBLs with the rest of an area-wide transportation network, we have to study the behavior of a single line; this is the aim of this thesis.

In order to be consistent with the principle of proceeding from the simplest situations, our methodology will be to begin with the analysis of the one call-box case. The complete understanding of this simple case will allow us to derive results for the two call-boxes case and then to generalize them to the n-call boxes case. As we will see in subsequent chapters the derivations of our results will depend on the assumptions we shall make on such features as the demand process and the line's operational characteristics. These simplified assumptions are discussed in Chapter II which also describes some analytically difficult features of the real-world RDBLs.

CHAPTER II

ISSUES IN THE PROBABILISTIC ANALYSIS OF RDBLS

II.1 Introduction

The need for a probabilistic approach for analyzing Route Deviation Bus Lines has been pointed out in the previous chapter; besides the variabilities we face with all real world transportation systems (travel speed, headways, traffic congestion, fluctuations in demand, etc.) the principal idea behind the use of probabilistic analysis comes from the fact that "with high probability" the number of route deviations will not be large (motivation for setting up a RBDL).

The model we would like to develop must include three types of input variables:

1. The "geometry" of the situation (e.g. distances between stops, distances between the fixed line route and each of the deviation checkpoints, the number and location of the candidate sites for establishment of deviation checkpoints, etc.)

2. The average demand rates at each of the candidate deviation checkpoints as a function of time of the day.

3. The operational characteristics of the line (e.g. travel speed on the route, amount of traffic congestion,

planned frequency and headways at the starting points of the bus lines during the course of a day, etc.)

To describe the operation of an urban bus system fully, information must be supplied about the last two kinds of input variables; specifically we need assumptions about:

1. the arrival process of passengers at the system;
2. the service process.

Given these input variables and this information, we will calculate through our model the probabilistic perturbations introduced by call boxes.

The rest of this chapter discusses current practices regarding the assumptions about the two generic elements of the system (arrival process of passengers and service process) and evaluates them for the case of RDBLs.

II.2 Arrival Process of Passengers

II.2.1 General Assumptions on the Arrival of Passengers at a Bus System

In urban service systems a lot of demand processes have been modeled through a Poisson process; Poisson processes are processes in which "arrivals" of demands are distributed completely randomly in time. As Larson and Odoni observed [1] the Poisson process can be used as a reasonable model for the generation of fire alarms, police calls and ambulance calls and is very often applied to occurrences of events such as requests for service, arrivals

of vehicles at an intersection and so on.

The assumption of random arrivals of passengers has been widely used in most of the research relating to transit systems and above all in the field of transit reliability where these arrivals must be modeled in order to determine the expected passenger waiting time. The increasing concern with improving the quality of transit service in urban areas through improvements in the "reliability" of this service stimulated the development of explicit relationships between passenger arrival behavior at stations and the service characteristics. Empirical evidence suggests that passengers often coordinate their arrival times with the bus schedule in order to reduce their waiting time; O'Flaherty and Manoan [9] and Seddon and Day [10] have provided data showing that under certain conditions (published timetable for the service, buses running at fixed times, "commuters") the bus and passenger arrival times at the stop will be associated so as to reduce the average waiting time. The expression for average waiting time under the assumption of random arrivals of passengers is

$$E[W] = \frac{E[H]}{2} \left(1 + \frac{\sigma_H^2}{E[H]^2} \right) \quad (2.1)$$

where:

$E[W]$: expected wait time for a randomly arriving
passenger

$E[H]$: mean headway between passage of buses

σ_H : standard deviation of the headway

The "awareness" of the schedule of service by some passengers have been investigated in the studies of Okrent [2], Jackson [3], Jolliffe and Hutchinson [4]. Turnquist [5], and Bowman and Turnquist [6].

Okrent and Jackson estimated continuous distributions (beta and gamma distributions) to fit some observed data, (Okrent), attempted to estimate the share of "aware arrivals" and tried to estimate arrival rate functions for the proportion who are aware of the schedule (Jackson).

Jolliffe and Hutchinson proposed a discrete model considering passengers to be of three types: a proportion q whose arrival time is coincidental with the bus; a proportion $p(1 - q)$ who arrive at the optimal time (the time at which the expected waiting time is smallest); and a proportion $(1 - p)(1 - q)$ who arrive at random.

Turnquist tried to incorporate the effect of service reliability on passenger into the study of passenger waiting times.

Bowman and Turnquist highlighted the problems of the previous models (lack of explanation of the mechanism underlying the "aware" passenger's choice of arrival time for the continuous distribution fit by Okrent and Jackson; no guarantee of accurately predicting the magnitude of the change in wait time resulting from changes in system

performance for the discrete nature of the models developed by Jolliffe and Hutchinson) and related the parameters of the distribution of "aware" passenger arrivals to the service level provided by the bus system. They derived a model based on the limiting form of the discrete choice logit model

$$f(t) = - \frac{e^{U(t)}}{\int_0^H e^{U(t)} dt} \quad (2.2)$$

where:

$U(t)$ = utility of arrival at time t

H = scheduled headway

They calibrated their model testing the form

$$U(t) = a E[W(t)]^b \quad (2.3)$$

for the utility.

where:

$E[W(t)]$ is the expected waiting time for an arrival at t .

They concluded that their "passenger-choice arrival model" indicates a much greater sensitivity to schedule deviation, and a much lower sensitivity to frequency, than does the random arrival model. According to them it also provides a more plausible behavioral hypothesis than do previous models.

Before concluding this general presentation we give a schematic representation of the difference in the mean waiting time between a random passenger and an "aware" passenger (see Figure II.1). This is just an abstraction of a possible situation and it is based on the following assumptions:

1. For a random passenger the expected waiting time is given by equation (2.1); it is well known that:

- for buses maintaining perfect headway $\sigma_H^2 = 0$. Substituting in (2.1), we have

$$E[H] = \frac{E[H]}{2} \quad (2.4)$$

- for buses with completely random (Poisson) headway, $\sigma_H = E[H]$. By substitution in (2.1), we have

$$E[W] = E[H] \quad (2.5)$$

We assume that for bus systems the range of plausible values of the expected waiting time of random passenger goes from $\frac{E[H]}{2}$ ("perfectly scheduled" system) to $E[H]$ (Poisson headway). Thus we exclude systems with irregularity greater than the Poisson process (that is, $\sigma_H > E[H]$). When the mean headway $E[H]$ increases the buses tend to adhere better to perfectly regular headways; thus we assume that as $E[H]$ goes to infinity $\frac{\sigma_H^2}{E[H]^2}$ goes to zero so $E[W] \sim \frac{E[H]}{2}$ for very large values of $E[H]$.

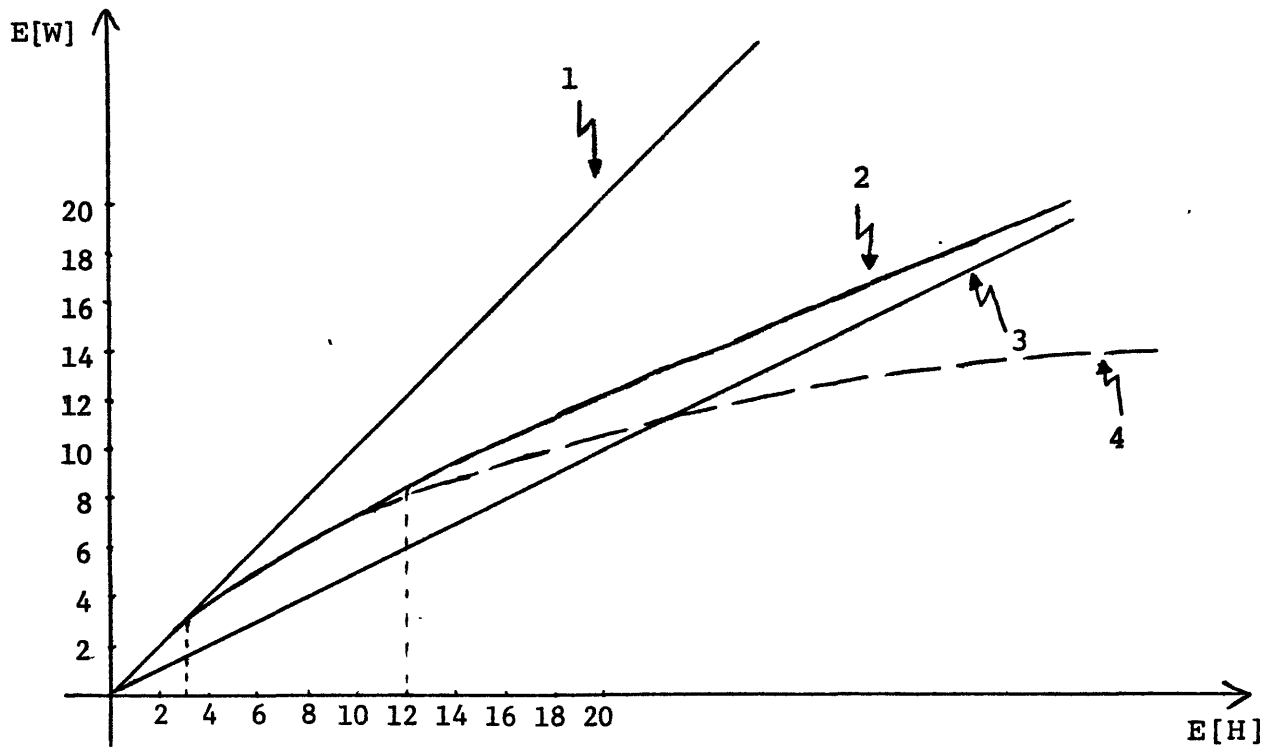


Figure II.1 Schematic Representation of the One Call Box Case

- 1 = $E[W]$ function of $E[H]$ for random passengers and random headway for bus system (equation 2.5).
- 2 = $E[W]$ function of $E[H]$ for random passengers given by equation 2.1 and with the coefficient of variation $\sigma_H^2 / E^2[H]$ approaching 0 as $E[H]$ increases bus system between random headway and "perfectly scheduled" headway)
- 3 = $E[W]$ function of $E[H]$ for random passengers and "perfectly scheduled" bus system (equation 2.4)
- 4 = $E[W]$ function of $E[H]$ for "aware" passengers and bus system as in 2.

Inversely when the mean headway $E[H]$ decreases, the buses tend to have completely random headways; thus we assume that as $E[H]$ goes to zero, σ_H approaches $E[H]$ (random headway) so the coefficient of variation $\frac{\sigma_H^2}{E[H]^2}$ goes to 1 and $E[W] \sim E[H]$ for very small values of $E[H]$ (typical values are under 5 minutes (Turnquist)).

2. We have seen that passengers may coordinate their arrival times with the bus schedule and reduce their waiting time. We expect that for these "aware" passengers the mean waiting time will be less than the expression given in equation (2.1). However, the ability of users to do so will be related to the reliability of the service; as this reliability decreases with the mean headway (indeed, in this case we have seen that the buses tend to have completely random headway) we assume that for small values of $E[H]$ all passengers are random passengers. It is only when $E[H]$ will be large enough to ensure some reliability, that we begin to have "aware" passengers and thus a distinguishable curve for them. (12 minutes is a value for $E[H]$ given by Okrent).

To conclude we can say that in the case of "aware" passengers we face a much more complex process which in fact becomes a clustered process: the next arrival of a prospective passenger is much more likely to occur just after the previous passenger's arrival; this is due to the tendency of passenger arrivals at the stops to be

grouped around the time corresponding to the known schedule. One example of a process with such probabilistic properties is the Negative Binomial Process.

II.2.2 Specification of Checkpoints

We have seen in Chapter I that by definition RDBLs are most appropriate for adoption in areas with demand of intermediate density such as in suburban locations, or in denser urban environments for infrequent demands by special-needs passengers. In both cases we can expect that the number of route deviations will not be too large. We can also reasonably expect that the behavior of these passengers will be different from those observed at fixed stops; indeed the conditions required by the checkpoints system for its possible passengers are closer to those for demand responsive systems rather than to those set by conventional fixed route line. That is, the prospective passenger is more likely to consider this service as a demand responsive one (according to whether he will make a call or not, the bus will make a deviation to pick him up or not) rather than a prescheduled fixed route service. Thus, we feel that in this case it is appropriate to ignore the complications arising from the existence of "aware" passengers and to assume quite reasonably that passengers at checkpoints arrive randomly. Though this cannot be completely true in a real-world system, we think that a Poisson process for

modeling the arrival of passengers at checkpoints can remain the basis for our model.

The average demand rates at each of the candidate deviation checkpoints will be a function of time of the day; in an urban context we can assume that these average demand rates $\lambda(t)$ are periodic with period $\tau = 24$ Hr. Thus we can define subperiods during which the variations of $\lambda(t)$ are weak enough to allow us to derive a model for each subperiod with the assumption of a homogeneous Poisson process with rates λ .

The last assumption we will make is the independence of the arrival processes at different checkpoints. Though convenient and somewhat reasonable for such a system (where the demand this system typically addresses is relatively low), we must be aware that it may not be entirely true. In practice, we may indeed find statistical correlations (positive or negative) between demands at various checkpoints.

II.3 The Service Process

This section presents the assumptions we will make for the operation of buses on the fixed route. We then discuss the perturbations introduced by the checkpoints and finally the general assumptions we will make for the construction of our model.

II.3.1 Assumptions About the Fixed Route

a. Time Spent by a Bus on a Route

To be consistent with the notation generally used in the literature (e.g. Chapman [7]), the time spent by buses in service on the road is referred to as the bus journey time. Bus movements along a route can be analysed in terms of three components: time at bus stops, time between bus stops (travel time) and time at terminals. Time at terminals is usually less than 10 per cent of the journey time (Chapman) and a major part of it is intended to allow for variations in the other components of journey time; so we will assume that the bus journey time refers to time at and between bus stops, but not time at terminals.

Mean Time at Stops

The time spent by a bus on a route at stops is to allow for passenger boarding and passenger alighting and for opening and closing bus doors. Depending on the number of passengers boarding or alighting it has been found that the time spent at a stop by a bus is usually a linear function of the number of passengers boarding or alighting.

We have the following formula:

$$T = C + Bn \quad (2.6)$$

where:

T = stop time

C = Constant (often called dead time)

n = number of passengers boarding or alighting
B = estimate of the marginal boarding or alighting time taken by an extra passenger to board or alight.

The proportion of bus journey time spent by a bus at a bus stop will depend on the number of stops on the route, the boarding and alighting time of passengers and the passengers loadings (vary throughout the day).

Mean Time Spent Between Stops

The mean time spent between stops is the bus travel time (excluding time at stops). The proportion of bus journey time spent by a bus between stops will depend on the speed of other traffic on link, road design, probability of delay, amount of time in each delay.

The overall speed of buses, including stopping time at bus stops, is referred to as the journey speed.

The travel speed (mean travel speed) is the length of a section of route divided by the bus travel time (excluding time at bus stops) over that section. Chapman found that the journey speed is typically about 4 km/h lower than the travel speed (according to a summary of bus surveys in various areas).

b. Sources of Irregularity

From our discussion so far, it is clear that we have several sources of variation for the time spent by a bus on

a route:

- at bus stops = different numbers of passengers boarding (alighting), different boarding (alighting) time;
- between stops = different travel time, different amounts of traffic delay.

All these cause variations in bus headways which are amplified along the route (bus bunching mechanisms).

Chapman showed that during the morning peak period on a Newcastle-upon-Tyne bus route the main features of the variations included in order:

1. variation in time spent between stops (variation in queue delay);
2. variation due to the probability of buses stopping at bus stops
3. variation at bus stops.

c. Assumptions About the Speed of the Vehicles

We have seen two definitions relating to the speed: the journey speed and the travel speed; journey speed is likely to have greater variations than the travel speed due to the added variations in time spent at stops, and constitutes what we will call the speed of the bus.

For any given fixed route, we shall define a random variable S representing the speed of the bus, and its probability density function $f_S(s_0)$. This random variable S (for one bus) depends on the location of the bus along

the route and the time during the day when it makes its trip.

Formally if the variables ℓ, t represent respectively the position along the line ($0 < \ell < L$) and the time of the day, then the speed of the bus is given by the pdf $f_{S(\ell, t)}(s_0)$.

The pdf $f_{S(\ell, t)}(s_0)$ can be assumed to be identical for any bus on the system (in other words, we do not include variabilities due to bus types, driver behavior, etc.)

If we consider that we have a "homogeneous" fixed route (that is, either CBD line or suburban line but not a mixture of those) then we can reasonably assume that the random variable $S(\ell, t)$ does not depend strongly on the location along the route and then can be assumed to be constant according to ℓ : $S(\ell, t) = S(t)$.

We can also assume that we can divide the day in "homogeneous" periods (e.g. morning peak period, off peak period, evening peak period) and then estimate for each period only one random variable S which does not depend on t .

Finally, with these conditions, we can derive for each bus a probability density function $f_S(s_0)$.

Furthermore, we have seen that the assumption of an identical distribution of random variable $S_1, S_2, S_3 \dots S_3, \dots, S_k \dots$ representing respectively the speed of the first, second, third, ..., k^{th} , ..., bus is reasonable (all of those functions being derived for a given fixed route and

a given period of the day).

The assumption of independence between these random variables is stronger; indeed the speed of a given bus will be correlated with the speed of the previous bus because of the components of time spent by a bus on route (see above) and their variabilities (for example, because of the variation of passenger loads, the time spent at bus stops will not be independent; in other words, the probability of stopping at a fixed stop for a given bus will depend on what the previous bus did and thus its journey speed will be affected.).

But if we assume that the spacing of bus stops is such that the occurrence of passenger arrivals and of passenger alighting are sufficiently high to ensure that almost all buses stop (which is a possible way of reducing variability in journey time caused by time at stops) the assumption of independence will depend mainly on the traffic conditions between stops. Here again the "homogeneous" of the route and the application to a specific period of the day will allow us to consider the within-variation sufficiently small to keep the assumption of independence valid.

Under these conditions we can go a step further and assume for a given area and a given subperiod of the day that these random variables can be reasonably estimated by using the average journey speeds = s_0 (identical for all buses).

d. Assumptions about the Headways

We have seen that variations in journey time due to the sources discussed previously cause irregularities in bus headways; the kind of variations we are talking about are not systematic and cannot be predicted to a great extent. Then we can think of bus headways as also being random variables. We will use H_k to denote the random variable that represents the time interval between the passage of the k^{th} and $k-1^{\text{th}}$ bus from a given point.

Using the same arguments as for the speed we can assume that for a given area (given route) and given period of the day (both being "homogeneous") these random variables are identically distributed with a pdf $f_H(h)$.

The assumption of independence between successive headways is more difficult. Indeed in addition to the same considerations as for the journey speed, the dependence between bus headways is amplified along the route by means of the bus bunching mechanism. However, the assumption of independence between headways is very often indispensable to obtain a tractable model.

Newell and Potts [8] have suggested that the pairing of buses is primarily caused by the variation in the time taken to load passengers. They considered a service of buses which leave at regular intervals to pick up passengers in a city during a busy period. In the mathematical model of the pairing of buses, they showed that two "passenger rates"

play an important role, namely the rate at which passengers arrive at a stop, and the rate at which passengers load on a bus. The ratio of these two rates, a dimensionless constant, determines the strength of the tendency of the bus to pair. The smaller this ratio the slower will be the amplification of this tendency, and the better will be the approximation given by the assumption of independence between headways.

In the derivation of our model we will consider a general distribution, if possible, for our headways. The assumption of perfect scheduled headways H_0 will be a convenient one for beginning our analysis but is very difficult to support even if we assume the conditions given above about the homogeneity of the operation of the line. Its interest resides in the tradeoffs between the difficulties of the real world and the simplifications needed for a tractable model. However it should be emphasized that this last assumption (perfectly scheduled headways) must be viewed only as a way of obtaining insights into the proposed problem and that the need for a general distribution is indispensable to represent the reality to an acceptable degree of approximation.

II.3.2-Irregularity Introduced by the Checkpoints

We have seen (Chapter I) that checkpoints are established at some distance from the regular route and are visited

by a bus only in response to a demand. So each time a request arises at these checkpoints the bus which will respond to this demand will undergo a change in its mode of operation.

Since the system is designed to make a small number of deviations, the number of buses affected by this will have to be not too large. However it is worthwhile for us to understand the two main effects of these deviations.

First, we will have to consider the time spent at route deviation stops; we know that the time spent by a bus on a route at fixed stops is due to three factors: passengers boarding, passengers alighting, and opening and closing of doors. In the case of checkpoints stops, where boarding events govern the stop time, the formula given by equation (2.6) can be approximated by a constant: T_1 ; indeed, by definition of a RDBL the ratio indicated by Newell and Potts (ratio of passenger arrival rate to passenger boarding rate) will be small and so will be the variations of time spent at these checkpoints for different buses.

The second effect is introduced by the penalty for stopping. If the cruising speed is v , the average deceleration b , and the average acceleration a , then the penalty for stopping is given by

$$\frac{v}{2} \left(\frac{1}{a} + \frac{1}{b} \right) \quad (2.7)$$

(assuming that the bus stops are not closer together than

$\frac{v^2}{2} (\frac{1}{a} + \frac{1}{b})$; otherwise buses could not reach their cruising speed v before having to slow down for the next stop). Here again we can assume that this is a constant T_2 for all buses.

Thus for the derivation of our models, each time a bus makes a deviation, in addition to the time needed to go to the checkpoints (travel time), we will add what we call a dwell time which will be $t_d = T_1 + T_2$.

II.4 Overall Assumptions

Before concluding Chapter II we summarize our overall assumptions concerning the conditions under which we will derive our models.

II.4.1 Ridership

We will consider the demand variables as a given input in our model; in other words, we will assume a fixed ridership, that is, not influenced by any characteristics such as fares, levels of service, etc. Obviously, this assumption can be seen as reasonable in the short run, lapse of time in which we will explore the RDBLs.

II.4.2 Bus Capacity

We will assume that the capacity of buses is sufficiently large so that a prospective passenger, at any checkpoint along the route, during any time of the day, can board the first vehicle which responds to this demand.

This assumption avoids the difficulties which could arise in case of capacity constraints; namely, the waiting time of the passenger at the checkpoint would increase, the operating strategy would become more complicated and above all the behavior of buses would become less "predictible" even in probabilistic terms (indeed, in case of a capacity constraint, the probability of making a deviation for a bus would not be simply related to the probability of having a request) introducing more complexities in the deviation of our results.

In the real world the determination of capacity (system capacity) is the result of tradeoffs between users ("level of service") and operator ("cost of providing this level of service") and it is true that such guarantee (no capacity constraints) is not always met along the whole route ("overload" section) or during all periods of the day ("peak" period).

However for our purposes, by assuming a "homogeneous" line (e.g. CBD only) and by isolating "homogeneous" periods during the day (e.g. peak period) we can assume that the level of service for each period may approach our theoretical situation of infinite capacity.

II.4.3 Control Strategy

Our "standard" operating behavior will be the following: in case of a request at one checkpoint, this checkpoint will be visited by the next bus on the route (this bus being

either on the fixed route, or proceeding on a deviation toward this checkpoint in response to a previous call, or already at this checkpoint).

We may note that implicit in this strategy is the assumption of infinite capacity of the bus; we must emphasize also that this "strategy" implies that a single and isolated call (in time, at a given checkpoint) makes the deviation necessary for the next bus.

Alternative strategies might be to place a lower bound on the number of calls at a checkpoint before a deviation can be made toward this checkpoint (decreasing the operating cost but increasing the waiting time of passengers at call boxes); or one might place an upper limit on the number of deviation checkpoints that can be visited during any single trip no matter what the number of requests is; or, one might establish that some combinations of checkpoints will be given priority over other combinations.

However it seems reasonable to first look at our simple "standard" strategy before going through these alternative strategies (certainly more appropriate for a real world system).

II.5 Conclusion

We have reviewed in that chapter common problems for the operation of buses and specifically studied the two generic processes of a bus system:

- arrival process of passenger at the system
- service process.

We have noted the strong interdependence between these two processes: the arrival process of the passenger depends on the reliability and frequency of the bus schedule and this reliability depends on the arrival process of the passengers.

By decomposing each process we have been able to explain the assumption we are going to use in the next chapters.

We will derive closed form expressions for the one call box case in Chapter III and will describe the way of solving the two call boxes case (Chapter IV) and then the n call boxes case (Chapter V) next.

CHAPTER III

ANALYSIS OF THE ONE-CALL BOX CASE

III.1 The Problem; Introduction, Notation

III.1.1 Introduction

A schematic representation of a simple route-deviation bus line with one call box (or "deviation checkpoint") is given in Figure III.1. The general operating characteristics of RDBLs are described in Chapter I (I.2).

Assuming that we know the characteristics of the fixed route bus line without deviations (that is, the effective travel speed of the vehicles, the headways between buses) we would like to study the probabilistic effects introduced by the perturbations of a call box. As pointed out in Chapters I and II the effects introduced can be estimated by deriving the probability mass function of the random variable "time between A and B for a given bus".

Before introducing our notation it is important to clarify one additional point regarding the operation of RDBLs. We have seen in I.2 that the call box is visited by a bus on route only in response to a demand made known to the bus operator (and then bus driver) through a communication device. It will be assumed in all the following chapters that the technology chosen allows direct

communications between the demand and the bus driver. It will be assumed that a demand is known instantly by the bus driver currently driving the bus. Thus, the bus driver will be able to respond to this demand as long as he is still either waiting at A to depart on his route or driving between A and G_1 (see Figure III.1). If this demand were not known instantly we would have to consider only the length AG_1' to be a possible field of reception (G_1' being the point along the fixed route after which a bus driver would not be able to respond to a demand; that is, if the bus driver is between G_1' and G_1 when a demand arises, this demand would not be accepted by this bus). The assumption of instantaneous communications allows us to consider $G_1'G_1$ as negligible.

III.1 Notation

a. The Geometry of the Situation (see Figure III.2)

Without loss of generality we can assume that the bus goes to C, picks up passengers there and then comes back through the same route; with reference to Figure III.1, we have assumed that G_1 and F_1 are the same (G in Figure III.2). The reason is very simple: the only thing that matters (with respect to these distances) is the distance added through a deviation. In terms of probabilistic behavior this added distance is a deterministic value (that is, the route chosen to go to the call box is defined

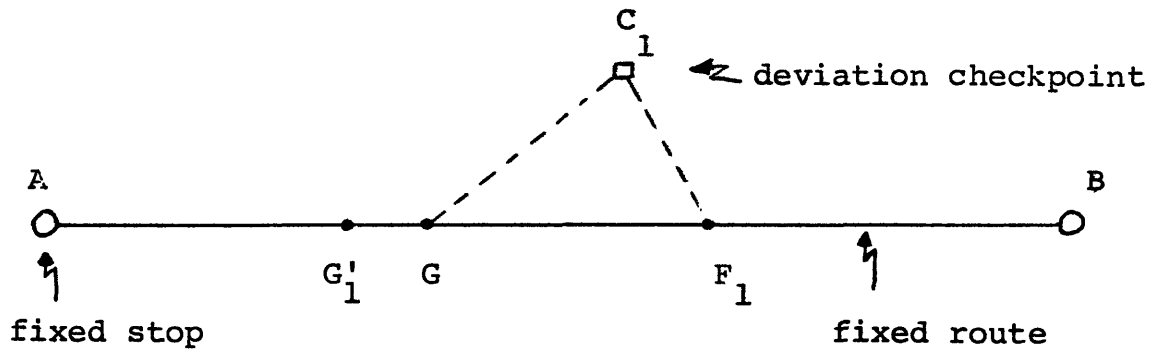


Figure III.1 Schematic Representation of the One Call Box Case

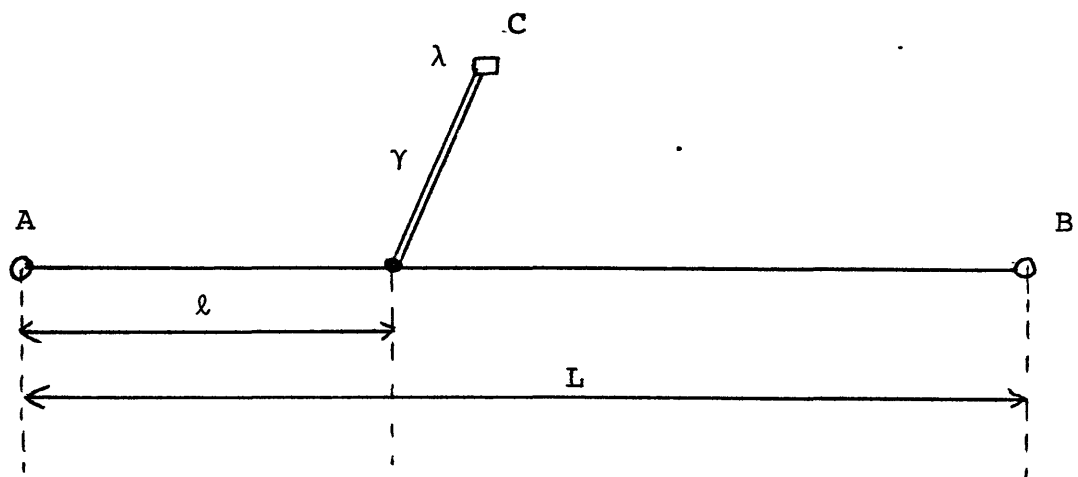


Figure III.2 Geometry of the Situation in the One Call Box Case

as fixed) and thus the above assumption is only made for the purpose of making the derivation as simple as possible.

Our geometrical parameters are then reduced to the following three:

- L = distance of the fixed route between the two fixed stops A and B;
- l = distance of the fixed route between the fixed stop A and the point G where the possible deviation begins;
- γ = distance added by a deviation; γ is two times the distance between G and C.

b. The Demand Characteristics at the Call Box

As explained in Chapter II (II.2) we assume that the arrival of requests at the call box is described by an homogeneous (non-time dependent) Poisson process with an average demand rate λ .

c. The Operational Characteristics

As described in Chapter II (II.3) our assumptions are the following:

- We assume a constant effective travel speed s_0 for all the buses; thus, we will call $t_0 = \frac{L}{s_0}$ the time needed by a bus to go from A to B without deviation.

- The headways between buses are described by the random variable H_k $k = 2, 3, 4, \dots$. H_k = number of units of time the k^{th} bus will start after the $k-1^{\text{th}}$ bus from A.

We assume that the random variables H_k $k = 2, 3, 4, \dots$ are independently and identically distributed with a probability density function $f_H(h)$.

d. The Probabilistic Modeling Experiment

We start observing the system at $t = 0$; we assume that at this time the first bus starts from A and that before $t = 0$ no requests were registered.

The random variable of interest is T_k : this is the time duration of the k^{th} trip between A and B (including or not including a deviation).

So what we want to find is the probability mass function $f_{T_k}(t)$ of the random variable T_k for $k = 1, 2, 3, 4, \dots$. We denote as V_k the event: "there is a deviation during the k^{th} trip"; and $p(V_k)$ represents the probability that the event V_k occurs; we will note \bar{V}_k the complement of event V_k .

e. Summary of all the notations: (see Figure III.2)

L = length (distance) of the fixed route between the two fixed stops A and B;

l = distance between A and G;

γ = average demand rate of the Poisson process modeling the arrival of requests at the call box C.

s_0 = constant effective travel speed for all buses.

H_k , $k = 2, 3, 4, \dots$ = headway between the k^{th} trip

and the $k-1^{\text{th}}$ trip at A;

- $f_H(h)$ = probability density function of the headways;
 T_k = random variable time between A and B for the k^{th} trip (including or not a deviation);
 V_k = event: "there is a deviation for the k^{th} trip";
 $p(V_k)$ = probability of occurrence of V_k
 t_d = dwell time at call box C (time to account for the process of picking up passengers);
 $E []$ = expectation of the random variable used in argument.

III.2 Case With a Deterministic Headway

It will be assumed in this section that $H_k = H_0$ $k = 2, 3, 4, \dots$. In other words, the headways between successive buses are considered as deterministic variables, all equaled to the constant H_0 .

III.2.1 Derivation for the First Bus

We have two possibilities for T_1 the random variable representing the time between A and B for the first bus.

Either we have no deviation and then $t = t_0 = \frac{L}{s_0}$ or we have a deviation and then $t = t_0 + \frac{\gamma}{s_0} + t_d = \frac{L+\gamma}{s_0} + t_d$ (when we have a deviation the bus covers the additional distance γ at a speed s_0 and the dwell time t_d must be also added). We have now to determine the probability of these events: the first one corresponds to \bar{V}_1 and the second to V_1 .

We have:

$$p(V_1) = \Pr(\text{there is a deviation}) = \Pr \left(\begin{array}{l} \text{there is at least} \\ \text{one call during the} \\ \text{time the bus is} \\ \text{between A and } G_1 \end{array} \right)$$

The time between A and G_1 is $\frac{\ell}{s_0}$ and as we have a Poisson Process with rate λ .

$$p(V_1) = 1 - \Pr \left\{ \begin{array}{l} \text{there are no calls} \\ \text{during } \ell/s_0 \end{array} \right\} = 1 - e^{-\frac{\lambda \ell}{s_0}} \quad (3.1)$$

$$p(\bar{V}_1) = 1 - p(V_1) = e^{-\frac{\lambda \ell}{s_0}} \quad (3.2)$$

So the probability mass function (pmf) for T_1 is described by:

$$f_{T_k}(t) = \begin{cases} e^{-\frac{\lambda \ell}{s_0}} & \text{if } t = \frac{L}{s_0} \\ 1 - e^{-\frac{\lambda \ell}{s_0}} & \text{if } t = \frac{L+\gamma}{s_0} + t_d \\ 0 & \text{otherwise} \end{cases} \quad (3.3)$$

The expected value of T_1 is given by:

$$E[T_1] = \frac{L}{s_0} (e^{-\frac{\lambda \ell}{s_0}}) + \left(\frac{L+\gamma}{s_0} + t_d \right) (1 - e^{-\frac{\lambda \ell}{s_0}})$$

$$\text{or } E[T_1] = \left(\frac{\gamma}{s_0} + t_d\right) (1 - e^{-\frac{\lambda \ell}{s_0}}) + \frac{L}{s_0} \quad (3.4)$$

We can easily check that if:

1. $\lambda = 0$ (never requests at C)

then

$$E[T_1] = \frac{L}{s_0} = t_0 \quad (\text{time without deviation})$$

2. $\lambda \rightarrow +\infty$ (always requests at C)

$$\lim_{\lambda \rightarrow +\infty} E[T_1] = \frac{L+\gamma}{s_0} + t_d \quad (\text{time without deviation})$$

The variance of T_1 is given by $\sigma_{T_1}^2 = E[T_1^2] - E[T_1]^2$
and we obtain

$$\sigma_{T_1}^2 = \left(\frac{\gamma}{s_0} + t_d\right)^2 (e^{-\frac{\lambda \ell}{s_0}}) (1 - e^{-\frac{\lambda \ell}{s_0}}) \quad (3.5)$$

III.2.2 Derivation of f_{T_k} , $k = 2, 3, 4, \dots$

According to the previous section (III.1 2)d) and since we have a constant headway H_0 between successive buses, we know that the second bus starts H_0 units of time after the first bus from A. Then the third one starts H_0

time units after the second and so on. In order to fully describe the behavior of the second bus (or more precisely of the second trip) we have to introduce conditional probabilities. Indeed, depending on what was done by the first bus the results for the second will differ. So we will have to determine $p(V_2|V_1)$ and $p(V_2|\bar{V}_1)$ where:

$p(V_2|V_1)$ = probability that the second bus makes a deviation (given that the first bus made one)

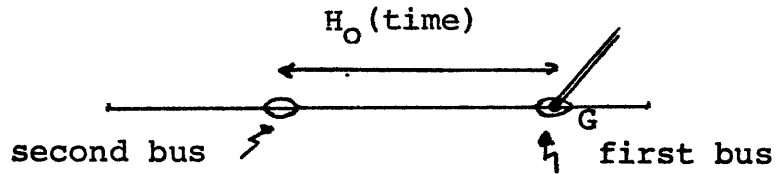
$p(V_2|\bar{V}_1)$ = probability that the second bus makes a deviation (given that the first bus did not did not make one)

To understand why the event V_2 depends on what the first bus did let's look at the difference of time which separates the two buses at point G: (see Figure III.3 for a graphical explanation).

- if the first bus did not make a deviation, the second (which behaves like the first in terms of its speed) will arrive at G exactly H_0 times after the first one, and since we have a Poisson process modeling the demand at the call box, the probability of deviation for this second bus will simply be the probability that we have at least one request at C during these H_0 units of time.

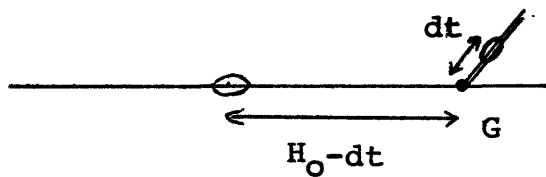
At a given t the first bus arrives at G

(H_0 units of time separate the second bus from the first)



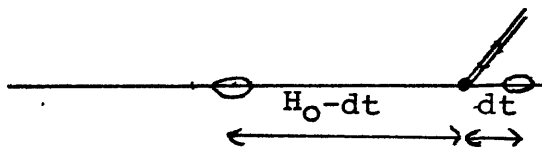
At $t + dt$

if we have a deviation:



(A new call at this time is still answered by the first bus)

if we have no deviation:



(The new request will be answered by the second bus)

Figure III.3 Schematic Representation of the Dependence Between Successive Buses

so $p(V_2|\bar{V}_1) = \Pr \left\{ \text{at least one request during } H_0 \right\} =$
 $1 - \Pr \left\{ \text{no requests during } H_0 \right\}$

thus $p(V_2|\bar{V}_1) = 1 - e^{-\lambda H_0}$ (3.6)

- if the first bus made a deviation, the second will still arrive at G exactly H_0 units of time after the first one did; but two situations can be possible:

1. either H_0 is smaller than the time required to go from G to C (that is $\frac{\gamma}{2s_0}$) plus the dwell time required at C (that is t_d). In that case the second bus will pass G before the first one will leave C and so all the possible requests for the second bus will be answered by the first one

thus $p(V_2|V_1) = 0$ (3.7a)

2. or H_0 is greater than $\frac{\gamma}{2s_0} + t_d$. In that case the probability of a deviation for the second bus will be the probability of having at least a request during $H_0 - \frac{\gamma}{2s_0} - t_d$ (length of time between the instant the first bus leaves C and the instant the second bus reaches G).

thus $p(V_2|V_1) = 1 - e^{-\lambda(H_0 - \frac{\gamma}{2s_0} - t_d)}$ (3.7b)

New assumption:

We have seen (equations (3.7a) and (3.7b)) that

$$p(V_2|V_1) = \begin{cases} 1 - e^{-\lambda(H_0 - \frac{\gamma}{2s_0} - t_d)} & \text{if } H_0 > \frac{\gamma}{2s_0} + t_d \\ 0 & \text{if } H_0 \leq \frac{\gamma}{2s_0} + t_d \end{cases}$$

Let's assume that $H_0 \leq \frac{\gamma}{2s_0} + t_d$:

That means that the 2nd bus will never make a deviation if the first one made one; so for the third bus we have only one possibility: $p(V_3) = p(V_3|\bar{V}_2)$. But this probability depends again on what did the first bus and on the value of H_0 ($2H_0 > \frac{\gamma}{2s_0} + t_d$ or $2H_0 \leq \frac{\gamma}{2s_0} + t_d$).

For the purpose of this thesis we will address only the case where $H_0 > \frac{\gamma}{2s_0} + t_d$,

As V_1 and \bar{V}_1 are complementary and collectively exhaustive we have

$$p(V_2) = p(V_2|V_1)p(V_1) + p(V_2|\bar{V}_1)p(\bar{V}_1)$$

or, using $p(\bar{V}_1) = 1 - p(V_1)$

$$p(V_2) = p(V_1)(p(V_2|V_1) - p(V_2|\bar{V}_1)) + p(V_2|\bar{V}_1) \quad (3.8)$$

The pmf for T_2 is then described by:

$$f_{T_2}(t) = \begin{cases} p(V_2) & \text{if } t = \frac{L + \gamma}{s_0} + t_d \\ p(\bar{V}_2) = 1 - p(V_2) & \text{if } t = \frac{L}{s_0} \\ 0 & \text{otherwise} \end{cases} \quad (3.9)$$

where $P(V_2)$ is given by substitution of (3.1), (3.6) and (3.7b) in (3.8).

The reasoning that we have just applied between the second bus and the first one can be applied between the third bus and the second bus and so on. Because of the way the system is operated (every bus starts, from A, H_0 time units after the previous one) we can easily see that

$$p(V_3|V_2) = p(V_2|V_1)$$

$$p(V_3|\bar{V}_2) = p(V_2|\bar{V}_1)$$

and that for $k = 2, 3, 4, \dots$

$$p(V_k|V_{k-1}) = p(V_2|V_1)$$

$$p(V_k|\bar{V}_{k-1}) = p(V_2|\bar{V}_1) \quad (3.10)$$

Thus in writing equations (3.8) between the k^{th} bus and the $k-1^{\text{th}}$ bus we obtain

$$p(V_k) = p(V_{k-1}) (p(V_k|V_{k-1}) - p(V_k|\bar{V}_{k-1})) + p(V_k|\bar{V}_{k-1})$$

and in using (3.10) we get for $k = 2, 3, 4, \dots$

$$p(V_k) = p(V_{k-1}) [p(V_2|V_1) - p(V_2|\bar{V}_1)] + p(V_2|\bar{V}_1) \quad (3.11)$$

To simplify the notation, let us call

$$p(V_2|V_1) = Q_0$$

$$p(V_2|\bar{V}_1) = R_0$$

Using (3.11) for: $k = 2$, we obtain

$$p(V_2) = p(V_1)(Q_0 - R_0) + R_0$$

$k = 3$, we obtain

$$p(V_3) = p(V_2)(Q_0 - R_0) + R_0 = (p(V_1)(Q_0 - R_0) + R_0) \times (Q_0 - R_0) + R_0$$

or

$$p(V_3) = p(V_1)(Q_0 - R_0)^2 + R_0(1 + Q_0 - R_0)$$

$k = 4$, we obtain

$$p(V_4) = p(V_3)(Q_0 - R_0) + R_0 = (p(V_1)(Q_0 - R_0)^2 + R_0(1 + Q_0 - R_0))(Q_0 - R_0) + R_0$$

or

$$p(V_4) = p(V_1)(Q_0 - R_0)^3 + R_0(1 + (Q_0 - R_0) + (Q_0 - R_0)^2)$$

Thus we can easily see that

$$p(V_k) = p(V_1)(Q_0 - R_0)^{k-1} + R_0 \left(\sum_{m=0}^{k-2} (Q_0 - R_0)^m \right) \quad (3.12)$$

If we now recall that

$$p(V_1) = 1 - e^{-\lambda \frac{L}{s_0}} \quad (3.1)$$

$$Q_0 = 1 - e^{-\lambda (H_0 - \frac{\gamma}{2s_0} - t_d)} \quad (3.7b)$$

$$R_0 = 1 - e^{-\lambda H_0} \quad (3.6)$$

we can check that:

1. if $\lambda = 0$ (no demand at the call box at any time)

$$p(V_1) = 0 \quad Q_0 = 0 \quad R_0 = 0 \quad s_0, p(V_k) = 0 \quad (\text{reasonable})$$

2. if $\lambda \rightarrow +\infty$ (always a demand at the call box)

$$\lim_{\lambda \rightarrow +\infty} p(V_1) = 1 \quad \lim_{\lambda \rightarrow +\infty} Q_0 = 1 \quad \lim_{\lambda \rightarrow +\infty} R_0 = 1$$

$$\lim_{\lambda \rightarrow +\infty} p(V_k) = 1 \quad (\text{reasonable})$$

Having derived $p(V_k)$ the probability of deviation of the k^{th} bus (or k^{th} trip) we have

$$f_{T_K}(t) = \begin{cases} 1 - p(V_k) & \text{if } t = \frac{L}{s_0} \\ p(V_k) & \text{if } t = \frac{L + \gamma}{s_0} + t_d \\ 0 & \text{otherwise} \end{cases} \quad (3.13)$$

III.2.3 Study of $p(V_k)$ as k Goes to Infinity

One important question would be to know if $P(V_k)$ reaches a limit as k goes to infinity; in other words, does the system reach a kind of steady state where the probability of deviation would be this limit?

Subtracting equation (3.7b) from (3.6) gives us:

$$Q_0 - R_0 = e^{-\lambda H_0} - e^{-\lambda(H_0 - \frac{\gamma}{2s_0} - t_d)}$$

thus $-1 \leq Q_0 - R_0 \leq 0$

If we define $W_0 = R_0 - Q_0$ we have:

$$p(V_k) = (-1)^{k-1} p(V_1) W_0^{k-1} + R_0 \sum_{m=0}^{k-2} (-1)^m W_0^m$$

$(0 \leq W_0 \leq 1)$

but,

$$\lim_{k \rightarrow +\infty} (-1)^{k-1} W_0^{k-1} = 0 \quad (0 \leq W_0 \leq 1) \quad (3.14)$$

and as

$$\lim_{k \rightarrow +\infty} \sum_{m=0}^k X_0^k = \frac{1}{1-X_0} \quad \text{for } |X_0| \leq 1 \quad (3.15)$$

we finally obtain (taking $X_0 = -W_0$)

$$\lim_{k \rightarrow \infty} p(V_k) = \frac{R_0}{1 + W_0} = \frac{R_0}{1 + R_0 - Q_0}$$

Furthermore, since $0 \leq Q_0 \leq 1$, then $1 - Q_0 \geq 0$ and

$$R_0 + 1 - Q_0 \geq R_0 \geq 0 \text{ meaning that}$$

$$0 \leq \frac{R_0}{1 + R_0 - Q_0} \leq 1$$

So when the "steady state" is reached the probability of making a deviation for a given bus is given by $\lim_{k \rightarrow \infty} p(V_k) = P$ where

$$P = \frac{R_0}{1 - R_0 - Q_0} = \frac{1 - e^{-\lambda H_0}}{1 - e^{-\lambda H_0} + e^{-\lambda (H_0 - \frac{\gamma}{2s_0} - t_d)}} \quad (3.16)$$

We can check that if:

1. $\lambda = 0$ $P = 0$ (no demand, so no deviation)
2. $\lambda \rightarrow +\infty$ $P \rightarrow 1$ (always a demand, always a deviation)
3. $\frac{\gamma}{2s_0} + t_d \ll H_0$ ($\frac{\gamma}{2s_0} + t_d$ negligible compared to H_0)

then P is approximately equal to $1 - e^{-\lambda H_0}$ which is the probability that a given bus makes a deviation given that the previous one did not make a deviation; this makes sense (the importance of the added distance in case of a

deviation becomes negligible).

The pmf for the time T between A and B of a trip when this steady state is reached is given by

$$f_T(t) = \begin{cases} P & \text{if } t = \frac{L + \gamma}{s_0} + t_d \\ 1-P & \text{if } t = \frac{L}{s_0} \\ 0 & \text{otherwise} \end{cases} \quad (3.17)$$

III.3 Case with a General Distribution for the Headways

As explained in Chapter II we are next going to study a more general case for the headways H_k , $k = 2, 3, 4, \dots$. It will be assumed here that all the random variables H_k , $k = 2, 3, 4, \dots$ are independent and identically distributed with a probability density function (pdf) $f_H(h)$. As we did in the previous section for H_0 , we are going to assume that the possible values of H_k , $k = 2, 3, 4, \dots$ are greater than $\frac{\gamma}{2s_0} + t_d$; that is, $f_H(h)$ is defined for $h > \frac{\gamma}{2s_0} + t_d$. The reasons for such an assumption are the same as for the deterministic case (see III.2). The system is still described by Figure III.1

For the first bus nothing has changed and the probability of making a deviation for this first bus is still

$$p(V_1) = 1 - e^{-\frac{\lambda l}{s_0}} \quad (3.1)$$

For buses after the first one, however, the assumption that headways are random variables is going to introduce additional complications compared to the deterministic case. We first examine the case of the second bus.

In order to understand the situation we have to break down the procedure into several different steps. (All notations employed below have been defined in III.1.)

First, let's assume that the random variable H_2 took the experimental value h ; that is given $H_2 = h$:

$$p(V_2 | \bar{V}_1 \text{ and } H_2 = h) = 1 - e^{-\lambda h} \quad (3.18)$$

$$p(V_2 | V_1 \text{ and } H_2 = h) = 1 - e^{-\lambda(h - \frac{\gamma}{2s_0} - t_d)} \quad (3.19)$$

and the probability of making a deviation for the second bus given $H_2 = h$ is

$$p(V_2 | H_2 = h) = p(V_2 | \bar{V}_1 \text{ and } H_2 = h)p(\bar{V}_1) + p(V_2 | V_1 \text{ and } H_2 = h) p(V_1) \quad (3.20)$$

then the pmf of T_2 given $H_2 = h$ is given by:

$$f_{T_2 | H_2=h}(t | H_2=h) = \begin{cases} p(V_2 | H_2 = h) & \text{if } t = \frac{L + \gamma}{s_0} + t_d \\ 1 - p(V_2 | H_2 = h) & \text{if } t = \frac{L}{s_0} \\ 0 & \text{otherwise} \end{cases} \quad (3.21)$$

And the expected value of T_2 given $H_2 = h$ is

$$E[T_2 | H_2 = h] = \left(\frac{\gamma}{s_0} + t_d\right) P(V_2 | H_2 = h) + \frac{L}{s_0} \quad (3.22)$$

To find the unconditional expectation (this is the second step) we integrate over all possible values of H_2 to obtain:

$$E[T_2] = \int_{\alpha}^{+\infty} E[T_2 | H_2 = h] f_H(h) dh \quad (3.23)$$

$$\text{(where } \alpha = \frac{\gamma}{2s_0} + t_d \text{)}$$

Using (3.22) we have

$$E[T_2] = \left(\frac{\gamma}{s_0} + t_d\right) \int_{\alpha}^{+\infty} P(V_2 | H_2 = h) f_H(h) dh + \frac{L}{s_0} \int_{\alpha}^{+\infty} f_H(h) dh \quad (3.24)$$

And since $\int_{\alpha}^{+\infty} f_H(h) dh = 1$

$$E[T_2] = \left(\frac{\gamma}{s_0} + t_d\right) \int_{\alpha}^{+\infty} P(V_2 | H_2 = h) f_H(h) dh + \frac{L}{s_0}$$

To compute $\int_{\alpha}^{+\infty} P(V_2 | H_2 = h) f_H(h) dh$ we use (3.20) and

through substitution obtain:

$$\int_{\alpha}^{\infty} p(V_2 | \bar{V}_1 \text{ and } H_2 = h) f_H(h) dh = \int_{\alpha}^{+\infty} (1 - e^{-\lambda h}) f_H(h) dh$$

(3.25)

and =

$$\int_{\alpha}^{\infty} p(V_2 | V_1 \text{ and } H_2 = h) f_H(h) dh = \int_{\alpha}^{\infty} (1 - e^{-\lambda(h-\alpha)}) f_H(h) dh$$

(3.26)

Now that we understand how to derive the result for the second bus let's look at the following ones. An additional difficulty will arise from the fact that now we will have to take into account in the derivation of $P(V_3)$ the fact that $P(V_2)$ is a function of h as well, which was not the case for $P(V_1)$ when we derived $P(V_2)$.

Given $H_3 = h$:

$$p(V_3 | \bar{V}_2 \text{ and } H_3 = h) = 1 - e^{-\lambda h}$$

$$p(V_3 | V_2 \text{ and } H_3 = h) = 1 - e^{-\lambda(h - \frac{\gamma}{2so} - t_d)}$$

Since $f_{H_2}(h) = f_{H_3}(h) = f_H(h)$

$$E[p(V_3 | \bar{V}_2)] = \int_{\alpha}^{\infty} (1 - e^{-\lambda h}) f_H(h) dh = E[p(V_2 | \bar{V}_1)]$$

$$E[p(V_3|V_2)] = \int_{\alpha}^{\infty} (1 - e^{-\lambda(h - \alpha)}) f_H(h) dh = E[p(V_2|V_1)]$$

Applying the same reasoning and using the important feature that all the random variables H_k , $k = 2, 3, 4, \dots$ are identically distributed we find that

$$E[p(V_k|V_{k-1})] = E[p(V_2|V_1)] \quad k = 2, 3, 4, \dots \quad (3.27)$$

$$E[p(V_k|\bar{V}_{k-1})] = E[p(V_2|\bar{V}_1)] \quad k = 2, 3, 4, \dots$$

These relations are similar to those we derived between $p(V_k|V_{k-1})$ and $p(V_2|V_1)$ in the deterministic case; but it is important to notice that we cannot derive the same formula as in this deterministic case for (V_k) as a function of (V_1) (see Eq. (3.12)). The equalities given above (3.27) suggest that we could find a similar relationship between $E[p(V_k)]$ and $E[p(V_1)]$:

Given $H_k = h_k$, $H_{k-1} = h_{k-1}$:

$$\begin{aligned} p(V_k|H_k = h_k) &= p(V_{k-1}|H_{k-1} = h_{k-1})p(V_k|V_{k-1} \text{ and } H_k = h_k) \\ &\quad + p(\bar{V}_{k-1}|H_{k-1} = h_{k-1})p(V_k|\bar{V}_{k-1} \text{ and } H_k = h_k) \end{aligned} \quad (3.28)$$

Since H_k and H_{k-1} are independent and identically distributed we have linear independence between

$$p(V_{k-1}|H_{k-1} = h_{k-1}) \quad \text{and} \quad \bar{p}(V_k|V_{k-1} \text{ and } H_k = h_k)$$

and between

$$p(\bar{V}_{k-1} | H_{k-1} = h_{k-1}) \text{ and } p(V_k | \bar{V}_{k-1} \text{ and } H_k = h_k)$$

That is:

$$E[p(V_{k-1})p(V_k | V_{k-1})] = E[p(V_{k-1})] \cdot E[p(V_k | V_{k-1})] \quad (3.29)$$

$$E[p(\bar{V}_{k-1})p(V_k | \bar{V}_{k-1})] = E[p(\bar{V}_{k-1})] \cdot E[p(V_k | \bar{V}_{k-1})]$$

(the formal demonstration is given in Appendix A)

Since

$$E[p(\bar{V}_k)] = \int_{\alpha}^{+\infty} p(\bar{V}_k | H_k = h) f_H(h) dh = \int_{\alpha}^{+\infty} (1 - p(V_k | H_k = h) f_H(h) dh$$

$$\text{We have } E[p(\bar{V}_k)] = 1 - E[p(V_k)] \quad (3.30)$$

Thus taking the expected value of the equation (3.28) and taking advantage of equations (3.29) we derive by the same reasoning as for eq. (3.12)

$$E[p(V_k)] = E[p(V_1)] [E[p(V_2 | V_1)] - E[p(V_2 | \bar{V}_1)]]^{k-1} + E[p(V_2 | \bar{V}_1)] \left[\sum_{m=0}^{k-2} (E[p(V_2 | V_1)] - E[p(V_2 | \bar{V}_1)])^m \right] \quad (3.31)$$

In III.2 we called $P(V_2 | \bar{V}_1) = R_0$ $P(V_2 | V_1) = Q_0$. This

time we will call $E[p(V_2|V_1)] = \tilde{R}_0$ $E[p(V_2|V_1)] = \tilde{Q}_0$ and since $E[p(V_1)] = p(V_1)$, equation (3.31) becomes

$$E[p(V_k)] = p(V_1) [\tilde{Q}_0 - \tilde{R}_0]^{k-1} + \tilde{R}_0 \left[\sum_{m=0}^{k-2} (\tilde{Q}_0 - \tilde{R}_0)^m \right] \quad (3.32)$$

where

$$P(V_1) = 1 - e^{-\frac{\lambda L}{s_0}} \quad (3.1)$$

$$\tilde{Q}_0 = \int_{\alpha}^{+\infty} (1 - e^{-\lambda(h - \alpha)}) f_H(h) dh \quad (3.33)$$

$$\tilde{R}_0 = \int_{\alpha}^{+\infty} (1 - e^{-\lambda h}) f_H(h) dh \quad (3.34)$$

The expected value of T_k given $H_k = h$ is

$$E[T_k | H_k = h] = \left(\frac{\gamma}{s_0} + t_d \right) p(V_k | H_k = h) + \frac{L}{s_0} \quad (3.35)$$

So as for eq. (3.23) the unconditional expectation is

$$E[T_k] = \left(\frac{\gamma}{s_0} + t_d \right) E[p(V_k)] + \frac{L}{s_0} \quad (3.36)$$

Once again the nice result we obtained for $\lim_{k \rightarrow \infty} P(V_k)$ in the deterministic case can still be obtained in this more general case:

Since $0 \leq p(V_2 | V_1 \text{ and } H_2 = h) \leq p(V_2 | \bar{V}_1 \text{ and } H_2 = h) \leq 1$

for all $h > \frac{\gamma}{2s_0} + t_d$

then $0 \leq E[p(V_2|V_1)] \leq E[p(V_2|\bar{V}_1)] \leq 1$

or $0 \leq \tilde{Q}_0 \leq \tilde{R}_0 \leq 1$

so $0 \leq \tilde{R}_0 - \tilde{Q}_0 \leq 1$ (3.37)

Applying this result and using eq. (3.14) eq. (3.15)

$$\lim (-1)^{k-1} [\tilde{R}_0 - \tilde{Q}_0]^{k-1} = 0$$

$$\lim \tilde{R}_0 \left[\sum_{m=0}^{k-2} (-1)^m (\tilde{R}_0 - \tilde{Q}_0)^m \right] = \frac{\tilde{R}_0}{1 + (\tilde{R}_0 - \tilde{Q}_0)} = \tilde{P}$$

As we did in the deterministic case we can check that \tilde{P} behaves correctly; that is:

1. if $\frac{\gamma}{2s_0} + t_d$ becomes negligible then $\tilde{R}_0 \sim \tilde{Q}_0$ and

$$\tilde{P} \sim \tilde{R}_0$$

2. if $\lambda = 0$ $\tilde{P} = 0$

3. if $\lambda \rightarrow \infty$ $\tilde{P} \rightarrow 1$

III.4 Conclusion; Introduction to the Two Call Boxes Case

We have been able to derive $f_{T_k}(t)$, $E[T_k]$ and $\sigma_{T_k}^2$ both the deterministic headway and for the more general assumption of a general distribution function for the headways.

It must be emphasized that for the case of non deterministic headways all our derivations have been strongly based on the assumptions of independent and identically distributed random variables. If we eliminate this assumption our results are not true any more. To be convinced of that, one only has to look at $p(V_k)$ which must be conditioned by $H_{k-1} = h_{k-1}$ which is not independent of the fact that $H_{k-2} = h_{k-2}$ and so on; we thus cannot establish a tractable relationship between $E[p(V_k)]$ and $E[p(V_{k-1})]$ as we did before.

Considering that we have understood pretty well the one call box case, we are going to look next at a more complicated problem = the two call boxes case. The added complexity comes from the fact that in addition to the dependence between successive buses, we also have a dependence between deviations at the two call boxes. That is, the probability of deviation of the second bus at the second call box will depend not only on what the first bus has done during the previous trip but also on what this second bus did at the first call box. We shall address this new problem in Chapter IV.

CHAPTER IV

THE TWO CALL BOXES CASE

IV.1 The Problem; Notation, Approach

A schematic representation of a simple route-deviation bus line with two call boxes is given in Figure IV.1.

In this chapter we would like to study the probabilistic effects introduced by the perturbations of two call boxes. Here again the assumption of instantaneous communications allows us to consider $G_1'G_1'$ and $G_2'G_2'$ as negligible.

IV.1.1 Notation

a. Geometry of the Situation: (see Figure IV.2)

By analogy to the one call-box case we assume, without loss of generality, that $G_1(G_2)$ and $F_1(F_2)$ are the same and we call these points G_1 for the first call box and G_2 for the second call box.

Our "geometrical" parameters are then:

L = distance of the fixed route between the two fixed stops A and B

l_1 = distance of the fixed route between A and G_1

l_2 = distance of the fixed route between A and G_2

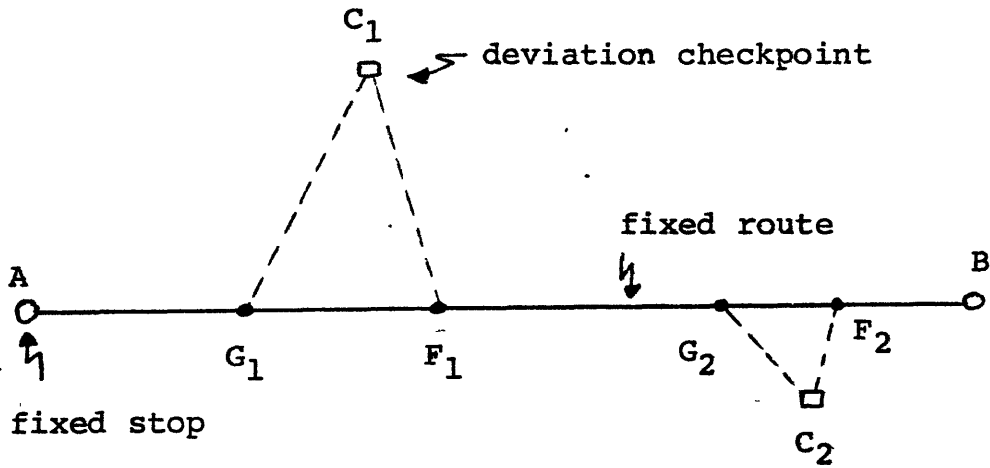


Figure IV.1 The Two Call Boxes Case

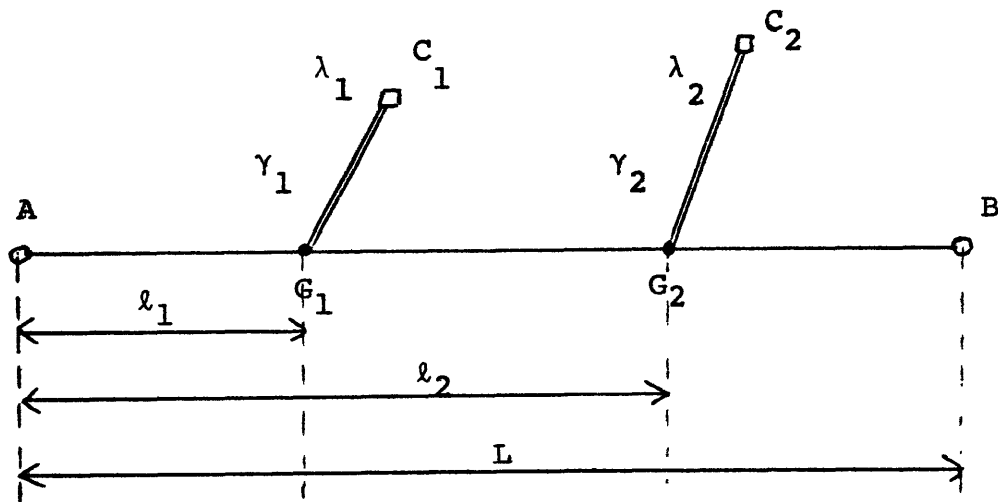


Figure IV.2 Geometry of the Situation in the Two Call Boxes Case

γ_1 = distance added by a deviation at the first call box
box C_1

γ_2 = distance added by a deviation at the second call
box C_2

b. The Demand Characteristics at the Call Boxes

We assume that the arrival of requests at the two call boxes are described by two independent homogeneous (non-time dependent) Poisson Processes with average demand rates λ_1 for C_1 and λ_2 for C_2 .

c. The Operational Characteristics

They are the same as for the one call box (see section III.2 c). We consider the added time given by possible fixed stops between G_1 and G_2 as already included in what we call the time t_0 needed by a bus to go from A to B without deviations.

d. The Probabilistic Modeling Experiment

We start observing the system at $t = 0$; we assume that at this time the first bus starts from A and that before $t = 0$ no requests were registered. The random variable of interest is T_k : this is the time duration of the k^{th} trip between A and B (including or not including a deviation at C_1 and C_2).

We denote as:

V_{k1} the event: "There is a deviation at the call box C_1
during the k^{th} trip"

V_{k2} the event: "there is a deviation at the call box C_2
during the k^{th} trip"

and $p(V_{k1}, V_{k2})$ represents the probability of the event
 $V_{k1} \cap V_{k2}$, intersection of the two events V_{k1}, V_{k2} .

e. Other Notations: (See Figure IV.2 for a summary)

t_d = dwell time at call box C_1 or C_2 (time to account
for the process of picking up passengers; assumed
to be the same for C_1 and C_2)

$E[.]$ = expectation of ".".

IV.1.2 Approach

What we want to find is the probability mass function
 $f_{T_k}(t)$ of the random variable T_k for $k = 1, 2, 3, 4, \dots$

In the two call boxes case we have four different
situations (events) for describing the trip of the k^{th} bus
($k = 1, 2, 3, \dots$) between A and B:

1. No deviations; then the random variable T_k takes on
the experimental value t_0 . This event is described by
 $\bar{V}_{k1} \cap \bar{V}_{k2}$ and its probability of occurrence is $p(\bar{V}_{k1}, \bar{V}_{k2})$

2. A deviation only at C_1 ; then T_k takes on the

experimental value $t_o + \frac{\gamma_1}{s_o} + t_d$ and the probability of occurrence of such an event is $p(V_{k1}, \bar{V}_{k2})$

3. A deviation only at C_2 ; then T_k takes on the experimental value $t_o + \frac{\gamma_2}{s_o} + t_d$ and the probability of occurrence of such an event is $p(\bar{V}_{k1}, V_{k2})$

4. Deviations both at C_1 and C_2 ; then T_k takes on the experimental value $t_o + \frac{\gamma_1 + \gamma_2}{s_o} + 2t_d$ and the probability of occurrence of such an event is $p(V_{k1}, V_{k2})$

Thus, to sum up we have:

$$f_{T_k}(t) = \begin{cases} p(\bar{V}_{k1}, V_{k2}) & \text{if } t = t_o \\ p(V_{k1}, V_{k2}) & \text{if } t = t_o + \frac{\gamma_1}{s_o} + t_d \\ p(\bar{V}_{k1}, V_{k2}) & \text{if } t = t_o + \frac{\gamma_2}{s_o} + t_d \\ p(V_{k1}, V_{k2}) & \text{if } t = t_o + \frac{\gamma_1 + \gamma_2}{s_o} + 2t_d \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

In order to solve this problem we have to determine the following four probabilities: $p(\bar{V}_{k1}, \bar{V}_{k2})$; $p(V_{k1}, \bar{V}_{k2})$; $p(\bar{V}_{k1}, V_{k2})$; $p(V_{k1}, V_{k2})$. It should be emphasized that the events $V_{k1} \cap V_{k2}$, $\bar{V}_{k1} \cap V_{k2}$ and $V_{k1} \cap \bar{V}_{k2}$ form a mutually exclusive and collectively exhaustive list of events. (They

form an event space.)

Similarly to what we did in the one call box case we are going to derive our result in the deterministic case ($H_k = H_0 = \text{constant } k = 2, 3, 4, \dots$), then we will address the problem of a general distribution for H_k . However, as noted in the previous chapter, we shall limit our study to the case where $H_0 > \max\left\{\frac{\gamma_1}{2s_0} + t_d, \frac{\gamma_2}{2s_0} + t_d\right\}$ in the deterministic case and to a pdf $f_H(h)$ defined for $h > \max\left\{\frac{\gamma_1}{2s_0} + t_d, \frac{\gamma_2}{2s_0} + t_d\right\}$ in the general case.

IV.2 Derivation with Deterministic Headways

It will be assumed in this section that $H_k = H_0$, $k = 2, 3, 4, \dots$

IV.2.1 Derivation with Deterministic Headways

The four probabilities of interest for this first bus $p(\bar{V}_{11}, \bar{V}_{12})$, $p(\bar{V}_{11}, V_{12})$, $p(V_{11}, \bar{V}_{12})$, and $p(V_{11}, V_{12})$ can be derived by taking advantage of the relations:

$$p(\bar{A}) = 1 - p(A) \tag{4.2}$$

$$p(A, B) = p(A|B)p(B)$$

1. $p(V_{11})$ is derived by using the equation (3.1)

$$p(V_{11}) = 1 - e^{-\frac{\lambda_1 \ell_1}{s_0}} \quad (4.3)$$

2. $p(V_{12}|\bar{V}_{11})$

$$p(V_{12}|\bar{V}_{11}) = \text{pr} \left\{ \text{there is a call from } C_2 \text{ during } \frac{\ell_2}{s_0} \right\} = 1 - \text{pr} \left\{ \text{no calls from } C_2 \text{ during } \frac{\ell_2}{s_0} \right\}$$

then

$$p(V_{12}|\bar{V}_{11}) = 1 - e^{-\frac{\lambda_2 \ell_2}{s_0}} \quad (4.4)$$

3. $p(V_{12}|V_{11})$ (The bus makes deviation at C_1 so its time of arrival at G_2 is increased by $\frac{\gamma_1}{s_0} + t_d$)

$$p(V_{12}|V_{11}) = \left\{ \begin{array}{l} \text{there is at least a call from } C_2 \text{ during} \\ \text{the length of time } \frac{\ell_2}{s_0} + \frac{\gamma_1}{s_0} + t_d \end{array} \right\}$$

$$= 1 - \text{pr} \left\{ \text{no calls from } C_2 \text{ during } \frac{\ell_2}{s_0} + \frac{\gamma_1}{s_0} + t_d \right\}$$

$$\text{then } p(V_{12}|V_{11}) = 1 - e^{-\lambda_2 \left(\frac{\ell_2 + \gamma_1}{s_0} + t_d \right)} \quad (4.5)$$

So the probability mass function (pmf) for T_1 is described by:

$$\left[\begin{array}{ll}
 e^{-\lambda_1 \frac{l_1}{s_0}} e^{-\lambda_2 \frac{l_2}{s_0}} & \text{if } t = t_0 \\
 (1 - e^{-\lambda_1 \frac{l_1}{s_0}}) e^{-\lambda_2 (\frac{l_2 + \gamma_1}{s_0} + t_d)} & \text{if } t = t_0 + \frac{\gamma_1}{s_0} + t_d \\
 e^{-\lambda_1 \frac{l_1}{s_0}} (1 - e^{-\lambda_2 \frac{l_2}{s_0}}) & \text{if } t = t_0 + \frac{\gamma_2}{s_0} + t_d \\
 (1 - e^{-\lambda_1 \frac{l_1}{s_0}}) (1 - e^{-\lambda_2 (\frac{l_2 + \gamma_1}{s_0} + t_d)}) & \text{if } t = t_0 + \frac{\gamma_1 + \gamma_2}{s_0} + t_d \\
 0 & \text{otherwise}
 \end{array} \right. \quad (4.6)$$

IV.2.2 Derivation for Successive Buses

We know that the second bus starts H_0 units of time after the first bus from A. Then the third one starts H_0 units of time after the second and so on.

Here again, in order to fully understand the behavior of the second bus we have to introduce conditional probabilities. (depending on what the first bus did on its trip, the results for the second one will differ.)

What we would like to find is a relationship between what the bus does and what its previous bus did (by analogy of the one call box case, see equations (3.8), (3.10), and

(3.11)†

To simplify the notation we will note:

$$\begin{aligned} A_k &= (\bar{V}_{k1} \cap \bar{V}_{k2}), & B_k &= (V_{k1} \cap \bar{V}_{k2}) \\ C_k &= (\bar{V}_{k1} \cap V_{k2}), & D_k &= (V_{k1} \cap V_{k2}) \end{aligned} \quad (4.7)$$

and we would like to determine

$$p(A_k), \quad p(B_k), \quad p(C_k), \quad \text{and} \quad p(D_k)$$

We have seen in section IV.1 that for $k = 1, 2, 3, \dots$ the events $A_k, B_k, C_k,$ and D_k are mutually exclusive and collectively exhaustive, so:

$$p(A_k) + p(B_k) + p(C_k) + p(D_k) = 1; \quad k, = 1, 2, 3, \dots \quad (4.8)$$

Thus we can find four relationships between the k^{th} bus and the $k-1^{\text{th}}$ bus:

$$\begin{aligned} p(A_k) &= p(A_k | A_{k-1})p(A_{k-1}) + p(A_k | B_{k-1})p(B_{k-1}) + \\ &\quad p(A_k | C_{k-1})p(C_{k-1}) + p(A_k | D_{k-1})p(D_{k-1}) \\ p(B_k) &= p(B_k | A_{k-1})p(A_{k-1}) + p(B_k | B_{k-1})p(B_{k-1}) + \\ &\quad p(B_k | C_{k-1})p(C_{k-1}) + p(B_k | D_{k-1})p(D_{k-1}) \end{aligned} \quad (4.9)$$

$$\begin{aligned} p(C_k) &= p(C_k | A_{k-1})p(A_{k-1}) + p(C_k | B_{k-1})p(B_{k-1}) + \\ &\quad p(D_k | C_{k-1})p(C_{k-1}) + p(D_k | D_{k-1})p(D_{k-1}) \end{aligned} \quad (\text{cont'd.})$$

$$p(D_k) = p(D_k | A_{k-1})p(A_{k-1}) + p(D_k | B_{k-1})p(B_{k-1}) + \\ p(D_k | C_{k-1})p(C_{k-1}) + p(D_k | D_{k-1})p(D_{k-1})$$

and using the relation (4.8) for k and $k-1$ we see that the system of equations (4.9) is equivalent to three of the four equations of (4.9) plus the relation (4.8).

The operation of the system allows us to say that:

$$p(A_k | A_{k-1}) = p(A_k | A_1) \quad (4.10)$$

and this is true for the 15 other conditional probabilities of system (4.9) (that is, $p(A_k | B_{k-1}) = p(A_2 | B_1)$, $p(D_k | C_{k-1}) = p(D_2 | C_1)$, etc.

We denote all these conditional probabilities in the form

$$P_{A,A} = p(A_k | A_{k-1}) = p(A_2 | A_1) \quad (4.11)$$

The system (4.9) becomes

$$p(A_k) = P_{A,A} p(A_{k-1}) + P_{A,B} p(B_{k-1}) + P_{A,C} p(C_{k-1}) + \\ P_{A,D} p(D_{k-1}) \\ p(B_k) = P_{B,A} p(A_{k-1}) + P_{B,B} p(B_{k-1}) + P_{B,C} p(C_{k-1}) + \\ P_{B,D} p(D_{k-1}) \quad (4.12) \\ p(C_k) = P_{C,A} p(A_{k-1}) + P_{C,B} p(B_{k-1}) + P_{C,C} p(C_{k-1}) + \\ P_{C,D} p(D_{k-1}) \\ p(D_k) = 1 - p(A_k) = p(B_k) = p(C_k) \\ p(D_{k-1}) = 1 - p(A_{k-1}) - p(B_{k-1}) - p(C_{k-1})$$

The derivation of closed form expressions for $p(A_k)$, $p(B_k)$, $p(C_k)$, or $p(D_k)$ as functions of $p(A_1)$, $p(B_1)$, $p(C_1)$, and $p(D_1)$, from this system is a cumbersome procedure. So instead, we are going to use a different method in order to find these probabilities in the steady state; that is, when k goes to infinity (if they exist). This method is quite simple and could also have been used in order to derive P in the one call box case:

Let us assume that $p(A_k)$, $p(B_k)$, $p(C_k)$, and $p(D_k)$ tend to a limit when k goes to infinity and let us note these limits $\lim_{k \rightarrow \infty} p(A_k) = P_A$, $\lim_{k \rightarrow \infty} p(B_k) = P_B$, $\lim_{k \rightarrow \infty} p(C_k) = P_C$ and $\lim_{k \rightarrow \infty} p(D_k) = P_D$. Then taking the limits of the system (4.12) gives us at infinity:

$$\begin{aligned}
 P_A &= P_{A,A} P_A + P_{A,B} P_B + P_{A,C} P_C + P_{A,D} P_D \\
 P_B &= P_{B,A} P_A + P_{B,B} P_B + P_{B,C} P_C + P_{B,D} P_D \\
 P_C &= P_{C,A} P_A + P_{C,B} P_B + P_{C,C} P_C + P_{C,D} P_D \\
 P_D &= 1 - P_A - P_B - P_C
 \end{aligned}
 \tag{4.13}$$

And solving this system of four equations with four unknowns allows us to find P_A , P_B , P_C , and P_D .

Recalling that

$$\begin{aligned}
 P_A &= \lim_{k \rightarrow \infty} p(A_k) = \lim_{k \rightarrow \infty} p(\bar{V}_{k1}, \bar{V}_{k2}) \\
 P_B &= \lim_{k \rightarrow \infty} p(B_k) = \lim_{k \rightarrow \infty} p(V_{k1}, \bar{V}_{k2}) \\
 P_C &= \lim_{k \rightarrow \infty} p(C_k) = \lim_{k \rightarrow \infty} p(\bar{V}_{k1}, V_{k2}) \\
 P_D &= \lim_{k \rightarrow \infty} p(D_k) = \lim_{k \rightarrow \infty} p(V_{k1}, V_{k2})
 \end{aligned}
 \tag{4.14}$$

we find that the pmf for the time T between A and B of a trip when this steady state is reached is given by:

$$f_T(t) = \begin{cases} P_A & \text{if } t = t_0 \\ P_B & \text{if } t = t_0 + \frac{\gamma_1}{s_0} + t_d \\ P_C & \text{if } t = t_0 + \frac{\gamma_1}{s_0} + t_d \\ P_D & \text{if } t = t_0 + \frac{\gamma_1 + \gamma_2}{s_0} + 2t_d \\ 0 & \text{otherwise} \end{cases} \quad (4.15)$$

As noted earlier this method could have been used for the one call box case in order to find P (see section III.2.3)

Assuming that $\lim_{k \rightarrow \infty} p(V_k) = P$ we could have found P by taking the limit in equation (3.11); that is:

$$\lim_{k \rightarrow \infty} p(V_k) = \lim_{k \rightarrow \infty} p(V_{k-1}) [Q_0 - R_0] + R_0$$

or

$$P = P(Q_0 - R_0) + R_0$$

which gives $P = \frac{R_0}{1 + R_0 - Q_0}$ as before.

Note:

Though we are not going to solve the system (4.13), we could show how to derive $P_{D,D}$ in this two call boxes case before going to the non deterministic case:

$$P_{D,D} = p(D_2|D_1) = p(V_{12}, V_{22}|V_{11}, V_{21}) \quad (\text{using 4.11 and 4.7})$$

Also

$$p(V_{12}, V_{22}|V_{11}, V_{21}) = P \frac{p(V_{12}, V_{22}, V_{11}, V_{21})}{p(V_{11}, V_{21})} \quad (4.16)$$

Using

$$p(V_{12}, V_{22}, V_{11}, V_{21}) = p(V_{22}|V_{12}, V_{21}, V_{11})p(V_{12}, V_{21}, V_{11})$$

and

$$p(V_{12}, V_{21}, V_{11}) = p(V_{21}|V_{12}, V_{11}) p(V_{12}, V_{11})$$

and through substitution in (4.16) we get

$$P_{DD} = \frac{p(V_{22}|V_{12}, V_{21}, V_{11})p(V_{21}|V_{12}, V_{11})p(V_{12}, V_{11})}{p(V_{21}, V_{11})} \quad (4.17)$$

Noting that $p(V_{21}|V_{12}, V_{11}) = p(V_{21}|V_{11})$

and $p(V_{12}, V_{11}) = p(V_{12}|V_{11})p(V_{11})$

we finally found that

$$P_{D,D} = p(V_{22}|V_{12}, V_{21}, V_{11})p(V_{12}|V_{11}) \quad (4.18)$$

where $p(V_{12}|V_{11})$ is given by (4.5)

$$\text{and } p(V_{22}|V_{12}, V_{21}, V_{11}) = 1 - e^{-\lambda_2(H_0 - \frac{\gamma_2}{2s_0} - t_d)} \quad (4.19)$$

The fifteen other probabilities are derived in the same way ($P_{A,A}$; $P_{A,D}$, etc.)

Thus, using a computer to determine the numerical values of all these probabilities and then using an algorithm for solving the system of equations (4.13) will allow us to determine the exact pmf for T , the time for the trip between A and B when steady state is reached. From the pmf we can also obtain $E[T]$ and σ_T^2 . As we can note from our work, the simplified derivation in case of this "steady state" raises the interesting practical issue of the speed of convergence toward steady state.

In the next section of this chapter we are going to address the case of a general distribution $f_H(h)$ for the headways.

IV.3 General Distribution for the Headways

Instead of giving closed-form derivation of the results for this section we are going to describe its outline relying heavily on what we did for the one call box case and pointing out the differences (if any).

Recall that all the random variables, H_k , $k = 2, 3, 4, \dots$ are independently and uniformly distributed with pdf $f_H(h)$.

Given $H_k = h$ we can, by conditioning all the probabilities on this event, derive a system of equations which is analogous to (4.9).

Using the same approach as for the one call box case we can show (Appendix A) that for finding the unconditional expectation we can apply

$$E[p(A_k | A_{k-1}) p(A_{k-1})] = E[p(A_k | A_{k-1})] E[p(A_{k-1})]$$

The same is true for all the other conditional probabilities in (4.9).

Then by noting that $E[p(A_k | A_{k-1})] = E[p(A_2 | A_1)]$ (see 3.27) we obtain the following system (using the notation of $P_{A,A}$ for $E[p(A_2 | A_1)]$ as we used $P_{A,A}$ for $p(A_2 | A_1)$)

$$E[p(A_k)] = P_{A,A} E[p(A_{k-1})] + P_{A,B} E[p(B_{k-1})] + P_{A,C} E[p(C_{k-1})] + P_{A,D} E[p(D_{k-1})]$$

$$E[p(B_k)] = P_{B,A} E[p(A_{k-1})] + P_{B,B} E[p(B_{k-1})] + P_{B,C} E[p(C_{k-1})] + P_{B,D} E[p(D_{k-1})]$$

$$E[p(C_k)] = P_{C,A} E[p(A_{k-1})] + P_{C,B} E[p(B_{k-1})] + P_{C,C} E[p(C_{k-1})] + P_{C,D} E[p(D_{k-1})]$$

$$E[p(D_{k-1})] = 1 - E[p(A_{k-1})] - E[p(B_{k-1})] - E[p(C_{k-1})]$$

$$E[p(D_k)] = 1 - E[p(A_k)] - E[p(B_k)] - E[p(C_k)]$$

Letting k go to infinity and assuming that

$$\lim_{k \rightarrow \infty} E[p(A_k)] = \tilde{P}_A$$

$$\lim_{k \rightarrow \infty} E[p(B_k)] = \tilde{P}_B$$

$$\lim_{k \rightarrow \infty} E[p(C_k)] = \tilde{P}_C$$

$$\lim_{k \rightarrow \infty} E[p(D_k)] = \tilde{P}_D$$

We obtain a system of four equations (4.20) with four unknowns \tilde{P}_A , \tilde{P}_B , \tilde{P}_C , and \tilde{P}_D .

As we pointed out in section III.3 this derivation allows us to find $E[T]$ the expectation of the time between A and B for a trip when steady state is reached.

$$E[T] = t_o \tilde{P}_A + (t_o + \frac{\gamma_1}{s_o} + t_d) \tilde{P}_B + (t_o + \frac{\gamma_2}{s_o} + t_d) \tilde{P}_C + (t_o + \frac{\gamma_1 + \gamma_2}{s_o} + 2t_d) \tilde{P}_D$$

IV.4 Conclusion

We have been able to derive $f_T(t)$ in the deterministic case when the system reaches the steady state; we have highlighted the way we could find this expression for a more general distribution for the headways. Before concluding this thesis we are going to generalize these results to the case of n call boxes with deterministic headways; this is done in Chapter V.

CHAPTER V

GENERALISATION TO N CALL BOXES
IN THE DETERMINISTIC CASE

V.1 Notation

The notation used in this Chapter is just an extension, to the general case, of the notation used in Chapter IV. Here again a schematic representation of a simple route-deviation bus line with n call boxes has been drawn and is given in Figure V.1.

The assumptions we made about the instantaneous communications between the call box requests and the bus drivers are still valid.

Examination of the geometry of the situation, the demand characteristics at the call boxes, the operational characteristics, and the probabilistic modeling experiment leads us to the following notation:

L = length (distance) of the fixed route between the two fixed stops A and B

t_0 = time needed by a bus to go from A to B without deviations

l_i = distance of the fixed route between A and the i^{th} call box

γ_i = added distance by a deviation at C_i ($2 \times G_i C_i$)

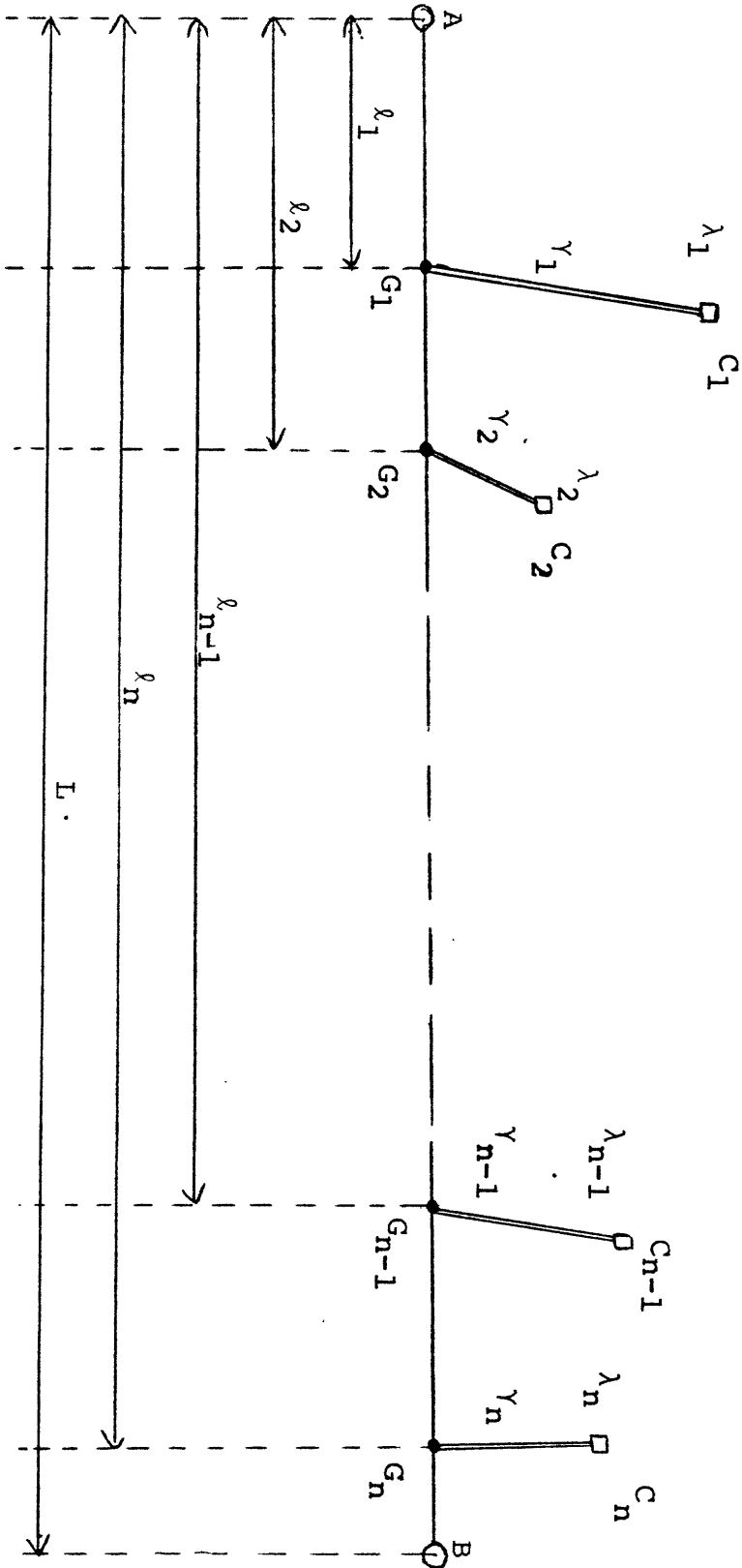


Figure V.1 The nCall Boxes Case; Schematic Representation

λ_i = average demand rates at C_i (independent)

s_o = constant effective travel speed for all buses

$H_k, k = 2, 3, 4$ = headways between the k^{th} bus and the $k-1^{\text{th}}$ bus at A

$f_H(h)$ = probability density function of the headways.

T_k = random variable "time between A and B for the k^{th} trip"

V_{ki} = event: "there is a deviation at C_i for the k^{th} trip"

t_d = dwell time at C_i (same for all C_i $i = 1, 2, 3, \dots, m$)

To be consistent with what we have already done (see section III.2.2) we are going to suppose that we will restrict our study to the case where the headways (H_o in the deterministic case, or the experimental value h of H_k in the general case) are always greater than the maximum of $\frac{\gamma_i}{2s_o} + t_d$ that is:

$$H_o > \max_i \left\{ \frac{\gamma_i}{2s_o} + t_d \right\} \quad i = 1, 2, 3, \dots, n.$$

V.2 Derivation

Our approach is very close to the two call boxes and consists, in fact, of extending the earlier notation such that derivations are not too cumbersome.

V.2.1 Extension of the Two Call Boxes Case

We saw in Chapter IV that we had four different events to describe the trip behavior of the k^{th} bus (see IV.1.2).

This was due to the fact that we had 2 call boxes and for each call box 2 possible events: "deviation or not", giving a total of $2 \times 2 = 4$ different events.

This time we have n call boxes and we still have 2 possible events for each of them, so the characteristic of a trip for the k^{th} bus are described by a set of $2 \times 2 \times 2 \times 2 \dots \times 2 = 2^n$ different events. For each of these n times 2^n events we have the corresponding probabilities. For example, we will write $P(V_{k1}, V_{k2}, V_{k3}, \dots, V_{kn})$ to denote the probability of $V_{k1} \cap V_{k2} \dots \cap V_{kn}$, that is, the probability that the bus makes a deviation at all the n call boxes.

Corresponding to these 2^n events and associated probabilities we are going to have specific and known values of the random variable T_k (time of the trip between A and B) (Possibly different events may result in identical times for the whole trip but for the time being we are going to keep the probabilities of these different events distinct in order to get our pmf).

Thus, in the most general case we can say that the pmf of T_k will be given by the knowledge of the probabilities of these 2^n different events and is described as follows:

$$f_{T_k}(t) = \begin{cases} p(\bar{v}_{k1}, \bar{v}_{k2}, \bar{v}_{k3}, \dots, \bar{v}_{kn}) & \text{if } t = t_0 \\ p(v_{k1}, \bar{v}_{k2}, \bar{v}_{k3}, \dots, \bar{v}_{kn}) & \text{if } t = t_0 + \frac{\gamma_1}{s_0} + t_d \\ p(\bar{v}_{k1}, v_{k2}, \bar{v}_{k3}, \dots, \bar{v}_{kn}) & \text{if } t = t_0 + \frac{\gamma_2}{s_0} + t_d \\ \dots\dots\dots \\ p(\bar{v}_{k1}, \bar{v}_{k2}, \bar{v}_{k3}, \dots, v_{kn}) & \text{if } t = t_0 + \frac{\gamma_n}{s_0} + t_d \\ p(v_{k1}, v_{k2}, \bar{v}_{k3}, \dots, \bar{v}_{kn}) & \text{if } t = t_0 + \frac{\gamma_1 + \gamma_2}{s_0} + 2t_d \quad (5.1) \\ p(v_{k1}, \bar{v}_{k2}, v_{k3}, \dots, \bar{v}_{kn}) & \text{if } t = t_0 + \frac{\gamma_1 + \gamma_3}{s_0} + 2t_d \\ \dots\dots\dots \\ p(\bar{v}_{k1}, \bar{v}_{k2}, \bar{v}_{k3}, \dots, v_{kn}) & \text{if } t = t_0 + \frac{\gamma_{n-1} + \gamma_n}{s_0} + 2t_d \\ " \quad " \quad " \quad " & \\ " \quad " \quad " \quad " & \\ " \quad " \quad " \quad " & \\ " \quad " \quad " \quad " & \\ p(v_{k1}, v_{k2}, v_{k3}, \dots, v_{kn}) & \text{if } t = t_0 + \frac{\sum_{i=1}^n \gamma_i}{s_0} + nt_d \\ 0 & \text{otherwise} \end{cases}$$

We can easily see that this way of ordering our different events gives the right number of events that is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \sum_{k=0}^n \binom{n}{k} 1^n 1^k = (1 + 1)^n = 2^n$$

In fact, using the fact that these 2^n events are mutually exclusive and collectively exhaustive we only have $2^n - 1$ probabilities to determine.

V.2.2 Derivation for the First Bus

We are going to use the same method as in Chapter IV.

We begin with the first call box C_1 ; the probability for the first bus of making a deviation at C_1 is still given by the equation (4.3)

$$(p(V_{11}) = 1 - e^{-\frac{\lambda_1 s_1}{s_0}}).$$

We then consider the second call box C_2 ; the necessary probabilities $p(V_{12}|\bar{V}_{11})$ and $p(V_{12}|V_{11})$ in order to fully describe the behavior of this bus are given by the equations (4.4) and (4.5). Using $p(V_{11})$, $p(V_{12}|\bar{V}_{11})$, and $p(V_{12}|V_{11})$ we are able to derive the four probabilities of interest.

We go to the next step which is the call box C_3 ; we condition V_{13} by the four different events $\bar{V}_{11} \cap \bar{V}_{12}$, $V_{11} \cap \bar{V}_{12}$, $\bar{V}_{11} \cap V_{12}$, and $V_{11} \cap V_{12}$ and we evaluate these four conditional probabilities; then we can evaluate the eight joint probabilities (four for V_{13} and four for \bar{V}_{13}) which fully describe this new state.

We then go on to the fourth call box and we keep on using the same method until we arrive at the n^{th} call boxes. This "algorithm" allows us to determine systematically our 2^n probabilities and gives us the pmf for T_k for the first bus.

V.2.3 Successive Buses

We first examine the case of deterministic headways, that is each bus starts H_0 units of time after the previous

one (from A).

The same analysis we did in the two call boxes case still applies (in principle) in this general n call boxes case; indeed depending on what the first bus did on its trip, the results for the second one will differ and so on. Here again we will consider the case when k goes to infinity, that is when the "steady state" is reached. At this point we should make an important simplification of notation.

We wish to find a notation for describing these 2^n events.

Assuming that we have prespecified the order in which these events are listed, we can choose to say that

X_{1k} represents the event of the first event

X_{2k} represents the event of the second event

and so on. We can use for this purpose the order of events presented in equation (5.1) Then $p(X_{j,k})$ represents the probability of the event associated with the j^{th} line of (5.1).

We will use $p(X_j) = \lim_{k \rightarrow \infty} p(X_{j,k}) = P_j$

Each of the 2^n events $X_{j,k}$ for the k^{th} trip can be conditioned by one of the 2^n mutually exclusive collectively exhaustive events $X_{j,k-1}$ for the $(k-1)^{\text{th}}$ trip and we obtain the following system:

$$\begin{aligned}
 p(X_{1,k}) &= p(X_{1,k}|X_{1,k-1})p(X_{1,k-1}) + \dots + p(X_{1,k}|X_{2^n,k-1})p(X_{2^n,k-1}) \\
 p(X_{2,k}) &+ p(X_{2,k}|X_{2,k-1})p(X_{2,k-1}) + \dots + p(X_{2,k}|X_{2^n,k-1})p(X_{2^n,k-1}) \\
 &\dots\dots\dots \\
 p(X_{2^n,k}) &= p(X_{2^n,k}|X_{2,k-1})p(X_{2,k-1}) + \dots + p(X_{2^n,k}|X_{2^n,k-1})p(X_{2^n,k-1})
 \end{aligned}
 \tag{5.2}$$

Which is very similar (and in fact is the extension to n call boxes) of the system (4.9) obtained in Chapter IV.

The operations of the system allows us to say that

$$p(X_{1,k}|X_{1,k-1}) = p(X_{1,2}|X_{1,1}) \tag{5.3}$$

for all the conditional probabilities of the system (5.8). Here again, as we did for the two call boxes (see (4.11)) we will denote these probabilities

$$p(X_{i,2}|X_{j,1}) = P_{ij} \quad i \text{ and } j = 1, 2, 3, \dots, 2^n$$

Then assuming that all our $p(X_{j,k})$ tend to a limit P_j when k goes to infinity the system (5.2) becomes (the reader is referred to (4.13) to see the analogy):

$$\begin{aligned}
 P_1 &= P_{1,1}P_1 + P_{1,2}P_2 + \dots + P_{1,2^n}P_{2^n} \\
 P_2 &= P_{2,1}P_1 + P_{2,2}P_2 + \dots + P_{2,2^n}P_{2^n} \\
 &\dots\dots\dots
 \end{aligned}
 \tag{5.4}$$

$$P_{2^{n-1}} = P_{2^{n-1},1} P_1 + P_{2^{n-1},2} P_2 + \dots + P_{2^{n-1},2^n} P_{2^n}$$

$$P_1 + P_2 + P_3 + \dots + P_{2^n} = 1$$

which is a system of 2^n equations with 2^n unknowns.

Thus using a computer and a package for solving systems of linear equations we can obtain the 2^n probabilities which fully describe the behavior of a trip when the steady state is reached; that is the pmf of the Time T between A and B.

V.3 Conclusion

Before concluding this Chapter it should be emphasized that the case of general distributions for the headways is still derivable in the n call boxes case and requires the same analysis but is much more cumbersome in terms of notations. Here again the important assumption is that the headways are independently and identically distributed. As in the two call boxes case we end up at the last step of the analysis with the expectation of the time needed between A and B when the steady state is reached.

CHAPTER VI

CONCLUSION

The major concern of this thesis has been to study the probabilistic behavior of a single Route Deviation Bus Line. The need for a probabilistic analysis has been pointed out in Chapter I, and Chapter II allowed us to clarify the issues in such an analysis. In Chapter III we have been able to derive closed form expressions for the probability distribution for the time length of a typical bus run; this derivation has been done for both deterministic headways between successive buses and for a general probability distribution function for the headways. For the case of non-deterministic headways all our derivations have been strongly based on the assumptions of independent and identically distributed random variables; we demonstrated the difficulties associated with the elimination of such assumptions. Chapter IV. was concerned with the two call boxes case. The closed-form derivation turned out to be a more cumbersome procedure than in the one call box case; however we indicated a way of obtaining results in the case when a "steady state" is reached. (This steady state has been defined to characterize the behavior of the k^{th} trip as k goes to infinity; it is achieved when a finite limit is obtained for the probabilities of deviation at the call boxes). We

generalized the simple method developed in Chapter IV to the n call box case in Chapter V.

On the basis of our work so far, the promising directions for new research must include the following considerations.

First, a numerical analysis of the two call boxes and n call boxes cases through computer work seems indispensable in order to determine the sensitivity of the time length of a typical bus run with respect to the distance of checkpoints from the fixed route, to the demand density at call boxes, to vehicle speed, etc.

It would then be interesting to study the single RDBL under much more general assumptions such as dependence in the headways between successive buses, or a general distribution function for the speed of the vehicles, or under the utilization of a more realistic process for modeling the demand at checkpoints, considering and including dependence between checkpoints. The utilization of simulation techniques will certainly turn out to be useful in order to fully treat these problems. The study of probabilistic performance characteristics other than the length of time for a bus run could be useful (e.g. the probability density function for the number of deviation checkpoints that will be visited during a typical bus run, passenger waiting times and ride times, etc.)

In addition to pursuing the analysis of the single

Route Deviation Bus Line, other topics of interest exist. They correspond to the operational implications of RDBLs and a systematic approach to this topic would ideally explore the following types of questions:

(a) What are the possible savings in operating costs that may result from declaring a particular point to be a "deviation checkpoint" -- as opposed to a stop that is on the fixed route and will always be visited? Clearly the answer will depend on such considerations as the location of the demand point, the intensity of demand, the bus operating costs and the cost associated with deviation routing (communications equipment requirements, control and dispatching, etc.)

(b) At what level of demand should a deviation checkpoint become a permanently visited stop on the route and vice-versa? For example it is conceivable that the bus line shown in Figure I.1 might best be operated as a fixed route consisting of stops A-C₁-C₂-B during the morning and evening peak-demand periods and then be changed to a RDBL with stops A-B (and deviation checkpoints at C₁ and C₂) during the rest of the day.

(c) What are good operating strategies for serving deviation checkpoints? Note that in the case of RDBL, the line's operator retains some control regarding the way that service is provided to deviation checkpoints. It would be interesting to see if we can identify some general

characteristics of good strategies under various sets of circumstances.

(d) Finally, how do RDBLs interface with the rest of an area-wide transportation network? This clearly calls for an investigation of how RDBLs can best be operated within the context of an integrated transportation system.

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APPENDIX A

Formal demonstration of the linear independence between $p(\bar{V}_{k-1} | H_{k-1} = h_{k-1})$ and $p(V_k | \bar{V}_{k-1} \text{ and } H_k = h_k)$ given that H_k and H_{k-1} are independent and identically distributed random variables.

1. $p(\bar{V}_{k-1} | H_{k-1} = h_{k-1})$ is a random variable function of h_{k-1} so let $p(\bar{V}_{k-1} | H_{k-1} = h_{k-1})$ be represented by $g(h_{k-1})$. The expected value of this random variable is given by

$$E[p(V_{k-1})] = \int_{\alpha}^{+\infty} g(h_{k-1}) f_{H_{k-1}}(h_{k-1}) dh_{k-1} \quad (A.1)$$

2. $p(V_k | \bar{V}_{k-1} \text{ and } H_k = h_k) = 1 - e^{-\lambda h_k}$ so the expected value of this random variable is given by

$$E[p(V_k | \bar{V}_{k-1})] = \int_{\alpha}^{+\infty} (1 - e^{-\lambda h_k}) f_{H_k}(h_k) dh_k \quad (A.2)$$

3. $p(\bar{V}_{k-1} | H_{k-1} = h_{k-1}) p(V_k | \bar{V}_{k-1} \text{ and } H_k = h_k) = (1 - e^{-\lambda h_k}) g(h_{k-1})$ so the expected value of this random variable is given by

$$E[-] = \iint [(1 - e^{-\lambda h_k}) g(h_{k-1})] f_{H_{k-1}, H_k}(h_{k-1}, h_k) dh_{k-1} dh_k$$

(A.3)

A.1, A.2 and A.3 have been derived using the fact that the $H_k, k = 2, 3, 4, \dots$ are independent. Using again this

characteristic and the fact that H_k and H_{k-1} are identically distributed we have:

$$E[p(\bar{V}_{k-1})] = \int_{\alpha}^{+\infty} g(h) f_H(h) dh$$

$$E[p(V_k | \bar{V}_{k-1})] = \int_{\alpha}^{+\infty} (1 - e^{-\lambda h}) f_H(h) dh$$

$$E[p(\bar{V}_{k-1})p(V_k | \bar{V}_{k-1})] = \int_{\alpha}^{+\infty} g(h) f_H(h) dh \int_{\alpha}^{+\infty} (1 - e^{-\lambda h}) x f_H(h) dh$$

and this demonstrates the linear independence we wanted.

The demonstration for the case between $p(V_{k-1} | H_{k-1} = h_{k-1})$ and $p(V_k | V_{k-1} \text{ and } H_k = h_k)$ is identical.

APPENDIX B

Solution of the System of Equations (4.13)
in Chapter IV

We have a system of four equations with four unknowns P_A ,
 P_B , P_C , P_D .

$$\begin{aligned} P_A &= P_{A,A} P_A + P_{A,B} P_B + P_{A,C} P_C + P_{A,D} P_D \\ P_B &= P_{B,A} P_A + P_{B,B} P_B + P_{B,C} P_C + P_{B,D} P_D \\ P_C &= P_{C,A} P_A + P_{C,B} P_B + P_{C,C} P_C + P_{C,D} P_D \\ P_D &= 1 - P_A - P_B - P_C \end{aligned} \quad (B.1)$$

Before solving (B.1) we find the value of $p_{i,j}$ $i = A, B, C, D$
 $j = A, B, C, D$.

By the same argument we have:

$$\begin{aligned} P_{A,A} &= p(\bar{V}_{22} | \bar{V}_{21}, \bar{V}_{12}, \bar{V}_{11}) p(\bar{V}_{21} | \bar{V}_{11}) = (1 - \beta_2)(1 - R_1) \\ P_{B,A} &= p(\bar{V}_{22} | V_{21}, \bar{V}_{12}, \bar{V}_{11}) p(V_{21} | \bar{V}_{11}) = (1 - \beta_4) R_1 \\ P_{C,A} &= p(V_{22} | \bar{V}_{21}, \bar{V}_{12}, \bar{V}_{11}) p(\bar{V}_{21} | \bar{V}_{11}) = \beta_2(1 - R_1) \\ P_{D,A} &= p(V_{22} | V_{21}, \bar{V}_{12}, \bar{V}_{11}) p(V_{21} | \bar{V}_{11}) = \beta_4 R_1 \\ P_{A,C} &= p(\bar{V}_{22} | \bar{V}_{21}, V_{12}, \bar{V}_{11}) p(\bar{V}_{21} | \bar{V}_{11}) = (1 - \beta_1)(1 - R_1) \\ P_{B,C} &= p(\bar{V}_{22} | V_{21}, V_{12}, \bar{V}_{11}) p(V_{21} | \bar{V}_{11}) = (1 - \beta_3) R_1 \\ P_{C,C} &= p(V_{22} | \bar{V}_{21}, V_{12}, \bar{V}_{11}) p(\bar{V}_{21} | \bar{V}_{11}) = \beta_1(1 - R_1) \end{aligned}$$

$$P_{A,B} = P(\bar{V}_{22} | \bar{V}_{21}, \bar{V}_{12}, V_{11}) P(\bar{V}_{21} | V_{11}) = (1 - \beta_6)(1 - Q_1)$$

$$P_{B,B} = P(\bar{V}_{22} | V_{21}, \bar{V}_{12}, V_{11}) P(V_{21} | V_{11}) = (1 - \beta_2)Q_1$$

$$P_{C,B} = P(V_{22} | \bar{V}_{21}, \bar{V}_{12}, V_{11}) P(\bar{V}_{21} | V_{11}) = \beta_6(1 - Q_1)$$

$$P_{D,B} = P(\bar{V}_{22} | V_{21}, \bar{V}_{12}, V_{11}) P(V_{21} | V_{11}) = \beta_2 Q_1$$

$$P_{A,D} = P(\bar{V}_{22} | \bar{V}_{21}, V_{12}, V_{11}) P(\bar{V}_{21} | V_{11}) = (1 - \beta_5)(1 - Q_1)$$

$$P_{B,D} = P(\bar{V}_{22} | V_{21}, V_{12}, V_{11}) P(V_{21} | V_{11}) = (1 - \beta_1)Q_1$$

$$P_{C,D} = P(V_{22} | \bar{V}_{21}, V_{12}, V_{11}) P(\bar{V}_{21} | V_{11}) = \beta_5(1 - Q_1)$$

$$P_{D,D} = P(V_{22} | V_{21}, V_{12}, V_{11}) P(V_{21} | V_{11}) = \beta_1 Q_1$$

where: $-\lambda_2(H_0 - \frac{\gamma_2}{2s_0} - t_d)$

$$\beta_1 = 1 - e$$

$$\beta_2 = 1 - e^{-\lambda_2 H_0}$$

$$\beta_3 = 1 - e^{-\lambda_2(H_0 + \frac{\gamma_1}{s_0} - \frac{\gamma_2}{2s_0})}$$

$$\beta_4 = 1 - e$$

$$\beta_5 = 1 - e^{-\lambda_2(H_0 + \frac{\gamma_1}{s_0} + t_d)}$$

$$\beta_6 = 1 - e$$

$$\beta_7 = 1 - e^{-\lambda_2(H_0 - \frac{\gamma_1}{s_0} - \frac{\gamma_2}{2s_0} - 2t_d)}$$

$$Q_1 = 1 - e$$

$$Q_2 = 1 - e^{-\lambda_2(H_0 - \frac{\gamma_1}{s_0} - t_d)}$$

$$Q_3 = 1 - e$$

$$Q_4 = 1 - e^{-\lambda_1(H_0 - \frac{\gamma_1}{2s_0} - t_d)}$$

$$Q_5 = 1 - e$$

$$R_1 = 1 - e^{-\lambda_1 H_0}$$

In solving the system (B.1) we implicitly assumed that all our probabilities P_A, P_B, P_C , and P_D are greater than 0 (otherwise some of the $P_{i,j}$ would not be defined; indeed a conditional event must be of strictly positive probability).

By transforming B1 (essentially through substitution) we obtain the equivalent system;

$$x_1 P_A + y_1 P_B + z_1 = 0$$

$$x_2 P_A + y_2 P_B + z_2 = 0$$

$$P_C = \frac{1}{P_{A,D} - P_{A,C}} [P_A (P_{A,A}^{-1} - P_{A,D}) + P_B (P_{A,B} - P_{A,D} + P_{A,D})]$$

(B2)

$$P_D = 1 - P_A - P_B - P_C$$

The solution of this system is the following:

$$P_A = \frac{y_1 z_2 - y_2 z_1}{x_1 y_2 - x_2 y_1}$$

$$P_B = \frac{x_1 x_2 - x_2 z_1}{x_2 y_1 - y_2 x_1}$$

$$P_C = \frac{1}{(P_{A,D} - P_{A,C})} [P_A (P_{A,A}^{-1} - P_{A,D}) + P_B (P_{A,B} - P_{A,D} + P_{A,D})]$$

where:

$$x_1 = (P_{B,A}^{-1} - P_{B,D}) (P_{A,D} - P_{A,C}) + (P_{A,A}^{-1} - P_{A,D}) (P_{B,C} - P_{B,D})$$

$$y_1 = (P_{B,B}^{-1} - P_{B,D}) (P_{A,D} - P_{A,C}) + (P_{A,B} - P_{A,D}) (P_{B,C} - P_{B,D})$$

$$z_1 = P_{B,D} (P_{A,D} - P_{A,C}) + P_{A,D} (P_{B,C} - P_{B,D})$$

$$x_2 = (P_{C,A} - P_{C,D}) (P_{A,D} - P_{A,C}) + (P_{C,C}^{-1} - P_{C,D}) (P_{A,A}^{-1} - P_{A,D})$$

$$y_2 = (P_{C,B} - P_{C,D}) (P_{A,D} - P_{A,C}) + (P_{C,C}^{-1} - P_{C,D}) (P_{A,B} - P_{A,D})$$

$$z_2 = P_{C,D} (P_{A,D} - P_{A,C}) + P_{A,D} (P_{C,C}^{-1} - P_{C,D})$$