Comparing products of Schur functions and quasisymmetric functions

by

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Abstract

In this thesis a conjecture of Okounkov, a conjecture of Fomin-Fulton-Li-Poon, and a special case of Lascoux-Leclerc-Thibon's conjecture on Schur positivity of certain differences of products of Schur functions are proved. In the first part of the work a combinatorial method is developed that allows to prove weaker versions of those conjectures. In the second part a recent result of Rhoades and Skandera is used to provide a proof of actual Schur positivity results. Several further generalizations are stated and proved. In particular, an intriguing log-concavity property of Schur functions is observed. In addition, a stronger conjecture is stated in language of alcoved polytops. A weaker version of this conjecture is proved using a characterization of Klyachko cone and the theory of Temperley-Lieb immanants.

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Contents

1. Introduction

This thesis is based on [LP, LP3, LPP, DP].

The ring of symmetric functions has a linear basis of *Schur functions* s_λ labelled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$, see [Mac]. These functions appear in representation theory as characters of irreducible representations of GL_n and in geometry as representatives of Schubert classes for Grassmannians. A symmetric function is called *Schur nonnegative* if it is a linear combination with nonnegative coefficients of the Schur functions, or, equivalently, if it is the character of a certain representation of GL_n. In particular, *skew Schur functions* $s_{\lambda/\mu}$ are Schur nonnegative. Recently, a lot of work has gone into studying whether certain expressions of the form $s_\lambda s_\mu - s_\nu s_\rho$ are Schur nonnegative. Schur positivity of an expression of this form is equivalent to some inequalities between Littlewood-Richardson coefficients. In a sense, characterizing such inequalities is a "higher analogue" of the Klyachko problem on nonzero Littlewood-Richardson coefficients. Let us mention several Schur positivity conjectures due to Okounkov, Fomin-Fulton-Li-Poon, and Lascoux-Leclerc-Thibon of the above form.

Okounkov [Oko] studied branching rules for classical Lie groups and proved that the multiplicities were "monomial log-concave" in some sense. An essential combinatorial ingredient in his construction was the theorem about monomial nonnegativity of some symmetric functions. He conjectured that these functions are Schur nonnegative as well. For a partition λ with all even parts, let $\frac{\lambda}{2}$ denote the partition $(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \ldots)$. For two symmetric functions f and g, the notation $f \geq_s g$ means that $f - g$ is Schur nonnegative.

Conjecture 1.1. Okounkov [Oko, p. 269] *For two skew shapes λ/μ and ν/ρ such that $\lambda + \nu$ and $\mu + \rho$ both have all even parts, we have $(s_{\frac{(\lambda+\nu)}{2}/\frac{(\mu+\rho)}{2}})^2 \geq_s s_{\lambda/\mu} s_{\nu/\rho}$.*

Fomin, Fulton, Li, and Poon [FFLP] studied the eigenvalues and singular values of sums of Hermitian and of complex matrices. Their study led to two combinatorial conjectures concerning differences of products of Schur functions. Let us formulate one of these conjectures, which was also studied recently by Bergeron and McNamara [BM]. For two partitions λ and μ, let $\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \dots)$ be the partition obtained by rearranging all parts of λ and μ in weakly decreasing order. Let $\mathrm{sort}_1(\lambda, \mu) := (\nu_1, \nu_3, \nu_5, \dots)$ and $\mathrm{sort}_2(\lambda, \mu) := (\nu_2, \nu_4, \nu_6, \dots)$.

Conjecture 1.2. Fomin-Fulton-Li-Poon [FFLP, Conjecture 2.7] *For two partitions λ and μ, we have $s_{\mathrm{sort}_1(\lambda,\mu)} s_{\mathrm{sort}_2(\lambda,\mu)} \geq_s s_\lambda s_\mu$.*

Lascoux, Leclerc, and Thibon [LLT] studied a family of symmetric functions $\mathcal{G}_\lambda^{(n)}(q, x)$ arising combinatorially from ribbon tableaux and algebraically from the Fock space representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. They conjectured that $\mathcal{G}_{n\lambda}^{(n)}(q, x) \geq_s \mathcal{G}_{m\lambda}^{(m)}(q, x)$ for $m \leq n$. For the case $q = 1$, their conjecture can be reformulated, as follows. For a partition λ and $1 \leq i \leq n$, let $\lambda^{[i,n]} := (\lambda_i, \lambda_{i+n}, \lambda_{i+2n}, \dots)$. In particular, $\mathrm{sort}_i(\lambda, \mu) = (\lambda \cup \mu)^{[i,2]}$, for $i = 1, 2$.

Conjecture 1.3. Lascoux-Leclerc-Thibon [LLT, Conjecture 6.4] *For integers $1 \leq m \leq n$ and a partition λ, we have $\prod_{i=1}^n s_{\lambda^{[i,n]}} \geq_s \prod_{i=1}^m s_{\lambda^{[i,m]}}$.*

Theorem 1.4. *Conjectures 1.1, 1.2 and 1.3 are true.*

Our approach is based on the following result. For two partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, let us define partitions $\lambda \vee \mu := (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \dots)$ and $\lambda \wedge \mu := (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \dots)$. The Young diagram of $\lambda \vee \mu$ is the set-theoretical union of the Young diagrams of λ and μ. Similarly, the Young diagram of $\lambda \wedge \mu$ is the set-theoretical intersection of the Young diagrams of λ and μ. For two skew shapes, define $(\lambda/\mu) \vee (\nu/\rho) := \lambda \vee \nu / \mu \vee \rho$ and $(\lambda/\mu) \wedge (\nu/\rho) := \lambda \wedge \nu / \mu \wedge \rho$. We call this operation of forming two new skew shapes out of two old ones *the cell tranfer operation.*

Theorem 1.5. *Let λ/μ and ν/ρ be any two skew shapes. Then we have*

$$s_{(\lambda/\mu)\vee(\nu/\rho)}\, s_{(\lambda/\mu)\wedge(\nu/\rho)} \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}.$$

We begin however with proving weaker versions of Theorem 1.5. Namely, one can ask for inequality in terms of monomial and fundamental quasisymetric functions.

Okounkov for example has provided a proof of monomial version of his conjecture in [Oko]. It appears that when we ask for a weaker versions of positivity we can start with objects more general than Schur functions. Namely, we can derived monomial positivity for generating functions of so called $\mathbb{T}$-labeled posets, and fundamental quasisymmetric function positivity for Stanley's P-partitions. We summarize the results we obtain in the following table.

Ring	$\mathbb{Z}[[x_1, x_2, \ldots]]$	QSym	Sym
Basis	x^α	L_α	s_λ
Skew fcns.	$K_{P,O}$	$K_{P,\theta}$	$s_{\lambda/\mu}$
Posets	$\mathbb{T}$-labeled posets (P, O)	Stanley's (P, θ)	Young diagrams λ/μ

In all three cases, the difference of products of "skew functions" arising from the cell transfer operation on the "posets" is positive in the corresponding "basis". The definition of cell tranfer must be generalised to convex subposets of any poset. For example in case of P-partitions we study positivity of expressions of the form

$$K_{Q\wedge R,\theta} K_{Q\vee R,\theta} - K_{Q,\theta} K_{R,\theta} \tag{1}$$

where $K_{P,\theta}$ denotes P-partition generating function of poset P with labeling θ.

The methods employed differ for the case of Schur-positivity statements. For the case of monomial and fundamental quasisymmetric function positivity we use purely combinatorial *cell transfer injection*. The Schur positivity results rely however on theory of *Temperley-Lieb immanants*, developed by Rhoades and Skandera in [RS1, RS2]. Thus, the whole work is split into more combinatorial part dealing with monomial and fundamental quasisymmetric function positivity, and more algebraic part dealing with Schur positivity. The question of finding purely combinatorial proof of the Schur positivity version of cell transfer remains open.

The thesis proceeds as follows. In Section 2 we give some background on quasisymmetric functions. In Section 3 we recall basic notions related to posets and tableaux. We define $\mathbb{T}$-labelled posets and corresponding tableaux generating functions. In Section 4 we define cell transfer operation on convex subposets of an arbitrary poset. Then we state and prove monomial version of the Cell Transfer theorem. In Section 5 we briefly review the main definitions in the theory of symmetric functions. Then we relate the definition of cell transfer from Section 4 to the definition given above in terms of min and max operators. Is a result we deduce the monomial version of Theorem 1.5. In Section 6 we describe cell transfer injection as an algorithm. In Section 7 we remind the reader basic notions from the theory of P-partitions. Then we state and prove fundamental quasisymmetric function version of the Cell Transfer theorem. In Section 9 we study in detail the meaning of cell transfer on chain poset. This leads us to defining in Section 10 *wave Schur functions*, generalising the usual Schur functions. We prove a Jacobi-Trudi like formula for wave Schur functions, where the role of complete homogenous symmetric functions is played by (more general) fundamental quasisymmetric functions. In Section 11 we prove certain algebraic properties of QSym which are useful to know for some of the posed questions. In Section 12 we give a review of results of Haiman and Rhoades-Skandera on Temperley-Lieb immanants and their properties. In Section 13 we apply the theory of Temperley-Lieb immanants to prove Theorem 1.5. In Section 14, we present and prove more general versions of Conjectures 1.1, 1.2 and 1.3. In Section 15 we state a general conjecture on comparing products of $\mathfrak{sl}_n$ characters. In Section 16 we review Horn-Klyachko theory. In Section 17 we give a proof of a weaker version of conjecture from Section 15. Remarkably, the main tool used is again theory of Temperley-Lieb immanants. In Section 18 we study a poset on pairs of partitions defined by Stembridge, and describe its maximal elements.

2. Quasisymmetric functions

We refer to [Sta] for more details of the material in this section.

2.1. **Monomial and fundamental quasi-symmetric functions.** Let n be a positive integer. A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. We write $|\alpha| = n$. Denote the set of compositions of n by $\mathrm{Comp}(n)$. Associated to a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n is a subset $D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}$ of $[n-1]$. The map $\alpha \mapsto D(\alpha)$ is a bijection between compositions of n and subsets of $[n-1] = \{1, 2, \ldots, n-1\}$. We will denote the inverse map by $\mathbb{C} : 2^{[n-1]} \to \mathrm{Comp}(n)$ so that $\mathbb{C}(D(\alpha)) = \alpha$.

Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ denote the set of positive integers. A formal power series $f = f(x) \in \mathbb{Z}[[x_1, x_2, \ldots]]$ with bounded degree is called *quasi-symmetric* if for any $a_1, a_2, \ldots, a_k \in \mathbb{P}$ we have

$$\left[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}\right] f = \left[x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}\right] f$$

whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. Here $[x^\alpha]f$ denotes the coefficient of x^α in f. Denote by $\mathrm{QSym} \subset \mathbb{Z}[[x_1, x_2, \ldots]]$ the ring of quasi-symmetric functions.

Let α be a composition. Then the *monomial quasi-symmetric function* M_α is given by

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_k} \cdots x_{i_k}^{\alpha_k}.$$

The *fundamental quasi-symmetric function* L_α is given by

$$L_\alpha = \sum_{D(\beta) \subset D(\alpha)} M_\beta,$$

where the summation is over compositions β satisfying $|\beta| = |\alpha|$. The set of fundamental quasi-symmetric functions (resp. monomial quasi-symmetric functions) form a basis of QSym. We say that a quasi-symmetric function $f \in \mathrm{QSym}$ is *L-positive* (resp. *M-positive*) if it is a non-negative linear combination of fundamental quasi-symmetric functions (resp. monomial quasi-symmetric functions). Note that L-positivity implies M-positivity.

Two fundamental quasi-symmetric functions L_α and L_β multiply according to the *shuffle product*. Let $u = u_1 u_2 \cdots u_k$ and $v = v_1 v_2 \cdots v_l$ be two words. Then a word

$w = w_1 w_2 \cdots w_{k+l}$ is a shuffle of u and v if there exist disjoint ordered subsets $A, B \subset [k+l]$ such that $A = \{a_1, a_2, \ldots, a_k\}$, $B = \{b_1, b_2, \ldots, b_l\}$, $w_{a_i} = u_i$ for all $1 \leq i \leq k$, $w_{b_i} = v_i$ for all $1 \leq i \leq l$ and $A \cup B = [k+l]$. We denote the set of shuffles of u and v by $u \odot v$.

For a composition α with $|\alpha| = n$ let $w(\alpha) = w = w_1 w_2 \cdots w_n$ denote any word with *descent set* $D(w) = \{i : w_i > w_{i+1}\} \subset [n-1]$ equal to $D(\alpha)$. Suppose $w(\alpha)$ and $w(\beta)$ are chosen to have disjoint letters. Then

$$L_\alpha L_\beta = \sum_{u \in w(\alpha) \odot w(\beta)} L_{\mathbb{C}(u)},$$

where $\mathbb{C}(u)$ is by definition the composition $\mathbb{C}(D(u))$ associated to u.

2.2. **Two involutions on** QSym. If $D \subset [n-1]$ we let $\bar{D} = \{i \in [n-1] \mid i \notin D\}$ denote its complement. For a composition α, define $\bar{\alpha} = \mathbb{C}(\overline{D(\alpha)})$. Let ω denote the linear endomorphism of QSym given by $\omega(L_\alpha) = L_{\bar{\alpha}}$.

Let α^* denote α read backwards: $\alpha^* = (\alpha_k, \ldots, \alpha_1)$. Let ν be the linear endomorphism of QSym which sends $L_\alpha \mapsto L_{\alpha^*}$.

Proposition 2.1. *The maps ω and ν are algebra involutions of* QSym, *and we have $\nu(M_\alpha) = M_{\alpha^*}$.*

Proof. We will check the first statement for ν; the proof for ω is similar. Let $w = w_1 w_2 \cdots w_r \in S_r$ be a permutation with descent set $D(w) = D(\alpha)$. Then $w^* \in S_r$ given by $w^* = (r+1-w_r)(r+1-w_{r-1}) \cdots (r+1-w_1)$ has descent set $D(w^*) = D(\alpha^*)$. If $u \in S_{r+l}$ is a shuffle of $w \in S_r$ and $v \in S_l$, where $v \in S_l$ uses the letters $r+1, r+2, \ldots, r+l$, then u^* is a shuffle of $v^* \in S_l$ and $w^* \in S_r$ where $w^* \in S_r$ uses the letters $l+1, l+2, \ldots, r+l$. Thus $\nu(L_{\mathbb{C}(v)})\, \nu(L_{\mathbb{C}(w)}) = L_{\mathbb{C}(v)^*}\, L_{\mathbb{C}(w)^*} = \nu(L_{\mathbb{C}(w)}\, L_{\mathbb{C}(v)})$, showing that ν is an algebra map. That ν is an involution is clear from the definition.

The second statement can be deduced from the fact that ν commutes with the map $\alpha \mapsto \{\beta \mid D(\beta) \subset D(\alpha)\}$.

□

3. Posets and Tableaux

Let $(P, \leq)$ be a possibly infinite poset. Let $s, t \in P$. We say that s *covers* t and write $s \gtrdot t$ if for any $r \in P$ such that $s \geq r \geq t$ we have $r = s$ or $r = t$. The *Hasse diagram* of a poset P is the graph with vertex set equal to the elements of P and edge set equal to the set of covering relations in P. If $Q \subset P$ is a subset of the elements of P then Q has a natural induced subposet structure. If $s, t \in Q$ then $s \leq t$ in Q if and only if $s \leq t$ in P. Call a subset $Q \subset P$ *connected* if the elements in Q induce a connected subgraph in the Hasse diagram of P.

An *order ideal* I of P is an induced subposet of P such that if $s \in I$ and $s \geq t \in P$ then $t \in I$. A subposet $Q \subset P$ is called *convex* if for any $s, t \in Q$ and $r \in P$ satisfying $s \leq r \leq t$ we have $r \in Q$. Alternatively, a convex subposet is one which is closed under taking intervals. A convex subset Q is determined by specifying two order ideals J and I so that $J \subset I$ and $Q = \{s \in I \mid s \notin J\}$. We write $Q = I/J$. If $s \notin Q$ then we write $s < Q$ if $s < t$ for some $t \in Q$ and similarly for $s > Q$. If $s \in Q$ or s is incomparable with all elements in Q we write $s \sim Q$. Thus for any $s \in P$, exactly one of $s < Q$, $s > Q$ and $s \sim Q$ is true.

Let $\mathbb{P}$ denote the set of positive integers and $\mathbb{Z}$ denote the set of integers. Let $\mathbb{T}$ denote the set of all weakly increasing functions $f : \mathbb{P} \to \mathbb{Z} \cup \{\infty\}$.

Definition 3.1. *A $\mathbb{T}$-labelling O of a poset P is a map $O : \{(s,t) \in P^2 \mid s \gtrdot t\} \to \mathbb{T}$ labelling each edge (s,t) of the Hasse diagram by a weakly increasing function $O(s,t) : \mathbb{P} \to \mathbb{Z} \cup \{\infty\}$. A $\mathbb{T}$-labelled poset is an an ordered pair (P, O) where P is a poset, and O is a $\mathbb{T}$-labelling of P.*

We shall refer to a $\mathbb{T}$-labelled poset (P, O) as P when no ambiguity arises. If $Q \subset P$ is a convex subposet of P then the covering relations of Q are also covering relations in P. Thus a $\mathbb{T}$-labelling O of P naturally induces a $\mathbb{T}$-labelling $O|_Q$ of Q. We denote the resulting $\mathbb{T}$-labelled poset by $(Q, O) := (Q, O|_Q)$.

Definition 3.2. *A (P, O)-tableau is a map $\sigma : P \to \mathbb{P}$ such that for each covering relation $s \lessdot t$ in P we have*

$$\sigma(s) \leq O(s,t)(\sigma(t)).$$

If $\sigma : P \to \mathbb{P}$ is any map, then we say that σ respects O if σ is a (P, O)-tableau.

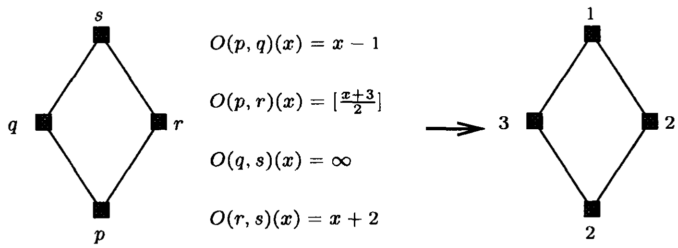


FIGURE 1. An example of a $\mathbb{T}$-labelled poset (P, O) and a (P, O)-tableaux.

Figure 1 contains an example of a $\mathbb{T}$-labelled poset (P, O) and a corresponding (P, O)-tableau.

Denote by $\mathcal{A}(P, O)$ the set of all (P, O)-tableaux. If P is finite then one can define the formal power series $K_{P,O}(x_1, x_2, \ldots) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ by

$$K_{P,O}(x_1, x_2, \ldots) = \sum_{\sigma \in \mathcal{A}(P,O)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots .$$

The composition $\mathrm{wt}(\sigma) = (\#\sigma^{-1}(1), \#\sigma^{-1}(2), \ldots)$ is called the weight of σ.

Our (P, O)-tableaux can be viewed as a generalization of Stanley's (P, ω)-partitions and also of McNamara's oriented posets; see [Sta, McN].

Example 3.3. *Any Young diagram $P = \lambda$ can be considered as a $\mathbb{T}$-labelled poset. Indeed, consider its cells to be elements of the poset, and let O be the labelling of the horizontal edges with the function $f^{\text{weak}}(x) = x$ and label the vertical edges with the function $f^{\text{strict}}(x) = x - 1$. A (λ, O)-tableau is just a semistandard Young tableaux and $K_{\lambda,O}(x_1, x_2, \cdots)$ is the Schur function $s_\lambda(x_1, x_2, \cdots)$.*

Example 3.4. *Another interesting example are cylindric tableaux and cylindric Schur functions. Let $1 \leq k < n$ be two positive integers. Let $\mathbb{C}_{k,n}$ be the quotient of $\mathbb{Z}^2$ given by*

$$\mathbb{C}_{k,n} = \mathbb{Z}^2/(k-n, k)/Z.$$

In other words, the integer points (a, b) and $(a+k-n, b+k)$ are identified in $\mathbb{C}_{k,n}$. We can give $\mathbb{C}_{k,n}$ the structure of a poset by the generating relations $(i, j) \lessdot (i+1, j)$ and $(i, j) \lessdot (i, j+1)$. We give $\mathbb{C}_{k,n}$ a $\mathbb{T}$-labelling O by labelling the edges $(i, j) \lessdot (i+1, j)$ with the function $f^{\text{weak}}(x) = x$ and the edges $(i, j) \lessdot (i, j+1)$ with the function $f^{\text{strict}}(x) = x - 1$. A finite convex subposet P of $\mathbb{C}_{k,n}$ is known as a cylindric skew *shape; see* [GK, Pos, McN]. *The* (P, O)*-tableau are known as semistandard cylindric tableaux of shape* P *and the generating function* $K_{P,O}(x_1, x_2, \cdots)$ *is the cylindric Schur function defined in* [BS, Pos].

Example 3.5. *Let* N *be the number of elements in a poset* P*, and let* $\omega : P \longrightarrow [N]$ *be a bijective labelling of elements of* P *with numbers from* 1 *to* N*. Recall that a* (P, ω)*-partition (see* [Sta]*) is a map* $\sigma : P \longrightarrow \mathbb{P}$ *such that* $s \le t$ *in* P *implies* $\sigma(s) \le \sigma(t)$*, while if in addition* $\omega(s) > \omega(t)$ *then* $\sigma(s) < \sigma(t)$*. Label now each edge* (s, t) *of the Hasse diagram of* P *with* f^{weak} *or* f^{strict}*, depending on whether* $\omega(s) \le \omega(t)$ *or* $\omega(s) > \omega(t)$ *correspondingly. It is not hard to see that for this labelling* O *the* (P, O)*-tableaux are exactly the* (P, ω)*-partitions. Similarly, if we allow any labelling of the edges of* P *with* f^{weak} *and* f^{strict}*, we get the oriented posets of McNamara; see* [McN].

4. The Cell Transfer Theorem

A generating function $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is *monomial-positive* if all coefficients in its expansion into monomials are non-negative. If f is actually a symmetric function then this is equivalent to f being a non-negative linear combination of monomial symmetric functions.

Let (P, O) be a $\mathbb{T}$-labelled poset. Let Q and R be two finite convex subposets of P. The subset $Q \cap R$ is also a convex subposet. Define two more subposets $Q \wedge R$ and $Q \vee R$ by

$$Q \wedge R = \{s \in R \mid s < Q\} \cup \{s \in Q \mid s \sim R \text{ or } s < R\} \tag{2}$$

and

$$Q \vee R = \{s \in Q \mid s > R\} \cup \{s \in R \mid s \sim Q \text{ or } s > Q\}. \tag{3}$$

Observe that the operations $\vee$, $\wedge$ are not commutative, and that $Q \cap R$ is a convex subposet of both $Q \vee R$ and $Q \wedge R$. On Figure 2 an example is given for $Q = (6, 5, 5, 5)/(3, 3)$ and $R = (6, 6, 4, 4, 4)/(6, 1, 1, 1, 1)$ being skew shapes.

Recall that if A and B are sets then $A \backslash B = \{a \in A \mid a \notin B\}$ denotes the *set difference*.

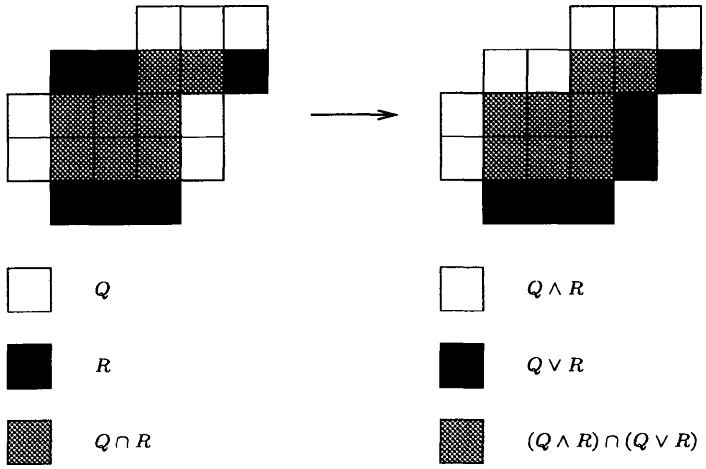


FIGURE 2. An example for semistandard Young tableaux.

Lemma 4.1. *The subposets $Q \wedge R$ and $Q \vee R$ are both convex subposets of P. We have $(Q \wedge R) \cup (Q \vee R) = Q \cup R$ and $(Q \wedge R) \cap (Q \vee R) = Q \cap R$.*

Proof. Suppose $s < t$ lie in $Q \wedge R$ and $s < r < t$ for some $r \in P$ but $r \notin Q \wedge R$. Then $t \in Q \wedge R$ implies either $t < Q$ or $t \in Q$. Then $r < t$ implies that either $r < Q$ or $r \in Q$. If $r < Q$ then $s < Q$ and therefore $s \in R$. Then either $r \in R$ or $r > R$. If $r \in R$ then since $r < Q$, we get $r \in Q \wedge R$ - contradiction. If $r > R$, then $t > r$ implies $t > R$ which contradicts $t \in Q \wedge R$. Now, if $r \in Q$ then $r \notin Q \wedge R$ implies $r > R$, and we proceed as above. The second statement of the lemma is straightforward. □

Note that the operations $\wedge$ and $\vee$ are stable so that $(Q \wedge R) \wedge (Q \vee R) = Q \wedge R$ and $(Q \wedge R) \vee (Q \vee R) = Q \vee R$.

Let ω be a (Q, O)-tableau and σ be a (R, O)-tableau. We now describe how to construct a $(Q \wedge R, O)$-tableau $\omega \wedge \sigma$ and a $(Q \vee R, O)$-tableau $\omega \vee \sigma$. Define a subset of $Q \cap R$, depending on ω and σ, by

$$(Q \cap R)^+ = \{x \in Q \cap R \mid \omega(x) < \sigma(x)\}.$$

We give $(Q \cap R)^+$ the structure of a graph by inducing from the Hasse diagram of $Q \cap R$.

Let $\mathrm{bd}(R) = \{x \in Q \cap R \mid x \gtrdot y \text{ for some } y \in R \backslash Q\}$ be the "lower boundary" of $Q \cap R$ which touches elements in R. Let $\mathrm{bd}(R)^+ \subset (Q \cap R)^+$ be the union of the connected components of $(Q \cap R)^+$ which contain an element of $\mathrm{bd}(R)$. Similarly, let $\mathrm{bd}(Q) = \{x \in Q \cap R \mid x \lessdot y \text{ for some } y \in Q \backslash R\}$ be the "upper boundary" of $Q \cap R$ which touches elements in Q. Let $\mathrm{bd}(Q)^+ \subset (Q \cap R)^+$ be the union of the connected components of $(Q \cap R)^+$ which contain an element of $\mathrm{bd}(Q)$. The elements in $\mathrm{bd}(Q)^+ \cup \mathrm{bd}(R)^+$ are amongst the cells that we might "transfer".

Let $S \subset Q \cap R$. Define $(\omega \wedge \sigma)_S : Q \wedge R \to \mathbb{P}$ by

$$(\omega \wedge \sigma)_S(x) = \begin{cases} \sigma(x) & \text{if } x \in R \backslash Q \text{ or } x \in S, \\ \omega(x) & \text{otherwise.} \end{cases}$$

Similarly, define $(\omega \vee \sigma)_S : Q \vee R \to \mathbb{P}$ by

$$(\omega \vee \sigma)_S(x) = \begin{cases} \omega(x) & \text{if } x \in Q \backslash R \text{ or } x \in S, \\ \sigma(x) & \text{otherwise.} \end{cases}$$

One checks directly that $\mathrm{wt}(\sigma) + \mathrm{wt}(\omega) = \mathrm{wt}((\omega \wedge \sigma)_S) + \mathrm{wt}((\omega \vee \sigma)_S)$.

Proposition 4.2. *When $S = S^* := \mathrm{bd}(Q)^+ \cup \mathrm{bd}(R)^+$, both $(\omega \wedge \sigma)_{S^*}$ and $(\omega \vee \sigma)_{S^*}$ respect O.*

Proof. We check this for $(\omega \wedge \sigma)_{S^*}$ and the claim for $(\omega \vee \sigma)_{S^*}$ follows from symmetry. Let $s \lessdot t$ be a covering relation in $Q \wedge R$. Since σ and ω are assumed to respect O, we need only check the conditions when $(\omega \wedge \sigma)_{S^*}(s) = \omega(s) (\neq \sigma(s))$ and $(\omega \wedge \sigma)_{S^*}(t) = \sigma(t) (\neq \omega(t))$; or when $(\omega \wedge \sigma)_{S^*}(s) = \sigma(s) (\neq \omega(s))$ and $(\omega \wedge \sigma)_{S^*}(t) = \omega(t) (\neq \sigma(t))$.

In the first case, we must have $s \in Q$ and $t \in R$. If $t \in R$ but $t \notin Q$ then by the definition of $Q \wedge R$ we must have $t < Q$ and so $t < t'$ for some $t' \in Q$. This is impossible since Q is convex. Thus $t \in Q \cap R$ and so $t \in S^*$. We compute that $\omega(s) \leq O(s,t)(\omega(t)) \leq O(s,t)(\sigma(t))$ since $\omega(t) < \sigma(t)$ and $O(s,t)$ is weakly increasing.

In the second case, we must have $s \in R$ and $t \in Q$. By the definition of $Q \wedge R$ we must have $t \in R$ as well. So $t \in Q \cap R$ but $t \notin S^*$ which means that $\omega(t) > \sigma(t)$. Thus $\sigma(s) \leq O(s,t)(\sigma(t)) \leq O(s,t)(\omega(t))$ and $(\omega \wedge \sigma)_{S^*}$ respects O here. $\square$

For each (ω, σ), say a subset $S \subseteq S^*$ is *transferrable* if both $(\omega \wedge \sigma)_S$ and $(\omega \vee \sigma)_S$ respect O.

Lemma 4.3. *If S' and S'' are both transferrable then so is $S' \cap S''$.*

Proof. Let $s \lessdot t$ be a covering relation as above. Then pair $((\omega \wedge \sigma)_{S' \cap S''}(s), (\omega \wedge \sigma)_{S' \cap S''}(t))$ coincides with one of the two pairs $((\omega \wedge \sigma)_{S'}(s), (\omega \wedge \sigma)_{S'}(t))$ or $((\omega \wedge \sigma)_{S''}(s), (\omega \wedge \sigma)_{S''}(t))$, depending on which of S', S'' elements s, t do or do not belong. Since both those pairs agree with O, so must $((\omega \wedge \sigma)_{S' \cap S''}(s), (\omega \wedge \sigma)_{S' \cap S''}(t))$. Same argument applies for $((\omega \vee \sigma)_{S' \cap S''}(s), (\omega \vee \sigma)_{S' \cap S''}(t))$. $\square$

The Lemma implies that there exists a unique smallest transferrable subset $S^\diamond \subseteq S^*$. The set $S^\diamond$ is going to be the key ingridient in the proof of Cell Transfer theorem below.

Define $\eta : \mathcal{A}(Q, O) \times \mathcal{A}(R, O) \to \mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$ by

$$(\omega, \sigma) \longmapsto ((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond}).$$

Note that $S^\diamond$ depends on ω and σ, though we have suppressed the dependence from the notation.

We call this η the *cell transfer procedure.* The name comes from our main examples where elements of a poset are cells of a Young diagram. For convenience, in this paper we call elements of any poset *cells.* We say that a cell s is *transfered* if $s \in S^\diamond$ The map η is applicable to any two tableaux (ω, σ) in $\mathcal{A}(Q, O) \times \mathcal{A}(R, O)$.

Lemma 4.4. *The map η is injective.*

Proof. Given $(\alpha, \beta) \in \eta(\mathcal{A}(Q, O) \times \mathcal{A}(R, O))$, we show how to recover ω and σ. As before, for a subset $S \subset Q \cap R$, define $\omega_S = \omega(\alpha, \beta)_S : Q \to \mathbb{P}$ by

$$\omega_S(x) = \begin{cases} \beta(x) & \text{if } x \in (Q \backslash R) \cap (Q \vee R) \text{ or } x \in S, \\ \alpha(x) & \text{otherwise.} \end{cases}$$

And define $\sigma_S = \sigma(\alpha, \beta)_S : R \to \mathbb{P}$ by

$$\sigma_S(x) = \begin{cases} \alpha(x) & \text{if } x \in (R \backslash Q) \cap (Q \wedge R) \text{ or } x \in S, \\ \beta(x) & \text{otherwise.} \end{cases}$$

Note that if $(\alpha, \beta) = ((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond}))$ then $\omega = \omega_{S^\diamond}$ and $\sigma = \sigma_{S^\diamond}$. Let $S^\square \subset Q \cap R$ be the unique smallest subset such that $\omega_{S^\square}$ and $\sigma_{S^\square}$ both respect O. Since we have assumed that $(\alpha,\beta) \in \eta(\mathcal{A}(Q,O) \times \mathcal{A}(R,O))$, such a $S^{\Box}$ must exist. (As before the intersection of two transferrable subsets with respect to (α,β) is transferrable.)

We now show that if $(\alpha,\beta) = ((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond}))$ then $S^{\Box} = S^{\diamond}$. We know that $S^{\Box} \subset S^{\diamond}$ since we've chosen $S^{\Box}$ to be the smallest (by containment) in the set which contains $S^{\diamond}$. Let $C \subset S^{\diamond}\backslash S^{\Box}$ be a connected component of $S^{\diamond}\backslash S^{\Box}$, viewed as an induced subgraph of the Hasse diagram of P. We claim that $S^{\diamond}\backslash C$ is a transferrable set for (ω,σ); this means that changing $\alpha|_C$ to $\omega|_C$ and $\beta|_C$ to $\sigma|_C$ gives a pair in $\mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$. Suppose first that $c \in C$ and $s \in S^{\Box}$ is so that $c \lessdot s$. By the definition of $S^{\Box}$, we must have $\alpha(c) \leq O(c,s)(\beta(s))$ and $\beta(c) \leq O(c,s)(\alpha(s))$. Now suppose that $c \in C$ and $s \in Q\backslash R$ such that $c \lessdot s$. Then we must have $O(c,s)(\omega(s)) = O(c,s)(\beta(s)) \geq \alpha(c) = \sigma(c)$. Similar conclusions hold for $c \gtrdot s$. Thus we have checked that $S^{\diamond}\backslash C$ is a transferrable set for (ω,σ), which is impossible by definition of $S^{\diamond}$: it is the minimal set with this property. Therefore the set $S^{\diamond}\backslash S^{\Box}$ is empty and thus $S^{\diamond} = S^{\Box}$.

Thus the maps $\omega_{S^\Box}$ and $\sigma_{S^\Box}$, which are well-defined in the sence that they depend only on α and β, provide an inverse to η. This shows that the map $(\omega,\sigma) \mapsto ((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond}))$ is injective, completing the proof. $\Box$

We call a map between pairs of tableaux *weight-preserving* if the multiset of their values over all $s \in P$ is not changed by the map.

Theorem 4.5 (Cell Transfer Theorem). *The difference*

$$K_{Q \wedge R,O} K_{Q \vee R,O} - K_{Q,O} K_{R,O}$$

is monomial-positive.

Proof. The map

$$\eta : \mathcal{A}(Q,O) \times \mathcal{A}(R,O) \longrightarrow \mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$$

defined above is weight-preserving. Indeed, in fact for each element $s \in P \cup Q$ we have $\{\omega(s), \sigma(s)\} = \{(\omega \wedge \sigma)_{S^\diamond}(s), (\omega \vee \sigma)_{S^\diamond}(s)\}$ as multisets, where the value of tableaux is zero outside of its range of definition. Then since the map η is injective and since

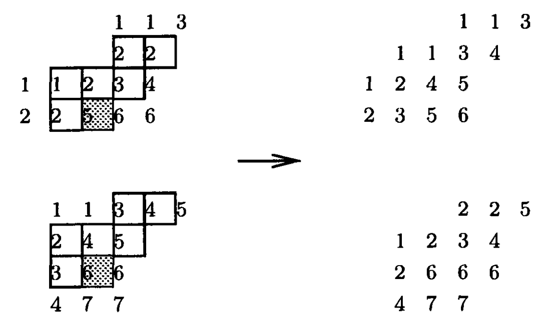


FIGURE 3. An example for semistandard Young tableaux, cells in $S^\diamond$ are marked on the left, the cell in $S^*/S^\diamond$ is marked on the right.

$K_{Q\wedge R,O}K_{Q\vee R,O}$ and $K_{Q,O}K_{R,O}$ are the generating functions of common weights of pairs of the tableaux of corresponding shapes, the statement follows. □

On Figure 3 an example of cell transfer is given for the tableaux shapes we have seen on Figure 2. Note that $S^\diamond$ does not contain one cell which is in S^*: the cell labelled 5 in Q and 6 in R.

Note that $(\omega, \sigma) \mapsto ((\omega \wedge \sigma)_{S^*}, (\omega \vee \sigma)_{S^*})$ also defines a weight-preserving map $\eta^* : \mathcal{A}(Q, O) \times \mathcal{A}(R, O) \to \mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$. Unfortunately, η^* is not always injective.

Suppose P is a locally-finite poset with a unique minimal element. Let $J(P)$ be the lattice of finite order ideals of P; see [Sta]. If $I, J \in J(P)$ then the subposets $I \wedge J$ and $I \vee J$ of P defined in (2) and (3) are finite order ideals of P and agree with the the meet $\wedge_{J(P)}$ and join $\vee_{J(P)}$ of I and J respectively within $J(P)$. In fact, by defining $Q \wedge' R = \{s \in R \mid s < Q\} \cup \{s \in Q \mid s \in R \text{ or } s < R\}$ and $Q \vee' R = \{s \in Q \mid s \sim R \text{ or } s > R\} \cup \{s \in R \mid s \sim Q \text{ or } s > Q\}$, the order ideals $I \wedge' J = I \wedge_{J(P)} J$ and $I \vee' J = I \vee_{J(P)} J$ agree with the meet and join in $J(P)$ even when P does not contain a minimal element.

Corollary 4.6. *Let P be a locally-finite poset, let I, J be elements of $J(P)$, and let O be a $\mathbb{T}$-labelling of P. Then the generating function*

$$K_{I\wedge_{J(P)}J,O}K_{I\vee_{J(P)}J,O} - K_{I,O}K_{J,O}$$

is monomial-positive.

Proof. The elements altered going from $(Q \wedge R)$ to $Q \wedge' R$ are exactly the elements $s \in Q$ which are incomparable with elements of R. Those cells end up in $Q \vee' R$ instead of $Q \wedge R$. In both cases those cells respect O since they respect it in Q and are never compared with cells in R. They also do not make the difference in defining η' since they are never compared with cells in $Q \cap R \subset R$. In fact, the definition of the map η' extends verbatim from that of the map η, and so do all the proofs. □

5. Symmetric functions

Define two functions $f^{\text{weak}}, f^{\text{strict}} : \mathbb{P} \to \mathbb{N} \cup \{\infty\}$ by $f^{\text{weak}}(n) = n$ and $f^{\text{strict}}(n) = n - 1$.

Proposition 5.1. *Let (P, O) be a finite $\mathbb{T}$-labelled poset. Suppose*

$$O(s,t) \in \{f^{\text{weak}}, f^{\text{strict}}\}$$

for each covering relation $s \lessdot t$. Then $K_{P,O}(x)$ is a quasi-symmetric function.

A $\mathbb{T}$-labelled poset satisfying the conditions of the proposition is called *oriented* in [McN]. Stanley's (P,ω)-partitions are special cases of (P,O)-tableaux, for such posets. If $f \in \mathcal{Q}sym$ then f is m-positive if and only if is a non-negative linear combination of the M_α.

A formal power series $f = f(x) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ with bounded degree is called *symmetric* if for any $a_1, a_2, \ldots, a_k \in \mathbb{P}$ we have

$$\left[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}\right] f = \left[x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}\right] f$$

whenever $i_1, \ldots, i_k$ are all distinct and $j_1, \ldots, j_k$ are all distinct. Denote by $\Lambda \subset \mathbb{Q}[[x_1, x_2, \ldots]]$ the algebra of symmetric functions. Every symmetric function is quasisymmetric.

Given $\lambda = (\lambda_1, \lambda_2, \ldots)$, the monomial symmetric functions m_λ is given by

$$m_\lambda(x) = \sum_\alpha x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$

where the sum is over all distinct permutations α of the entries of the (infinite) vector $(\lambda_1, \lambda_2, \ldots)$. As λ ranges over all partitions, the m_λ form a basis of Λ. If $f \in \Lambda$

then f is *monomial-positive* if and only if is a non-negative linear combination of the monomial symmetric functions. If $f \in \Lambda$ is a non-negative linear combination of Schur functions then we call f *Schur-positive.*

Theorem 5.2. *The symmetric function $s_{\mu\wedge\lambda}s_{\mu\vee\lambda} - s_\mu s_\lambda$ is monomial-positive.*

Proof. This follows immediately from Theorem 4.5. □

Let $\lambda/\mu = (\lambda_1,\ldots,\lambda_k)/(\mu_1,\ldots,\mu_k)$ and $\nu/\rho = (\nu_1,\ldots,\nu_k)/(\rho_1,\ldots,\rho_k)$. Define

$$\max(\lambda/\mu,\nu/\rho) := (\max(\lambda_1,\nu_1),\ldots,\max(\lambda_k,\nu_k))/(\max(\mu_1,\rho_1),\ldots,\max(\mu_k,\rho_k))$$

and

$$\min(\lambda/\mu,\nu/\rho) := (\min(\lambda_1,\nu_1),\ldots,\min(\lambda_k,\nu_k))/(\min(\mu_1,\rho_1),\ldots,\min(\mu_k,\rho_k)).$$

These shapes are nearly but not always the same as $\lambda/\mu \vee \nu/\rho$ and $\lambda/\mu \wedge \nu/\rho$ respectively. This is because occasionally we may for example have $\lambda_i = \mu_i = a$ for some i and then the shape λ/μ does not depend on the exact value of a. However, the shapes $\max(\lambda/\mu,\nu/\rho)$ and $\min(\lambda/\mu,\nu/\rho)$ do depend on the choice of a. The event of $\lambda_i = \mu_i$ for some i-s is however, as it is not hard to see, the only possible reason for $\lambda/\mu \vee \nu/\rho$ and $\lambda/\mu \wedge \nu/\rho$ to be different from $\max(\lambda/\mu,\nu/\rho)$ and $\min(\lambda/\mu,\nu/\rho)$.

Theorem 5.3. *The symmetric function $s_{\max(\lambda/\mu,\nu/\rho)}s_{\min(\lambda/\mu,\nu/\rho)} - s_{\lambda/\mu}s_{\nu/\rho}$ is monomial-positive.*

Proof. This follows using a minor modification of the proof of Theorem 4.5. Write $A = \min(\lambda/\mu,\nu/\rho)$ and $B = \max(\lambda/\mu,\nu/\rho)$. As mentioned above, if $\lambda/\mu \vee \nu/\rho$ and $\lambda/\mu \wedge \nu/\rho$ are different from $\max(\lambda/\mu,\nu/\rho)$ and $\min(\lambda/\mu,\nu/\rho)$, then for some i-s we must have $\lambda_i = \mu_i$. Let $i = p,\ldots,q$ be indexes such that $\lambda_i = \mu_i$ for each, however $\lambda_{p-1} \neq \mu_{p-1}$, $\lambda_{q+1} \neq \mu_{q+1}$. Obviously, the set of such i-s can be split into such segments $[p,q]$. The result of the map $(\lambda/\mu,\nu/\rho) \mapsto (\max(\lambda/\mu,\nu/\rho),\min(\lambda/\mu,\nu/\rho))$ then depends on the sequence $\lambda_{q+1} \leq \lambda_q = \mu_q \leq \cdots \leq \lambda_p = \mu_p \leq \mu_{p-1}$. Note that we get pair $(\max(\lambda/\mu,\nu/\rho),\min(\lambda/\mu,\nu/\rho))$ to be equal to $(\lambda/\mu \vee \nu/\rho, \lambda/\mu \wedge \nu/\rho)$ if $\lambda_{q+1} = \lambda_q = \mu_q = \cdots = \lambda_p = \mu_p$. Then for general sequence $\lambda_{q+1} \leq \lambda_q = \mu_q \leq \cdots \leq \lambda_p = \mu_p \leq \mu_{p-1}$ cells of $\max(\lambda/\mu,\nu/\rho)$ form a proper subset of cells of $\lambda/\mu \vee \nu/\rho$, and therefore are consistent with semistandard labelling of the letter. As for the cells of $\min(\lambda/\mu, \nu/\rho)$ in rows from p to q, all of them are former cells of ν/ρ and neither of them is comparable with any cell in $\lambda/\mu \wedge \nu/\rho$. Therefore when we form $\min(\lambda/\mu, \nu/\rho)$ as a union of those cells and $\lambda/\mu \wedge \nu/\rho$, we never violate the semistandard property of the tableaux.

Thus the set of such sequences $\lambda_{q+1} \leq \lambda_q = \mu_q \leq \cdots \leq \lambda_p = \mu_p \leq \mu_{p-1}$, one for each enterval $[p, q]$, produces an altered map $\bar{\eta}$ which consists of η and *cut and glue* map κ. The map κ is charactrised as follows: we cut cells (together with their fillings) in $\min(\lambda/\mu, \nu/\rho)/(\lambda/\mu \wedge \nu/\rho)$ from $\lambda/\mu \vee \nu/\rho$ and glue them to $\lambda/\mu \wedge \nu/\rho$. As shown above, the map $\bar{\eta} = \kappa \circ \eta$ always produces a valid pair of semistandard tableaux. It is also invertible since so is κ, as it is easy to see. Therefore, the map $\bar{\eta}$ allowes us to conclude the needed inequality same way as we used η to prove Theorem 4.5. ☐

Conjecture 5.4. *The symmetric function* $s_{\max(\lambda/\mu,\nu/\rho)} s_{\min(\lambda/\mu,\nu/\rho)} - s_{\lambda/\mu} s_{\nu/\rho}$ *is Schur-positive.*

This conjecture is proved in joint work with Alex Postnikov [LPP].

6. Cell transfer as an algorithm

Let (P, O) be a $\mathbb{T}$-labelled poset. We now give an algorithmic description of cell transfer. Let Q and R be two finite convex subposets of P. We construct step-by-step an injection

$$\eta : \mathcal{A}(Q, O) \times \mathcal{A}(R, O) \longrightarrow \mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$$

which is weight-preserving. Let ω be a (Q, O)-tableau and σ be an (R, O)-tableau. Let us recursively define $\bar{\omega} : Q \wedge R \to \mathbb{P}$ and $\bar{\sigma} : Q \vee R \to \mathbb{P}$ as follows.

(1) Define $\bar{\omega} : Q \wedge R \to \mathbb{P}$ and $\bar{\sigma} : Q \vee R \to \mathbb{P}$ as follows:

$$\bar{\omega}(s) = \begin{cases} \omega(s) & \text{if } s \in Q, \\ \sigma(s) & \text{if } s \in R/Q. \end{cases}$$

$$\bar{\sigma}(s) = \begin{cases} \sigma(s) & \text{if } s \in R, \\ \omega(s) & \text{if } s \in Q/R. \end{cases}$$

Note that $\bar{\omega}$ and $\bar{\sigma}$ do not necessarily respect O. Indeed, the parts of ω and σ which we glued together might not agree with each other, e.i. a covering relation $s \lessdot t$ might fail to respect O, where label of one of s, t comes from ω, and that of the other - from σ.

(2) We say that we *transfer* a cell $s \in Q \cap R$ when we swap the values at s of $\bar{\omega}$ and $\bar{\sigma}$. We say that a cell s in $Q \cap R$ is *critical* if one of the following condition holds

(a) for some $t \in R$ and $s \gtrdot t$ we have $O(s,t)(\bar{\omega}(s)) < \bar{\omega}(t)$,

(b) for some $t \in Q \cap R$ and $t \gtrdot s$ we have $O(s,t)(\bar{\sigma}(t)) < \bar{\sigma}(s)$,

(c) for some $t \in Q$ and $t \gtrdot s$ we have $O(s,t)(\bar{\sigma}(t)) < \bar{\sigma}(s)$,

(d) for some $t \in Q \cap R$ and $s \gtrdot t$ we have $O(s,t)(\bar{\omega}(s)) < \bar{\omega}(t)$,

and s was not transferred in a previous iteration. We now transfer all critical cells if there are any.

(3) Repeat step (2) until no critical cells are transferred.

Theorem 6.1 (Cell Transfer Algorithm). *The algorithm described above terminates in a finite number of steps. The resulting maps $\bar{\omega}$ and $\bar{\sigma}$ are (P,O)-tableaux and coincide with $(\omega \wedge \sigma)_{S^\diamond}$ and $(\omega \vee \sigma)_{S^\diamond}$ defined in the proof of Theorem 4.5.*

Proof. As for the first claim, there is a finite number of cells in $Q \cap R$ and each gets transferred at most once, thus the process terminates.

We say that an edge $a \lessdot b$ in the Hasse diagram of P respects O if $\bar{\omega}(a) \leq O(a,b)(\bar{\omega}(b))$ and $\bar{\sigma}(a) \leq O(a,b)(\bar{\sigma}(b))$, whenever these inequalities make sense. Note that a cell s is critical only if the cell t (from the definition of a critical cell) was transferred in previous iteration of step (2), or if it is the first iteration of step (2) and t belongs to $\{s \in R \mid s < Q\}$ or $\{s \in Q \mid s > R\}$ – the parts which were "glued" in step (1). Indeed, if s, t have both not been transferred then $s \lessdot t$ must respect O since ω and σ were (P,O)-tableaux to begin with. Similarly, two cells $s \lessdot t$ which have both been transferred must also respect O.

We thus see that after the second step every edge between $\{s \in R \mid s < Q\}$ and $Q \cap R$, as well as between $\{s \in Q \mid s > R\}$ and $Q \cap R$ respects O. After the algorithm terminates every edge in $Q \cap R$ must respect O, since if there exists an edge $s \lessdot t$ which does not then one of s and t must have already been transferred, and the other has not been transferred and thus is critical. This contradicts the termination condition of the algorithm. Therefore, the only possible edges which might fail to respect O are the ones between $\{s \in Q \mid s < R\}$ and $Q \cap R$, and the ones between $\{s \in R \mid s > Q\}$ and $Q \cap R$. However, it is easy to see that during the whole process values of $\bar{\omega}$ on $Q \cap R$ are increasing, and therefore cannot be not large enough for values of $\bar{\omega}$ on $\{s \in Q \mid s < R\}$. Similarly, values of $\bar{\sigma}$ on $Q \cap R$ are decreasing and cannot be too large for values of $\bar{\sigma}$ on $\{s \in R \mid s > Q\}$. Thus, we do obtain two (P, O)-tableaux $\bar{\omega}$ and $\bar{\sigma}$.

Let $\bar{S} \subset Q \cap R$ be the set of cells we transferred during the algorithm. The fact that values of $\bar{\omega}$ on $Q \cap R$ increase and the values of $\bar{\sigma}$ on $Q \cap R$ decrease implies that $\bar{S}$ is contained in S^* (as defined in the proof of Theorem 4.5). We claim that all transferrable sets contain $\bar{S}$. Indeed, in each iteration we transfer only those cells that must be transferred in order for the result to respect O. On the other hand, as shown above the set $\bar{S}$ is transferrable itself. Thus, it is exactly the set $S^\diamond$ – the minimal transferrable set. This completes the proof of the theorem. □

The algorithmic description above provides another way to verify injectivity of η. Let $\bar{\omega} : Q \wedge R \to \mathbb{P}$ and $\bar{\sigma} : Q \vee R \to \mathbb{P}$ be in the image of η. Then one can define maps $\omega' : Q \to \mathbb{P}$ and $\sigma' : R \to \mathbb{P}$ by

$$\omega'(s) = \begin{cases} \bar{\omega}(s) & \text{if } s \in Q \cap (Q \wedge R), \\ \bar{\sigma}(s) & \text{otherwise.} \end{cases}$$

$$\sigma(s) = \begin{cases} \bar{\sigma}(s) & \text{if } s \in R \cap (Q \vee R), \\ \bar{\omega}(s) & \text{otherwise.} \end{cases}$$

We now iterate step (2) of the cell transfer algorithm with ω' and σ' replacing $\bar{\omega}$ and $\bar{\sigma}$.

One can verify that for each step the set of transferred cells is identical to the corresponding step of the original algorithm for $\bar{\omega}$ and $\bar{\sigma}$. This produces the inverse of η.

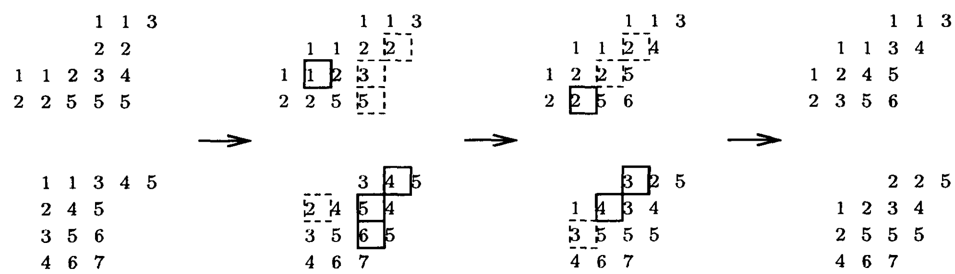


FIGURE 4. An example of the cell transfer algorithm. The critical cells are marked with squares.

We now give an example of the cell transfer algorithm for semistandard Young tableaux. Recall that we consider all horizontal edges to be labelled by $f^{\text{weak}}(x) = x$, and all vertical edges by $f^{\text{strict}}(x) = x - 1$. Convex subposets in this case are skew shapes. Recall the Figure 2 on which it is shown how the shapes $Q \wedge R$ and $Q \vee R$ are formed. On Figure 4 we give an example of step-by-step application of η for a particular pair ω and σ.

7. P-PARTITIONS

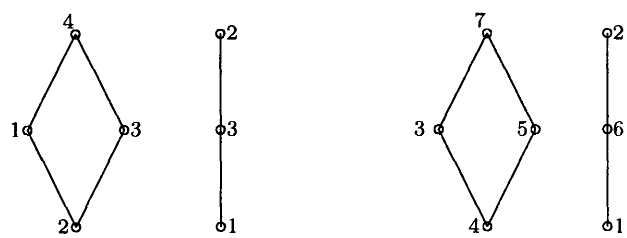
7.1. **Labeled posets.** Let P be a poset. A *labeling* θ of P is an injection $\theta : P \to \mathbb{P}$ into the positive integers. A descent of the labeling θ of P is a pair $p \lessdot p'$ in P such that $\theta(p) > \theta(p')$. Let us say that two labeled posets (P, θ_P), (Q, θ_Q) are isomorphic if there is an isomorphism of posets $f : P \to Q$ so that descents are preserved. That is if $p \lessdot p'$ then $\theta(p) < \theta(p')$ if and only if $\theta'(f(p)) < \theta'(f(p'))$. We say that two labelings θ_1 and θ_2 of P are *equivalent* if the identity map on P is an isomorphism of (P, θ_1) and (P, θ_2).

Let (P, θ) be a labeled poset. If $Q \subset P$ is a subposet, then it inherits a labeling $\theta|_Q$ by restriction. When no confusion can arise, we will often denote $\theta|_Q$ simply by θ. Note however, that the descents of $\theta|_Q$ are not completely determined by the descents of θ, unless Q is a convex subset of P.

Let (P, θ_P) and (Q, θ_Q) be labeled posets. Then the *disjoint sum* $(P \oplus Q, \theta^{\oplus})$ is the labeled poset (defined up to equivalence of labelings) where $\theta^{\oplus}$ has the same descents as the function

$$f(a) = \begin{cases} \theta_P(a) & \text{if } a \in P, \text{ and} \\ \theta_Q(a) & \text{if } a \in Q. \end{cases}$$

Example 7.1. *Let P be the diamond poset with elements $a < b, c < d$ and labeling θ_P given by $\theta_P(a) = 2$, $\theta_P(b) = 1$, $\theta_P(c) = 3$, and $\theta_P(d) = 4$. Let Q be the chain with elements $e < f < g$ and labeling θ_Q given by $\theta_Q(e) = 1$, $\theta_Q(f) = 3$, and $\theta_Q(g) = 2$. The one possible labeling $\theta^{\oplus}$ for the disjoin sum $P \oplus Q$ is given by $\theta^{\oplus}(a, b, c, d, e, f, g) = 4, 3, 5, 7, 1, 6, 2$. In Figure 5, the three labeled posets (P, θ_P), (Q, θ_Q), and $(P \oplus Q, \theta^{\oplus})$ are shown. Note that we have some freedom in choosing the labelling $\theta^{\oplus}$.*



(P, θ_P) (Q, θ_Q) $(P \oplus Q, \theta^{\oplus})$

FIGURE 5

7.2. P-partitions.

Definition 7.2. *A (P, θ)-partition is a map $\sigma : P \to \mathbb{P}$ such that for each covering relation $s \lessdot t$ in P we have*

$$\sigma(s) \leq \sigma(t) \qquad \text{if } \theta(s) < \theta(t),$$

$$\sigma(s) < \sigma(t) \qquad \text{if } \theta(t) < \theta(s).$$

If $\sigma : P \to \mathbb{P}$ is any map, then we say that σ respects θ if σ is a (P, θ)-partition.

Denote by $\mathcal{A}(P, \theta)$ the set of all (P, θ)-partitions. Clearly $\mathcal{A}(P, \theta)$ depends on (P, θ) only up to isomorphism. If P is finite then one can define the formal power series $K_{P,\theta}(x_1, x_2, \ldots) \in \mathbb{Z}[[x_1, x_2, \ldots]]$ by

$$K_{P,\theta}(x_1, x_2, \ldots) = \sum_{\sigma \in \mathcal{A}(P,\theta)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots .$$

The composition $\mathrm{wt}(\sigma) = (\#\sigma^{-1}(1), \#\sigma^{-1}(2), \ldots)$ is called the *weight* of σ.

Let P be a poset with n elements. Recall that a linear extension of P is a bijection $e : P \longrightarrow \{1, 2, \ldots, n\}$ satisfying $e(x) \leq e(y)$ if $x \leq y$ in P. The Jordan-Holder set $\mathcal{J}(P, \theta)$ of (P, θ) is the set

$$\{\theta(e^{-1}(1))\theta(e^{-1}(1)) \cdots \theta(e^{-1}(n)) \mid e \text{ is a linear extension of } P\}.$$

It is a subset of the set $\mathfrak{S}(\theta(P))$ of permutations of $\theta(P) \subset \mathbb{P}$.

Example 7.3. *Suppose C is a chain $c_1 < c_2 < \ldots < c_n$ with n elements and $w = w_1 w_2 \ldots w_n \in \mathfrak{S}_n$ a permutation of $\{1, 2, \ldots, n\}$. Then (C, w) can be considered a labeled poset, where $w(c_i) = w_i$. In this case we have $K_{C,w} = L_{\mathbb{C}(w)}$.*

Theorem 7.4 ([Sta]). *The generating function $K_{P,\theta}$ is quasi-symmetric. We have $K_{P,\theta} = \sum_{w \in \mathcal{J}(P,\theta)} L_{D(w)}$.*

In particular, $K_{P,\theta}$ is L-positive. This motivates our treatment of $K_{P,\theta}$ as "skew"-analogues of the functions L_α. Let Q and R be two finite convex subposets of (P, θ).

The following is the direct consequence of Theorem 4.5

Theorem 7.5. *The difference $K_{Q \wedge R,\theta} K_{Q \vee R,\theta} - K_{Q,\theta} K_{R,\theta}$ is M-positive.*

8. Cell transfer for P-partitions

By Theorem 7.4, the expression $K_{Q \wedge R,\theta} K_{Q \vee R,\theta} - K_{Q,\theta} K_{R,\theta}$ is always a quasi-symmetric function. We now show that this difference is L-positive.

Let (P, θ) be a labeled poset and let Q and R be convex subsets. In Section 4, we gave a weight preserving injection

$$\eta : \mathcal{A}(Q, \theta) \times \mathcal{A}(R, \theta) \longrightarrow \mathcal{A}(Q \wedge R, \theta) \times \mathcal{A}(Q \vee R, \theta).$$

Let $S \subset Q \cap R$. Then $(\omega \wedge \sigma)_S : Q \wedge R \to \mathbb{P}$ was defined by

$$(\omega \wedge \sigma)_S(x) = \begin{cases} \sigma(x) & \text{if } x \in R \backslash Q \text{ or } x \in S, \\ \omega(x) & \text{otherwise,} \end{cases}$$

and $(\omega \vee \sigma)_S : Q \vee R \to \mathbb{P}$ by

$$(\omega \vee \sigma)_S(x) = \begin{cases} \omega(x) & \text{if } x \in Q \backslash R \text{ or } x \in S, \\ \sigma(x) & \text{otherwise.} \end{cases}$$

Let $S^\diamond$ be the *smallest* set such that $((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond})$ is an element of $\mathcal{A}(Q \wedge R, \omega) \times \mathcal{A}(Q \vee R, \omega)$. In [LP], we showed that a smallest such set exists. Now define $\eta(\omega, \sigma)$ to be $((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond})$.

The injection η satisfies additional crucial properties. First let us say that $i \neq j$ are *adjacent* in a multiset T (of integers) if $i, j \in T$ and for any other $t \in T$ both $i \leq t \leq j$ and $j \leq t \leq i$ fail to hold.

Proposition 8.1. *Suppose* $\omega \in \mathcal{A}(Q, \theta)$ *and* $\sigma \in \mathcal{A}(R, \theta)$ *and* $\eta(\omega, \sigma) = (\omega \wedge \sigma, \omega \vee \sigma)$. *Let* $p \in Q \cup R$.

(1) *If* $p \in Q \cap R$, *then* $\{\omega(p), \sigma(p)\} = \{(\omega \wedge \sigma)(p), (\omega \vee \sigma)(p)\}$. *Furthermore, suppose* $\omega(p)$ *and* $\sigma(p)$ *are adjacent in the multiset* $\omega(Q) \cup \sigma(R)$. *Then* $(\omega \wedge \sigma)(p) = \omega(p)$ *and* $(\omega \vee \sigma)(p) = \sigma(p)$.

(2) *If* $p \in Q \wedge R$ *but* $p \notin Q \cap R$ *then* $(\omega \wedge \sigma)(p) = \omega(p)$ *if* $p \in Q$ *and* $(\omega \wedge \sigma)(p) = \sigma(p)$ *if* $p \in R$.

(3) *If* $p \in Q \vee R$ *but* $p \notin Q \cap R$ *then* $(\omega \vee \sigma)(p) = \omega(p)$ *if* $p \in Q$ *and* $(\omega \vee \sigma)(p) = \sigma(p)$ *if* $p \in R$.

Roughly speaking, Proposition 8.1(1) says that if $p \in Q \cap R$, then one obtains $((\omega \wedge \sigma)(p), (\omega \vee \sigma)(p))$ by possibly "swapping" $\omega(p)$ with $\sigma(p)$; in addition, no swapping occurs if $\omega(p)$ and $\sigma(p)$ are adjacent in $\omega(Q) \cup \sigma(R)$.

Proof. All except the last statement of (1) follows from the fact that $\eta(\omega, \sigma)$ has the form $((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond})$ for some set $S^\diamond$.

The last statement of (1) follows from the fact that $S^\diamond$ is defined to be the *smallest* set such that $((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond})$ is an element of $\mathcal{A}(Q \wedge R, \omega) \times \mathcal{A}(Q \vee R, \omega)$. More precisely, if $p \in Q \cap R$ is such that $\omega(p)$ and $\sigma(p)$ are adjacent then $p \notin S^\diamond$. $\square$

Now consider the labeled posets $(Q \oplus R, \theta^\oplus)$ and $((Q \vee R) \oplus (Q \wedge R), \theta^{\vee\wedge})$, where we shall pick $\theta^\oplus$ and $\theta^{\vee\wedge}$ as follows.

For each $p \in Q \cap R$, we "duplicate" $\theta(p)$ by picking $\theta(p)' > \theta(p)$ so that for every $x \in Q \cup R$ such that $x \neq p$ we have $\theta(p)' < \theta(x)$ if and only if $\theta(p) < \theta(x)$; also the duplicates satisfying the same inequalities as the originals so that $\theta(p)' < \theta(x)'$ if and only if $\theta(p) < \theta(x)$. This describes a total order on $\{\theta(p) \mid p \in Q \cup R\} \cup \{\theta(p)' \mid p \in Q \cap R\}$. Note that we may need to replace θ with an equivalent labeling so that there is enough "space" to insert the primed letters.

Now suppose $p \in Q \cap R$. Denote the copy of p inside $Q \subset Q \oplus R$ by p_Q and the copy of p inside $R \subset Q \oplus R$ by p_R. Similarly, denote the elements of $(Q \vee R) \oplus (Q \wedge R)$. We define

$$\theta^{\oplus}(p) = \begin{cases} \theta(p) & \text{if } p \notin Q \cap R, \\ \theta(p)' & \text{if } p = p_Q, \\ \theta(p) & \text{if } p = p_R \end{cases}$$

and

$$\theta^{\vee\wedge}(p) = \begin{cases} \theta(p) & \text{if } p \notin Q \cap R, \\ \theta(p)' & \text{if } p = p_{Q\wedge R}, \\ \theta(p) & \text{if } p = p_{Q\vee R}. \end{cases}$$

Clearly the descents of $\theta^{\oplus}$ (or $\theta^{\vee\wedge}$) on either component agree with the descents of that component as a convex subposet of (P, θ).

Theorem 8.2. *The difference $K_{Q\wedge R,\theta}K_{Q\vee R,\theta} - K_{Q,\theta}K_{R,\theta}$ is L-positive.*

Proof. Let $|Q| + |R| = n = |Q \vee R| + |Q \wedge R|$ and suppose $\alpha : Q \oplus R \to [n]$ is a linear extension. Then α in particular gives an element $(\alpha|_Q, \alpha|_R)$ of $\mathcal{A}(Q,\theta) \times \mathcal{A}(R,\theta)$. Using Proposition 8.1, we see that $\eta(\alpha|_Q, \alpha|_R) = (\beta|_{Q\wedge R}, \beta|_{Q\vee R})$ arises from a linear extension $\beta : (Q \wedge R) \oplus (Q \vee R) \to [n]$ (in other words the union $\beta|_{Q\wedge R} \cup \beta|_{Q\vee R}$ is exactly the interval $[n]$).

We claim that the two words

$$a_\alpha = \theta^{\oplus}(\alpha^{-1}(1))\theta^{\oplus}(\alpha^{-1}(2))\ldots\theta^{\oplus}(\alpha^{-1}(n))$$

$$b_\beta = \theta^{\vee\wedge}(\beta^{-1}(1))\theta^{\vee\wedge}(\beta^{-1}(2))\ldots\theta^{\vee\wedge}(\beta^{-1}(n))$$

have the same descent set. Again by Proposition 8.1, the word $b_\beta = b_1 b_2 \dots b_n$ is obtained from $a_\alpha = a_1 a_2 \dots a_n$ by swapping certain pairs (a_i, a_j) where $a_i = \theta^{\oplus}(p_Q)$ and $a_j = \theta^{\oplus}(p_R)$ for some $p \in Q \cap R$.

By definition $\theta^{\oplus}(p_Q) = \theta^{\vee\wedge}(p_{Q\wedge R})$ and $\theta^{\oplus}(p_R) = \theta^{\vee\wedge}(p_{Q\vee R})$ so swapping occurs if and only if $(\alpha(p_Q), \alpha(p_R)) = (\beta(p_{Q\vee R}), \beta(p_{Q\wedge R}))$. By the last statement of Proposition 8.1 (1), this never happens if $\alpha|_Q(p_Q)$ and $\alpha|_R(p_R)$ are adjacent in $[n]$, which is equivalent to $|i-j| = 1$. Thus swapping (a_i, a_j) is the same as swapping a pair of non-neighboring letters $(\theta(p), \theta(p)')$ in the word $a_1 a_2 \dots a_n$, which preserves descents by our choice of $\theta(p)'$.

We have $K_{Q,\theta} K_{R,\theta} = \sum_\alpha L_{D(a_\alpha)}$ and $K_{Q\wedge R,\theta} K_{Q\vee R,\theta} = \sum_\beta L_{D(b_\beta)}$, where the summations are over linear extensions of $Q \oplus R$ and $(Q \wedge R) \oplus (Q \vee R)$. Since η induces an injection from the first set of linear extensions to the second, we conclude that $K_{Q\wedge R,\theta} K_{Q\vee R,\theta} - K_{Q,\theta} K_{R,\theta}$ is L-positive. □

Example 8.3. *Let P be the poset on the 5 elements A, B, C, D, E given by the cover relations $A < B$, $A < C$, $B < D$, $B < E$, $C < D$, $C < E$. Take the following labeling θ of P: $\theta(A) = 2$, $\theta(B) = 1$, $\theta(C) = 4$, $\theta(D) = 5$, $\theta(E) = 3$. Take the two ideals $Q = \{A, B, C, D\}$, $R = \{A, B, C, E\}$ of P. Form the disjoint sum poset $Q \oplus R$. The elements $A, B, C \in Q \cap R$ have two images in the newly formed poset: A_Q, B_Q, C_Q and A_R, B_R, C_R. The labels of $Q \oplus R$ are formed according to the rule above: for $X = A, B, C$ we have $\theta^{\oplus}(X_Q) = \theta(X)'$ while $\theta^{\oplus}(X_R) = \theta(X)$. The resulting labeling is shown in Figure 6, with $\theta^{\oplus}$ taking the values $\{1 < 1' < 2 < 2' < 3 < 4 < 4' < 5\}$.*

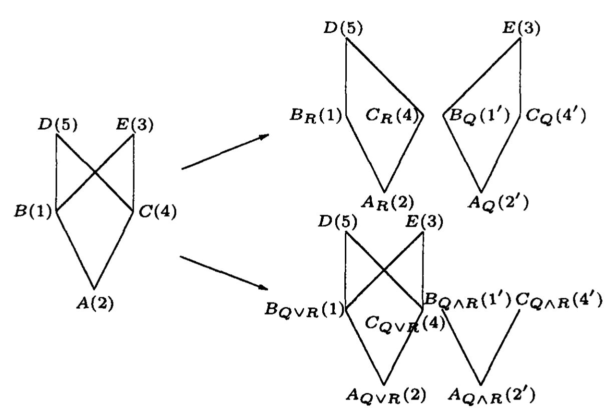


FIGURE 6. The labelings $\theta^{\oplus}$ of $Q \oplus R$ and $\theta^{\wedge\vee}$ of $(Q \wedge R) \oplus (Q \vee R)$ formed from a labeling θ of P. Labels are shown in parentheses.

Similarly, we obtain the labeling $\theta^{\wedge\vee}$ of $(Q \wedge R) \oplus (Q \vee R)$, as shown on Figure 6. Clearly each edge in the Hasse diagrams of $Q \oplus R$ and $(Q \wedge R) \oplus (Q \vee R)$ is a descent if and only if it is in the Hasse diagram of P.

Now, to illustrate the proof of Theorem 8.2 take a particular extension of $Q \oplus R$, namely α defined by $\alpha^{-1}([8]) = (A_Q, A_R, B_Q, C_Q, E, B_R, C_R, D)$. Performing cell transfer we get $\beta = \eta(\alpha)$ such that

$$\beta^{-1}([8]) = (A_{Q\wedge R}, A_{Q\vee R}, B_{Q\vee R}, C_{Q\vee R}, E, B_{Q\wedge R}, C_{Q\wedge R}, D)$$

In this case $\mathbf{a}_\alpha = (2', 2, 1', 4', 3, 1, 4, 5)$ and $\mathbf{b}_\beta = (2', 2, 1, 4, 3, 1', 4', 5)$. The pairs that got swapped are $(1, 1')$ and $(4, 4')$. Note also that the pair $(2, 2')$ did not get swapped, which we know cannot happen since those labels are neighbors are in the word $\mathbf{a}_\alpha$. It is clear that the descents in a_α are indeed the same as in b_β.

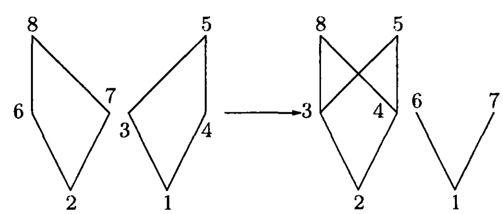


FIGURE 7. The linear extension α of $Q \oplus R$ and the linear extension β of $(Q \wedge R) \oplus (Q \vee R)$ obtained by cell transfer.

Comparing Theorem 8.2 and Theorem 7.4, we obtain the following question.

Question 8.4. *When is the difference $K_{Q\wedge R,\theta} K_{Q\vee R,\theta} - K_{Q,\theta} K_{R,\theta}$ itself of the form $K_{S,\xi}$ for some labeled poset (S, ξ)?*

In other words, we are asking for another (hopefully natural) operation $\sharp$ on convex subsets Q and R of a labeled poset (P, θ) so that

$$K_{Q\sharp R,\theta^\sharp} = K_{Q\wedge R,\theta} K_{Q\vee R,\theta} - K_{Q,\theta} K_{R,\theta}.$$

We will give an affirmative answer to Question 8.4 for the case of chains in Section 10. As the following example shows, Question 8.4 is not true in general.

Example 8.5. *Let P be the poset with four elements $\{a, b, c, d\}$ and relations $a < b, a < c, a < d$. Give P the labeling $\theta(a) = 4$, $\theta(b) = 1$, $\theta(c) = 2$, and $\theta(d) = 3$. Let Q*

be the ideal $\{a,b\}$ *and* R *be the ideal* $\{a,c,d\}$. *Then the difference* $K_{Q\wedge R,\theta}K_{Q\vee R,\theta} - K_{Q,\theta}K_{R,\theta}$ *is given by*

$$d = L_1(L_{1111} + 2L_{112} + 2L_{121} + L_{13}) - L_{11}(L_{12} + L_{111}). \tag{4}$$

We will argue that d *is not equal to* K_{S,θ_S} *for any* (S,θ_S). *First we claim that no term* L_α *in the* L*-expansion of* d *has* $\alpha_1 > 1$. *It is not difficult to see directly from the shuffle product that the expansion of each term in* d *has six* L_α *terms with* $\alpha_1 > 1$ *(in fact* $\alpha_1 = 2$*) and these cancel out by Theorem 8.2.*

Thus using Theorem 7.4 we conclude that if $d = K_{S,\theta_S}$ *then* S *must be a five element poset with a unique minimal element. Also one computes from (4) that* S *must have exactly 10 linear extensions. No poset* S *has these properties.*

Remark 8.6. *By carefully studying the cell transfer injection* η *of* [LP], *one can also give an affirmative answer to Question 8.4 for the case where* P *is a tree, and* Q *and* R *are order ideals so that both* $Q/Q\cap R$ *and* $R/Q\cap R$ *are connected.*

Remark 8.7. *Question 8.4 can be asked for the* $\mathbb{T}$*-labeled posets of* [LP] *and also for the differences of products of skew Schur functions studied in* [LPP]. *However, we will not investigate these questions in the current article.*

Remark 8.8. *Let* $K'_{P,\omega}$ *denote the generating function of Stembridge's enriched* P*-partitions of a labeled poset* (P,ω) *(*[Ste]*). Then*

$$K'_{Q\wedge R,\omega}K'_{Q\vee R,\omega} - K'_{Q,\omega}K'_{R,\omega}$$

is positive in the basis of peak functions. This follows immediately from [Ste, Theorem 3.1] *and Theorem 8.2.*

9. Chains and fundamental quasi-symmetric functions

9.1. **Cell transfer for compositions.** Let (C_n, w) be the labeled chain corresponding to the permutation $w \in \mathfrak{S}_n$. Let us consider C_n to consist of the elements $\{c_1 < c_2 < \cdots < c_n\}$, so that $w : C_n \to \mathbb{P}$ is given by $w(c_i) = w_i$. The convex subsets $C[i,j]$ of C_n are in bijection with intervals $[i,j] \subset [n]$.

Let $Q = [a, b]$ and $R = [c, d]$ and assume that $a \leq c$. Then we have the following two cases:

(1) If $b \leq d$ then $Q \wedge R = Q$ and $Q \vee R = R$.

(2) If $b \geq d$ then $Q \wedge R = [a, d]$ and $Q \vee R = [b, c]$.

Thus to obtain a non-trivial cell transfer we assume that $a < c \leq d < b$. Let $w[i, j]$ denote the word $w_i w_{i+1} \dots w_j$. Theorem 8.2 then says that the difference

$$L_{\mathbb{C}(w[a,d])} L_{\mathbb{C}(w[c,b])} - L_{\mathbb{C}(w[a,b])} L_{\mathbb{C}(w[c,d])} \tag{5}$$

is L-positive.

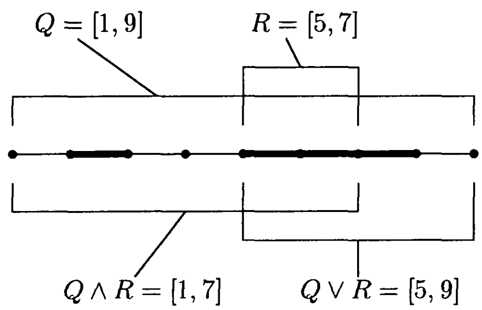


FIGURE 8. An example of the cell transfer operation for chains, here $w = (2, 1, 5, 4, 3, 7, 8, 9, 6)$, $a = 1, b = 9, c = 5, d = 7$.

We now make the difference (5) more precise by translating into the language of compositions and descent sets. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_l)$ be an ordered pair of compositions. Say β can be *found inside* α if there exists a non-negative integer $m \in [0, |\alpha| - |\beta|]$ so that $D(\beta) + m$ coincides with $D(\alpha)$ restricted to $[m + 1, m + |\beta| - 1]$. We then say that β can be found inside α at m. A composition can be found inside another in many different ways. For example if $\beta = (1)$ then one may pick m to be any integer in $[0, |\alpha| - 1]$.

Now for a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \vdash n$ and an integer $x \in [1, n]$ we define two new compositions $\alpha^{x\leftarrow}, \alpha^{x\rightarrow} \vdash x$ as follows. We define $\alpha^{x\leftarrow} = (\alpha_1, \alpha_2, \dots, \alpha_{r-1}, a)$ where a, r are the unique integers satisfying $1 \leq a \leq \alpha_r$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_{r-1} + a = x$. Similarly, define $\alpha^{x\rightarrow} = (b, \alpha_{s+1}, \dots, \alpha_k)$ where b, s are the unique integers satisfying $1 \leq b \leq \alpha_s$ and $b + \alpha_{s+1} + \cdots + \alpha_k = x$. If β can be found inside α at m, we set $\alpha \wedge_m \beta = \alpha^{(m+|\beta|)\leftarrow}$ and $\alpha \vee_m \beta = \alpha^{(|\alpha|-m)\rightarrow}$.

The L-positive expressions in (5) give the following theorem.

Theorem 9.1. *Let α and β be compositions such that β can be found inside α at m. Then the difference*

$$L_{\alpha\wedge_m\beta}\,L_{\alpha\vee_m\beta}-L_\alpha\,L_\beta$$

is L-positive.

Example 9.2. *Let us take chain (C_9,w) with $w=(2,1,5,4,3,7,8,9,6)$ and $Q=[1,9]$ and $R=[5,7]$ so that $a=1$, $b=9$, $c=5$, $d=7$. Then we get the situation shown in Figure 8, the thinner edges indicating descents. If $\alpha=(1,2,1,4,1)$ and $\beta=(3)$ then there are two ways to find β inside α, and Figure 8 shows the way to find it at $m=5$. In this case Theorem 9.1 says that $L_{(1,2,1,3)}L_{(4,1)}-L_{(1,2,1,4,1)}L_{(3)}$ is L-positive.*

Remark 9.3. *The operation $(\alpha,\beta)\mapsto(\alpha\wedge_m\beta,\alpha\vee_m\beta)$ interacts well with the involutions ν and ω of* QSym. *More precisely, if β can be found inside α at m then β^* can be found inside α^* at m and $\bar{\beta}$ can be found inside α^* at $|\alpha|-|\beta|-m$.*

9.2. **The L-positivity poset.** Fix a positive integer n. Now define a poset structure $(PC_n,\leq)$ ("Pairs of Compositions") on the set PC_n of unordered pairs $\{\alpha,\beta\}$ of compositions satisfying $|\alpha|+|\beta|=n$ by letting $\{\alpha,\beta\}\leq\{\gamma,\delta\}$ if $L_\gamma\,L_\delta-L_\alpha\,L_\beta$ is L-nonnegative. The following result relies on factorization properties of QSym which we prove in Section 11.

Proposition 9.4. *The relation $\{\alpha,\beta\}\leq\{\gamma,\delta\}$ if $L_\gamma\,L_\delta\geq_L L_\alpha\,L_\beta$ defines a partial order on the set PC_n.*

Proof. Reflexivity and transitivity of $\leq$ are clear. Suppose we have both $\{\alpha,\beta\}\leq\{\gamma,\delta\}$ and $\{\gamma,\delta\}\leq\{\alpha,\beta\}$ then we must have $L_\gamma\,L_\delta=L_\alpha\,L_\beta$. By Corollary 11.6 and Proposition 11.11 we must have $\{\alpha,\beta\}=\{\gamma,\delta\}$. Thus $\leq$ satisfies the symmetry condition of a partial order. □

For an unordered pair of compositions $\{\alpha,\beta\}$ we unambiguously define another unordered pair $\{\alpha\vee\beta,\alpha\wedge\beta\}$ as follows. Suppose $|\alpha|\geq|\beta|$. If β can be found inside of α, we pick the smallest $m\in(0,|\alpha|-|\beta|)$ where this is possible and set $\alpha\wedge\beta=\alpha\wedge_m\beta$ and $\alpha\vee\beta=\alpha\vee_m\beta$. Otherwise we set $\{\alpha\vee\beta,\alpha\wedge\beta\}=\{\alpha,\beta\}$.

Conjecture 9.5. *The maximal elements of PC_n are exactly the pairs $\{\alpha,\beta\}$ for which $\{\alpha,\beta\}=\{\alpha\wedge\beta,\alpha\vee\beta\}$.*

Note that Conjecture 9.5 is compatible with the two involutions ω and ν of QSym.

Remark 9.6. *(i) Conjecture 9.5 has been verified by computer up to* $n = 10$.

(ii) A result similar to Conjecture 9.5 holds for the case of Schur functions: the pairs of partitions corresponding to Schur-maximal products $s_\lambda s_\mu$ *are exactly those partitions fixed by "skew cell transfer"; see* [LPP2].

Example 9.7. *In Figure 9 the poset* PC_4 *is shown, compositions being represented by corresponding oriented line posets. The elements of the bottom row are single compositions of size* 4 *since the second composition in this case is empty.*

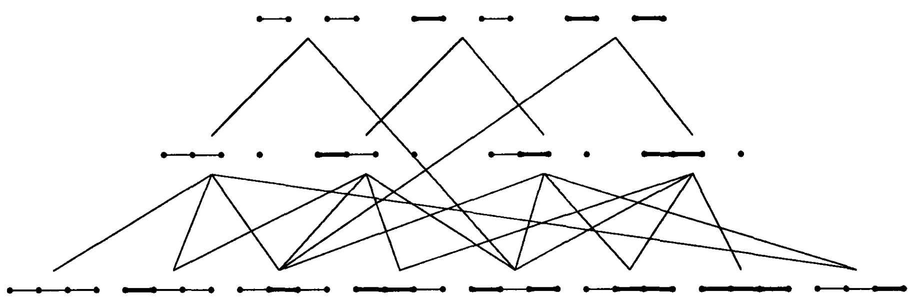


FIGURE 9. Partial order PC_4 on pairs of compositions, descents are drawn as thin edges.

One can see that maximal elements are exactly the ones for which one of the two compositions cannot be found inside the other. In this case those are exactly pairs (α, β) *such that* $|\alpha| = |\beta| = 2$.

10. WAVE SCHUR FUNCTIONS

In this section we define new generating functions called wave Schur functions. We first show that they are L-positive, and then prove a determinantal formula for them.

10.1. **Wave Schur functions as P-partition generating functions.** The poset $(\mathbb{N}^2, \leq)$ of (positive) points in a quadrant has cover relations $(i,j) \gtrdot (i-1,j)$ and $(i,j) \gtrdot (i,j-1)$. To agree with the "English" notation for Young diagrams the first coordinate i increases as we go down while the second coordinate j increases as we go to the right. Let us fix a sequence of "strict–weak" assignments $\mathbf{p} = (p_i \in \{\text{weak}, \text{strict}\} \mid i \in \mathbb{Z})$. Let $\overline{\text{weak}} = \text{strict}$ and $\overline{\text{strict}} = \text{weak}$. Define an *edge-labeling* (or *orientation* in the language of [McN]) $O_\mathbf{p}$ as a function from the covers of $\mathbb{N}^2$ to

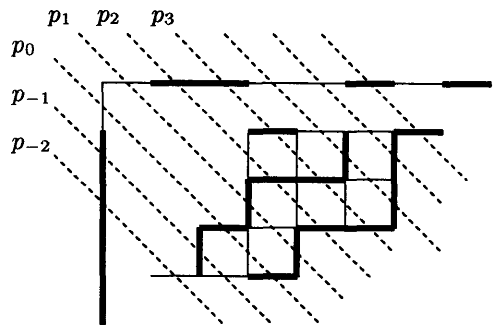


FIGURE 10. An edge labeling $O_{\mathbf{p}}$.

{weak, strict} by

$$O_{\mathbf{p}}((i,j) \gtrdot (i-1,j)) = \overline{p_{j-i+1}}$$

$$O_{\mathbf{p}}((i,j) \gtrdot (i,j-1)) = p_{j-i}.$$

An example of an such an edge-labeling $O_{\mathbf{p}}$ is given in Figure 10, where

$$\ldots p_{-3}, p_{-2}, p_{-1}, p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, \ldots =$$

$$\ldots \text{strict}, \text{strict}, \text{strict}, \text{weak}, \text{strict}, \text{weak}, \text{weak}, \text{strict}, \text{strict}, \text{weak}, \text{strict}, \text{weak} \ldots.$$

The lines show the diagonals along which $O_{\mathbf{p}}$ alternates between weak and strict edges. We have labeled weak edges thick and strong edges thin (agreeing with the way we labeled chains in Section 9).

In the following definition, λ/μ denotes a Young diagram $\{(i,j) \mid \mu_i \leq j \leq \lambda_i\}$ considered as a subposet of $(\mathbb{N}^2, \leq)$.

Definition 10.1. *A wave* **p***-tableau of shape* λ/μ *is a function* $T : \lambda \to \mathbb{P}$ *such that for each cover* $s \lessdot t$ *we have*

$$T(s) < T(t) \qquad \text{if } O_{\mathbf{p}}(s \lessdot t) = \text{strict},$$

$$T(s) \leq T(t) \qquad \text{if } O_{\mathbf{p}}(s \lessdot t) = \text{weak}.$$

The wave Schur function $s^{\mathbf{p}}_{\lambda/\mu}$ *is given by the weight generating function*

$$s^{\mathbf{p}}_{\lambda/\mu}(x_1, x_2, \ldots) = \sum_T x_1^{\#T^{-1}(1)} x_2^{\#T^{-1}(2)} \ldots$$

of all wave **p***-tableaux of shape* λ/μ.

We use the name *wave* since the pattern of strict and weak edges visually resembles waves. The *standard* "strict-weak" assignment is given by $\mathbf{p} = \{p_i\}$ where $p_i = \text{weak}$ for all i. In this case a wave $\mathbf{p}$-tableau is a usual semistandard tableau, and the wave Schur function is the usual Schur function. Note, however, that in general a wave Schur function is not symmetric. However, wave Schur functions are always (P,θ)-partition generating functions.

Proposition 10.2. *Let λ/μ be a skew shape. There exists a (vertex) labeling $\theta_{\mathbf{p}} : \lambda/\mu \to \mathbb{P}$ such that $(s \lessdot t)$ is a descent of $\theta_{\mathbf{p}}$ if and only if $O_{\mathbf{p}}(s \lessdot t) = \text{strict}$. Thus $s^{\mathbf{p}}_{\lambda/\mu} = K_{\lambda/\mu,\theta_{\mathbf{p}}}$.*

Proof. We shall prove the result by induction on the number of boxes in λ/μ. Let (i,j) be any outer corner of λ/μ. In other words there are no boxes to the bottom right of (i,j), and if we remove (i,j) from λ/μ we still obtain a valid skew shape $(\lambda/\mu)^-$. Suppose $\theta^-_{\mathbf{p}}$ has been defined for $(\lambda/\mu)^-$. If at most one of $(i-1,j)$ or $(i,j-1)$ is in $(\lambda/\mu)^-$ then one can define $\theta_{\mathbf{p}}$ by making $\theta_{\mathbf{p}}(i,j)$ either 1 or a very big value, letting $\theta_{\mathbf{p}}(i',j') = \theta^-_{\mathbf{p}}(i',j')$ for other boxes (i',j') (we may have to shift the values of $\theta^-_{\mathbf{p}}$ to be able to set $\theta_{\mathbf{p}}(i,j) = 1$).

So assume that $(i-1,j),(i,j-1) \in (\lambda/\mu)^-$. If $O_{\mathbf{p}}((i-1,j) \lessdot (i,j)) = O_{\mathbf{p}}((i,j-1) \lessdot (i,j))$, then $\theta_{\mathbf{p}}$ can be defined as in the previous case. So assume $O_{\mathbf{p}}((i-1,j) \lessdot (i,j)) = \overline{O_{\mathbf{p}}((i,j-1) \lessdot (i,j))}$. If $(i-1,j-1) \notin (\lambda/\mu)^-$ then $(\lambda/\mu)^-$ is disconnected. In this case, we may pick labelings $\theta^1_{\mathbf{p}}, \theta^2_{\mathbf{p}}$ for the two components C_1, C_2 of $(\lambda/\mu)^-$ so that we can set $\theta_{\mathbf{p}}(C_1) = \theta^1_{\mathbf{p}}(C_1) > \theta_{\mathbf{p}}(i,j) > \theta_{\mathbf{p}}(C_2) = \theta^1_{\mathbf{p}}(C_2)$.

Finally, suppose that $(i-1,j-1) \in (\lambda/\mu)^-$ and assume without loss of generality that $O_{\mathbf{p}}((i-1,j) \lessdot (i,j)) = \text{strict} = O_{\mathbf{p}}((i-1,j-1) \lessdot (i,j-1))$ and $O_{\mathbf{p}}((i,j-1) \lessdot (i,j)) = \text{weak} = O_{\mathbf{p}}((i-1,j-1) \lessdot (i-1,j))$ (we have used the definition of $O_{\mathbf{p}}$). Suppose $\theta^-_{\mathbf{p}}$ is defined. Then $\theta^-_{\mathbf{p}}(i-1,j) > \theta^-_{\mathbf{p}}(i-1,j-1) > \theta^-_{\mathbf{p}}(i,j-1)$. It suffices to define $\theta_{\mathbf{p}}(i,j)$ to be an integer very close to $\theta^-_{\mathbf{p}}(i-1,j-1)$ and $\theta_{\mathbf{p}}(i',j') = \theta^-_{\mathbf{p}}(i',j')$ for other boxes (i',j'), possibly shifting the values so that $\theta^{\mathbf{p}}(i,j)$ can be inserted.

□

Example 10.3. *In Figure 11 an edge-labeling $O_{\mathbf{p}}$ of the shape $\lambda = (2,2,1)$ is given. Note that here the vertices of the poset correspond to centers of boxes of the Young* *diagram. Here* p_{-1} = weak, p_0 = strict, p_1 = weak. *The corresponding wave Schur function* $s_\lambda^{\mathbf{p}}$ *can be computed to be equal to* $L_{(2,1,2)} + L_{(2,1,1,1)} + L_{(3,2)} + L_{(3,1,1)} + L_{(2,2,1)}$. *It is easy to check that this edge-labeling does come from a vertex labeling of the underlying poset.*

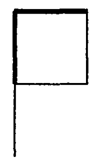


FIGURE 11. An edge-labeling $O_{\mathbf{p}}$ of the shape $\lambda = (2,2,1)$

Remark 10.4. *Proposition 10.2 implies a formula for* $s^{\mathbf{p}}_{\lambda/\mu}(1, q, q^2, q^3, \ldots)$ *similar to that of Proposition 7.19.11 in* [Sta]. *Indeed, we can consider the descent set* $D_{\mathbf{p}}(T)$ *of a standard tableau* T *with respect to* $\theta_{\mathbf{p}}$. *Then if we define a generalization of comajor index* $\mathrm{comaj}_{\mathbf{p}}(T) = \sum_{i \in D_{\mathbf{p}}(T)} (n - i)$, *we obtain the formula*

$$s^{\mathbf{p}}_{\lambda/\mu}(1, q, q^2, q^3, \ldots) = \frac{\sum_T q^{\mathrm{comaj}_{\mathbf{p}}(T)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

However, it seems unlikely that an analog of hook content formula (see [Sta71, Theorem 15.3]*) exists because the number of wave* $\mathbf{p}$*-tableaux filled with entries from* 1 *to* n *does not appear to factor nicely. In Example 10.3 the number of wave* $\mathbf{p}$*-tableaux with entries from* 1 *to* 4 *is the prime number* 23.

Corollary 10.5 (Cell transfer for wave Schur functions). *Let* λ/μ *and* ν/ρ *be two skew shapes and* $\mathbf{p}$ *be any "strict-weak" assignment. Then the difference* $s^{\mathbf{p}}_{\lambda/\mu \wedge \nu/\rho}\, s^{\mathbf{p}}_{\lambda/\mu \vee \nu/\rho} - s^{\mathbf{p}}_{\lambda/\mu}\, s^{\mathbf{p}}_{\nu/\rho}$ *is* L*-positive.*

Proof. Follows immediately from Theorem 8.2 and Proposition 10.2. □

In [LPP] it is shown that the difference in Corollary 10.5 is in fact *Schur-positive* when $\mathbf{p}$ is the standard assignment.

10.2. **Jacobi-Trudi formula for wave Schur functions.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ be two partitions satisfying $\mu \subset \lambda$. Now, for each pair $1 \le i, j \le l$ such that $\mu_j - j + 1 < \lambda_i - i$, define the set

$$D_{ij}(\lambda, \mu) = \{\mu_j - j + 1 < a \le \lambda_i - i \mid p_a = \text{strict}\} - (\mu_j - j + 1),$$

where the notation means that we subtract the number $(\mu_j - j + 1)$ from each element of the set. Set $\alpha_{ij}(\lambda, \mu) = \mathbb{C}(D_{ij}(\lambda, \mu))$ to be the corresponding composition of $\lambda_i - \mu_j - i + j$. If $\mu_j - j + 1 = \lambda_i - i$, set $\alpha_{ij}(\lambda, \mu) = (1)$. If $\mu_j - j = \lambda_i - i$ set $\alpha_{ij}(\lambda, \mu) = (0)$. Finally, if $\mu_j - j > \lambda_i - i$ set $\alpha_{ij}(\lambda, \mu) = \emptyset$. Let $L_{(0)} = 1$, $L_\emptyset = 0$.

Theorem 10.6 (Jacobi-Trudi expansion for wave Schur functions). *Let λ/μ be a skew shape. Then*

$$s^{\mathbf{p}}_{\lambda/\mu} = \det(L_{\alpha_{ij}(\lambda,\mu)})_{i,j=1}^{n}$$

where n is the number of rows in λ.

Example 10.7. *Let $\lambda = (7, 6, 6, 4)$, $\mu = (2, 2, 1, 0)$. Then for $\mathbf{p}$ given by*

$$\ldots p_{-3}, p_{-2}, p_{-1}, p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, \ldots =$$

$$\ldots \text{strict}, \text{strict}, \text{strict}, \text{weak}, \text{strict}, \text{weak}, \text{weak}, \text{strict}, \text{strict}, \text{weak}, \text{strict}, \text{weak} \ldots$$

we get the shape in Figure 10, and

$$s^{\mathbf{p}}_{\lambda/\mu} = \begin{vmatrix} L_{(2,1,2)} & L_{(3,1,2)} & L_{(2,3,1,2)} & L_{(1,1,2,3,1,2)} \\ L_{(2,1)} & L_{(3,1)} & L_{(2,3,1)} & L_{(1,1,2,3,1)} \\ L_{(2)} & L_{(3)} & L_{(2,3)} & L_{(1,1,2,3)} \\ 0 & 1 & L_{(2)} & L_{(1,1,2)} \end{vmatrix}.$$

Proof of Theorem 10.6. Let us construct an oriented network $N_{\mathbf{p}}$, which depends on the choice of $\mathbf{p}$. Namely, we begin with the square grid built on the points in the upper half plane, with row 1 being the bottom row, and orient all edges to the right or upwards. Then we alter the all the crossings in each column C_i such that $p_i =$ strict as shown in Figure 12. Namely, we arrange these crossings so that it is impossible to move from left to right through them, but other directions that were possible before are still possible (see Figure 13). We assign to each edge in row i weight x_i, and every other edge weight 1. Now mark the points M_k with coordinates $(\mu_k - k + 1, 1)$ on our grid. Mark exit directions N_k in the columns numbered $\lambda_k - k + 1$.

Now we apply the Gessel-Viennot method to this path network; see for example [Sta, Chapter 7] for the application of this method in the case of Schur functions.

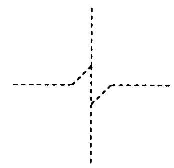


FIGURE 12. A local picture of an altered crossing.

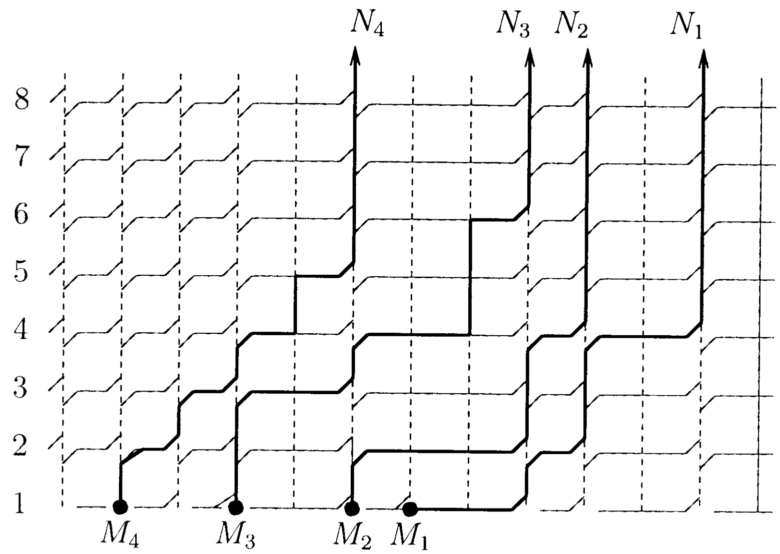


FIGURE 13. A family of paths on the altered grid corresponding to the wave **p**-tableau in Figure 14.

For each pair $1 \leq i, j \leq n$ the weight generating function of the paths from M_i to N_j is equal to $L_{\alpha_{ij}}(\lambda, \mu)$. Thus the determinant $\det(L_{\alpha_{ij}(\lambda,\mu)})_{i,j=1}^n$ is equal to the weight generating function of families of non-crossing paths starting at the M_i-s and ending in the columns N_i. These families of non-crossing paths are in (a weight-preserving) bijection with wave **p**-tableau of shape λ/μ. The bijection is obtained by placing the row numbers of the horizontal edges of the path from M_i to N_i in the i-th row of the wave **p**-tableau. This is illustrated in Figures 13 and 14. □

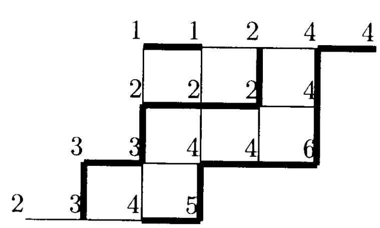


FIGURE 14. A wave **p**-tableau of shape $\lambda/\mu = (7,6,6,4)/(2,2,1,0)$ and edge labeling $O_{\mathbf{p}}$ as in Figure 10.

Remark 10.8. *(i) We have* $\omega(s^{\mathbf{p}}_{\lambda/\mu}) = s^{\bar{\mathbf{p}}}_{\lambda/\mu}$ *where* $\bar{\mathbf{p}} = (\ldots, \overline{p_{-2}}, \overline{p_{-1}}, \overline{p_0}, \overline{p_1}, \ldots)$.

(ii) Let us denote $\widetilde{\lambda/\mu}$ *the rotated on* 180 *degrees* $\mathbf{p}$*-tableau* λ/μ *with* $\tilde{p}_i = p_{-i}$. *Then* $\nu(s^{\mathbf{p}}_{\lambda/\mu}) = s^{\tilde{\mathbf{p}}}_{\widetilde{\lambda/\mu}}$.

The following theorem, combined with Proposition 10.2, answers Question 8.4 for the case that Q and R are convex subsets of a chain.

Theorem 10.9. *The differences* $L_{\alpha\wedge_m\beta} L_{\alpha\vee_m\beta} - L_\alpha L_\beta$ *of Theorem 9.1 are equal to wave Schur functions.*

Proof. We may suppose that $m \geq 1$ for otherwise the difference is 0. Pick a sequence $\mathbf{p} = \mathbf{p}^\alpha$ such that $p_i =$ strict if and only if $i \in D(\alpha)$ (this determines $p_1, p_2, \ldots, p_{|\alpha|-1}$). Then set $\lambda = (|\alpha|, m + |\beta|)$ and $\mu = (m-1, 1)$.

We can compute that

$$L_{\alpha_{11}(\lambda,\mu)} = L_{\alpha\vee_m\beta} \qquad L_{\alpha_{12}(\lambda,\mu)} = L_\alpha$$

$$L_{\alpha_{21}(\lambda,\mu)} = L_{\alpha\wedge_m\beta} \qquad L_{\alpha_{22}(\lambda,\mu)} = L_\beta.$$

Theorem 10.6 tells us that $s^{\mathbf{p}}_{\lambda/\mu}$ is exactly $\det\begin{pmatrix} L_{\alpha\vee_m\beta} & L_\alpha \\ L_\beta & L_{\alpha\wedge_m\beta} \end{pmatrix}$. □

We illustrate the choice of $\mathbf{p}^\alpha$, λ and μ of Theorem 10.9 in Figure 15. Here $\mathbf{p}^\alpha$ is such that $\ldots p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, \ldots =$

$$\ldots, \text{strict}, \text{weak}, \text{strict}, \text{strict}, \text{weak}, \text{weak}, \text{weak}, \text{strict},$$

$\alpha = (1,2,1,4,1)$, $\beta = (3)$, $m = 5$. Then $\lambda = (9,8)$, $\mu = (4,1)$ and the corresponding $s^{\mathbf{p}}_{\lambda/\mu}$ is equal to $L_{(1,2,1,3)}L_{(4,1)} - L_{(3)}L_{(1,2,1,4,1)}$.

11. Algebraic properties of QSym

We prove in this section some algebraic results concerning QSym used earlier.

11.1. **A factorization property of quasi-symmetric functions.** Denote by $K = \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$ the ring of formal power series in infinitely many variables with bounded degree. Clearly the units in K or in the polynomial ring in n variables $K^{(n)}$ are 1 and -1. In this subsection, we prove the following property of QSym.

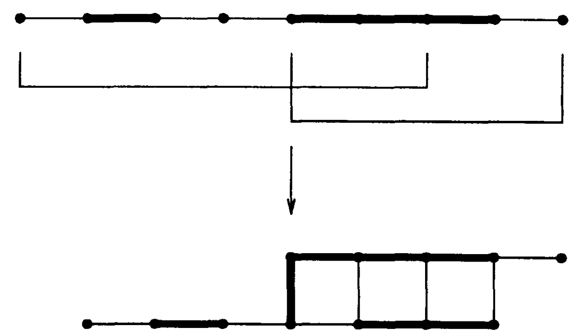


FIGURE 15. The skew shape corresponding to the difference $L_{(1,2,1,3)}L_{(4,1)} - L_{(3)}L_{(1,2,1,4,1)}$.

Theorem 11.1. *Suppose $f \in \mathrm{QSym}$ and $f = \prod_i f_i$ is a factorization of f into irreducibles in K. Then $f_i \in \mathrm{QSym}$ for each i.*

Now let $\mathbf{a} = (1 \leq a_1 < a_2 < \ldots <)$ be an increasing sequence of positive integers and let A denote the set of such sequences. Define the algebra homomorphism $A_{\mathbf{a}} : K \to K$ by

$$A_{\mathbf{a}} f := f(x_{\mathbf{a}}) := f(0, \ldots, 0, x_1, 0, \ldots, 0, x_2, 0, \ldots)$$

where x_i is placed in the a_i-th position. For a sequence $\mathbf{a}$, we shall also write $a(i) = a_i$ in function notation. Thus $a : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

As an example, take $a = (2, 3, 4, \cdots)$, $b = (1, 3, 5, \cdots)$. Then we have $A_a f = f(0, x_1, x_2, \ldots)$ and $A_b \circ A_a f = A_b(A_a f) = f(0, x_1, 0, x_2, 0, x_3, \cdots)$.

The following lemma is essentially the definition.

Lemma 11.2. *An element $f \in K$ is quasi-symmetric if and only if $f(x_{\mathbf{a}}) = f$ for each $\mathbf{a} \in A$.*

Let $k \geq 1$ be an integer. Define $\mathbf{a}^{(k)}$ by

$$a^{(k)}(i) = \begin{cases} i & \text{if } i < k, \\ i+1 & \text{if } i \geq k. \end{cases}$$

Lemma 11.3. *Suppose $f \in K$ has degree n. Then f is quasi-symmetric if and only if $f(x_{\mathbf{a}}) = f$ for the sequences $\mathbf{a}^{(k)}$ for $1 \leq k \leq n$.*

Proof. The only if direction is clear. Assume that $f(x_{\mathbf{a}}) = f$ for each $\mathbf{a}^{(k)}$ for $1 \leq k \leq n$. To show that the coefficients of $x_1^{c_1} \cdots x_n^{c_n}$ and $x_{b_1}^{c_1} \cdots x_{b_n}^{c_n}$ in f are the same we use the (coefficient of $x_1^{c_1} \cdots x_n^{c_n}$ in the equality)

$$A_{\mathbf{a}^{(n)}}^{b_n - b_{n-1}} \cdots A_{\mathbf{a}^{(2)}}^{b_2 - b_1} A_{\mathbf{a}^{(1)}}^{b_1 - 1} f = f.$$

□

The following lemma is a simple calculation.

Lemma 11.4. *We have $A_{\mathbf{b}} \circ A_{\mathbf{a}} = A_{\mathbf{c}}$ where $c(i) = a(b(i))$.*

Lemma 11.5. *Let $f \in K$. Suppose f has finite order with respect to $A_{\mathbf{a}}$ for every $\mathbf{a} \in A$. Then there exists $\mathbf{b} \in A$ so that $A_{\mathbf{b}} f \in \mathrm{QSym}$.*

Proof. Given $f \in K$ invariant under $A_{\mathbf{a}^{(k)}}$ for $1 \leq k \leq t$, with t possibly 0, we will produce an $f' = A_{\mathbf{b}} f$ invariant under $A_{\mathbf{a}^{(k)}}$ for $1 \leq k \leq t+1$. Using Lemma 11.3 and the fact that f has bounded degree this is sufficient.

So let f be invariant under $A_{\mathbf{a}^{(k)}}$ for $1 \leq k \leq t$. By assumption $A_{\mathbf{a}^{(t+1)}}$ has finite order d on f. Define $\mathbf{b} \in A$ by $b(i) = 1 + (i-1)d$ and let $f' = A_{\mathbf{b}} f$. We claim that f' is invariant under $A_{\mathbf{a}^{(k)}}$ for $1 \leq k \leq t+1$. We have

$$b(a^{(k)}(i)) = \begin{cases} 1 + (i-1)d & \text{if } i < k, \\ 1 + id & \text{if } i \geq k. \end{cases}$$

In the following we will repeatedly use Lemma 11.4.

Define $\mathbf{b}^{(j)} \in A$ for $1 \leq j < k$ by

$$b^{(j)}(i) = \begin{cases} i & \text{if } i < j, \\ j + (i-j)d & i \geq j. \end{cases}$$

Note that $A_{\mathbf{b}^{(j)}} \circ (A_{\mathbf{a}^{(j)}})^{d-1} = A_{\mathbf{b}^{(j-1)}}$. Similarly define $\mathbf{c}^{(j)} \in A$ for $1 \leq j < k$ by

$$c^{(j)}(i) = \begin{cases} i & \text{if } i \leq j, \\ j + (i-j)d & j < i < k, \\ j + (i-j+1)d & i \geq k. \end{cases}$$

Note that $A_{\mathbf{c}(j)} \circ (A_{\mathbf{a}(j)})^{d-1} = A_{\mathbf{c}(j-1)}$. We also have the equalities

$$A_{\mathbf{c}(k-1)} = A_{\mathbf{b}(k-1)} \circ (A_{\mathbf{a}(k)})^d,$$
$$A_{\mathbf{b}(1)} = A_{\mathbf{b}},$$
$$A_{\mathbf{c}(1)} = A_{\mathbf{a}(k)} \circ A_{\mathbf{b}}.$$

Finally using our assumptions and $1 \le k \le t+1$, we have

$$A_{\mathbf{b}} f = A_{\mathbf{b}(1)} f = \cdots = A_{\mathbf{b}(k-1)} f = A_{\mathbf{c}(k-1)} f = \cdots = A_{\mathbf{c}(1)} f = A_{\mathbf{a}(k)} \circ A_{\mathbf{b}} f.$$

□

Proof of Theorem 11.1. Let $\mathbf{a} \in A$. Applying $A_{\mathbf{a}}$ to $f = \prod_i f_i$ and using Lemma 11.2, we have $f = \prod_i A_{\mathbf{a}} f_i$. By Lemma 11.7, each $A_{\mathbf{a}} f_i$ must be equal to $\pm f_j$. In other words, $A_{\mathbf{a}}$ has finite order on each f_i and application of $A_{\mathbf{a}}$ to f_i produces (up to sign) another f_j. Using Lemma 11.5, we see that f_j must lie in QSym for some j. Now divide both sides by f_j and proceed by induction. □

Corollary 11.6. QSym *is a unique factorization domain.*

Corollary 11.6 also follows from work of Hazewinkel [Haz], who shows that QSym is a polynomial ring.

Proof. If $f \in$ QSym then two irreducible factorizations of f in QSym will also be irreducible factorizations in K, by Theorem 11.1. The theorem follows from Lemma 11.7, proven below. □

Lemma 11.7. *The ring K is a unique factorization domain.*

Proof. We start by recalling the well known fact that the polynomial rings $K^{(n)} = \mathbb{Z}[x_1, x_2, \ldots, x_n]$ are unique factorization domains. An element $f(x_1, x_2, \ldots) \in K$ is determined by its images

$$f^{(n)} = f(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in K^{(n)}.$$

We may write $f = (f^{(n)})$ for a compatible sequence of $f^{(n)} \in K^{(n)}$.

We first claim that f is irreducible if and only if there exists $N > 0$ such that $f^{(n)}$ is irreducible for all $n > N$. Let $f = \prod f_i$ be a decomposition of f into irreducibles.

Then there exists $M > 0$ so that $\deg(f_i^{(n)}) = \deg(f_i)$ for all i and $n > M$. Thus $f^{(n)} = \prod f_i^{(n)}$ is reducible for $n > M$ if f is. Conversely, suppose that $f^{(n)}$ is reducible for infinitely many values of n. If $n > M$ and $f^{(n)}$ is reducible then $f^{(m)}$ is also reducible for $n > m > M$. Thus we may assume $f^{(n)}$ is reducible for all $n > N$ for some $N > M$. Restriction of $f^{(n)}$ to $f^{(m)}$ for $n > m > N$ will not change the degree of any of the factors. Thus the factorizations of $f^{(n)}$ are compatible for each $n > N$. For sufficiently large $n \gg N$, the number k of irreducible factors of $f^{(n)}$ will be constant and greater than 1. Ordering the irreducible factorizations $f^{(n)} = \prod_{i=1}^{k} f_i^{(n)}$ compatibly, we conclude that $f = \prod_{i=1}^{k} f_i$ where $f_i = (f_i^{(n)})$ is irreducible.

Now suppose that $f = \prod_i f_i = \prod_j g_j$ are two factorizations of f into irreducibles. By our claim, there exists some huge N so that $\prod_i f_i^{(n)} = \prod_j g_j^{(n)}$ are factorizations of $f^{(n)}$ into irreducibles in $K^{(n)}$, for each $n > N$. Since $K^{(n)}$ is a UFD, these factorizations are the same up to permutation and sign: $g_i^{(n)} = \epsilon_i f_{\sigma(i)}^{(n)}$. If N is chosen large enough the same permutation σ and signs ϵ_i will work for all $n > N$. This shows that $g_i = \epsilon_i f_{\sigma(i)}$. □

Remark 11.8. *(i) Note that Corollary 11.6 is not true in finitely many variables. For example, in the two variables* x_1 *and* x_2 *we have* $(x_1^2 x_2)(x_1 x_2^2) = (x_1 x_2)^3$. *The quasi-symmetric functions* $x_1^2 x_2$, $x_1 x_2^2$ *and* $x_1 x_2$ *are all irreducible.*

(ii) It seems interesting to ask whether the r-quasi-symmetric functions defined by Hivert [Hiv] *also form a unique factorization domain. The m-quasi-invariants* [EG] *occurring in representation theory do not in general form unique factorization domains.*

11.2. **Irreducibility of fundamental quasi-symmetric functions.** In this section, we show that the fundamental quasi-symmetric functions $\{L_\alpha\}$ and the monomial quasi-symmetric functions $\{M_\alpha\}$ are irreducible in QSym and in K.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$ be two compositions. Define the *lexicographic* order on compositions by $\alpha > \beta$ if and only if for some i we have $\alpha_j = \beta_j$ for $1 \le j \le i-1$ and $\alpha_i > \beta_i$. Using this order, we obtain lexicographic orders on monomials $\{x^\alpha\}$, monomial quasi-symmetric functions $\{M_\alpha\}$ and fundamental quasi-symmetric functions $\{L_\alpha\}$. Note that the lexicographically maximal monomial in M_α or L_α is x^α.

In the following proofs we say that a quasi-symmetric function f contains a term L_α (and similarly for M_α) if the coefficient of L_α is non-zero when f is written in the basis of fundamental quasi-symmetric functions. The following lemma is immediate from the definitions.

Lemma 11.9. *The lexicographically maximal monomial in the product $f\,g$ of two quasi-symmetric functions f and g is the product of the lexicographically maximal monomials in f and g.*

Proposition 11.10. *The monomial quasi-symmetric function M_α is irreducible in* QSym *and in $K = \mathbb{Z}[[x_1, x_2, \ldots]]$.*

Proof. We proceed by induction on the size $n = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. For $n = 1$ the statement is obvious.

Assume now that $M_\alpha = f\,g$ is not irreducible. Note first that f and g must be homogeneous. Otherwise, the homogeneous components of maximal and minimal degree in the product would not cancel out, and thus we would never get the homogeneous function M_α. Also note that according to Theorem 11.1 both f and g must be quasi-symmetric. First we suppose that $k = 1$.

Now, take the specialization $x_i = q^{i-1}$. It is known ([Sta, Proposition 7.19.10]) that under this specialization we have $L_\alpha(1, q, q^2, \ldots) = \frac{q^{e(\alpha)}}{(1-q)(1-q^2)\cdots(1-q^{|\alpha|})}$, where $e(\alpha)$ is the "comajor" statistic. That means that if $\deg(f) = p$, $\deg(g) = \alpha_1 - p$ and $0 < p < \alpha_1$, fg will never have a pole at primitive α_1-th root of unity. On the other hand $M_\alpha(1, q, \ldots) = \frac{1}{1-q^{\alpha_1}}$, and thus M_α has a pole at a primitive α_1-th root of unity, which is a contradiction.

Now suppose that $k \neq 1$. We write each of the participating functions as polynomials in x_1:

$$M_\alpha = x_1^{\alpha_1}\tilde{M}_1 + \cdots, \qquad f = x_1^r \tilde{f}_1 + \cdots, \qquad g = x_1^{\alpha_1 - r}\tilde{g}_1 + \cdots.$$

Here the leading term is the one with the highest power of x_1, and the notation $\tilde{f}$ denotes a power series $f(x_2, x_3, \ldots)$ quasi-symmetric in the variables $x_2, x_3, \cdots$.

Note that $\tilde{M}_1 = M_{(\alpha_2,\cdots,\alpha_k)}(x_2, x_3, \cdots)$ is the monomial quasi-symmetric function corresponding to the composition obtained from α by removing the first part. Since $M_\alpha = f\,g$, we must have $M_1 = f_1\,g_1$. By induction one of f_1 or g_1 is equal to a unit, ± 1. Without loss of generality we can assume $f_1 = 1$. Thus the monomial quasi-symmetric function $M_{(r)}$ occurs in f. By Lemma 11.9 above we conclude that the lexicographically maximal monomial quasi-symmetric function in g is $M_{(\alpha_1 - r,\alpha_2,\cdots,\alpha_k)}$.

Now apply the involution ν of Proposition 2.1 to the equality $M_\alpha = f\,g$ to obtain $M_{\alpha^*} = \nu(f)\nu(g)$. By Proposition 2.1, $\nu(M_{(r)}) = M_{(r)}$ and so the monomial $M_{(r)}$ is still the lexicographically maximal monomial in $\nu(f)$. Similarly, the monomial symmetric function $M_{(\alpha_k,\cdots,\alpha_2,\alpha_1-r)}$ occurs in $\nu(g)$ with non-zero coefficient. Since $k \neq 1$, the lexicographically maximal monomial in the product $\nu(f)\nu(g)$ is at least as large as $M_{(\alpha_k+r,\cdots,\alpha_1-r)}$. This however is lexicographically larger than $M_{\alpha^*} = M_{(\alpha_k,\cdots,\alpha_1)}$ unless $r = 0$.

We conclude that $f = 1$ and that M_α is irreducible. □

Proposition 11.11. *The fundamental quasi-symmetric function L_α is irreducible in both* QSym *and in $K = \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$.*

Proof. The trick used for the case $k = 1$ in the proof of Proposition 11.10 also works here. □

12. Immanants and Schur positivity

In this section we give an overview of some results of Rhoades-Skandera [RS1, RS2], mostly paraphrasing [RS2, Sections 2–3]. The only new ingredient of this section is an alternative proof of one of their results.

12.1. **Haiman's Schur positivity result.** Let $H_n(q)$ be the *Hecke algebra* associated with the symmetric group S_n. The Hecke algebra has the standard basis $\{T_w \mid w \in S_n\}$ and the *Kazhdan-Lusztig basis* $\{C'_w(q) \mid w \in S_n\}$ related by

$$q^{l(v)/2} C'_v(q) = \sum_{w \le v} P_{w,v}(q)\, T_w \quad \text{and} \quad T_w = \sum_{v \le w} (-1)^{l(vw)} Q_{v,w}(q)\, q^{l(v)/2} C'_v(q),$$

where $P_{w,v}(q)$ are the *Kazhdan-Lusztig polynomials* and $Q_{v,w}(q) = P_{w_\circ w, w_\circ v}(q)$, for the longest permutation $w_\circ \in S_n$, see [Hum] for more details.

For $w \in S_n$ and an $n \times n$ matrix $X = (x_{ij})$, the *Kazhdan-Lusztig immanant* was defined in [RS2] as

$$\mathrm{Imm}_w(X) := \sum_{v \in S_n} (-1)^{l(vw)} Q_{w,v}(1)\, x_{1,v(1)} \cdots x_{n,v(n)}.$$

Let $h_k = \sum_{i_1 \le \cdots \le i_k} x_{i_1} \cdots x_{i_k}$ be the k-th homogeneous symmetric function, where $h_0 = 1$ and $h_k = 0$ for $k < 0$. A *generalized Jacobi-Trudi matrix* is an $n \times n$ matrix of the form $\left(h_{\mu_i - \nu_j}\right)_{i,j=1}^n$, for partitions $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n \ge 0)$ and $\nu = (\nu_1 \ge \nu_2 \ge \cdots \ge \nu_n \ge 0)$. Rhoades-Skandera reformulated Haiman's result as follows.

Theorem 12.1. Haiman [Hai, Theorem 1.5], Rhoades-Skandera [RS2, Proposition 3] *The immanants* Imm_w *of a generalized Jacobi-Trudi matrix are Schur nonnegative.*

Haiman's proof of this result is based on the Kazhdan-Lusztig conjecture proven by Beilinson-Bernstein and Brylinski-Kashiwara, e.g., see [Hai, Proposition 7.1] and the references following. This conjecture expresses the characters of Verma modules as sums of the characters of some irreducible highest weight representations of $\mathfrak{sl}_n$ with multiplicities equal to $P_{w,v}(1)$. One can derive from this conjecture that the coefficients of Schur functions in Imm_w are certain tensor product multiplicities of irreducible representations.

12.2. **Temperley-Lieb algebra.** The *Temperley-Lieb algebra* $TL_n(\xi)$ is the $\mathbb{C}[\xi]$-algebra generated by $t_1, \ldots, t_{n-1}$ subject to the relations $t_i^2 = \xi\, t_i$, $t_i t_j t_i = t_i$ if $|i-j| = 1$, $t_i t_j = t_j t_i$ if $|i-j| \geq 2$. The dimension of $TL_n(\xi)$ equals the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. A *321-avoiding permutation* is a permutation $w \in S_n$ that has no reduced decomposition of the form $w = \cdots s_i s_j s_i \cdots$ with $|i-j| = 1$. (These permutations are also called *fully-commutative*.) A natural basis of the Temperley-Lieb algebra is $\{t_w \mid w \text{ is a 321-avoiding permutation in } S_n\}$, where $t_w := t_{i_1} \cdots t_{i_l}$, for a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$.

The map $\theta : T_{s_i} \mapsto t_i - 1$ determines a homomorphism $\theta : H_n(1) = \mathbb{C}[S_n] \to TL_n(2)$. Indeed, the elements $t_i - 1$ in $TL_n(2)$ satisfy the Coxeter relations.

Theorem 12.2. Fan-Green [FG] *The homomorphism θ acts on the Kazhdan-Lusztig basis $\{C'_w(1)\}$ of $H_n(1)$ as follows:*

$$\theta(C'_w(1)) = \begin{cases} t_w & \text{if } w \text{ is 321-avoiding,} \\ 0 & \text{otherwise.} \end{cases}$$

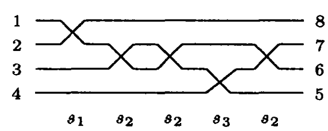
For any permutation $v \in S_n$ and a 321-avoiding permutation $w \in S_n$, let $f_w(v)$ be the coefficient of the basis element $t_w \in TL_n(2)$ in the basis expansion of $\theta(T_v) = (t_{i_1} - 1) \cdots (t_{i_l} - 1) \in TL_n(2)$, for a reduced decomposition $v = s_{i_1} \cdots s_{i_l}$. Rhoades and Skandera [RS1] defined the *Temperley-Lieb immanant* $\mathrm{Imm}_w^{\mathrm{TL}}(x)$ of an $n \times n$ matrix $X = (x_{ij})$ by

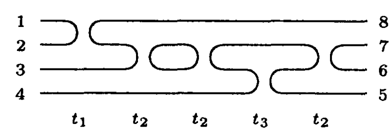
$$\mathrm{Imm}_w^{\mathrm{TL}}(X) := \sum_{v \in S_n} f_w(v)\, x_{1,v(1)} \cdots x_{n,v(n)}.$$

Theorem 12.3. Rhoades-Skandera [RS2, Proposition 5] *For a 321-avoiding permutation $w \in S_n$, we have $\mathrm{Imm}_w^{\mathrm{TL}}(X) = \mathrm{Imm}_w(X)$.*

Proof. Applying the map θ to $T_v = \sum_{w \leq v} (-1)^{l(vw)} Q_{w,v}(1)\, C'_w(1)$ and using Theorem 12.2 we obtain $\theta(T_v) = \sum (-1)^{l(vw)} Q_{w,v}(1)\, t_w$, where the sum is over 321-avoiding permutations w. Thus $f_w(v) = (-1)^{l(vw)} Q_{w,v}(1)$ and $\mathrm{Imm}_w^{\mathrm{TL}} = \mathrm{Imm}_w$. □

A product of generators (decomposition) $t_{i_1} \cdots t_{i_l}$ in the Temperley-Lieb algebra TL_n can be graphically presented by a *Temperley-Lieb diagram* with n non-crossing strands connecting the vertices $1, \ldots, 2n$ and, possibly, with some internal loops. This diagram is obtained from the wiring diagram of the decomposition $w = s_{i_1} \cdots s_{i_l} \in S_n$ by replacing each crossing " $\times$ " with a *vertical uncrossing* ") (". The left endpoints are assumed to be labeled $1, \dots, n$ from top to bottom and the right endpoints are assumed to be labeled $2n, \dots, n+1$ from top to bottom. For example, the following figure shows the wiring diagram for $s_1 s_2 s_2 s_3 s_2 \in S_4$ and the Temperley-Lieb diagram for $t_1 t_2 t_2 t_3 t_2 \in TL_4$.





Pairs of vertices connected by strands of a wiring diagram are $(2n+1-i, w(i))$, for $i = 1, \dots, n$. Pairs of vertices connected by strands in a Temperley-Lieb diagram form a *non-crossing matching*, i.e., a graph on the vertices $1, \dots, 2n$ with n disjoint edges that contains no pair of edges (a, c) and (b, d) with $a < b < c < d$. If two Temperley-Lieb diagrams give the same matching and have the same number of internal loops, then the corresponding products of generators of TL_n are equal to each other. If the diagram of a is obtained from the diagram of b by removing k internal loops, then $b = \xi^k a$ in TL_n.

The map that sends t_w to the non-crossing matching given by its Temperley-Lieb diagram is a bijection between basis elements t_w of TL_n, where w is 321-avoiding, and non-crossing matchings on the vertex set $[2n]$. For example, the basis element $t_1 t_3 t_2$ of TL_4 corresponds to the non-crosssing matching with the edges $(1, 2), (3, 4), (5, 8), (6, 7)$.

12.3. **An identity for products of minors.** Following [RS1], for a subset $S \subset [2n]$, let us say that a Temperley-Lieb diagram (or the associated element in TL_n) is S-*compatible* if each strand of the diagram has one end-point in S and the other end-point in its complement $[2n] \setminus S$. Coloring vertices in S black and the remaining vertices white, a basis element t_w is S-compatible if and only if each edge in the associated matching has two vertices of different colors. Let $\Theta(S)$ denote the set of all 321-avoiding permutation $w \in S_n$ such that t_w is S-compatible.

For two subsets $I, J \subset [n]$ of the same cardinality let $\Delta_{I,J}(X)$ denote the *minor* of an $n \times n$ matrix X in the row set I and the column set J. Let $\bar{I} := [n] \setminus I$ and let $I^\wedge := \{2n + 1 - i \mid i \in I\}$.

Theorem 12.4. Rhoades-Skandera [RS1, Proposition 4.3], cf. Skandera [Ska] *For two subsets $I, J \subset [n]$ of the same cardinality and $S = J \cup (\bar{I})^\wedge$, we have*

$$\Delta_{I,J}(X) \cdot \Delta_{\bar{I},\bar{J}}(X) = \sum_{w \in \Theta(S)} \mathrm{Imm}_w^{\mathrm{TL}}(X).$$

The proof given in [RS1] employs planar networks. Here we give an alternative proof directly from the definition of Temperley-Lieb immanants.

Proof. Let us fix a permutation $v \in S_n$ with a reduced decomposition $v = s_{i_1} \cdots s_{i_l}$. The coefficient of the monomial $x_{1,v(1)} \cdots x_{n,v(n)}$ in the expansion of the product of two minors $\Delta_{I,J}(X) \cdot \Delta_{\bar{I},\bar{J}}(X)$ equals

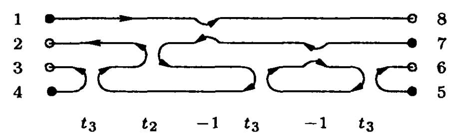
$$\begin{cases} (-1)^{\mathrm{inv}(I)+\mathrm{inv}(\bar{I})} & \text{if } v(I) = J, \\ 0 & \text{if } v(I) \neq J, \end{cases}$$

where $\mathrm{inv}(I)$ is the number of inversions $i < j$, $v(i) > v(j)$ such that $i, j \in I$.

On the other hand, by the definition of $\mathrm{Imm}_w^{\mathrm{TL}}$, the coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in the right-hand side of the identity equals the sum $\sum (-1)^r \, 2^s$ over all diagrams obtained from the wiring diagram of the reduced decomposition $s_{i_1} \cdots s_{i_l}$ by replacing each crossing " $\times$ " with either a *vertical uncrossing* ") (" or a *horizontal uncrossing* " $\asymp$ " so that the resulting diagram is S-compatible, where r is the number of horizontal uncrossings " $\asymp$ " and s is the number of internal loops in the resulting diagram. Indeed, the choice of ") (" corresponds to the choice of "t_{i_k}" and the choice of " $\asymp$ " corresponds to the choice of "-1" in the k-th term of the product $(t_{i_1} - 1) \cdots (t_{i_l} - 1) \in TL_n(2)$, for $k = 1, \ldots, l$.

Let us pick directions of all strands and loops in such diagrams so that the initial vertex in each strand belongs to S (and, thus, the end-point is not in S). There are 2^s ways to pick directions of s internal loops. Thus the above sum can be written as the sum $\sum (-1)^r$ over such *directed Temperley-Lieb* diagrams.

Here is an example of a directed diagram for $v = s_3 s_2 s_1 s_3 s_2 s_3$ and $S = \{1, 4, 5, 7\}$ corresponding to the term $t_3 t_2 (-1) t_3 (-1) t_3$ in the expansion of the product $(t_3 - 1)(t_2 - 1)(t_1 - 1)(t_3 - 1)(t_2 - 1)(t_3 - 1)$. This diagram comes with the sign $(-1)^2$.



Let us construct a sign reversing partial involution ι on the set of such directed Temperley-Lieb diagrams. If a diagram has a *misaligned uncrossing*, i.e., an uncrossing of the form "$)($", "$)($", "$\asymp$", or "$\asymp$", then ι switches the leftmost such uncrossing according to the rules $\iota :)(\leftrightarrow \asymp$ and $\iota :)(\leftrightarrow \asymp$. Otherwise, when the diagram involves only *aligned uncrossings* "$)($", "$)($", "$\asymp$", "$\asymp$", the involution ι is not defined.

For example, in the above diagram, the involution ι switches the second uncrossing, which has the form "$)($", to "$\asymp$". The resulting diagram corresponds to the term $t_3(-1)(-1)t_3(-1)t_3$.

Since the involution ι reverses signs, this shows that the total contribution of all diagrams with at least one misaligned uncrossing is zero. Let us show that there is at most one S-compatible directed Temperley-Lieb diagram with all aligned uncrossings. If we have a such diagram, then we can direct the strands of the wiring diagram for $v = s_{i_1} \dots s_{i_l}$ so that each segment of the wiring diagram has the same direction as in the Temperley-Lieb diagram. In particular, the end-points of strands in the wiring diagram should have different colors. Thus each strand starting at an element of J should finish at an element of $I^\wedge$, or, equivalently, $v(I) = J$. The directed Temperley-Lieb diagram can be uniquely recovered from this directed wiring diagram by replacing the crossings with uncrossings, as follows: $\times \to \asymp$, $\times \to)($, $\times \to)($, $\times \to \asymp$. Thus the coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in the right-hand side of the needed identity is zero, if $v(I) \neq J$, and is $(-1)^r$, if $v(I) = J$, where r is the number of crossings of the form "$\times$" or "$\times$" in the wiring diagram. In other words, r equals the number of crossings such that the right end-points of the pair of crossing strands have the same color. This is exactly the same as the expression for the coefficient in the left-hand side of the needed identity. $\square$

The following result of Rhoades-Skandera follows immediately from Theorems 12.1, 12.3, and 12.4.

Theorem 12.5. Rhoades-Skandera [RS2], cf. [RS2, Theorem 9], [RS1, Corollary 4.5] *Let $I, J, K, L \subseteq [n]$ be subsets such that $|I| = |J|$, $|K| = |L|$, let $S = J \cup (\bar{I})^\wedge$ and $T = L \cup (\bar{K})^\wedge$, and let X be a generalized Jacobi-Trudi matrix. If $\Theta(S) \subseteq \Theta(T)$, then the difference*

$$\Delta_{K,L}(X) \cdot \Delta_{\bar{K},\bar{L}}(X) - \Delta_{I,J}(X) \cdot \Delta_{\bar{I},\bar{J}}(X).$$

is Schur nonnegative.

Actually, Rhoades-Skandera proved a stronger result [RS2, Theorem 9] saying that the above difference is a nonnegative combination of Temperley-Lieb immanants if and only if $\Theta(S) \subseteq \Theta(T)$.

13. Proof of Theorem 1.5

For two subsets $I, J \subseteq [n]$ of the same cardinality, let $\Delta_{I,J}(H)$ denote the minor of the Jacobi-Trudi matrix $H = (h_{j-i})_{1 \leq i,j \leq n}$ with row set I and column set J, where h_i is the i-th homogeneous symmetric function, as before. According to the Jacobi-Trudi formula, see [Mac], the minors $\Delta_{I,J}(H)$ are precisely the skew Schur functions

$$\Delta_{I,J}(H) = s_{\lambda/\mu},$$

where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, $\mu = (\mu_1 \geq \cdots \geq \mu_k \geq 0)$ and the associated subsets are $I = \{\mu_k + 1 < \mu_{k-1} + 2 < \cdots < \mu_1 + k\}$, $J = \{\lambda_k + 1 < \lambda_{k-1} + 2 < \cdots < \lambda_1 + k\}$.

For two sets $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$, let us define $I \vee J := \{\max(i_1, j_1) < \cdots < \max(i_k, j_k)\}$ and $I \wedge J := \{\min(i_1, j_1) < \cdots < \min(i_k, j_k)\}$.

Theorem 1.5 can be reformulated in terms of minors, as follows. Without loss of generality we can assume that all partitions λ, μ, ν, ρ in Theorem 1.5 have the same number k of parts, some of which might be zero. Note that generalized Jacobi-Trudi matrices are obtained from H by skipping or duplicating rows and columns.

Theorem 13.1. *Let I, J, I', J' be k element subsets in $[n]$. Then we have*

$$\Delta_{I \vee I', J \vee J'}(X) \cdot \Delta_{I \wedge I', J \wedge J'}(X) \geq_s \Delta_{I,J}(X) \cdot \Delta_{I',J'}(X),$$

for a generalized Jacobi-Trudi matrix X.

Proof. Let us denote $\bar{I} := [n] \setminus I$ and $\check{S} := [2n] \setminus S$. By skipping or duplicating rows and columns of the matrix X, we may assume that $I' = \bar{I}$ and $J' = \bar{J}$. Then $I \vee I' = \overline{I \wedge I'}$ and $J \vee J' = \overline{J \wedge J'}$. Let $S := J \cup (\bar{I})^\wedge$ and $T := (J \vee J') \cup (\overline{I \vee I'})^\wedge$. Then we have $T = S \vee \check{S}$ and $\check{T} = S \wedge \check{S}$.

By Theorem 12.5, the containment $\Theta(S) \subseteq \Theta(T)$ implies the desired inequality. Thus we must show that every S-compatible non-crossing matching on $[2n]$ is also T-compatible. Let $S = \{s_1 < \cdots < s_n\}$ and $\check{S} = \{\check{s}_1 < \cdots < \check{s}_n\}$. Then $T = \{\max(s_1, \check{s}_1), \ldots, \max(s_n, \check{s}_n)\}$ and $\check{T} = \{\min(s_1, \check{s}_1), \ldots, \min(s_n, \check{s}_n)\}$. Let M be an S-compatible non-crossing matching on $[2n]$ and let $(a < b)$ be an edge of M. Without loss of generality we may assume that $a = s_i \in S$ and $b = \check{s}_j \in \check{S}$. We must show that either ($a \in T$ and $b \in \check{T}$) or ($a \in \check{T}$ and $b \in T$). Since no edge of M can cross (a, b), the elements of S in the interval $[a+1, b-1]$ are matched with the elements of $\check{S}$ in this interval. Let $k = \#(S \cap [a+1, b-1]) = \#(\check{S} \cap [a+1, b-1])$. Suppose that $a, b \in T$, or, equivalently, $\check{s}_i < s_i$ and $s_j < \check{s}_j$. Since there are at least k elements of $\check{S}$ in the interval $[\check{s}_i + 1, \check{s}_j - 1]$, we have $i + k + 1 \leq j$. On the other hand, since there are at most $k - 1$ elements of S in the interval $[s_i + 1, s_j - 1]$, we have $i + k \geq j$. We obtain a contradiction. The case $a, b \in \check{T}$ is analogous. $\square$

14. Proof of conjectures and generalizations

In this section we prove generalized versions of Conjectures 1.1-1.3, which were conjectured by Kirillov [Kir, Section 6.8]. Corollary 14.3 was also conjectured by Bergeron-McNamara [BM, Conjecture 5.2] who showed that it implies Theorem 14.4.

Let $\lfloor x \rfloor$ be the maximal integer $\leq x$ and $\lceil x \rceil$ be the minimal integer $\geq x$. For vectors v and w and a positive integer n, we assume that the operations $v + w$, $\frac{v}{n}$, $\lfloor v \rfloor$, $\lceil v \rceil$ are performed coordinate-wise. In particular, we have well-defined operations $\lfloor \frac{\lambda+\nu}{2} \rfloor$ and $\lceil \frac{\lambda+\nu}{2} \rceil$ on pairs of partitions.

The next claim extends Okounkov's conjecture (Conjecture 1.1).

Theorem 14.1. *Let λ/μ and ν/ρ be any two skew shapes. Then we have*

$$s_{\lfloor \frac{\lambda+\nu}{2} \rfloor / \lfloor \frac{\mu+\rho}{2} \rfloor} \, s_{\lceil \frac{\lambda+\nu}{2} \rceil / \lceil \frac{\mu+\rho}{2} \rceil} \geq_s s_{\lambda/\mu} \, s_{\nu/\rho}.$$

Proof. We will assume that all partitions have the same fixed number k of parts, some of which might be zero. For a skew shape $\lambda/\mu = (\lambda_1, \ldots, \lambda_k)/(\mu_1, \ldots, \mu_k)$, define

$$\overrightarrow{\lambda/\mu} := (\lambda_1 + 1, \ldots, \lambda_k + 1)/(\mu_1 + 1, \ldots, \mu_k + 1),$$

that is, $\overrightarrow{\lambda/\mu}$ is the skew shape obtained by shifting the shape λ/μ one step to the right. Similarly, define the left shift of λ/μ by

$$\overleftarrow{\lambda/\mu} := (\lambda_1 - 1, \ldots, \lambda_k - 1)/(\mu_1 - 1, \ldots, \mu_k - 1),$$

assuming that the result is a legitimate skew shape. Note that $s_{\lambda/\mu} = s_{\overleftarrow{\lambda/\mu}} = s_{\overrightarrow{\lambda/\mu}}$.

Let θ be the operation on pairs of skew shapes given by

$$\theta : (\lambda/\mu, \nu/\rho) \longmapsto ((\lambda/\mu) \vee (\nu/\rho), (\lambda/\mu) \wedge (\nu/\rho)).$$

According to Theorem 1.5, the product of the two skew Schur functions corresponding to the shapes in $\theta(\lambda/\mu, \nu/\rho)$ is $\geq_s s_{\lambda/\mu}\, s_{\nu/\rho}$. Let us show that we can repeatedly apply the operation θ together with the left and right shifts of shapes and the flips $(\lambda/\mu, \nu/\rho) \mapsto (\nu/\rho, \lambda/\mu)$ in order to obtain the pair of skew shapes $(\lfloor\frac{\lambda+\nu}{2}\rfloor/\lfloor\frac{\mu+\rho}{2}\rfloor, \lceil\frac{\lambda+\nu}{2}\rceil/\lceil\frac{\mu+\rho}{2}\rceil)$ from $(\lambda/\mu, \nu/\rho)$.

Let us define two operations ϕ and ψ on ordered pairs of skew shapes by conjugating θ with the right and left shifts and the flips, as follows:

$$\phi : (\lambda/\mu, \nu/\rho) \longmapsto ((\lambda/\mu) \wedge \overrightarrow{(\nu/\rho)}, \overleftarrow{(\lambda/\mu) \vee \overrightarrow{(\nu/\rho)}}),$$

$$\psi : (\lambda/\mu, \nu/\rho) \longmapsto (\overleftarrow{\overrightarrow{(\lambda/\mu)} \vee (\nu/\rho)}, \overrightarrow{(\lambda/\mu)} \wedge (\nu/\rho)).$$

In this definition the application of the left shift "$\leftarrow$" always makes sense. Indeed, in both cases, before the application of "$\leftarrow$", we apply "$\rightarrow$" and then "$\vee$", which guarantees all parts of the partitions to be positive. Again by Theorem 1.5, both products of skew Schur functions for shapes in $\phi(\lambda/\mu, \nu/\rho)$ and in $\psi(\lambda/\mu, \nu/\rho)$ are $\geq_s s_{\lambda/\mu}\, s_{\nu/\rho}$.

It is convenient to write the operations ϕ and ψ in the coordinates $\lambda_i, \mu_i, \nu_i, \rho_i$, for $i = 1, \ldots, k$. These operations independently act on the pairs (λ_i, ν_i) by

$$\begin{aligned} \phi &: (\lambda_i, \nu_i) \mapsto (\min(\lambda_i, \nu_i + 1), \max(\lambda_i, \nu_i + 1) - 1), \\ \psi &: (\lambda_i, \nu_i) \mapsto (\max(\lambda_i + 1, \nu_i) - 1, \min(\lambda_i + 1, \nu_i)), \end{aligned}$$

and independently act on the pairs (μ_i, ρ_i) by exactly the same formulas. Note that both operations ϕ and ψ preserve the sums $\lambda_i + \nu_i$ and $\mu_i + \rho_i$.

The operations ϕ and ψ transform the differences $\lambda_i - \nu_i$ and $\mu_i - \rho_i$ according to the following piecewise-linear maps:

$$\bar{\phi}(x) = \begin{cases} x & \text{if } x \leq 1, \\ 2 - x & \text{if } x \geq 1, \end{cases} \qquad \text{and} \qquad \bar{\psi}(x) = \begin{cases} x & \text{if } x \geq -1, \\ -2 - x & \text{if } x \leq -1. \end{cases}$$

Whenever we apply the composition $\phi \circ \psi$ of these operations, all absolute values $|\lambda_i - \nu_i|$ and $|\mu_i - \rho_i|$ strictly decrease, if these absolute values are ≥ 2. It follows that, for a sufficiently large integer N, we have $(\phi \circ \psi)^N(\lambda/\mu, \nu/\rho) = (\tilde{\lambda}/\tilde{\mu}, \tilde{\nu}/\tilde{\rho})$ with $\tilde{\lambda}_i + \tilde{\nu}_i = \lambda_i + \nu_i$, $\tilde{\mu}_i + \tilde{\rho}_i = \mu_i + \rho_i$, and $|\tilde{\lambda}_i - \tilde{\nu}_i| \leq 1$, $|\tilde{\mu}_i - \tilde{\rho}_i| \leq 1$, for all i. Finally, applying the operation θ, we obtain $\theta(\tilde{\lambda}/\tilde{\mu}, \tilde{\nu}/\tilde{\rho}) = (\lceil \frac{\lambda+\nu}{2} \rceil / \lceil \frac{\mu+\rho}{2} \rceil, \lfloor \frac{\lambda+\nu}{2} \rfloor / \lfloor \frac{\mu+\rho}{2} \rfloor)$, as needed. □

Example 14.2. *Let us illustrate the proof above. Let* $\lambda = (4, 3)$, $\nu = (12, 0)$, $\mu = \rho = (0, 0)$. *So* λ/μ *and* ν/ρ *are regular Young shapes (non-skew). Note that the maps* θ, ϕ, ψ *send a pair of non-skew shapes to a pair of non-skew shapes. We have* $((4,3),(12,0)) \xrightarrow{\psi} ((11,3),(5,0)) \xrightarrow{\phi} ((6,1),(10,2)) \xrightarrow{\psi} ((9,1),(7,2)) \xrightarrow{\phi} ((8,1),(8,2)) \xrightarrow{\theta} ((8,2),(8,1))$. *Thus* $s_\lambda\, s_\nu \leq_s s_{(11,3)}\, s_{(5,0)} \leq_s \cdots \leq_s s_{(8,2)}\, s_{(8,1)} = s_{\lceil \frac{\lambda+\nu}{2} \rceil}\, s_{\lfloor \frac{\lambda+\nu}{2} \rfloor}$.

The following conjugate version of Theorem 14.1 extends Fomin-Fulton-Li-Poon's conjecture (Conjecture 1.2) to skew shapes.

Corollary 14.3. *Let* λ/μ *and* ν/ρ *be two skew shapes. Then we have*

$$s_{\mathrm{sort}_1(\lambda,\nu)/\mathrm{sort}_1(\mu,\rho)}\, s_{\mathrm{sort}_2(\lambda,\nu)/\mathrm{sort}_2(\mu,\rho)} \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}.$$

Proof. This statement is obtained from Theorem 14.1 by conjugating the shapes. Indeed, $\lceil \frac{\lambda+\mu}{2} \rceil' = \mathrm{sort}_1(\lambda', \mu')$ and $\lfloor \frac{\lambda+\mu}{2} \rfloor' = \mathrm{sort}_2(\lambda', \mu')$. Here λ' denotes the partition conjugate to λ. $\square$

Theorem 14.4. *Let $\lambda^{(1)}/\mu^{(1)}, \ldots, \lambda^{(n)}/\mu^{(n)}$ be n skew shapes, let $\lambda = \bigcup \lambda^{(i)}$ be the partition obtained by the decreasing rearrangement of the parts in all $\lambda^{(i)}$, and, similarly, let $\mu = \bigcup \mu^{(i)}$. Then we have $\prod_{i=1}^{n} s_{\lambda^{[i,n]}/\mu^{[i,n]}} \geq_s \prod_{i=1}^{n} s_{\lambda^{(i)}/\mu^{(i)}}$.*

This theorem extends Corollary 14.3 and Conjecture 1.2. Also note that Lascoux-Leclerc-Thibon's conjecture (Conjecture 1.3) is a special case of Theorem 14.4 for the n-tuple of partitions $(\lambda^{[1,m]}, \ldots, \lambda^{[m,m]}, \emptyset, \ldots, \emptyset)$.

Proof. Let us derive the statement by applying Corollary 14.3 repeatedly. For a sequence $v = (v_1, v_2, \ldots, v_l)$ of integers, the *anti-inversion number* is $\mathrm{ainv}(v) := \#\{(i,j) \mid i < j,\ v_i < v_j\}$. Let $L = (\lambda^{(1)}/\mu^{(1)}, \ldots, \lambda^{(n)}/\mu^{(n)})$ be a sequence of skew shapes. Define its anti-inversion number as

$$\begin{aligned}\mathrm{ainv}(L) = {} & \mathrm{ainv}(\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_1^{(n)}, \lambda_2^{(1)}, \ldots, \lambda_2^{(n)}, \lambda_3^{(1)}, \ldots, \lambda_3^{(n)}, \ldots) \\ & + \mathrm{ainv}(\mu_1^{(1)}, \mu_1^{(2)}, \ldots, \mu_1^{(n)}, \mu_2^{(1)}, \ldots, \mu_2^{(n)}, \mu_3^{(1)}, \ldots, \mu_3^{(n)}, \ldots).\end{aligned}$$

If $\mathrm{ainv}(L) \neq 0$ then there is a pair $k < l$ such that $\mathrm{ainv}(\lambda^{(k)}/\mu^{(k)}, \lambda^{(l)}/\mu^{(l)}) \neq 0$. Let $\tilde{L}$ be the sequence of skew shapes obtained from L by replacing the two terms $\lambda^{(k)}/\mu^{(k)}$ and $\lambda^{(l)}/\mu^{(l)}$ with the terms

$$\mathrm{sort}_1(\lambda^{(k)}, \lambda^{(l)})/\mathrm{sort}_1(\mu^{(k)}, \mu^{(l)}) \quad \text{and} \quad \mathrm{sort}_2(\lambda^{(k)}, \lambda^{(l)})/\mathrm{sort}_2(\mu^{(k)}, \mu^{(l)}),$$

correspondingly. Then $\mathrm{ainv}(\tilde{L}) < \mathrm{ainv}(L)$. Indeed, if we rearrange a subsequence in a sequence in the decreasing order, the total number of anti-inversions decreases. According to Corollary 14.3, we have $s_{\tilde{L}} \geq_s s_L$, where $s_L := \prod_{i=1}^{n} s_{\lambda^{(i)}/\mu^{(i)}}$. Note that the operation $L \mapsto \tilde{L}$ does not change the unions of partitions $\bigcup \lambda^{(i)}$ and $\bigcup \mu^{(i)}$. Let us apply the operations $L \mapsto \tilde{L}$ for various pairs (k, l) until we obtain a sequence of skew shapes $\hat{L} = (\hat{\lambda}^{(1)}/\hat{\mu}^{(1)}, \ldots, \hat{\lambda}^{(n)}/\hat{\mu}^{(n)})$ with $\mathrm{ainv}(\hat{L}) = 0$, i.e., the parts of all partitions must be sorted as $\hat{\lambda}_1^{(1)} \geq \cdots \geq \hat{\lambda}_1^{(n)} \geq \hat{\lambda}_2^{(1)} \geq \cdots \geq \hat{\lambda}_2^{(n)} \geq \hat{\lambda}_3^{(1)} \geq \cdots \geq \hat{\lambda}_3^{(n)} \geq \cdots$, and the same inequalities hold for the $\hat{\mu}_j^{(i)}$. This means that $\hat{\lambda}^{(i)}/\hat{\mu}^{(i)} = \lambda^{[i,n]}/\mu^{[i,n]}$, for $i = 1, \ldots, n$. Thus $s_{\hat{L}} = \prod s_{\lambda^{[i,n]}/\mu^{[i,n]}} \geq_s s_L$, as needed. $\square$

Let us define $\lambda^{\{i,n\}} := ((\lambda')^{[i,n]})'$, for $i = 1, \ldots, n$. Here λ' again denotes the partition conjugate to λ. The partitions $\lambda^{\{i,n\}}$ are uniquely defined by the conditions $\lceil \frac{\lambda}{n} \rceil \supseteq \lambda^{\{1,n\}} \supseteq \cdots \supseteq \lambda^{\{n,n\}} \supseteq \lfloor \frac{\lambda}{n} \rfloor$ and $\sum_{i=1}^{n} \lambda^{\{i,n\}} = \lambda$. In particular, $\lambda^{\{1,2\}} = \lceil \frac{\lambda}{2} \rceil$ and $\lambda^{\{2,2\}} = \lfloor \frac{\lambda}{2} \rfloor$. If $\frac{\lambda}{n}$ is a partition, i.e., all parts of λ are divisible by n, then $\lambda^{\{i,n\}} = \frac{\lambda}{n}$ for each $1 \leq i \leq n$.

Corollary 14.5. *Let $\lambda^{(1)}/\mu^{(1)}, \ldots, \lambda^{(n)}/\mu^{(n)}$ be n skew shapes, let $\lambda = \lambda^{(1)} + \cdots + \lambda^{(n)}$ and $\mu = \mu^{(1)} + \cdots + \mu^{(n)}$. Then we have $\prod_{i=1}^{n} s_{\lambda^{\{i,n\}}/\mu^{\{i,n\}}} \geq_s \prod_{i=1}^{n} s_{\lambda^{(i)}/\mu^{(i)}}$.*

Proof. This claim is obtained from Theorem 14.4 by conjugating the shapes. Indeed, $(\bigcup \lambda^{(i)})' = \sum (\lambda^{(i)})'$. □

For a skew shape λ/μ and a positive integer n, define $s^{\langle n \rangle}_{\frac{\lambda}{n}/\frac{\mu}{n}} := \prod_{i=1}^{n} s_{\lambda^{\{i,n\}}/\mu^{\{i,n\}}}$. In particular, if $\frac{\lambda}{n}$ and $\frac{\mu}{n}$ are partitions, then $s^{\langle n \rangle}_{\frac{\lambda}{n}/\frac{\mu}{n}} = \left(s_{\frac{\lambda}{n}/\frac{\mu}{n}}\right)^n$.

Corollary 14.6. *Let c and d be positive integers and $n = c + d$. Let λ/μ and ν/ρ be two skew shapes. Then $s^{\langle n \rangle}_{\frac{c\lambda+d\nu}{n}/\frac{c\mu+d\rho}{n}} \geq_s s^c_{\lambda/\mu}\, s^d_{\nu/\rho}$.*

Theorem 14.1 is a special case of Corollary 14.6 for $c = d = 1$.

Proof. This claim follows from Corollary 14.5 for the sequence of skew shapes that consists of λ/μ repeated c times and ν/ρ repeated d times. □

Corollary 14.6 implies that the map $S : \lambda \mapsto s_\lambda$ from the set of partitions to symmetric functions satisfies the following "Schur log-concavity" property.

Corollary 14.7. *For positive integers c, d and partitions λ, μ such that $\frac{c\lambda+d\mu}{c+d}$ is a partition, we have $\left(S\left(\frac{c\lambda+d\mu}{c+d}\right)\right)^{c+d} \geq_s S(\lambda)^c S(\mu)^d$.*

This notion of Schur log-concavity is inspired by Okounkov's paper [Oko].

15. Comparing products of $\mathfrak{sl}_n$ characters

Let λ, μ be two dominant weights of $\mathfrak{sl}_n$. Recall that the weight lattice in this case is $\mathbb{Z}^n/(1, \ldots, 1)$. Thus dominant weights can be viewed as partitions with n-th part equal to zero. Equivalently, dominant weights can be associated with Young diagrams with $n-1$ rows. Let V_λ denote the highest weight module corresponding to λ.

Recall that as λ runs through all dominant weights, the V_λ-s constitute the set of irreducible $\mathfrak{sl}_n$-modules. Since $\mathfrak{sl}_n$ is semisimple, the tensor product $V_\lambda \otimes V_\mu = \bigoplus_\nu c^\nu_{\lambda,\mu} V_\nu$ decomposes into a direct sum of V_ν-s. The coefficients $c^\nu_{\lambda,\mu}$ which appear in this decomposition are the celebrated Littlewood-Richardson coefficients - the same ones that appear when we multiply Schur functions. Let s_λ denote the polynomial character of irreducible representation V_λ. Then $s_\lambda = s_\lambda(x_1, \ldots, x_n, 0, \ldots)$ is the evaluation of the Schur function s_λ modulo the relation $x_1 \cdots x_n = 1$. For the background on representation theory of $\mathfrak{sl}_n$ and Schur functions see [Hum], [Sta]. Note that under the substitution $x_{n+1} = \cdots = 0$ the Schur functions s_ν with ν having more than n parts vanish. This causes a subtle difference between multiplication of s_λ-s and multiplication of the usual Schur functions: some terms appearing in the latter vanish in the former.

Thus, instead of asking of when product of two Schur functions is bigger than the product of different two Schur functions, one can ask when $V_\lambda \otimes V_\mu$ is contained in $V_\nu \otimes V_\rho$ as an $\mathfrak{sl}_n$-module.

The first thing to note is that the highest weight appearing in $V_\lambda \otimes V_\mu$ is $\lambda + \mu$. Thus, in order for $V_\lambda \otimes V_\mu$ to be a submodule of $V_\nu \otimes V_\rho$ we need to have $\lambda + \mu \leq \nu + \rho$ in dominance order. It is natural to investigate what happens if we restrict our attention to the case when equality holds, i.e. $\lambda + \mu = \nu + \rho$. For this situation, we make a conjecture concerning a sufficient condition for $V_\lambda \otimes V_\mu$ to be a submodule of $V_\nu \otimes V_\rho$, or equivalently for $s_\nu s_\rho - s_\lambda s_\mu$ to be Schur-nonnegative.

Let $\alpha_{ij} = e_i - e_j$ be the roots of the type A root system. Call a polytope *alcoved* if its faces belong to hyperplanes given by the equations $\langle \alpha_{ij}, \tau \rangle = m$, where $\langle , \rangle$ is the standard inner product and $m \in \mathbb{Z}$. Given two weights λ, μ one can consider the *minimal alcoved polytope* $P_{\lambda,\mu}$ containing λ and μ. $P_{\lambda,\mu}$ is always a parallelepiped in which λ and μ are a pair of opposite vertices. An example for $\mathfrak{sl}_3$ is shown in Figure 15. The weights τ inside $P_{\lambda,\mu}$ can be characterized by the following condition: for all $1 \leq i, j \leq n$, the number $\tau_i - \tau_j$ lies weakly between $\lambda_i - \lambda_j$ and $\mu_i - \mu_j$. Let ν and ρ be another pair of weights.

Conjecture 15.1. [LPP2] *If $\lambda + \mu = \nu + \rho$ and $\nu, \rho \in P_{\lambda,\mu}$, then $s_\nu s_\rho - s_\lambda s_\mu$ is Schur-nonnegative.*

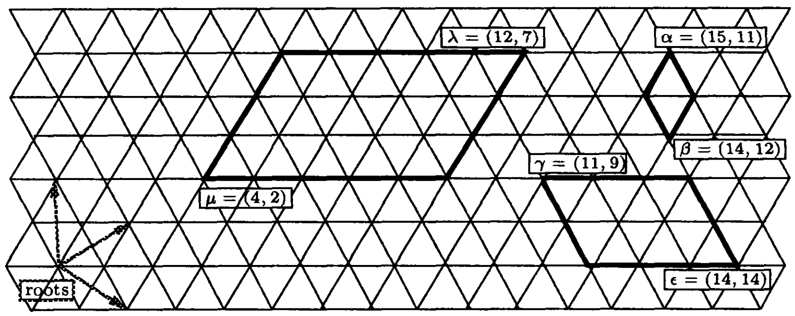


FIGURE 16

Example 15.2. *It is easy to see in Figure 15 that points* $\rho = (11, 7, 0)$, $\nu = (5, 2, 0)$ *lie inside marked* $P_{\lambda,\mu}$ *with* $\lambda = (12, 7, 0)$, $\mu = (4, 2, 0)$. *In this case*

$$s_\nu s_\rho - s_\lambda s_\mu = s_{(13,12,0)} + s_{(6,4,0)} + s_{(7,6,0)} + s_{(8,8,0)} + s_{(7,3,0)} + s_{(8,5,0)} + s_{(9,7,0)}$$

$$+ s_{(10,9,0)} + s_{(11,11,0)} + s_{(8,2,0)} + s_{(9,4,0)} + s_{(10,6,0)} + s_{(11,8,0)} + s_{(12,10,0)}.$$

We prove the following weaker statement.

Theorem 15.3. *If* $\lambda + \mu = \nu + \rho$ *and* $\nu, \rho \in P_{\lambda,\mu}$, *then every* s_κ *occuring in* $s_\lambda s_\mu$ *with a non-zero coefficient does also occur in* $s_\nu s_\rho$ *with a non-zero coefficient.*

16. Horn-Klyachko inequalities

For a finite set $I = \{i_1 > \cdots > i_r\}$ of positive integers, define the corresponding partition $\lambda(I)$ by

$$\lambda(I) = (i_1 - r, i_2 - (r-1), \ldots, i_r - 1).$$

Definition 16.1. *Define* T_r^n *to be the set of triples* (I, J, K) *of subsets of* $\{1, \ldots, n\}$ *of the same cardinality* r *such that the Littlewood-Richardson coefficient* $c^{\lambda(K)}_{\lambda(I)\lambda(J)}$ *is positive. A* Horn-Klyachko inequality *for a triple of partitions* α, β, γ *has the form*

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

for a triple (I, J, K) *in* T_r^n *and some* $r < n$.

The following fact was proved in [Kl, KT], see also [Ful] for a survey:

Theorem 16.2. *For a triple of partitions α, β, γ of length n, the Littlewood-Richardson coefficient $c^{\gamma}_{\alpha\beta}$ is positive if and only if $\sum_{i=1}^{n}\gamma_i = \sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\beta_i$ and Horn-Klyachko inequalities for α, β, γ are valid for all $(I, J, K) \in T^n_r$ and all $r < n$.*

Let partitions λ, μ, ν, ρ with at most n parts satisfy the conditions of Conjecture 15.1, and γ be a partiton such that $c^{\gamma}_{\lambda\mu} > 0$. Consider a triple

$$(I = (i_1, \dots, i_r), J = (j_1, \dots, j_r), K = (k_1, \dots, k_r))$$

in T^n_r. Given permutations $\{l_1, \dots, l_r\}$ of I and $\{m_1, \dots, m_r\}$ of J, switch l_p and m_p in some of the pairs $\{l_p, m_p\}$. This operation yields 2^r possible pairs (I', J').

Lemma 16.3. *Assume there exist permutations $\{l_1, \dots, l_r\}$ of I and $\{m_1, \dots, m_r\}$ of J such that all possible triples (I', J', K) are in T^n_r. Then the Horn-Klyachko inequality corresponding to the triple (I, J, K) holds for ν, ρ, γ.*

Proof. Since $\nu, \rho \in P_{\lambda,\mu}$, for $i, j \geq 1$ both $\nu_i - \nu_j$ and $\rho_i - \rho_j$ are between $\lambda_i - \lambda_j$ and $\mu_i - \mu_j$, which implies

$$|(\nu_i - \nu_j) - (\rho_i - \rho_j)| \leq |(\lambda_i - \lambda_j) - (\mu_i - \mu_j)|.$$

Rearranging terms, we obtain

$$|(\nu_i + \rho_j) - (\nu_j + \rho_i)| \leq |(\lambda_i + \mu_j) - (\lambda_j + \mu_i)|.$$

This inequality combined with the equality $(\nu_i + \rho_j) + (\nu_j + \rho_i) = (\lambda_i + \mu_j) + (\lambda_j + \mu_i)$ following from $\lambda + \mu = \nu + \rho$, shows that $\nu_i + \rho_j$ and $\nu_j + \rho_i$ are between $\lambda_i + \mu_j$ and $\lambda_j + \mu_i$. We use the fact that for all $i, j \geq 1$ we have

$$\nu_i + \rho_j \geq \min\{\lambda_i + \mu_j, \lambda_j + \mu_i\}.$$

For every p in $\{1, \dots, r\}$, choose (l'_p, m'_p) to be a permutation of $\{l_p, m_p\}$ such that $\lambda_{l'_p} + \mu_{m'_p} = \min\{\lambda_{l_p} + \mu_{m_p}, \lambda_{m_p} + \mu_{l_p}\}$, and let $I' = \{l'_1, \dots, l'_r\}$, $J' = \{m'_1, \dots, m'_r\}$ be the corresronding subsets of $\{1, \dots, n\}$. By the assumption of the lemma, $c^{\gamma}_{\lambda\mu} > 0$ and (I', J', K) is in T^n_r. Therefore, by Theorem 16.2 the Horn-Klyachko inequality for λ, μ, γ and the triple (I', J', K) holds:

$$\sum_{p=1}^{r} \lambda_{l'_p} + \sum_{p=1}^{r} \mu_{m'_p} \geq \sum_{k \in K} \gamma_k.$$

Observe that

$$\sum_{i \in I} \nu_i + \sum_{j \in J} \rho_j = \sum_{p=1}^{r} \nu_{l_p} + \sum_{p=1}^{r} \rho_{m_p} = \sum_{p=1}^{r} (\nu_{l_p} + \rho_{m_p}) \geq$$

$$\sum_{p=1}^{r} min\{\lambda_{l_p} + \mu_{m_p}, \lambda_{m_p} + \mu_{l_p}\} = \sum_{p=1}^{r} (\lambda_{l_p'} + \mu_{m_p'}) \geq \sum_{k \in K} \gamma_k.$$

Therefore, the Horn-Klyachko inequality for ν, ρ, γ and the triple (I, J, K) holds. □

17. Proof of the conjecture

The symmetric functions $h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}$ are called the *homogeneous symmetric functions.* For background on them, see [Sta]. Given two sets $V = (v_1 \geq v_2 \cdots \geq v_n \geq 0)$ and $U = (u_1 \geq u_2 \cdots \geq u_n \geq 0)$ one can construct the *generalized Jacobi-Trudi matrix* $X_{V,U} = \left(h_{v_i - u_j}\right)_{i,j=1}^{n}$. For example, for $V = (4, 3, 3, 2)$ and $U = (3, 2, 1, 0)$ we get

$$X_{V,U} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_1 & h_2 \end{bmatrix}$$

Note that for the operation $\lambda = \lambda(I) = (i_1 - r, \ldots, i_r - 1)$ defined in Section 16 we have the Jacobi-Trudi identity $s_{\lambda(I)} = detX_{I,\{r,\ldots,2,1\}}$. (See [Sta]).

We begin by proving the following lemma, which shows that the condition of Lemma 16.3 can be achieved.

Lemma 17.1. *In the setup preceding Lemma 16.3, there exist permutations $\{l_1, \ldots, l_r\}$ and $\{m_1, \ldots, m_r\}$ of I and J respectively such that all possible triples (I', J', K) are in T_r^n.*

Proof. Let $X_{V,U}$ be the generalized Jacobi-Trudi matrix for column set $U = (r, r, r-1, r-1, \ldots, 1, 1)$, and row set $V = I \cup J$ in some chosen non-increasing arrangement.

Let $\#I$ and $\#J$ denote the sets of numbers of the rows of I and J in the chosen non-increasing arrangement of $I \cup J$. Since $(I, J, K) \in T_r^n$, we have $c_{\lambda(I)\lambda(J)}^{\lambda(K)} > 0$. Hence $s_{\lambda(K)}$ is present in the decomposition of $s_{\lambda(I)}s_{\lambda(J)}$, which by Jacobi-Trudi identity equals to the product $\Delta_{\#I,\{2r,2r-2,\dots,2\}}\Delta_{\#J,\{2r-1,2r-3,\dots,1\}}$ of complementary minors of $X_{V,U}$. This product, in turns, by Theorem 12.4 equals to $\sum_{w\in\Theta(S)} \mathrm{Imm}_w^{\mathrm{TL}}(X_{V,U})$, where $S = \#J \cup \{4r, 4r-2, \dots, 2r+2\}$ is the subset of the vertices $1, \dots, 4r$ of the Temperley-Lieb diagram which are colored black.

Since $s_{\lambda(K)}$ is in the Schur function decomposition of $\sum_{w\in\Theta(S)} \mathrm{Imm}_w^{\mathrm{TL}}(X_{U,V})$, it is present in the Schur function decomposition of one of the immanants $\mathrm{Imm}_w^{\mathrm{TL}}(X_{V,U})$ for some 321-avoiding permutation $w \in \Theta(S)$. For this 321-avoiding permutation w, the basis element t_w and the corresponding non-crossing matching M_w of the Temperley-Lieb diagram with columns V and U are S-compatible. Therefore, all edges of M_w have endpoints of different color in the Temperley-Lieb diagram on vertices $\{1, 2, \dots, 4r\}$ where S is colored black and $[4r]/S$ colored white.

We proceed now to construct the needed permutations $\{l_1, \dots, l_r\}$ of I and $\{m_1, \dots, m_r\}$ of J based on S and M_w. We go along V from top to bottom (see Figure 17(i)) and label vertices in I that are connected to vertices in J by edges in M_w (suppose that there are k such vertices in I) with variables $l_1, \dots, l_k$ as we meet them. We also label the vertex in J connected to l_i ($i \le k$) by m_i.

Next, we remove the vertices $l_1, \dots, l_k, m_1, \dots, m_k$ from V and call the remaining set V^-. We also go along U and discard every pair of vertices in U connected by an edge in M_w, and call the remaining set U^-. We go along V^- from top to bottom and label the white vertices that we meet by $l_{k+1}, \dots, l_r$, and the black vertices we meet by $m_{k+1}, \dots, m_r$ from top to bottom. For $f \ge 1$, we also label the vertices in U^- connected by edges in M_w to l_{k+f} by p_{k+f}, and those connected to m_{k+f} by q_{k+f}. (See Figure 17(ii)). Note that every vertex in V between adjacent vertices of V' is connected by an edge in M_w to another vertex between the same vertices of V^- because M_w is a non-crossing, and the same is true about U. Therefore, in building V^- and U^- we discarded segments of even lengths from V and U.

Claim. For $f \geq 1$, vertices l_{k+f} and q_{k+f} are white and odd-numbered in the Temperley-Lieb diagram for S and M_w; vertices p_{k+f} and m_{k+f} are black and even-numbered. Also, $l_{k+f+1} > m_{k+f} > l_{k+f}$ and $p_{k+f+1} < q_{k+f} < p_{k+f}$. (See Figure 17($ii$))

Proof. Since we discarded segments of even lengths from U to obtain U^- and the colors in U were alternating from top to bottom beginning with the black even vertex $4r$, the colors in U^- are also alternating from top to bottom beginning with a black even vertex. Therefore, vertices in U^- from top to bottom are $p_{k+1} > q_{k+1} > p_{k+2} > \ldots > p_r > q_r$, where p_{k+f} is black and q_{k+f} is white for $f \geq 1$. Because the restriction of the matching M_w to $U^- \cup V^-$ is non-crossing, the inequalities $p_{k+1} > q_{k+1} > p_{k+2} > \ldots > p_r > q_r$ for U^- imply that $l_{k+1} < m_{k+1} < l_{k+2} < \ldots < l_r < m_r$ for V^-. The colors in V^- alternate and have a white odd vertex at the top because the colors in U^- alternate with a black even vertex at the top. Therefore, l_{k+f} is white and m_{k+f} is black for $f \geq 0$. The statements about being odd/even now follow from the fact that we discarded segments of even lengths from U and V to obtain U^- and V^-.

We now build a new coloring S' of $U \cup V$ based on the transpositions (l_p, m_p) that may have occured in going from I, J to I', J'. We only allow ourselves to recolor both elements in a pair $\{2m, 2m-1\} \in \{2r+1, \ldots, 4r\}$ of vertices in the second column of Temperley-Lieb diagram for $S'_0 = \#J' \cup \{4r, 4r-2, \ldots, 2r+2\}$, because the columns $4r+1-2m$ and $4r+2-2m$ of $X_{U,V}$ are identical and hence such a recoloring produces the same pair of complementary minors $\Delta_{\#I',\{4r-1,\ldots,2r+1\}}$, $\Delta_{\#J',\{4r,\ldots,2r+2\}}$ of $X_{V,U}$ as S'_0 does, and therefore by Jacobi-Trudi identity the product of these complementary minors is $s_{\lambda(I')} s_{\lambda(J')}$.

Rule of recoloring. For every pair l_{k+f} and m_{k+f} ($f \geq 1$) that exchanged colors in transition from I, J to I', J', recolor the pairs $(p_{k+f}, p_{k+f}-1), (p_{k+f}-2, p_{k+f}-3), \ldots, (q_{k+f}+1, q_{k+f})$. The recoloring is permissible because the vertex p_{k+f} is even by the Claim. (See Figure 17(iii))

Why the rule produces a coloring compatible with M_w. The vertices between p_{k+f} and q_{k+f} either all changed color or all stayed the same, so an edge in M_w that connected two vertices in U between p_{k+f} and q_{k+f} now has its endpoints changed or not changed simultaneously, so they are of different color in the new coloring.

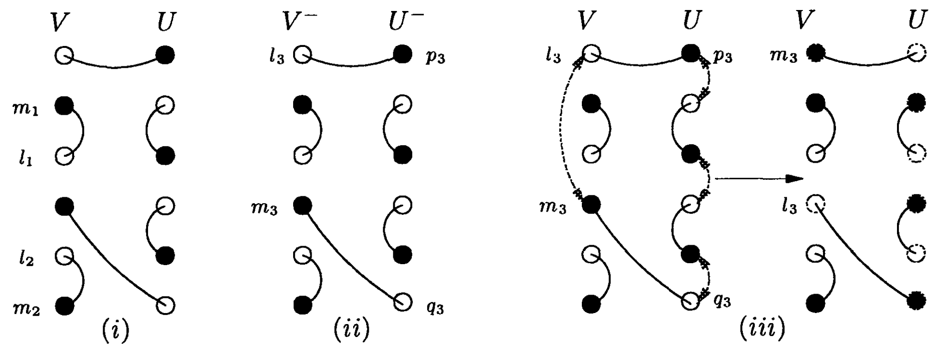


FIGURE 17

A pair (l_{k+f}, m_{k+f}) changes color simultaneously with the pair (p_{k+f}, q_{k+f}), so l_{k+f} and p_{k+f} , and m_{k+f} and q_{k+f} change or do not change their color simultaneously, so the endpoints of the edges between U^- and V^- remain colored differently in the new coloring.

A pair of vertices (l_p, m_p) in V connected by an edge in M_w changes color simultaneously when the corresponding transposition occurs, so the endpoints of such an edge remain colored differently. Finally, a pair of vertices in U between q_{k+f} and p_{k+f+1} connected by an edge in M_w never changes color, so such an edge has its endpoints colored differently in the new coloring. We considered all possibilities for an edge in M_w relative to U^- and V^- in a non-crossing matching, so M_w is compatible with the new coloring.

We already noticed that the new coloring produces the product of complementary minors of $X_{V,U}$ equal to $s_{\lambda(I')}s_{\lambda(J')}$. The fact that the new coloring is compatible with M_w implies that the immanant $\mathrm{Imm}_w^{\mathrm{TL}}(X_{V,U})$ is present in the decomposition $s_{\lambda(I')}s_{\lambda(J')} = \sum_{w\in\Theta(S')} \mathrm{Imm}_w^{\mathrm{TL}}(X_{V,U})$. Since $s_{\lambda(K)}$ is in the decomposition of $\mathrm{Imm}_w^{\mathrm{TL}}(X_{V,U})$ which is Schur-nonnegative by Theorem 12.1, $s_{\lambda(K)}$ is present in the Schur function decomposition of $s_{\lambda(I')}s_{\lambda(J')}$. Therefore $c_{\lambda(I')\lambda(J')}^{\lambda(K)} > 0$ and $(I', J', K) \in {T_r}^n$ for all I', J' that can be obtained by transposing pairs (l_p, m_p) in I,J. □

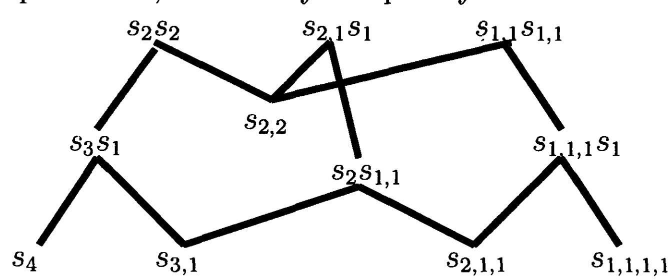
We are ready to prove Theorem 15.3.

Proof. From Lemma 16.3 and Lemma 17.1 it follows that whenever the Horn-Klyachko inequality for triple (I, J, K) holds for λ, μ, γ, it also holds for ν, ρ, γ. Thus all possible

γ-s for which all needed Horn-Klyachko inequlities hold or, equivalently, $c^{\gamma}_{\lambda,\mu} > 0$, also have the property that $c^{\gamma}_{\nu,\rho} > 0$. □

18. Stembridge's poset

John Stembridge introduced the following partial order $\mathfrak{T}_n$ on pairs of partitions of fixed total size n. We say that $(\lambda, \mu) < (\nu, \rho)$ iff $s_\lambda s_\mu < s_\nu s_\rho$. An example of $\mathfrak{T}_4$ is given on the Figure below. A natural question to ask is to describe all covering relations in $\mathfrak{T}_n$. This question seems to be hard, however certain partial results can be obtained. In particular, we classify completely the *maximal elements* of $\mathfrak{T}_n$.



Lemma 18.1. *The equality $s_\lambda s_\mu = s_{\bar\lambda} s_{\bar\mu}$ can hold only when (unordered) pair (λ, μ) is the same as $(\bar\lambda, \bar\mu)$.*

Proof. The ring of symmetric functions is a UFD since it is isomorphic to $k[e_1, e_2, \ldots]$ - the free ring in elementary symmetric functions. Thus, it is enough to show that Schur functions are irreducable in this order. Assume on the contrary that $s_\lambda = f_1 f_2 \ldots f_k$, where $k \geq 2$. Then each f_i is a linear combination of Schur polynomials. Let us pick the lex. maximal partition λ^i such that s_{λ^i} occurs in f_i with a nonzero coefficient. Then the product $f_1 \ldots f_k$ contains $s_{\lambda^1 + \ldots + \lambda^k}$ with a nonzero coefficient. Thus $\lambda = \lambda^1 + \ldots + \lambda^k$. Applying the same argument to the conjugate partitions, we obtain $\lambda' = (\mu^1)' + \ldots + (\mu^k)'$, where μ^i is lex minimal partition such that s_{μ^i} occurs in f_i. (In particular, μ^i is less than or equal to λ^i.) This is a contradiction since adding partitions horyzontally always produces result larger in lex. order than adding them verically.

□

Call λ and μ *similar* if one can be obtained from the other by adding some outer corner squares.

Theorem 18.2. *Pair (λ, μ) is maximal iff λ and μ are similar.*

Proof. First, observe that if one of the two diagrams is not contained in the other, than a non-trivial cell transfer can be performed. According to Lemma above and the cell transfer theorem, this implies that original pair was not maximal - contradiction. Thus, we can assume that say μ is contained in λ. Next, if the difference is not a horyzontal border strip we can perform shift μ one raw down and have a non-trivial cell transfer - contradiction as above. Thus the difference is horyzontal border strip. Similarly it is a vertical border strip. Then it must be just a callection of single cells in outer corners of μ.

Now we show that each such pair is indeed maximal. If it is not, it would be smaller than another such pair, say $(\lambda, \mu) < (\nu, \delta)$. Let ρ be defined by $\rho_1 = \rho_2 = \lambda_1 + \mu_1, \rho_3 = \rho_4 = \lambda_2 + \mu_2$, etc. Similarly define σ out of ν and δ. We claim that $\rho < \sigma$ and $\rho' < \sigma'$ in dominance order. This would imply that $\rho = \sigma$, and for our specific type of pairs that would mean that $(\lambda, \mu) = (\nu, \delta)$.

Since $(\lambda, \mu) < (\nu, \delta)$, we know that $\lambda + \mu \leq \nu + \delta$ in dominance order. Then for even k-s we have $\sum_{i=1}^{k} \rho_i = 2(\sum_{i=1}^{k} \lambda_i + \mu_i) \leq 2(\sum_{i=1}^{k} \nu_i + \delta_i) = \sum_{i=1}^{k} \sigma_i$.

To obtain needed enequality for odd k-s we just take the half-sum of the enequalities for $k-1$ and $k+1$.

To see that $\rho' < \sigma'$, note that for even k-s the enequality again follows from the fact that $\lambda' + \mu' \leq \nu' + \delta'$.

For odd $k = 2j+1$ we can still take half sum of inequalities of the form above for k-1 and k+1. What we get is $(\sum_{i=1}^{(k-1)} \rho'_i) + (\rho'_k + \rho'_{(k+1)})/2 \leq (\sum_{i=1}^{(k-1)} \sigma'_i) + (\sigma'_k + \sigma'_{(k+1)})/2$. Note that ρ'_k is either equal to $(\rho'_k + \rho'_{(k+1)})/2$ (if $\lambda'_j = \mu'_j$) or greater by 1 (if $\lambda'_j = \mu'_j + 1$). Same for σ'_k.

Then the only case when $\sum_{i=1}^{k} \rho'_i > \sum_{i=1}^{k} \sigma'_i$ is if we had an equality in $(\sum_{i=1}^{(k-1)} \rho'_i) + (\rho'_k + \rho'_{(k+1)})/2 \leq (\sum_{i=1}^{(k-1)} \sigma'_i) + (\sigma'_k + \sigma'_{(k+1)})/2$ [and thus, in both $\sum_{i=1}^{(k-1)} \rho'_i = \sum_{i=1}^{(k-1)} \sigma'_i$ and $\sum_{i=1}^{(k+1)} \rho'_i = \sum_{i=1}^{(k+1)} \sigma'_i$] and at the same time if $\lambda'_j = \mu'_j + 1$, while if $\nu'_j = \delta'_j$.

This is a contradiction however since that means that $2\nu'_j = \nu'_j + \delta'_j = (\sum_{i=1}^{(k+1)} \sigma'_i - \sum_{i=1}^{(k-1)} \sigma'_i)/2 = (\sum_{i=1}^{(k+1)} \rho'_i - \sum_{i=1}^{(k-1)} \rho'_i)/2 = \mu'_j + \lambda'_j = 2\mu_j + 1$.

□

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