Parabolic Equations without a Minimum Principle

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2007

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April 24, 2007

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Abstract

In this thesis, we consider several parabolic equations for which the minimum principle fails. We first consider a two-point boundary value problem for a one dimensional diffusion equation. We show the uniqueness and existence of the solution for initial data, which may not be continuous at two boundary points. We also examine the circumstances when these solutions admit a probabilistic interpretation. Some partial results are given for analogous problems in more than one dimension.

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Acknowledgments

My thanks go in first place to my advisor Prof.Daniel Stroock. Without his insightful guidance, this thesis would not be finished. I am deeply grateful to him for his encouragements, support and care throughout the preparation of this thesis and my study at MIT. I immensely benefited from seeing his great view of the mathematics and many other things. The frequent meetings with him are the most enjoyable and valuable events in my life. It's very hard to find a precise word to express my admiration of his work and his wonderful thinking. I was deeply touched by his kindness, generosity and encouragement. I am very fortunate to have been able to work with him.

I like to thank Prof.Jerison for his help on the proof of nonnegativity in third chapter and great suggestions during my thesis preparation. I want to thank Prof.Dudley for his time of reviewing the draft and his help and care of my teaching assistant work. I also benefited a lot from their excellent lectures and seminars.

Thanks to Linda Okun for her consistent help and kindness during my study here and my thesis preparation. Thanks also to Shan-Yuan Ho and Alan Leung for their great help on latex and thesis preparation.

Thanks to my friends Edward Fan, Yanir Rubinstein, Zhou Zhang, Xuhua He, Fangyun Yang, Chuying Fang, Zuoqin Wang, and Linan Chen, etc, for the help and care throughout my life and study at math department.

Special thanks to my loving wife, Zhen Wang, for her support and encouragement during my thesis preparation. I would like to dedicate this thesis to my parents, my wife and my brother. Thanks to them for their love and care during all these years.

Contents

Chapter 1

Introduction to Our Parabolic Equations

1.1 Introduction

The main purpose of this chapter is to introduce the family of parabolic equations we are interested in, where the minimal principle fails. We will review the main results from [12], [13] and summarize the main results we will discuss in later Chapters. In [12] and [13], Stroock and Williams studied a 1-dim diffusion equation on half line with a one point boundary condition, for which the minimal principle fails. They also tried to elucidate the general case with a Markov chain analog. Later, in [17], Williams and Andrews studied a special case of 1-dim diffusion equation with two-point boundary case by using the indefinite innner product method. In section 1.2, we will recall the equation and the main result in [12] and [13]. In section 1.3, we introduce the equations we will consider and the main results we get.

Note: All the content and notation in Chapter 2 and Chapter 3 are self-contained and we will specify all the notation in each chapter to avoid possible confusion.

1.2 Results from [12] ,[13]

Let F be the space of bounded functions on $[0,\infty)$ that are continuous on $(0,\infty)$ but not necessarily continuous at 0. Convergence of f_n to f in F means that $\{\|f_n\|_u\}_1^\infty$ is bounded, $f_n(0) \to f(0)$, and $f_n \to f$ uniformly on compact subsets of $(0,\infty)$. (We write $f_n \to f$ u.c.c. $(0,\infty)$ as a shorthand for the last requirement.) Note that we use probabilist conventions, writing $u(t,x)$, instead of $u(x,t)$. As usual,

$$\dot{u} := \frac{\partial u}{\partial t}, \quad u' := \frac{\partial u}{\partial x}, \quad u'' := \frac{\partial^2 u}{\partial x^2}.$$

Now let U be the space of functions u on $(0,\infty) \times [0,\infty)$ such that u is bounded on $(0,1] \times [0,\infty)$ and whenever $0 < T_1 < T_2 < \infty$ we have

$$u \restriction ([T_1,T_2] \times [0,\infty)) \in C_b^{1,2}([T_1,T_2] \times [0,\infty)).$$

This last statement means that $u, \dot{u}, u'$, and u'' are all bounded continuous functions on $[T_1,T_2] \times [0,\infty)$. Observe that we insist that u is $C^{1,2}$ right up to the boundary where $(t,x) \in (0,\infty) \times \{0\}$.

THEOREM 2.1[12] (i) Let $\mu, \sigma \in \mathbb{R}$ and let $f \in C_b(0,\infty)$. Suppose that $u \in U$ and that u satisfies the PDE

$$\dot{u} = \tfrac{1}{2}u'' + \mu u' \quad on \ (0,\infty)^2, \tag{2.1a}$$

$$\dot{u}(t,0) = \sigma u'(t,0) \quad for \ t \in (0,\infty), \tag{2.1b}$$

$$\lim_{t \searrow 0} u(t,x) = f(x) \quad u.c. \ (0,\infty). \tag{2.1c}$$

Then $f(0) := u(0+,0) := \lim_{t\searrow 0} u(t,0)$ exists, and we note that the extended function f is in F.

(ii) There exists a unique one-parameter semigroup $\{Q_t\}$ of continuous operators on F such that for $f \in F$, $u(t,x) := (Q_t f)(x)$ is the unique element of U solving (2.1) with $u(0+,0) = f(0)$.

It is helpful to think of Q_t as $\exp(t\mathcal{H})$ where

$$\mathcal{H}f = \tfrac{1}{2}f'' + \mu f', \quad \mathcal{D}(\mathcal{H}) = \{f \in C^2 : \tfrac{1}{2}f''(0) + \mu f'(0) = \sigma f'(0)\}.$$

If $\sigma \geq 0$, then $u(\cdot,\cdot) \geq 0$ if and only if $f \geq 0$, so that $\{Q_t\}$ is a semigroup of nonnegative operators on F. If $\sigma < 0$, then the minimal principle is lost, and indeed $\{Q_t\}$ is a semigroup of nonnegative operators only on a certain invariant subspace of F. The precise statements are given in the following

THEOREM 2.2[12] *Assume that $\sigma < 0$. Then*

$$u(\cdot,\cdot) \geq 0 \ \ \textit{if and only if both}$$

$$f \geq 0 \ \ \textit{and} \ \ f(0) \geq \langle J^{min}, f\rangle := \int_{(0,\infty)} J^{min}(x) f(x) dx,$$

where

$$J^{min}(x) = \begin{cases} 2|\sigma|e^{-2|\sigma|x} & \textit{if } \mu \geq \sigma, \\ 2|\sigma|e^{-2|\mu|x} & \textit{if } \mu < \sigma. \end{cases} \tag{2.2}$$

The function J^{min} may be characterized analytically as the minimal solution of the Riccati equation

$$\tfrac{1}{2}J''(x) - \mu J'(x) + \{\tfrac{1}{2}J'(0) - \mu J(0)\}J(x) = 0, \tag{2.3a}$$

$$J(0) = -2\sigma, \tag{2.3b}$$

$$J(\cdot) \geq 0, \quad J(0,\infty) := \int_0^\infty J(x)dx < \infty. \tag{2.3c}$$

The significance of Riccati equation is that for $J \in C_b^2([0,\infty))$ satisfying (2.3c), $\{Q_t : t \geq 0\}$ acts as a nonnegative semigroup on $F^J := \{f \in F : f(0) = \langle J, f\rangle\}$ if and only if J satisfies (2.3a) and (2.3b).

Clearly the $\sigma < 0$ case, where the minimal principle fails, is the most interesting case. In [13], the long-term behavior of solutions was discussed.

THEOREM 2.3[13] *Assume that $\sigma < 0$. Given $f \in F$, set $Df = f(0) - \langle J^{min}, f\rangle$. Then, as $t \nearrow \infty$,*

$$\mu < \sigma \Longrightarrow u_f(t,x) \to \frac{\mu}{\mu-\sigma}Df,$$

$$\mu = \sigma \Longrightarrow t^{-1}u_f(t,x) \to 2\sigma^2 Df,$$

$$\mu > \sigma \Longrightarrow e^{-2|(\mu-\sigma)\sigma|t}u_f(t,x) \to \frac{\mu-2\sigma}{\mu-\sigma}e^{-2(\mu-\sigma)x}Df,$$

the limits being uniform over x in compact subsets of $[0,\infty)$.

1.3 The Generalized Problem

A natural question is what is the picture if one generalize the problems discussed in [12] and [13]. Two generalizations will be discussed in this thesis: one is to add more boundary, the other one is to consider high dimension problem. In the following Chapter 2, we will consider the following two point boundary value problem .

Let F be the space of bounded functions on $[0,1]$ which are continuous on $(0,1)$ but not necessarily continuous at the boundary $\{0,1\}$. Convergence of $\{f_n\}_1^\infty \subseteq F$ to f in F means that $\{\|f_n\|_u\}_1^\infty$ is bounded, $f_n(x) \longrightarrow f(x)$ for each $x \in [0,1]$ and uniformly for x in compact subsets of $(0,1)$.

Now let U be the space of functions $u \in C^{1,2}\big((0,\infty)\times[0,1];\mathbb{R}\big)$ with the properties that u is bounded on $(0,1]\times[0,1]$ and, for each $0 < T_1 < T_2 < \infty$, $\dot{u}$, u' and u'' are bounded on $[T_1,T_2]\times[0,1]$. Note that we are insisting that u be $C^{1,2}$ right up to, and including, the spacial boundary $(0,\infty)\times\{0,1\}$. We consider the following boundary value problem:

$$\begin{aligned} &\dot{u} = \tfrac{1}{2}u'' + \mu u' \quad \text{on } (0,\infty)\times(0,1), \\ &\dot{u}(t,0) = -\sigma u'(t,0) \text{ and } \dot{u}(t,1) = \sigma u'(t,1) \quad \text{for } t \in (0,\infty). \end{aligned} \tag{3.1}$$

In Chapter 2 we will show that there exists one unqiue solution to (1.3) with any initial value $f \in F$. The minimal principle also fails when $\sigma > 0$. To clarify how far this problem is from satisfying the minimum principle, we need to understand the following Riccati system:

$$(R) \qquad \begin{array}{c} \frac{1}{2}J''(x) - \mu J'(x) + B(J)J(x) = 0 \quad \text{on } [0,1] \\ J(0) = \begin{pmatrix} 2\sigma \\ 0 \end{pmatrix} \text{ and } J(1) = \begin{pmatrix} 0 \\ 2\sigma \end{pmatrix} \end{array} .$$

where $J : [0,1] \longrightarrow \mathbb{R}^2$, $\sigma > 0$ and $\mu \in \mathbb{R}$, and

$$B(J) = \begin{pmatrix} -2\mu\sigma + \frac{J_0'(0)}{2} & -\frac{J_0'(1)}{2} \\ \frac{J_1'(0)}{2} & 2\mu\sigma - \frac{J_1'(1)}{2} \end{pmatrix} = \left(2\sigma\mu - \frac{1}{2}(J'(0), J'(1))\right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We will prove that there exists exactly one solution $J^{\sigma,\mu} = \begin{pmatrix} J_0^{\sigma,\mu} \\ J_1^{\sigma,\mu} \end{pmatrix}$ which satisfies

$$(3.2) \qquad \max_{k\in\{0,1\}} \int_0^1 |J_k(x)|\, dx \begin{cases} \le 1 & \text{if } \sigma \ge \mu \coth \mu \\ < 1 & \text{if } \sigma < \mu \coth \mu. \end{cases}$$

and that this solution is non-negative(which means each component is a nonnegative function).

Now we introduce the vector

$$D^{\sigma,\mu} f \equiv \begin{pmatrix} f(0) - \langle f, J_0 \rangle \\ f(1) - \langle f, J_1 \rangle \end{pmatrix} \quad \textit{for } f \in F.$$

There is an intimate connection between the representation of Df in terms of the eigenvectors of $B^{\sigma,\mu}$ and the properties of u_f. Namely, we have the following theorem:

Theorem 3.1 *Assume that $\sigma > 0$, and, for $f \in F$, define*

$$D^{\sigma,\mu} f = \begin{pmatrix} f(0) - \langle f, J_0^{\sigma,\mu} \rangle \\ f(1) - \langle f, J_1^{\sigma,\mu} \rangle \end{pmatrix},$$

where $\langle \varphi, \psi \rangle \equiv \int_0^1 \varphi(x)\psi(x)\, dx$. Then $u_f \ge 0$ if and only if $f \ge 0$ and $D^{\sigma,\mu} f = \alpha V_0^{\sigma,\mu}$ for some $\alpha \ge 0$. Moreover, if $F^{\sigma,\mu}$ denotes the subspace of $f \in F$ with $D^{\sigma,\mu} f = 0$, then $F^{\sigma,\mu}$ is invariant under $\{Q_t : t \ge 0\}$ and the restriction $\{Q_t \restriction F^{\sigma,\mu} : t \ge 0\}$ is a

Markov semigroup which is conservative (i.e., $Q_t\mathbf{1} = \mathbf{1}$) if and only if $\sigma \geq \mu \coth \mu$. Finally, if $f \in F$ and $D^{\sigma,\mu} f = a_0 V_0^{\sigma,\mu} + a_1 V_1^{\sigma,\mu}$, then, uniformly for $x \in [0,1]$

$$a_1 \neq 0 \implies \lim_{t\to\infty} e^{t\lambda_1^{\sigma,\mu}} u_f(t,x) = a_1 g_1^{\sigma,\mu}(x)$$

and

$$a_1 = 0 \neq a_0 \implies \begin{cases} \lim_{t\to\infty} e^{t\lambda_0^{\sigma,\mu}} u_f(t,x) = a_0 g_0^{\sigma,\mu}(x) & \text{if } \sigma > \mu\coth\mu \\ \lim_{t\to\infty} t^{-1} u_f(t,x) = a_0 g_0^{\sigma,\mu}(x) & \text{if } \sigma = \mu\coth\mu \\ \lim_{t\to\infty} u_f(t,x) = a_0 g_0^{\sigma,\mu}(x) & \text{if } \sigma < \mu\coth\mu, \end{cases}$$

where $g_1^{\sigma,\mu}$ takes both strictly positive and strictly negative values whereas $g_0^{\sigma,\mu}$ is always strictly positive and is constant when $\sigma \leq \mu \coth \mu$. (Explicit expressions are given for $g_k^{\sigma,\mu}$, $k \in \{0,1\}$, in Chapter 2).

Theorem 3.1 is in agreement with the guess made in [12] on the basis of the Markov chain situation. Moreover, it gives us some hints for the higher dimension case, where we will give some partial results.

Now we re-define the notation for higher dimension case. Let F be the space of bounded functions on $E := [0,+\infty) \times (-\infty,+\infty)$ that are continuous on $E^+ := (0,+\infty) \times (-\infty,+\infty)$ but not necessarily continuous at the boundary $E^- = \{0\} \times (-\infty,+\infty)$. Convergence of f_n to f in F means that $\{\|f_n\|_u\}_1^\infty$ is bounded, $f_n(x,y) \longrightarrow f(x,y)$ for each $(x,y) \in E$, and uniformly for (x,y) in compact subsets of E^+. Now, for $T > 0$, let U_T be the space of functions u on $(0,T) \times E$ such that u is bounded on $(0,T/2] \times E$ and whenever $0 < T_1 < T_2 < T$ we have

$$u \restriction ([T_1,T_2] \times E) \in C_b^{1,2}([T_1,T_2] \times E).$$

We let $U_\infty = \bigcap_{T\in\mathbb{R}^+} U_T$.

As usual,

$$\dot{u} := \frac{\partial u}{\partial t},\ u_x' := \frac{\partial u}{\partial x},\ u_{xx}'' := \frac{\partial^2 u}{\partial x^2},\ u_y' := \frac{\partial u}{\partial y},\ u_{yy}'' := \frac{\partial^2 u}{\partial y^2},\ \triangle u = u_{xx}'' + u_{yy}''.$$

We consider the following parabolic equation:

(3.3) $$\dot{u} = \tfrac{1}{2}\triangle u + \mu_0 u_x' + \mu_1 u_y' \quad \text{on } (0,\Theta) \times E^+,$$

$$\dot{u}(t,0,y) = \sigma u'(t,0,y) \quad for\ t \in (0,\Theta), y \in (-\infty,+\infty),$$

$$\lim_{t\searrow 0} u(t,x,y) = f(x,y) \quad u.c.\ E^+.\ and\ \lim_{t\searrow 0} u(t,0,y) = f(0,y)$$

To state our results, we need to introduce nonnegative finite measures $\{J(x,\cdot)\}$ by $J(0,\cdot) = -2\sigma\delta_0(\cdot)$ and $J(x,dy) = J^{\sigma,\mu_1,\mu_2}(x,y)dy$ for $x > 0$, where $J^{\sigma,\mu_0,\mu_1}(x,y) = \frac{-2\sigma x}{2\pi} e^{(\mu_0+\sigma)x-2\mu_1 y} \int_0^\infty \exp\{-\frac{[(\sigma-\mu_0)^2+2\mu_1^2]}{\xi} - \frac{(x^2+y^2)\xi}{2}\}d\xi$.
In fact, $x \in [0,\infty) \longrightarrow J(x,\cdot)$ is a convolution semigroup of finite measures over $\mathbb{R}$.

Let $Df = f(0,y) - \int_0^\infty J_p(x,\cdot) * f(x,\cdot)dx$. Suppose $u_f(t,x,y)$ is the solution of our PDE with initial data f, then we can get the following important equation:

$$Df = \frac{1}{-2\sigma} e^{2t\sigma\mu_0} J_p(-\sigma t,\cdot) * Du_f(t,\cdot). \tag{3.5}$$

By subordination, it is equivalent to:

$$Df(\cdot) = e^{t\sigma(-\sigma+\mu_0+K)}(Du_f(t,\cdot)).$$

where $K = \sqrt{-\partial_y^2 + 2\mu_1\partial_y + (\sigma+\mu_0)^2}$ in a sense we will explained in Chapter 3. From this equation, we can observe and show that the existence of the solution of our PDE depends on the behavior of Df. The solution may only exist in finite time, which is quite different from the one dimension case. We give the following necessary and sufficient condition for the existence and uniqueness in finite time.

THEOREM 3.2 *If $\sigma < 0$, for $\Theta \in \mathbb{R}^+, f \in F$, the following statements are equivalent:*
(i) There is a unique $u_f \in U_\Theta$ which satisfies (1.1) with $u(0+,0,y) = f(0,y)$ for $y \in (-\infty,+\infty)$.

(ii) There exists $h \in C_b(\mathbb{R})$ such that $Df = \tilde{J}(\Theta,\cdot) * h(\cdot)$, where

$$Df(y) = f(0,y) - \int_0^{+\infty} f(x,\cdot) * J(x,\cdot)dx,$$

$\{\frac{1}{-2\sigma}J(x,\cdot) : x \in [0,\infty)\}$ is a convolution semigroup given by: $J(0,\cdot) = -2\sigma\delta_0(\cdot)$ and

$J(x,dy) = J^{\sigma,\mu_1,\mu_2}(x,y)dy$ *for* $x > 0$,

$$J^{\sigma,\mu_0,\mu_1}(x,y) = \frac{-2\sigma x}{2\pi} e^{(\mu_0+\sigma)x - 2\mu_1 y} \int_0^\infty \exp\{-\tfrac{[(\sigma-\mu_0)^2+2\mu_1^2]}{\xi} - \tfrac{(x^2+y^2)\xi}{2}\} d\xi,$$

and $\tilde{J}(x,\cdot) = \frac{1}{-2\sigma} e^{2x\sigma\mu_0} J(-\sigma x,\cdot)$.

(iii) There exists $v(t,y) \in C([\sigma\Theta,\infty)\times\mathbb{R}) \bigcap C^\infty((\sigma\Theta,\infty)\times\mathbb{R})$ *satisfies:*

$$\partial_t^2 v(t,y) + 2(\mu_0-\sigma)\partial_t v(t,y) + \partial_y^2 v(t,y) - 2\mu_1\partial_y v(t,y) = 0, \tag{1.2}$$

$$with\ v(0,\cdot) = Df(\cdot),\ \sup_{t\geq\sigma\Theta} \|v(t,\cdot)\|_u < \infty,$$

$$\lim_{t\to\infty} \|v(t,\cdot)\|_u = 0 \quad if\ \mu_0 > \sigma.$$

In particular, if $Df \in H^p(\sigma\Theta, -\sigma\Theta)$, *for some* $p \in [1,2]$, *then for any* $0 < T < \Theta$, *there is a unique* $u_f \in U_T$ *which satisfies (1.1) with* $u(0+,0,y) = f(0,y)$ *for* $y \in (-\infty,+\infty)$.

Thus we have the following corollary for infinite time:

COROLLARY 3.2 *If* $\sigma < 0$, *for* $f \in F$, *the following statements are equivalent:*

(i) There is a unique $u_f \in U_\infty$ *which satisfies (1.1) with* $u(0+,0,y) = f(0,y)$ *for* $y \in (-\infty,+\infty)$

(ii) For any $\Theta \in (0,\infty)$, *there exists* $h_\Theta \in C_b(\mathbb{R})$ *such that* $Df = \tilde{J}(\Theta,\cdot) * h_\Theta(\cdot)$.

(iii) There exists $v(t,y) \in C^\infty(\mathbb{R}^2)$ *satisfies:*

$$\partial_t^2 v(t,y) + 2(\mu_0-\sigma)\partial_t v(t,y) + \partial_y^2 v(t,y) - 2\mu_1\partial_y v(t,y) = 0,$$

$$with\ v(0,\cdot) = Df(\cdot),\ \sup_{t\geq T} \|v(t,\cdot)\|_u < \infty\ for\ any\ T \in \mathbb{R},$$

$$\lim_{t\to\infty} \|v(t,\cdot)\|_u = 0 \quad if\ \mu_0 > \sigma.$$

In particular, if $Df \in H^p(\mathbb{R})$, *for some* $p \in [1,2]$, *there is a unique* $u_f \in U_\infty$ *which satisfies (1.1) with* $u(0+,0,y) = f(0,y)$ *for* $y \in (-\infty,+\infty)$.

The following criterion for non-negativity is obtained:

THEOREM 3.3 *Assume that $\sigma < 0$, and $u \in U_\infty$ satisfies our parabolic equation with $u(0+, x, y) = f(x, y)$ for $(x, y) \in E^+$. Then*

$$u(\cdot, \cdot, \cdot) \geq 0$$

if and only if

$$f(\cdot, \cdot) \geq 0 \text{ and } Df(y) \equiv C \text{ for some constant } C \in [0, \infty),$$

where

$$Df(y) = f(0, y) - \int_0^{+\infty} f(x, \cdot) * J(x, \cdot) dx.$$

The proof of this theorem will rely on the equation (3.5) and a representation formula for positive solutions to $\triangle u - u = 0$ in $\mathbb{R}^2$, proved in [4].

Chapter 2

A Two Point Boundary Problem

2.1 Introduction

In this chapter[1], we continue the study, started in [12] and [13], of a diffusion equation in one dimension with a boundary condition for which the minimum principle fails. The main distinction between the situation here and the one studied earlier is that we are now dealing with a problem in which there are two boundary points, not just one, and the addition of the second boundary point introduces some new phenomena which we find interesting.

Although the relationship is not immediate apparent, related considerations appear in [7] and [8].

2.1.1 The problem and a basic result

Let F be the space of bounded functions on $[0, 1]$ which are continuous on $(0, 1)$ but not necessarily continuous at the boundary $\{0, 1\}$. Convergence of $\{f_n\}_1^\infty \subseteq F$ to f in F means that $\{\|f_n\|_{\mathrm{u}}\}_1^\infty$ is bounded, $f_n(x) \longrightarrow f(x)$ for each $x \in [0, 1]$ and uniformly for x in compact subsets of $(0, 1)$.

In the next definition, and hereafter, we use the probabilistic convention of writing $u(t, x)$ where analysts would use $u(x, t)$. As usual,

$$\dot{u} \equiv \frac{\partial u}{\partial t}, \quad u' \equiv \frac{\partial u}{\partial x}, \quad \text{and } u'' \equiv \frac{\partial^2 u}{\partial x^2}.$$

Now let U be the space of functions $u \in C^{1,2}\big((0, \infty) \times [0, 1]; \mathbb{R}\big)$ with the properties that u is bounded on $(0, 1] \times [0, 1]$ and, for each $0 < T_1 < T_2 < \infty$, $\dot{u}$, u' and u'' are

[1]Chapter 2 has the same content as the paper [14], which is a joint work with Daniel Stroock.

bounded on $[T_1, T_2] \times [0,1]$ Note that we are insisting that u be $C^{1,2}$ right up to, and including, the spacial boundary $(0,\infty) \times \{0,1\}$.

Because its proof is more easily understood after seeing the proofs of the other results in this article, we have put the derivation of the following basic existence and uniqueness statement into an appendix at the end of this article.

Theorem 1.1 *Let $(\mu,\sigma) \in \mathbb{R}^2$ be given.*

(i) *Suppose that $u \in U$ satisfies*

$$\begin{gathered} \dot u = \tfrac{1}{2}u'' + \mu u' \quad \text{on } (0,\infty)\times(0,1), \\ \dot u(t,0) = -\sigma u'(t,0) \text{ and } \dot u(t,1) = \sigma u'(t,1) \quad \text{for } t \in (0,\infty). \end{gathered} \tag{1.2}$$

If, as $t \searrow 0$, $u(t,\cdot)$ converges uniformly on compact subsets of $(0,1)$, then both $u(t,0)$ and $u(t,1)$ converge as $t \searrow 0$, and so $u(t,\cdot)$ converges in F.

(ii) *Given $f \in F$, there is a unique $u_f \in U$ which satisfies* (1.2) *and the initial condition that, as $t \searrow 0$, $u(t,\cdot)$ converges to f in F.*
In particular, if $Q_t f \equiv u_f(t,\cdot)$, then $\{Q_t : t \geq 0\}$ is a semigroup of bounded, continuous operators on F. (See (3.2) *below for more information.)*

For semigroup enthusiasts, it may be helpful to think of the operator Q_t as $\exp(t\mathcal{H})$ where $\mathcal{H}f = \frac{1}{2}f'' + \mu f'$ with domain

$$\operatorname{dom}(\mathcal{H}) = \left\{ f \in C^2\big([0,1];\mathbb{R}\big) : \tfrac{1}{2}f''(k) + \mu f'(k) = (-1)^{1-k}\sigma f'(k) \text{ for } k \in \{0,1\} \right\}.$$

For probabilists, it may be helpful to remark that, unless $\sigma \leq 0$, $\{Q_t : t \geq 0\}$ is *not* a Markov semigroup.

2.1.2 Non-negativity and growth of solutions

If $\sigma \leq 0$, then $u_f(\cdot,\cdot) \geq 0$ if and only if $f \geq 0$, and therefore $\{Q_t : t \geq 0\}$ is a Markov (i.e., non-negativity preserving) semigroup. This may be proved by either an elementary minimum principle argument or the well-known probabilistic model.[2] However, when $\sigma > 0$, the minimum principle is lost, and, as a consequence $\{Q_t : t \geq 0\}$ is no longer Markov. Nonetheless, we will show that there is a certain $\{Q_t : t \geq 0\}$-invariant subspace of F on which the Q_t's do preserve non-negativity. In order to describe this subspace, we need the following.

[2]The corresponding diffusion is Brownian motion in $(0,1)$ with drift μ which, depending on whether $\sigma = 0$ or $\sigma < 0$, is either absorbed when it hits $\{0,1\}$ or has a "sticky" reflection there.

Theorem 1.3 *Given a continuously differentiable function $J : [0,1] \longrightarrow \mathbb{R}^2$, set*

$$B(J) = \begin{pmatrix} -2\mu\sigma + \frac{J_0'(0)}{2} & -\frac{J_0'(1)}{2} \\ \frac{J_1'(0)}{2} & 2\mu\sigma - \frac{J_1'(1)}{2} \end{pmatrix} = \left(2\sigma\mu - \frac{1}{2}(J'(0), J'(1))\right)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for each $\sigma > 0$ and $\mu \in \mathbb{R}$, there exist a unique solution $J^{\sigma,\mu}$ to

$$(R) \qquad \begin{gathered} \tfrac{1}{2}J''(x) - \mu J'(x) + B(J)J(x) = 0 \quad \text{on } [0,1] \\ J(0) = \begin{pmatrix} 2\sigma \\ 0 \end{pmatrix} \text{ and } J(1) = \begin{pmatrix} 0 \\ 2\sigma \end{pmatrix} \end{gathered}.$$

which satisfies

$$(1.4) \qquad \max_{k\in\{0,1\}} \int_0^1 |J_k(x)|\,dx \begin{cases} \le 1 & \text{if } \sigma \ge \mu\coth\mu \\ < 1 & \text{if } \sigma < \mu\coth\mu. \end{cases}$$

Moreover, $J^{\sigma,\mu} \ge 0$ in the sense that both of its components are non-negative. Finally, set $B^{\sigma,\mu} = B(J^{\sigma,\mu})$. Then $B^{\sigma,\mu}$ has real eigenvalues $\lambda_1^{\sigma,\mu} < \lambda_0^{\sigma,\mu} \le 0$, $\lambda_0^{\sigma,\mu} < 0$ if and only if $\sigma > \mu\coth\mu$, and the corresponding eigenvector $V_0^{\sigma,\mu}$ can be chosen to be strictly positive with $(V_0^{\sigma,\mu})_0 + (V_0^{\sigma,\mu})_1 = 1$, whereas the eigenvector $V_1^{\sigma,\mu}$ corresponding to $\lambda_1^{\sigma,\mu}$ can be chosen so that $(V_1^{\sigma,\mu})_0 > 0 > (V_1^{\sigma,\mu})_1$ and $(V_1^{\sigma,\mu})_0 - (V_1^{\sigma,\mu})_1 = 1$. (See Lemmas 2.9 and 2.10 below for more information.)

Referring to the quantities in Theorem 1.3, we have the following. When $\mu = 0$, some of the same conclusions were obtained in [17] using an entirely different approach, one which is based on the use of an inner product which is not definite. Also, the criterion given below for non-negativity is analogous to, but somewhat more involved, than the one given in [12], where the same sort of problem is considered on half line $[0,\infty)$,

Theorem 1.5 *Assume that $\sigma > 0$, and, for $f \in F$, define*

$$D^{\sigma,\mu}f = \begin{pmatrix} f(0) - \langle f, J_0^{\sigma,\mu}\rangle \\ f(1) - \langle f, J_1^{\sigma,\mu}\rangle \end{pmatrix},$$

where $\langle \varphi, \psi\rangle \equiv \int_0^1 \varphi(x)\psi(x)\,dx$. Then $u_f \ge 0$ if and only if $f \ge 0$ and $D^{\sigma,\mu}f = \alpha V_0^{\sigma,\mu}$ for some $\alpha \ge 0$. Moreover, if $F^{\sigma,\mu}$ denotes the subspace of $f \in F$ with $D^{\sigma,\mu}f = 0$, then $F^{\sigma,\mu}$ is invariant under $\{Q_t : t \ge 0\}$ and the restriction $\{Q_t \upharpoonright F^{\sigma,\mu} : t \ge 0\}$ is a Markov semigroup which is conservative (i.e., $Q_t\mathbf{1} = \mathbf{1}$) if and only if $\sigma \ge \mu\coth\mu$.

Finally, if $f \in F$ and $D^{\sigma,\mu} f = a_0 V_0^{\sigma,\mu} + a_1 V_1^{\sigma,\mu}$, then, uniformly for $x \in [0,1]$

$$(1.6) \qquad a_1 \neq 0 \implies \lim_{t\to\infty} e^{t\lambda_1^{\sigma,\mu}} u_f(t,x) = a_1 g_1^{\sigma,\mu}(x)$$

and

$$(1.7) \qquad a_1 = 0 \neq a_0 \implies \begin{cases} \lim_{t\to\infty} e^{t\lambda_0^{\sigma,\mu}} u_f(t,x) = a_0 g_0^{\sigma,\mu}(x) & \text{if } \sigma > \mu \coth \mu \\ \lim_{t\to\infty} t^{-1} u_f(t,x) = a_0 g_0^{\sigma,\mu}(x) & \text{if } \sigma = \mu \coth \mu \\ \lim_{t\to\infty} u_f(t,x) = a_0 g_0^{\sigma,\mu}(x) & \text{if } \sigma < \mu \coth \mu, \end{cases}$$

where $g_1^{\sigma,\mu}$ takes both strictly positive and strictly negative values whereas $g_0^{\sigma,\mu}$ is always strictly positive and is constant when $\sigma \leq \mu \coth \mu$. (Explicit expressions are given for $g_k^{\sigma,\mu}$, $k \in \{0,1\}$, in section 3 below).

Remark: It should be mentioned that the Harnack principle discussed in §5 of [13] transfers immediately to the setting here. Namely, if u is a non-negative solution to $\dot{u} = \frac{1}{2}u'' + \mu u'$ in a region of the form $[T_1, T_2] \times [0, R]$ and $\dot{u}(t, 0) = -\sigma u'(t, 0)$ for $t \in [T_1, T_2]$, then, for each $T_1 < t_1 < t_2 < T_2$ and $0 < r < R$, there is a constant $C < \infty$ such that $u(s, x) \leq Cu(t, y)$ for all (s, x), $(t, y) \in [t_1, t_2] \times [0, r]$, and an analogous result holds when the region is of the form $[T_1, T_2] \times [R, 1]$. The surprising aspect of this Harnack principle is that, because of the boundary condition, one can control $u(s, x)$ in terms of $u(t, y)$ even when $s \geq t$, whereas usual Harnack principles for non-negative solutions to parabolic equations give control only when $s < t$.

2.1.3 The Basic Probabilistic Model

The necessary stochastic calculus may be found, for example, in Revuz and Yor [6] or Rogers and Williams [9]. In particular, the second of these also contains the relevant "Markovian" results.

The probabilistic model associated with our boundary value problem can be described as follows. First, let X be Brownian motion with drift μ and reflection at the boundary $\{0, 1\}$. That is, if B a standard Brownian motion, then one description of X is as the solution to the Skorohod stochastic integral equation

$$0 \leq X_t = X_0 + B_t + \mu t + (L_0)_t - (L_1)_t \leq 1,$$

where L_0 and L_1 are the "local times" of X at 0 and 1, respectively. In particular, for $k \in \{0, 1\}$, $t \rightsquigarrow (L_k)_t$ is non-decreasing and increases only on $\{t : X_t = k\}$. Next, set

$$\Phi_t \equiv t - \sigma^{-1}(L_0)_t - \sigma^{-1}(L_1)_t, \quad \zeta_t \equiv \inf\{\tau > 0 : \Phi_\tau > t\} \\ \text{and } Y_t \equiv X(\zeta_t). \tag{1.8}$$

When $\sigma = 0$, the interpretation of ζ_t is that it is equal $t \wedge \inf\{\tau \geq 0 : X_\tau \in \{0,1\}\}$, and so Y is absorbed at the first time it leaves $(0,1)$. When $\sigma < 0$, Y is Brownian motion in $(0,1)$ with drift μ and a "sticky" (i.e., it spends positive time) reflection at $\{0,1\}$. When $\sigma > 0$, ζ_t may be infinite, in which case we send Y_t to a "graveyard" ∂ (i.e., an absorbing state outside of $[0,1]$).

The connection between (1.2) and these processes is that, for each $f \in F$ and $T \geq 0$, an application of standard Itô calculus shows that (note that $X_0 \in \{0,1\}$ & $\sigma > 0 \implies \zeta_0 > 0$ a.s.)

$$u_f\big(T - \Phi_t,\, X_t\big) \in \mathbb{R} \text{ is a continuous local martingale in } t. \tag{1.9}$$

In particular,

$$u_f \text{ bounded and } \mathbb{P}\left(\zeta_T = \infty \implies \lim_{t \nearrow \zeta_T} u_f\big(T - \Phi_t, X_t\big) = 0 \,\middle|\, X_0 = x\right) = 1 \\ \implies u_f(T,x) = \mathbb{E}\big[f(Y_T),\, \zeta_T < \infty \,\big|\, X_0 = x\big]. \tag{1.10}$$

Similarly,

$$u_f \geq 0 \implies u_f(T,x) \geq \mathbb{E}\big[f(Y_T),\, \zeta_T < \infty \,\big|\, X_0 = x\big]. \tag{1.11}$$

Remark: It should be emphasized that, although the process Y is a familiar, continuous diffusion when $\sigma \leq 0$, it is discontinuous when $\sigma > 0$. Indeed, when $\sigma > 0$, although Y behaves just like X as long as it stays away from $\{0,1\}$, upon approaching $\{0,1\}$, Y either jumps back inside or gets sent to ∂. In particular, even though it is right-continuous and has left-limits, Y is *not* a Hunt process because its jump times are totally accessible.

In order to make the connection between Y and the functions $J_k^{\sigma,\mu}$ in Theorem 1.3, we will need the following lemma about the behavior of Φ_t as $t \to \infty$.

Lemma 1.12 *Assume that* $\sigma > 0$ *and take* $\mu \coth \mu = 1$ *when* $\mu = 0$. *Then, al-*

most surely,

$$(1.13)\qquad \lim_{t\to\infty} \Phi_t = \begin{cases} \infty & \textit{if } \sigma > \mu\coth\mu \\ -\infty & \textit{if } \sigma < \mu\coth\mu \end{cases}$$

and

$$(1.14)\qquad \sigma = \mu\coth\mu \implies \overline{\lim_{t\to\infty}} \pm\Phi_t = \infty.$$

In particular, for all $T \geq 0$, $\sigma \geq \mu\coth\mu \implies \zeta_T < \infty$ *a.s. and* $\sigma < \mu\coth\mu \implies \lim_{t\to\infty}\Phi_t = -\infty$ *a.s. on* $\{\zeta_T = \infty\}$.

Proof Assume that $\mu \neq 0$, and set

$$\psi(x) = -\left(x + \frac{e^{-2\mu x}}{\mu(1+e^{-2\mu})}\right)\coth\mu.$$

Then, $\frac{1}{2}\psi'' + \mu\psi' = -\mu\coth\mu$ and $\psi'(0) = 1 = -\psi'(1)$, and so, by Itô's formula,

$$M_t \equiv \int_0^t \psi'(X_\tau)\,dB_\tau = \psi(X_t) + (\mu\coth\mu)t - (L_0)_t - (L_1)_t = \psi(X_t) - (\sigma - \mu\coth\mu)t + \sigma\Phi_t.$$

Since $\lim_{t\to\infty} t^{-1}|M_t| = 0$ a.s., this proves that

$$\lim_{t\to\infty}\frac{\Phi_t}{t} = 1 - \frac{\mu\coth\mu}{\sigma} \quad \text{a.s.,}$$

which completes the proof of (13) when $\mu \neq 0$ and $\sigma \neq \mu\coth\mu$. In addition, when $\mu \neq 0$ and $\sigma = \mu\coth\mu$, the preceding says that $\psi(X_t) + \sigma\Phi_t = M_t$, and so the desired result will follow once we check that $\overline{\lim}_{t\to\infty} \pm M_t = \infty$ a.s, which, in turn, comes down to showing that $\int_0^\infty \psi'(X_\tau)^2\,d\tau = \infty$ a.s. But, by standard ergodic theoretic considerations,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi'(X_\tau)^2\,d\tau = \int_{(0,1)} \psi'(y)^2\,\nu(dy) > 0, \quad \text{where } \nu(dy) = \frac{2\mu e^{2\mu y}}{e^{2\mu}-1}\,dy$$

is the stationary measure for X. Thus, the case when $\mu \neq 0$ is complete. The case $\mu = 0$ can be handled in the same way by considering the function $\psi(x) = x(1-x)$. □

As a consequence of Lemma 1.12, we can now make the connection alluded to above.

Theorem 1.15 *Assume that $\sigma > 0$. For all bounded, measurable $\varphi : (0,1) \longrightarrow \mathbb{R}$,*

$$\mathbb{E}\big[\varphi(X_{\zeta_0}),\, \zeta_0 < \infty \,|\, X_0 = k\big] = \langle \varphi, J_k^{\sigma,\mu} \rangle, \quad k \in \{0,1\}. \tag{1.16}$$

In particular, $\mathbb{P}(\zeta_0 < \infty \,|\, X_0 = k) = \langle 1, J_k^{\sigma,\mu} \rangle$ and $\frac{J_k^{\sigma,\mu}}{\langle 1, J_k^{\sigma,\mu} \rangle}$ is the density for the distribution of $Y_0 = X_{\zeta_0}$ given that $X_0 = k$ and $\zeta_0 < \infty$.

Proof Clearly, it suffices to treat the case when φ is continuous as well as bounded. Given such a φ, define $f \in F$ so that $f \restriction (0,1) = \varphi$ and $f(k) = \langle \varphi, J_k^{\sigma,\mu} \rangle$ for $k \in \{0,1\}$. Then, by Theorem 1.5, u_f is bounded and, as $t \to \infty$, $u_f(t,x) \longrightarrow 0$ uniformly for $x \in [0,1]$ when $\sigma < \mu \coth \mu$. Hence, by Lemma 1.12 and (1.10),

$$\langle \varphi, J_k^{\sigma,\mu} \rangle = f(k) = \mathbb{E}\big[\varphi(X_{\zeta_0}),\, \zeta_0 < \infty \big| X_0 = k\big]. \quad \square$$

2.2 The Riccati Equation

In this section we will prove Theorem 1.3 and the connection between solutions to (R) and solutions to (1.2). Throughout, we assume that $\sigma > 0$.

2.2.1 Uniqueness of solutions to (R)

Theorem 2.1 *Suppose that $J \in C^2\big([0,1];\mathbb{R}^2\big)$ is a solution to* (R)*, and define $B(J)$ accordingly, as in Theorem 1.2. Next, for $f \in F$, set*

$$D^J f \equiv \begin{pmatrix} f(0) - \langle f, J_0 \rangle \\ f(1) - \langle f, J_1 \rangle \end{pmatrix}.$$

Then, for any $f \in F$, $D^J u_f(t) = e^{-tB(J)} D^J f$, and so $D^J f = 0 \implies D^J u_f(t) = 0$ for all $t \geq 0$. In particular, if $m(J) \equiv \int_0^1 |J_0(x)|\, dx \vee \int_0^1 |J_1(x)|\, dx \leq 1$, then $D^J f = 0$ implies that $\|u_f\|_u \leq \|f\|_u$, and, if $m(J) < 1$, then $D^J f = 0$ implies $\|u_f(t)\|_u \longrightarrow 0$ as $t \to \infty$. Finally, if $J \geq 0$, then for any non-negative $f \in F$ with the property that $D^J f$ is a non-negative eigenvector of $B(J)$, $u_f \geq 0$.

Proof If J is any solution to (R), then,

$$\begin{aligned}
\frac{d}{dt}\langle u_f(t), J\rangle &= \big\langle \tfrac{1}{2}u_f''(t) + \mu u_f', J\big\rangle \\
&= \big\langle u_f(t), \tfrac{1}{2}J'' - \mu J\big\rangle + \tfrac{1}{2}\big(u_f'(t,1)J(1) - u_f'(t,0)J(0)\big) \\
&\quad - \tfrac{1}{2}\big(u_f(t,1)J'(1) - u_f(t,0)J'(0)\big) + \mu\big(u_f(t,1)J(1) - u_f(t,0)J(0)\big) \\
&= -B(J)\langle u_f(t), J\rangle + \frac{d}{dt}\begin{pmatrix} u_f(t,0) \\ u_f(t,1)\end{pmatrix} + B(J)\begin{pmatrix} u_f(t,0) \\ u_f(t,1)\end{pmatrix},
\end{aligned}$$

and so $\frac{d}{dt}D^J u_f(t) = -B(J)D^J u_f(t)$, which is equivalent to $D^J u_f(t) = e^{-tB(J)}D^J f$. Now assume that $m(J) \le 1$ and that $D^J f = 0$. To see that $\|u_f\|_{\mathrm{u}} \le \|f\|_{\mathrm{u}}$, let $\epsilon > 0$ be given and suppose that $\|u_f(t)\|_{\mathrm{u}} \ge \|f\|_{\mathrm{u}} + \epsilon$ for some $t \ge 0$. We can then find a $T > 0$ such that $\|u_f(T)\|_{\mathrm{u}} = \|f\|_{\mathrm{u}} + \epsilon > \|u_f(t)\|_{\mathrm{u}}$ for $0 \le t < T$. Clearly, there exists an $x \in [0,1]$ for which $|u_f(T,x)| = \|f\| + \epsilon$. If $x \in (0,1)$, then, by the strong maximum principle for the parabolic operator $\partial_t - \frac{1}{2}\partial_x^2 - \mu\partial_x$, $|u_f|$ must be constantly equal to $\|f\|_{\mathrm{u}} + \epsilon$ on $(0,T) \times (0,1)$, which is obviously impossible. Thus, it remains to check that x can always be chosen from $(0,1)$. To this end, simply note that if $|u_f(T,x)| < \|f\|_{\mathrm{u}} + \epsilon$ for all $x \in (0,1)$, then, for $k \in \{0,1\}$, $|u_f(T,k)| = |\langle u_f(T), J_k\rangle| < \|f\|_{\mathrm{u}} + \epsilon$ also.

Next assume that $m(J) < 1$ and that $D^J f = 0$. To see that $\|u_f(t)\|_{\mathrm{u}} \longrightarrow 0$ as $t \to \infty$, it suffices to show that $\|u_f(1)\|_{\mathrm{u}} \le \theta\|f\|_{\mathrm{u}}$ for some $\theta \in (0,1)$ which is independent of f. Indeed, by the semigroup property and the fact that $D^J u_f(t) = 0$ for all $t \ge 0$, one would then know that $\|u_f(t)\|_{\mathrm{u}} \le \theta^n\|f\|_{\mathrm{u}}$ for $t \ge n$. To produce such a θ, let ρ denote that first time that the process X leaves $(0,1)$. Then

$$u_f(1,x) = \mathbb{E}\big[f(X_1),\, \rho > 1 \,\big|\, X_0 = x\big] + \mathbb{E}\big[u_f(1-\rho, X_\rho),\, \rho \le 1 \,\big|\, X_0 = x\big].$$

Because $\|u_f\|_{\mathrm{u}} \le \|f\|_{\mathrm{u}}$ and $|u_f(t,k)| = |\langle u_f(t,\cdot), J_k\rangle| \le m(J)\|f\|_{\mathrm{u}}$, this leads to $\|u_f(1)\|_{\mathrm{u}} \le \theta\|f\|_{\mathrm{u}}$ with $\theta = 1 - \eta\big(1 - m(J)\big)$, where $\eta = \inf_{x\in[0,1]}\mathbb{P}(\rho \le 1 \,|\, X_0 = x) > 0$.

Finally, assume that $J \ge 0$ and that $D^J f$ is a non-negative eigenvector for $B(J)$. If $f > 0$ and u_f ever becomes negative, then there exists a $T > 0$ such that $u_f(t) > 0$ for $t \in [0,T)$ and $u_f(T,x) = 0$ for some $x \in [0,1]$. Again, from the strong maximum principle, we get a contradiction if $x \in (0,1)$. At the same time, because $u_f(T,k) \ge \langle u_f(T), J_k\rangle$ for $k \in \{0,1\}$, we see that the only way that $u_f(T)$ can vanish somewhere on $[0,1]$ is if vanishes somewhere on $(0,1)$. Thus, when $f>0$, $u_f \geq 0$. To handle the case when $f \geq 0$, define $g \in F$ so that $g=1$ in $(0,1)$ and $g(k) = \langle \mathbf{1}, J_k \rangle$ for $k \in \{0,1\}$. Next, apply the preceding result to see that $u_f + \epsilon u_g = u_{f+\epsilon g} \geq 0$ for all $\epsilon > 0$, and conclude that $u_f \geq 0$. □

Corollary 2.2 *Let J be a solution to* (R) *which satisfies* (1.4). *Then*

$$\langle f, J_k \rangle = \mathbb{E}[f(X_{\zeta_0}), \zeta_0 < \infty \,|\, X_0 = k] \quad \text{for } f \in F \text{ and } k \in \{0,1\}$$

if either $\sigma \geq \mu \coth \mu$ and (cf. the notation in Theorem 2.1) $m(J) \leq 1$ or $\sigma < \mu \coth \mu$ and $m(J) < 1$. In particular, in each of these cases, there is at most one such J, that J must be non-negative, and $\langle \mathbf{1}, J_k \rangle = \mathbb{P}(\zeta_0 < \infty \,|\, X = k)$ for $k \in \{1,2\}$.

Proof Given the results in Theorem 2.1, there is no difference between the proof of this result and the proof given earlier of Theorem 1.15. □

By combining Theorems 1.15 and 2.1 with (1.11), we have a proof of the first assertion in Theorem 1.5. Namely, if $u_f \geq 0$, then (1.11) says that $f(k) \geq \mathbb{E}[f(X_{\zeta_0}), \zeta_0 < \infty \,|\, X_0 = k]$ and Theorem 1.15 says that $\mathbb{E}[f(X_{\zeta_0}), \zeta_0 < \infty \,|\, X_0 = k] = \langle f, J_k^{\sigma,\mu} \rangle$. Hence, we now know that $u_f \geq 0 \implies D^{\sigma,\mu} f \geq 0$, and, by the semigroup property, this self-improves to $u_f \geq 0 \implies D^{\sigma,\mu} u_f(t) \geq 0$ for all $t \geq 0$. Now suppose (cf. Theorems 1.3 and 1.5) that $D^{\sigma,\mu} f = a_0 V_0 + a_1 V_1$. Then, by Theorem 2.1, $D^{\sigma,\mu} u_f(t) = a_0 e^{-\lambda_0^{\sigma,\mu} t} V_0 + a_1 e^{-\lambda_1^{\sigma,\mu} t} V_1$. Thus, if $a_1 \neq 0$, then the ratio of the components of $D^{\sigma,\mu} u_f(t)$ is negative for sufficiently large $t > 0$, and so $a_1 = 0$ if $u_f \geq 0$. Hence, $u_f \geq 0 \implies 0 \leq D^{\sigma,\mu} f = a_0 V_0$ and therefore that $a_0 \geq 0$.

2.2.2 Existence of Solution to (R)

In order to find solutions to (R), we will first look for solutions to

$$\text{(2.3)} \qquad \frac{1}{2} J'' - \mu J' + BJ = 0 \quad \text{with } J(0) = \begin{pmatrix} 2\sigma \\ 0 \end{pmatrix} \text{ and } J(1) = \begin{pmatrix} 0 \\ 2\sigma \end{pmatrix}$$

for any non-singular matrix B, and we will then see how to choose B so that $B = B(J)$. For this purpose, set $\Omega = \sqrt{\mu^2 - 2B}$[3] and

$$J(x) = 2\sigma e^{\mu x}\left[\frac{\sinh(1-x)\Omega}{\sinh\Omega}\begin{pmatrix}1\\0\end{pmatrix} + e^{-\mu}\frac{\sinh x\Omega}{\sinh\Omega}\begin{pmatrix}0\\1\end{pmatrix}\right], \tag{2.4}$$

where we take $\frac{\sinh x\omega}{\sinh\omega} \equiv x$ when $\omega = 0$. It is clear that the J in (2.4) solves (2.3). In addition,

$$B(J) = \sigma\left[\mu\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} - \Omega\coth\Omega + \frac{\Omega}{\sinh\Omega}\begin{pmatrix}0 & e^{\mu}\\ e^{-\mu} & 0\end{pmatrix}\right].$$

Hence, we are looking for B's such that the corresponding Ω satisfies

$$\frac{\mu^2 I - \Omega^2}{2\sigma} = \mu\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} - \Omega\coth\Omega + \frac{\Omega}{\sinh\Omega}\begin{pmatrix}0 & e^{\mu}\\ e^{-\mu} & 0\end{pmatrix}. \tag{2.5}$$

To solve (2.5), suppose that $W = (w_0, w_1)$ is a left eigenvector of Ω with eigenvalue ω. Then

$$\begin{aligned}\frac{\mu^2-\omega^2}{2\sigma}w_0 &= -(\mu + \omega\coth\omega)w_0 + \frac{e^{-\mu}\omega}{\sinh\omega}w_1\\ \frac{\mu^2-\omega^2}{2\sigma}w_1 &= (\mu - \omega\coth\omega)w_1 + \frac{e^{\mu}\omega}{\sinh\omega}w_0,\end{aligned}$$

and so

$$\begin{aligned}\frac{w_1}{w_0} &= \left(\frac{\mu^2-\omega^2}{2\sigma} + \omega\coth\omega + \mu\right)\frac{e^{\mu}\sinh\omega}{\omega}\\ \frac{w_0}{w_1} &= \left(\frac{\mu^2-\omega^2}{2\sigma} + \omega\coth\omega - \mu\right)\frac{e^{-\mu}\sinh\omega}{\omega}.\end{aligned}$$

In particular, ω must be a solution to

$$\frac{\mu^2-\omega^2}{2\sigma} + \omega\coth\omega = \pm\sqrt{\mu^2 + \frac{\omega^2}{\sinh^2\omega}} \tag{2.7($\pm$)}$$

[3]Because of potential problems coming from nilpotence, this assignment of Ω should be thought of as an *ansatz* which is justified, *ex post facto* by the fact that it works.

and

$$\tag{2.8($\pm$)} \begin{aligned} \frac{w_1}{w_0} &= \left(\pm\sqrt{\mu^2 + \frac{\omega^2}{\sinh^2 \omega}} + \mu \right) \frac{e^{\mu} \sinh \omega}{\omega} \\ \frac{w_0}{w_1} &= \left(\pm\sqrt{\mu^2 + \frac{\omega^2}{\sinh^2 \omega}} - \mu \right) \frac{e^{-\mu} \sinh \omega}{\omega}. \end{aligned}$$

Lemma 2.9 *There is a unique $\omega \geq 0$ which solves (2.7(-)). Moreover, if ω_1 denotes this unique solution, then $\omega_1 > |\mu|$. On the other hand, $|\mu|$ is always a solution to (2.7(+)), and there is a second solution $\omega \in (|\mu|, \omega_1)$ if $\sigma > \mu \coth \mu$.*

Proof Without loss in generality, we will assume that $\mu \geq 0$.

Clearly, $\omega \geq 0$ solves (2.7(-)) if and only if $g_1(\omega) = 0$, where

$$g_1(\omega) \equiv \omega^2 - 2\sigma\omega \coth \omega - 2\sigma\sqrt{\mu^2 + \frac{\omega^2}{\sinh^2 \omega}} - \mu^2.$$

Since $g_1(0) < 0$ and $\lim_{\omega \to \infty} g_1(\omega) = \infty$, it is clear that g_1 vanishes somewhere on $(0, \infty)$. In order to prove that it vanishes only once and that it can do so only in (μ, ∞), first note that

$$g_1(\omega) \geq 0 \implies (\omega - \sigma \coth \omega)^2 \geq \sigma^2 \coth^2 \omega + 2\sigma\sqrt{\mu^2 + \frac{\omega^2}{\sinh^2 \omega}} + \mu^2,$$

which is impossible unless $\omega \geq \sigma \coth \omega$, in which case $\omega > (2\sigma \coth \omega) \vee \mu$. Furthermore, if $\omega \geq 2\sigma \coth \omega$, then

$$\begin{aligned} \tfrac{1}{2} g_1'(\omega) &= \omega - \sigma \coth \omega - \sigma \frac{1}{\sqrt{\mu^2 + \frac{\omega^2}{\sinh^2 \omega}}} \frac{\omega}{\sinh^2 \omega} (1 - \omega \coth \omega) \\ &\geq \sigma \coth \omega - \frac{\sigma}{\sinh \omega} = \frac{\sigma}{\sinh \omega} (\cosh \omega - 1) > 0. \end{aligned}$$

Knowing that $g_1(\omega) \geq 0 \implies g_1'(\omega) > 0$ and that $\omega > \mu$, the first part of the lemma is now proved.

Turning to the second part, set

$$g_0(\omega) \equiv \omega^2 - 2\sigma\omega \coth \omega + 2\sqrt{\mu^2 + \frac{\omega^2}{\sinh^2 \omega}} - \mu^2.$$

Then ω satisfies (2.7(+)) if and only if $g_0(\omega) = 0$, and clearly $g_0(\mu) = 0$. In addition, since $g_1(\omega) \geq 0 \implies g_0(\omega) > 0$ and $g_1 \geq 0$ on $[\omega_1, \infty)$, we know that g_0 can vanish only on $(0, \omega_1)$. Finally, to show that it vanishes somewhere on (μ, ω_1) if $\sigma > \mu \coth \mu$, note that, since $g_0(\omega_1) > 0$ and $g_0(\mu) = 0$, it suffices to check that $\sigma > \mu \coth \mu \implies g_0'(\mu) < 0$. But $g_0'(\mu) = \big(\mu \coth \mu - \sigma\big) \tanh \mu$, and so this is clear. $\square$

From now on, we take ω_1 as in Lemma 2.9 and ω_0 to be a solution to (2.7(+)) which is equal to $|\mu|$ if $\sigma \leq \mu \coth \mu$ and is in $(|\mu|, \omega_1)$ if $\sigma > \mu \coth \mu$. The corresponding solution J to (R) is given by $\frac{2\sigma e^{\mu x}}{w_{00}w_{11} - w_{01}w_{10}}$ times

$$\begin{pmatrix} e^{-\mu} w_{01} w_{11} \left(\frac{\sinh x\omega_0}{\omega_0} - \frac{\sinh x\omega_1}{\omega_1} \right) + w_{00} w_{11} \frac{\sinh(1-x)\omega_0}{\omega_0} - w_{01} w_{10} \frac{\sinh(1-x)\omega_1}{\omega_1} \\ -w_{00} w_{10} \left(\frac{\sinh(1-x)\omega_0}{\omega_0} - \frac{\sinh(1-x)\omega_1}{\omega_1} \right) - e^{-\mu} w_{01} w_{10} \frac{\sinh x\omega_0}{\omega_0} + e^{-\mu} w_{00} w_{11} \frac{\sinh x\omega_1}{\omega_1} \end{pmatrix},$$

where $W_k = (w_{k0}, w_{k1})$ is a left eigenvector of Ω with eigenvalue ω_k.

Remark: For those readers who are wondering, the reason why, when $\sigma < \mu \coth \mu$, we take ω_0 to be the solution to (2.7(+)) which is greater than $|\mu|$ is to get a solution to (**R**) which satisfies (1.4).

Lemma 2.10 *The preceding J is a non-negative solution to* (R). *In addition, $\langle \mathbf{1}, J_0 \rangle = 1 = \langle \mathbf{1}, J_1 \rangle$ if $\sigma \geq \mu \coth \mu$ and $\langle \mathbf{1}, J_0 \rangle \vee \langle \mathbf{1}, J_1 \rangle < 1$ if $\sigma < \mu \coth \mu$. The eigenvalues of $B(J)$ are $\lambda_k = \frac{\mu^2 - \omega_k^2}{2}$, $k \in \{0, 1\}$, and associated right eigenvectors $V_k = \begin{pmatrix} v_{k0} \\ v_{k1} \end{pmatrix}$ satisfy*

$$\frac{v_{k1}}{v_{k0}} = (-1)^k \left(\sqrt{\mu^2 + \left(\frac{\sinh \omega_k}{\omega_k} \right)^2} + \mu \right) \frac{e^\mu \sinh \omega_k}{\omega_k}.$$

Hence, they can be chosen so that $v_{00} \wedge v_{01} > 0$ with $v_{01} + v_{01} = 1$ and $v_{10} > 0 > v_{11}$ with $v_{10} - v_{11} = 1$.

Proof To check that J is non-negative, we begin by remarking that $u(y) \equiv \frac{\sinh y\omega_0}{\omega_0} - \frac{\sinh y\omega_1}{\omega_1} \geq 0$ for $y \in [0,1]$. Indeed, $u(0) = 0 = u(1)$ and $u'' \leq \omega_1^2 u$. Hence, if u achieves a strictly negative minimum, it would have to do so at some $y \in (0,1)$, in which case we would have the contradiction $0 \leq u''(y) \leq \omega_1^2 u(y) < 0$. Because of this remark, it suffices to show that all the numbers

$$\frac{w_{00}w_{11} - w_{01}w_{10}}{w_{01}w_{11}}, \quad \frac{w_{00}w_{11} - w_{01}w_{10}}{-w_{00}w_{10}}, \quad \frac{w_{00}w_{11} - w_{01}w_{10}}{w_{00}w_{11}}, \quad \text{and } \frac{w_{00}w_{11} - w_{01}w_{10}}{-w_{01}w_{10}}$$

are positive. But, using (2.8(±)), this is an elementary, if somewhat tedious, task.

Next, from $B(J) = \frac{\mu^2 I - \Omega^2}{2}$, the identification of the eigenvalues of $B(J)$ is clear. In addition, if W_0 and W_1 are left eigenvectors of $B(J)$, then the columns of $\begin{pmatrix} W_0 \\ W_1 \end{pmatrix}^{-1}$ are associated right eigenvectors of $B(J)$. Hence, the calculation of $\frac{v_{k1}}{v_{k0}}$ is a consequence of (2.8(±)).

Turning to the calculation of $\langle \mathbf{1}, J_k \rangle$, observe that, by integrating (R), one sees that

$$B(J)\begin{pmatrix} 1 - \langle \mathbf{1}, J_0 \rangle \\ 1 - \langle \mathbf{1}, J_1 \rangle \end{pmatrix} = \mathbf{0}.$$

Hence, if $\omega_0 > |\mu|$, and therefore $B(J)$ is non-degenerate, $1 - \langle \mathbf{1}, J_k \rangle = 0$ for $k \in \{0, 1\}$. On the other hand, when $\omega_0 = |\mu|$, $\begin{pmatrix} 1 - \langle \mathbf{1}, J_0 \rangle \\ 1 - \langle \mathbf{1}, J_1 \rangle \end{pmatrix}$ must be a multiple of V_0. In particular, this means that either $\langle \mathbf{1}, J_0 \rangle$ and $\langle \mathbf{1}, J_1 \rangle$ are both equal 1, both strictly greater than 1, or both strictly less than 1. To determine which of these holds, note that, when $\omega_0 = |\mu|$, $\frac{w_{01}}{w_{00}} = e^{2\mu}$ and therefore that

$$\langle \mathbf{1}, J_0 \rangle + e^{2\mu} \langle \mathbf{1}, J_1(x) \rangle = 2\sigma \left[\int_0^1 e^{x\mu} \frac{\sinh(1-x)\mu}{\sinh \mu} \, dx + e^{\mu} \int_0^1 e^{x\mu} \frac{\sinh x\mu}{\sinh \mu} \right] = \frac{2\sigma e^{\mu} \sinh \mu}{\mu},$$

and so

$$1 - \langle \mathbf{1}, J_0 \rangle + e^{2\mu}\big(1 - \langle \mathbf{1}, J_1 \rangle\big) = 1 + e^{2\mu} - \frac{2\sigma e^{\mu} \sinh \mu}{\mu} = \frac{2e^{\mu} \sinh \mu}{\mu} \big(\mu \coth \mu - \sigma\big).$$

Thus, $\sigma = \mu \coth \mu \implies \langle \mathbf{1}, J_k \rangle = 1$ and $\sigma < \mu \coth \mu \implies \langle \mathbf{1}, J_k \rangle < 1$ for $k \in \{0, 1\}$. □

2.3 Growth of Solutions

In this section we will give the proof of the final part of Theorem 1.5. To this end, set

$$c_k = \frac{(-1)^k\sqrt{\mu^2\cosh^2\omega_k + \omega_k^2 - \mu^2} - \mu\cosh\omega_k}{\omega_k + \mu} \quad \text{for } k \in \{0,1\},$$

and define $h_0^{\sigma,\mu}$ and $h_1^{\sigma,\mu}$ by

$$(3.1) \qquad h_0^{\sigma,\mu}(x) = \begin{cases} \left(e^{x\omega_0} + c_0 e^{(1-x)\omega_0}\right)e^{-x\mu} & \text{if } \sigma > \mu\coth\mu \\ \frac{1}{|\mu|} + \frac{x}{\mu} + \frac{1}{2\mu^2}\left(1+\tanh\mu\right)e^{-x2\mu} & \text{if } \sigma = \mu\coth\mu \ \&\ \mu \neq 0 \\ 1 - x(1-x) & \text{if } \sigma = 1 \ \&\ \mu = 0 \\ 1 & \text{if } \sigma < \mu\coth\mu, \end{cases}$$

$$h_1^{\sigma,\mu}(x) = \left(e^{x\omega_1} + c_1 e^{(1-x)\omega_1}\right)e^{-x\mu}.$$

If $u_k^{\sigma,\mu}$ denotes $u_{h_k^{\sigma,\mu}}$, then

$$u_0^{\sigma,\mu}(t,x) = \begin{cases} e^{-t\lambda_0^{\sigma,\mu}} h_0^{\sigma,\mu}(x) & \text{if } \sigma > \mu\coth\mu \\ t + h_0^{\sigma,\mu}(x) & \text{if } \sigma = \mu\coth\mu \\ 1 & \text{if } \sigma < \mu\coth\mu, \end{cases}$$

and

$$u_1^{\sigma,\mu}(t,x) = e^{-t\lambda_1^{\sigma,\mu}} h_1^{\sigma,\mu}(x).$$

In addition, because $u_0^{\sigma,\mu} \geq 0$, the first part of Theorem 1.4 says that $D^{\sigma,\mu}h_0^{\sigma,\mu}$ is a non-negative, scalar multiple of V_0. At the same time, because, $u_0^{\sigma,\mu}$ is unbounded when $\sigma \geq \mu\coth\mu$ and when $\sigma < \mu\coth\mu$ it does not tend to 0 as $t \to \infty$, this scalar cannot be 0. Hence, there exists a $K_0^{\sigma,\mu} > 0$ so that $K_0^{\sigma,\mu}D^{\sigma,\mu}h_0^{\sigma,\mu} = V_0$. We next want to show that $K_1^{\sigma,\mu} \neq 0$ can be chosen so that $K_1^{\sigma,\mu}D^{\sigma,\mu}h_1^{\sigma,\mu} = V_1$. It is clear (cf. Theorem 2.1) that

$$-B^{\sigma,\mu}D^{\sigma,\mu}h_1^{\sigma,\mu} = \frac{d}{dt}D^{\sigma,\mu}u_1^{\sigma,\mu}(t)\Big|_{t=0} = -\lambda_1^{\sigma,\mu}D^{\sigma,\mu}h_1^{\sigma,\mu}.$$

Thus $D^{\sigma,\mu}h_1^{\sigma,\mu}$ is a scalar multiple of V_1, and, because $u_1^{\sigma,\mu}$ is unbounded, this scalar cannot be 0. That is, $K_1^{\sigma,\mu} \neq 0$ can be chosen to make $K_1^{\sigma,\mu}D^{\sigma,\mu}h_1^{\sigma,\mu} = V_1$. Finally, $h_1^{\sigma,\mu}$ must take both strictly positive and strictly negative values. If not, $u_1^{\sigma,\mu}$ would have to take only one sign, which would lead to that contradiction that $D^{\sigma,\mu}h_1^{\sigma,\mu}$ is a multiple of V_1.

To complete the program, set

$$g_0^{\sigma,\mu} = \begin{cases} K_0^{\sigma,\mu}h_0^{\sigma,\mu} & \text{if } \sigma > \mu\coth\mu \\ K_0^{\sigma,\mu} & \text{if } \sigma \le \mu\coth\mu \end{cases}$$

and $g_1^{\sigma,\mu} = K_1^{\sigma,\mu}h_1^{\sigma,\mu}$. Given $f \in F$, determine a_0 and a_1 by $D^{\sigma,\mu}f = a_0V_0 + a_1V_1$, and set $\tilde{f} = f - a_0g_0^{\sigma,\mu} - a_1g_1^{\sigma,\mu}$. Then $u_f = u_{\tilde{f}} + a_0K_0^{\sigma,\mu}u_0^{\sigma,\mu} + a_1K_1^{\sigma,\mu}u_1^{\sigma,\mu}$. Because $D^{\sigma,\mu}\tilde{f} = \mathbf{0}$, as $t \to \infty$, $u_{\tilde{f}}(t,\cdot)$ tends to 0 if $\sigma < \mu\coth\mu$ and, in any case, stays bounded. Clearly, the last part of Theorem 1.5 follows from these considerations.

As a consequence of the preceding, we see that $-\lambda_1^{\sigma,\mu}$ is the exact exponential rate constant governing the growth of the semigroup $\{Q_t : t \ge 0\}$. That is, there is a $C < \infty$ such that

$$\|Q_tf\|_{\mathrm{u}} \le Ce^{-t\lambda_1^{\sigma,\mu}}\|f\|_{\mathrm{u}}, \tag{3.2}$$

and there are f's for which $\lim_{t\to\infty} e^{t\lambda_1^{\sigma,\mu}}\|Q_tf\|_{\mathrm{u}} > 0$.

2.4 Proof of Theorem 1.1

This appendix is devoted to the proof of Theorem 1.1, and we begin by introducing a little notation. First, let $g(t,x) = (2\pi t)^{-\frac{1}{2}}e^{-\frac{x^2}{2t}}$ be the centered Gauss kernel with variance t, and set $G(t,x) = \sum_{k\in\mathbb{Z}} g(t,x+2k)$. Clearly, $G(t,\cdot)$ is even and is periodic with period 2. Next, set

$$Q^0(t,x,y) = e^{\mu(y-x)-\frac{\mu^2t}{2}}\big[G(t,y-x) - G(t,y+x)\big], \quad (t,x,y) \in (0,\infty)\times[0,1]^2. \tag{4.1}$$

As one can easily check, Q^0 is the fundamental solution to $\dot{u} = \frac{1}{2}u'' + \mu u'$ in $[0,\infty)\times(0,1)$ with boundary condition 0 at $\{0,1\}$. Equivalently, if τ_k denotes $\inf\{t \ge 0 : X_t = k\}$, then

$$\mathbb{P}\big(X_t \in dy \;\&\; \tau_0\wedge\tau_1 > t \,\big|\, X_0 = x\big) = Q^0(t,x,y)\,dy.$$

Next, set

$$q_k(t,x) = (-1)^k \frac{1}{2}\frac{d}{dy} Q^0(t,x,y)\Big|_{y=k}, \quad k \in \{0,1\}.$$

Then, by Green's Theorem, for $h_k \in C([0,\infty);\mathbb{R})$,

$$w(t,x) = \int_0^t q_0(t-\tau)h_0(\tau)\,d\tau + \int_0^t q_1(t-\tau)h_1(\tau)\,d\tau$$

is the solution to $\dot{u} = \frac{1}{2}u'' + \mu u'$ in $[0,\infty)\times(0,1)$ satisfying $\lim_{t\searrow 0} u(t,\cdot) = 0$ and $\lim_{x\to k} u(t,x) = h_k(t)$. Equivalently,

$$\mathbb{P}\big(\tau_1 > \tau_0 \in dt \,\big|\, X_0 = x\big) = q_0(t,x)\,dt \text{ and } \mathbb{P}\big(\tau_0 > \tau_1 \in dt \,\big|\, X_0 = x\big) = q_1(t,x)\,dt.$$

In particular, these lead to $q_k \geq 0$ and

$$\begin{aligned} Q^0(s+t,x,y) &= \int_{(0,1)} Q^0(s,x,z)Q^0(t,z,y)\,dx \\ q_k(s+t,x) &= \int_{(0,1)} Q^0(s,x,y)q_k(t,y)\,dy \quad \text{for } k \in \{0,1\} \\ \int_{(0,1)} Q^0(t,x,y)\,dy + \int_0^t q_0(\tau,x)\,d\tau &+ \int_0^t q_1(\tau,x)\,d\tau = 1, \end{aligned} \tag{4.2}$$

and

$$q_0(t,x) = -e^{-\mu x - \frac{\mu^2}{2}t} G'(t,x) \quad \text{and} \quad q_1(t,x) = -e^{\mu(1-x) - \frac{\mu^2 t}{2}} G'(t,1-x), \tag{4.3}$$

where the second of these comes from $G'(t,1+x) = -G'(t,-1-x) = -G'(t,1-x)$.

Clearly,

$$0 \leq Q^0(t,x,y) \leq g(t,x-y) \leq \frac{1}{\sqrt{2\pi t}}. \tag{4.4}$$

In order to estimate $q_k(t,x)$, first note that, from (4.3), it is clear that $G'(t,x) \leq 0$. Second,

$$G'(t,x) = -\frac{x}{t}G(t,x) + \frac{2}{t}\sum_{m=1} m\big(g(t,2m-x) - g(t,2m+x)\big) \geq -\frac{x}{t}G(t,x).$$

Hence,

$$|G'(t,x)| \le \frac{x}{t}G(t,x) \le C\frac{x}{t}g(t\wedge 1,x), \tag{4.5}$$

and so

$$0 \le q_0(t,x) \le C\frac{x}{t}g(t\wedge 1,x) \quad \text{and} \quad 0 \le q_1(t,x) \le C\frac{1-x}{t}g(t\wedge 1,1-x). \tag{4.6}$$

for some $C < \infty$.

In what follows, we will be using the notation

$$w_0(x) = \frac{e^{2\mu x} - e^{2\mu}}{1 - e^{2\mu}}, \quad w_1(x) = \frac{e^{2\mu x} - 1}{e^{2\mu} - 1}, \quad \text{and } \hat{f}_k = \langle f, w_k \rangle \quad \text{for } f \in F.$$

Note that if $u \in U$ satisfies (1.2), then, after integrating by parts, one finds that

$$\begin{aligned} \dot{\hat{u}}_0(t) &= -\tfrac{1}{2}u'(t,0) + \frac{\mu e^{2\mu}}{e^{2\mu}-1}u(t,1) - \frac{\mu e^{2\mu}}{e^{2\mu}-1}u(t,0), \\ \dot{\hat{u}}_1(t) &= \tfrac{1}{2}u'(t,1) - \frac{\mu}{e^{2\mu}-1}u(t,1) + \frac{\mu}{e^{2\mu}-1}u(t,0), \end{aligned}$$

and therefore

$$\frac{d}{dt}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = 2\sigma\frac{d}{dt}\begin{pmatrix} \hat{u}_0(t) \\ \hat{u}_1(t) \end{pmatrix} + A\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix},$$
$$\text{where } A \equiv \frac{2\sigma\mu}{e^{2\mu}-1}\begin{pmatrix} e^{2\mu} & -e^{2\mu} \\ -1 & 1 \end{pmatrix}.$$

Solving this, we see that

$$\begin{aligned} &e^{-tA}\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} - e^{-sA}\begin{pmatrix} u(s,0) \\ u(s,1) \end{pmatrix} \\ &\quad = 2\sigma e^{-tA}\begin{pmatrix} \hat{u}_0(t) \\ \hat{u}_1(t) \end{pmatrix} - 2\sigma e^{-sA}\begin{pmatrix} \hat{u}_0(s) \\ \hat{u}_1(s) \end{pmatrix} + 2\sigma\int_s^t e^{-\tau A}A\begin{pmatrix} \hat{u}_0(\tau) \\ \hat{u}_1(\tau) \end{pmatrix} d\tau, \end{aligned}$$

from which it is clear that if, as $s \searrow 0$, $u(s,\cdot) \restriction (0,1)$ converges pointwise to a function $f : (0,1) \longrightarrow \mathbb{R}$, then $\lim_{s\searrow 0} u(s,k)$ exists for $k \in \{0,1\}$. Thus, the first part of Theorem 1.1 is proved, and, in addition, we know that

$$\begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix} = e^{tA}\begin{pmatrix} f(0) - 2\sigma\hat{f}_0 \\ f(1) - 2\sigma\hat{f}_1 \end{pmatrix} + 2\sigma\begin{pmatrix} \hat{u}_0(t) \\ \hat{u}_1(t) \end{pmatrix} + 2\sigma\int_0^t e^{(t-\tau)A}A\begin{pmatrix} \hat{u}_0(\tau) \\ \hat{u}_1(\tau) \end{pmatrix} d\tau \tag{4.7}$$

if $u(t,\cdot) \longrightarrow f$ in F.

Because, for any $u \in U$ satisfying $\dot u = \frac{1}{2}u'' + \mu u'$ and, as $t \searrow 0$, $u(t,\cdot) \longrightarrow f$ pointwise on $(0,1)$,

$$\begin{aligned}u(t,x) &= \mathbb{E}\big[f(X_t),\, \sigma_0\wedge\sigma_1 > t \,\big|\, X(0)=x\big] + \mathbb{E}\big[u(t-\sigma_0,0),\, \sigma_0 < t\wedge\sigma_1 \,\big|\, X(0)=x\big]\\ &\qquad + \mathbb{E}\big[u(t-\sigma_1,0),\, \sigma_1 < t\wedge\sigma_0 \,\big|\, X(0)=x\big]\\ &= \int_{(0,1)} Q^0(t,x,y)f(y)\,dy + \int_0^t q_0(\tau,x)u(t-\tau,0)\,d\tau + \int_0^t q_1(\tau,x)u(t-\tau,1)\,d\tau,\end{aligned}$$

(4.7) tells us that if $u \in U$ satisfies (1.2) and $u(t,\cdot) \longrightarrow f$ in F, then

$$u(t,x) = r_f(t,x) + \int_0^t k(t-\tau,x)\begin{pmatrix}\hat u_0(\tau)\\ \hat u_1(\tau)\end{pmatrix} d\tau, \tag{4.8}$$

where

$$r_f(t,x) \equiv h_f(t,x) + \int_0^t q(t-\tau,x)e^{\tau A}\begin{pmatrix} f(0)-2\sigma\hat f_0\\ f(1)-2\sigma\hat f_1\end{pmatrix} d\tau$$

$$k(t,x) \equiv 2\sigma q(t,x) + 2\sigma\int_0^t q(t-\tau,x)e^{\tau A}A\,d\tau$$

with $h_f(t,x) = \int_{(0,1)} Q^0(t,x,y)f(y)\,dy$ and $q(t,x) = \big(q_0(t,x), q_1(t,x)\big)$.

Our proof of the existence and uniqueness statements in Theorem 1.1 will be based on an analysis of the integral equation (4.8). Clearly, given $f \in F$, finding a solution u to (4.8) comes down to finding a $t \in [0,\infty) \longmapsto v(t) = \begin{pmatrix} v_0(t)\\ v_1(t)\end{pmatrix} \in \mathbb{R}^2$ which satisfies

$$v(t) = \hat r_f(t) + \int_0^t \hat K(t-\tau)v(\tau)\,d\tau, \tag{4.9}$$

where

$$\hat r_f(t) = \begin{pmatrix}\langle r_f(t,\cdot), w_0\rangle\\ \langle r_f(t,\cdot), w_1\rangle\end{pmatrix} \quad\text{and}\quad \hat K(t) = \begin{pmatrix}\langle k(t,\cdot), w_0\rangle\\ \langle k(t,\cdot), w_1\rangle\end{pmatrix}.$$

Indeed, if v solves (4.9) and u is defined by

$$u(t,x) = r_f(t,x) + \int_0^t k(t-\tau,x)v(\tau)\,d\tau,$$

then u satisfies (4.8). Conversely, if u solves (4.8) and $v(t) = \begin{pmatrix} \hat{u}_0(t) \\ \hat{u}_1(t) \end{pmatrix}$, then v solves (4.9). Thus, existence and uniqueness for solutions to (4.8) is equivalent to existence and uniqueness for solutions to (4.9).

To prove that, for each $f \in F$, (4.9) has precisely one solution, we use the following simple lemma.

Lemma 4.10 *Suppose that $M : (0,T] \longrightarrow \mathbb{R} \otimes \mathbb{R}$ is a continuous, 2×2-matrix-valued function with the property that $L(T) = \sup_{t\in(0,T]} t^{\frac{1}{2}} \|M(t)\|_{op} < \infty$ and that $v^0 : (0,T] \longrightarrow \mathbb{R}^2$ is a continuous function for which $\|v^0\|_{\alpha,T} \equiv \sup_{t\in(0,T]} t^\alpha |v^0(t)| < \infty$, where $\alpha \in [0,1)$. If $\{v^n : n \geq 1\}$ is defined inductively by*

$$v^n(t) = v^0(t) + \int_0^t M(t-\tau) v^{n-1}(\tau)\, d\tau, \quad t \in (0,T],$$

then

$$\sup_{\tau\in[0,T]} \left| v^n(\tau) - v^{n-1}(\tau) \right| \leq \frac{(L(T)\sqrt{\pi})^n \Gamma(1-\alpha) \|v^0\|_{\alpha,T}}{\Gamma\left(\frac{n}{2}+1-\alpha\right)} T^{\frac{n}{2}-\alpha}.$$

In particular, $\{v^n - v^0 : n \geq 1\}$ converges uniformly on $(0,T]$ to a contiguous function which tends to 0 as $t \searrow 0$. Finally, if $v^\infty = v^0 + \lim_{n\to\infty}(v^n - v^0)$, then v^∞ is the unique $v : (0,T] \longrightarrow \mathbb{R}^2$ satisfying

$$v(t) = v^0(t) + \int_0^t M(t-\tau) v(\tau)\, d\tau \quad \text{with } \|v\|_{\alpha,T} < \infty.$$

In fact, there is a $C_\alpha < \infty$ such that $\|v^\infty\|_{\alpha,T} \leq C_\alpha L(T) \|v^0\|_{\alpha,T} e^{C_\alpha L(T) T}$.

Using the estimates in (4.5) and applying Lemma 4.10 with $\alpha = 0$, we now know that, for each $f \in F$, there is precisely one solution to (4.9), which, in view of the preceding discussion, means that there is precisely one solution to (4.8). Moreover, because every solution to (1.2) with initial data f is a solution to (4.8), this proves that, for each $f \in F$, the only solution to (1.2) is the corresponding unique solution to (4.8); and, for this reason, in spite of our not having shown yet that every solution to (4.8) is an admissible solution to (1.2), we will use u_f to denote this solution. Note that, from the last part of Lemma 4.10 and our construction,

$$\|u_f(t,\cdot)\|_{\mathrm{u}} \leq C \|f\|_{\mathrm{u}} e^{Ct} \tag{4.11}$$

for a suitable $C < \infty$.

What remains is to show that solutions to (4.8) have sufficient regularity to be an admissable solutions to (1.2) and that their dependence on f is sufficiently continuous. To this end, return to (4.9), set $v^0 = \widehat{r_f}(t)$, and

$$v^n(t) = v^0(t) + \int_0^t \hat{K}(t-\tau)v^{n-1}(\tau)\,d\tau.$$

Then

$$\dot{v}^n(t) = \dot{\hat{h}}_f(t) + \hat{q}(t)\begin{pmatrix} f(0) - 2\sigma\hat{f}_0 \\ f(1) - 2\sigma\hat{f}_1 \end{pmatrix} + \int_0^t \hat{K}(t-\tau)\dot{v}^{n-1}(\tau)\,d\tau,$$

where

$$\dot{\hat{h}}_f(t) = \begin{pmatrix} \langle \dot{h}_f(t,\cdot), w_0\rangle \\ \langle \dot{h}_f(t,\cdot), w_1\rangle \end{pmatrix} = \int_{(0,1)} \dot{\hat{Q}}^0(t,y)f(y)\,dy$$

with

$$\hat{Q}^0(t,y) = \begin{pmatrix} \langle Q^0(t,\cdot,y), w_0\rangle \\ \langle Q^0(t,\cdot,y), w_1\rangle \end{pmatrix}.$$

Using integration by parts, one sees that

$$\dot{\hat{Q}}^0(t,y) = \begin{pmatrix} e^{\mu y}G'(t,y) \\ e^{\mu(y-1)}G'(t,1-y) \end{pmatrix},$$

and therefore that the estimate in (4.5) together with Lemma 4.10 guarantee that $\hat{u}_f(t) = \begin{pmatrix} (\hat{u}_f)_0(t) \\ (\hat{u}_f)_1(t) \end{pmatrix}$ is continuously differentiable on $(0,\infty)$ and that

$$|\dot{\hat{u}}_f(t)| \le Ct^{-\frac{1}{2}}\|f\|_{\mathrm{u}}e^{Ct} \tag{4.12}$$

for some $C < \infty$. Combining this with (4.8), it follows that u_f is continuously differentiable with respect to $t \in (0,\infty)$ and that

$$\dot{u}_f(t,x) = \dot{h}_f(t,x) + k(t,x)\hat{f} + q(t,x)\begin{pmatrix} f(0) - 2\sigma\hat{f}_0 \\ f(1) - 2\sigma\hat{f}_1 \end{pmatrix} + \int_0^t k(t-\tau)\dot{\hat{u}}_f(\tau)\,d\tau.$$

Since elementary estimates show that $\sup_{t>0}|t\dot{Q}^0(t,x,y)| < \infty$, we have now shown that

$$\|\dot{u}_f(t,\cdot)\|_{\mathrm{u}} \le Ct^{-1}\|f\|_{\mathrm{u}}e^{Ct} \tag{4.13}$$

for a suitable $C < \infty$.

It is clear from (4.8) that u_f is differentiable on $(0,\infty) \times (0,1)$ and that

$$u_f'(t,x) = r_f'(t,x) + \int_0^t k'(t-\tau,x)\hat{u}_f(\tau)\,d\tau \quad \text{for } (t,x) \in (0,\infty) \times (0,1).$$

The contribution of h_f to r_f' poses no difficulty and can be extends without difficulty to $(0,\infty) \times [0,1]$ as a smooth function. Instead, the problems come from the appearance of integrals of the form $\int_0^t q_k'(t-\tau)\psi(\tau)\,d\tau$ as $x \to k$. To handle such terms, we use (4.3) to write

$$\begin{aligned} q_k'(t,x) &= -\mu q_k(t,x) + (-1)^k e^{\mu(k-x)-\frac{\mu^2 t}{2}} G''(t,k-x) \\ &= -\mu q_k(t,x) + (-1)^{1-k} 2 e^{\mu(k-x)-\frac{\mu^2 t}{2}} \dot{G}(t,k-x). \end{aligned}$$

The first term causes no problems. As for the second, we can integrate by parts to see that

$$\int_0^t \dot{G}(t-\tau,x)\psi(\tau)\,d\tau = G(t,x)\psi(0) + \int_0^t G(t-\tau,x)\dot{\psi}(\tau)\,d\tau.$$

Hence, by (4.12), the preceding expression for $u_f'(t,x)$ on $(0,\infty) \times (0,1)$ admits a continuous extension to $(0,\infty) \times [0,1]$. In addition, one can easily check from our earlier estimates, especially (4.12), that

$$(4.14) \qquad |u_f'(t,\cdot)\|_{\mathrm{u}} \le Ct^{-\frac{1}{2}}\|f\|_{\mathrm{u}} e^{Ct}$$

for an appropriate $C < \infty$. Finally, because u_f is smooth and satisfies $\dot{u}_f = \frac{1}{2}u'' + \mu u'$ on $(0,\infty) \times (0,1)$, we now see that u'' extends as a continuous function on $(0,\infty) \times [0,1]$ satisfying

$$(4.15) \qquad \|u''(t,\cdot)\|_{\mathrm{u}} \le Ct^{-1}\|f\|_{\mathrm{u}} e^{Ct}$$

for some $C < \infty$.

In view of the preceding, all that we have to do is check that $\dot{u}_f(t,k) = (-1)^{1-k}\sigma u_f'(t,k)$.

To this end, observe that (4.8) is designed so that its solutions will satisfy

$$\begin{pmatrix} \dot{u}(t,0) \\ \dot{u}(t,1) \end{pmatrix} = 2\sigma \begin{pmatrix} \dot{\hat{u}}_0(t) \\ \dot{\hat{u}}_1(t) \end{pmatrix} + A \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix}$$

and that, because $\dot{u} = \frac{1}{2}u'' + \mu u'$,

$$2\sigma \begin{pmatrix} \dot{\hat{u}}_0(t) \\ \dot{\hat{u}}_1(t) \end{pmatrix} = \sigma \begin{pmatrix} -u'(t,0) \\ u'(t,1) \end{pmatrix} - A \begin{pmatrix} u(t,0) \\ u(t,1) \end{pmatrix}.$$

Chapter 3

A Generalization to Dimension Higher than One

3.1 Introduction to Our PDE Case

3.1.1 Our Main PDE Result

Let F be the space of bounded functions on $E := [0, +\infty) \times (-\infty, +\infty)$ that are continuous on $E^+ := (0, +\infty) \times (-\infty, +\infty)$ but not necessarily continuous at the boundary $E^- = \{0\} \times (-\infty, +\infty)$. Convergence of f_n to f in F means that $\{\|f_n\|_u\}$ is bounded, $f_n(x, y) \longrightarrow f(x, y)$ for each $(x, y) \in E$, and uniformly for (x, y) in compact subsets of E^+. (We write $f_n \longrightarrow f$ u.c. E^+ as a shorthand for the last requirement.

In the next definition and hereafter, note that we use probabilistic convention of writing $u(t, x, y)$, not $u(x, y, t)$. As usual,

$$\dot{u} := \frac{\partial u}{\partial t},\ u'_x := \frac{\partial u}{\partial x},\ u''_{xx} := \frac{\partial^2 u}{\partial x^2},\ u'_y := \frac{\partial u}{\partial y},\ u''_{yy} := \frac{\partial^2 u}{\partial y^2},\ \triangle u = u''_{xx} + u''_{yy}.$$

Now, for $T > 0$, let U_T be the space of functions u on $(0, T] \times E$ such that u is bounded on $(0, T] \times E$ and whenever $0 < T_1 < T_2 < T$ we have

$$u \restriction ([T_1, T_2] \times E) \in C_b^{1,2}([T_1, T_2] \times E).$$

Recall that this last statement means that $u, \dot{u}, u'_x, u'_y, u''_{xx}$, and u''_{yy} are all bounded continuous functions on $[T_1, T_2] \times E$. Note that we insist that u is $C^{1,2}$ right up to and including the spacial boundary $(0,\infty)\times E^-$. We let $U_\infty = \bigcap_{T\in\mathbb{R}^+} U_T$.

Denote by $H^p(-a,a)$(for $1\le p\le 2$) the Hardy space over the band $|Imz|<a$, i.e., the space of functions g analytic for $|Imz|<a$ such that (let $z=\alpha+i\beta$)

$$\sup_{|\beta|<a}\left(\int_{\mathbb{R}}|g(\alpha+i\beta)|^p d\alpha\right)^{1/p}<+\infty.$$

We denote $H^p(\mathbb{R})=\bigcap_{a\in\mathbb{R}^+} H^p(-a,a)$.

THEOREM 1.1

Let $\mu_0,\mu_1,\sigma\in\mathbb{R},\Theta\in\mathbb{R}^+\bigcup\{\infty\}$, and let $f\in C_b(E^+)$. Suppose that $u\in U_\Theta$ and that u satisfies the PDE

$$\dot{u}=\tfrac{1}{2}\triangle u+\mu_0 u'_x+\mu_1 u'_y \quad on\ (0,\Theta)\times E^+, \tag{1.1a}$$

$$\dot{u}(t,0,y)=\sigma u'_x(t,0,y)\quad for\ t\in(0,\Theta), y\in(-\infty,+\infty), \tag{1.1b}$$

$$\lim_{t\searrow 0}u(t,x,y)=f(x,y)\quad u.c.\ E^+. \tag{1.1c}$$

Then $f(0,y):=u(0+,0,y):=\lim_{t\searrow 0}u(t,0,y)$ exist, and we note that the extended function f is in F.

THEOREM 1.2 *If $\sigma<0$, for $\Theta\in\mathbb{R}^+, f\in F$, the following statements are equivalent:*

(i) There is a unique $u_f\in U_\Theta$ which satisfies (1.1) with $u(0+,0,y)=f(0,y)$ for $y\in(-\infty,+\infty)$.

(ii) There exists $h\in C_b(\mathbb{R})$ such that $Df=\tilde{J}(\Theta,\cdot)*h(\cdot)$, where

$$Df(y)=f(0,y)-\int_0^{+\infty}f(x,\cdot)*J(x,\cdot)dx,$$

$\{\frac{1}{-2\sigma}J(x,\cdot) : x \in [0,\infty)\}$ is a convolution semigroup given by: $J(0,\cdot) = -2\sigma\delta_0(\cdot)$ and $J(x,dy) = J^{\sigma,\mu_1,\mu_2}(x,y)dy$ for $x > 0$,

$$J^{\sigma,\mu_0,\mu_1}(x,y) = \frac{-2\sigma x}{2\pi}e^{(\mu_0+\sigma)x-2\mu_1 y}\int_0^\infty \exp\{-\tfrac{[(\sigma-\mu_0)^2+2\mu_1^2]}{\xi} - \tfrac{(x^2+y^2)\xi}{2}\}d\xi,$$

and $\tilde{J}(x,\cdot) = \frac{1}{-2\sigma}e^{2x\sigma\mu_0}J(-\sigma x,\cdot)$.

(iii) There exists $v(t,y) \in C([\sigma\Theta,\infty)\times\mathbb{R})\bigcap C^\infty((\sigma\Theta,\infty)\times\mathbb{R}))$ satisfies:

$$\partial_t^2 v(t,y) + 2(\mu_0-\sigma)\partial_t v(t,y) + \partial_y^2 v(t,y) - 2\mu_1\partial_y v(t,y) = 0, \tag{1.2}$$

$$\textit{with } v(0,\cdot) = Df(\cdot),\ \sup_{t\geq\sigma\Theta}\|v(t,\cdot)\|_u < \infty,$$

$$\lim_{t\to\infty}\|v(t,\cdot)\|_u = 0 \ \ \textit{if}\ \mu_0 > \sigma.$$

In particular, if $Df \in H^p(\sigma\Theta,-\sigma\Theta)$, for some $p\in[1,2]$, then for any $0 < T < \Theta$, there is a unique $u_f \in U_T$ which satisfies (1.1) with $u(0+,0,y) = f(0,y)$ for $y \in (-\infty,+\infty)$.

By theorem 1.2, we immediately have:

COROLLARY 1.2 *If $\sigma < 0$, for $f \in F$, the following statements are equivalent:*
(i) There is a unique $u_f \in U_\infty$ which satisfies (1.1) with $u(0+,0,y) = f(0,y)$ for $y \in (-\infty,+\infty)$

(ii) For any $\Theta \in (0,\infty)$, there exists $h_\Theta \in C_b(\mathbb{R})$ such that $Df = \tilde{J}(\Theta,\cdot) * h_\Theta(\cdot)$.

(iii) There exists $v(t,y) \in C^\infty(\mathbb{R}^2)$ satisfies:

$$\partial_t^2 v(t,y) + 2(\mu_0-\sigma)\partial_t v(t,y) + \partial_y^2 v(t,y) - 2\mu_1\partial_y v(t,y) = 0,$$

$$\textit{with } v(0,\cdot) = Df(\cdot),\ \sup_{t\geq T}\|v(t,\cdot)\|_u < \infty \ \textit{for any}\ T\in\mathbb{R},$$

$$\lim_{t\to\infty}\|v(t,\cdot)\|_u = 0 \ \ \textit{if}\ \mu_0 > \sigma.$$

In particular, if $Df \in H^p(\mathbb{R})$, for some $p \in [1,2]$, there is a unique $u_f \in U_\infty$ which satisfies (1.1) with $u(0+,0,y) = f(0,y)$ for $y \in (-\infty,+\infty)$.

3.1.2 Non-negative Solutions

If $\sigma \geq 0$, then $u(\cdot,\cdot,\cdot) \geq 0$ if and only if $f \geq 0$, so that $\{Q_t : t \geq 0\}$ is a semigroup of nonnegative operators on F. This may be proved either by the use of the minimum principle or via supermartingales. We emphasize that by *nonnegative operator* we mean an operator which is non-negativity preserving in that it maps nonnegative functions to nonnegative functions(not a nonnegative definite operator).

Now assume that $\sigma < 0$. Then we lose the minimum principle, and, as a consequence $\{Q_t : t \geq 0\}\}$ is no longer a semigroup of nonnegative operators on F. We are interested in certain $\{Q_t : t \geq 0\}$-invariant subspaces of F on which each Q_t is non-negativity preserving.
We will show that:

THEOREM 1.3 *For each $\sigma < 0$ and $\mu_0, \mu_1 \in \mathbb{R}$, there exists a unique non-negative finite measure $J(\cdot,\cdot)$ on E, which satisfies the Riccati equation(in the sense of tempered distribution)*

$$\tfrac{1}{2}\triangle J(x,y) + \tfrac{1}{2}(\partial_x J(0,\cdot) * J(x,\cdot))(y) + 2\sigma\mu_0 J(x,y) - \mu_0 J'_x(x,y) - \mu_1 J'_y(x,y) = 0,$$

$$(R) \qquad for \quad (x,y) \in E, \ and$$

$$J(0,\cdot) = -2\sigma\delta_0(\cdot)$$

which satisfies

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} J(dx,dy) \begin{cases} = 1 & if\ \sigma \leq \mu_0 \\ < 1 & if\ \sigma > \mu_0, \end{cases}$$

In fact, $J(dx,dy) = J^{\sigma,\mu_0,\mu_1}(x,y)dxdy$ for $x > 0$, where

$$(1.3) \qquad J^{\sigma,\mu_0,\mu_1}(x,y) = \frac{-2\sigma x}{2\pi} e^{(\mu_0+\sigma)x - 2\mu_1 y} \int_0^\infty \exp\{-\tfrac{[(\sigma-\mu_0)^2 + 2\mu_1^2]}{\xi} - \tfrac{(x^2+y^2)\xi}{2}\} d\xi$$

and $\int_{-\infty}^{+\infty} J^{\sigma,\mu_0,\mu_1}(x,y)dy = (-2\sigma)e^{2(\mu_0\wedge\sigma)x}$ *for* $x \in (0,\infty)$. *Therefore, we have a convolution semigroup of finite measures* $x \in [0,\infty) \longrightarrow \frac{1}{-2\sigma}J(x,\cdot)$ *over* $\mathbb{R}$.

We use the following Fourier transform definition:

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{+\infty} f(y)e^{-iy\xi}dy,$$

and $\mathcal{F}^{-1}$ is the corresponding fourier inverse transform. If $f \in \mathcal{S}'(\mathbb{R})\backslash L^1$, $\mathcal{F}(f)$ is defined in the distribution sense. That is for any $g \in \mathcal{S}(R)$,

$$\langle \mathcal{F}(f), g\rangle = \langle f, \mathcal{F}(g)\rangle .$$

For any $\varphi \in C_b^2(\mathbb{R})$, we define linear operators $\mathcal{J}(x)$ as following:

$$\mathcal{J}(x)(\varphi)(y) = (\varphi(\cdot) * J(x,\cdot))(y) \ for\ any\ x \in [0,\infty).$$

Let $K = \sqrt{-\partial_y^2 + 2\mu_1\partial_y + (\sigma-\mu_0)^2}$ in the sense that: for any $\varphi \in \mathcal{S}(\mathbb{R})$ (Schwartz class),

$$\mathcal{F}(K(\varphi))(\xi) = \sqrt{\xi^2 - 2i\mu_1\xi + (\sigma-\mu_0)^2}\mathcal{F}(\varphi)(\xi)$$

where we choose the right branch of square root such that $Re(\sqrt{\xi^2 - 2i\mu_1\xi + (\sigma-\mu_0)^2}) \geq 0$.

By subbordination, $\mathcal{F}(J^{\sigma,\mu_0,\mu_1}(x,\cdot)) = (-2\sigma)e^{\left(\mu_0+\sigma-\sqrt{(\sigma-\mu_0)^2+\xi^2-2i\mu_1\xi}\right)x}$. Since $\{\frac{1}{-2\sigma}J(x,\cdot) : x \in [0,\infty)\}$ is convolution semigroup of finite measures and by Lévy-Khinchine formula(see Theorem 2.1.9 in [11]), we can extend the domain of operator K in the following sense: for any $\varphi \in C_b^2(\mathbb{R})$,

$$(-2\sigma(\mu_0+\sigma) + 2\sigma K)\varphi = \lim_{x\searrow 0} \frac{\mathcal{J}(x)\varphi - (-2\sigma)\varphi}{x}.$$

We immediately have the following corollary:

COROLLARY 1.3 $\mathcal{J}(x) = -2\sigma e^{x(\sigma+\mu_0)-xK}$, and for any $\varphi \in C_b^2(\mathbb{R})$,

$$\frac{1}{2}\partial_x^2(\mathcal{J}(x)(\varphi)) - \mu_0\partial_x(\mathcal{J}(x)(\varphi)) = -\frac{1}{2}\mathcal{J}(x)(\partial^2\varphi) + \mu_1\mathcal{J}(x)(\partial\varphi) \tag{1.4}$$

$$-\sigma\mathcal{J}(x)(K(\varphi)) + (\sigma^2 - \mu_0\sigma)\mathcal{J}(x)(\varphi).$$

Now, we have the following result:

THEOREM 1.4 *Assume that $\sigma < 0$, and $u \in U_\infty$ satisfies (1.1) with $u(0+, x, y) = f(x, y)$ for $(x, y) \in E^+$. Then*

$$u(\cdot, \cdot, \cdot) \geq 0$$

if and only if

$$f(\cdot, \cdot) \geq 0 \text{ and } Df(y) \equiv C \text{ for some constant } C \in [0, \infty),$$

where

$$Df(y) = f(0, y) - \int_0^{+\infty} f(x, \cdot) * J(x, \cdot)dx.$$

3.1.3 The Basic Probabilistic Model

The probabilistic model associated with our PDE is the following: Suppose that X is Brownian Motion on $[0, \infty)$ with drift μ_0 and reflection at 0. That is, with B a standard Brownian Motion,

$$X_t = X_0 + B_t + \mu_0 t + L_t \geq 0,$$

Here L are "local time" of X at 0 given by $L_t = max\{(X_0 + B_s + \mu_0 s)^- : s \leq t\}$. In particular, it is a non-decreasing process which satisfies

$$L_0 = 0, \quad \int_0^t \mathbf{1}_{\{0\}}(X_s)dL_s = L_t,$$

and so L grows only when X is at 0. Let Y be Brownian motion on $(-\infty, +\infty)$ with drift μ_1 and independent of X.

Let

$$(1.5)\quad \Phi_t := \Phi_0 + t + \sigma^{-1}L_t,\ \tau_t^+ := \inf\ \{r : \Phi_r > t\},\ X_t^+ := X(\tau_t^+),\ Y_t^+ := Y(\tau_t^+).$$

If $\tau_t^+ = \infty$(equivalently, if $sup\ \Phi_r \le t$), we set $(X_t^+, Y_t^+) = \partial$ as usual, where ∂ is an absorbing point not in $[0,\infty] \times [-\infty, +\infty]$. In the case when $\sigma = 0$, we define $\tau_t^+ := H_0$ for $t > H_0$, where $H_0 := \inf\{t : X_t = 0\}$.

The connection between (1.1)(with $\Theta = \infty$) and these processes is that, for each $f \in F$ and $T \ge 0$, an application of standard Itô calculus shows that(note that $X_0 = 0 \ \&\ \sigma < 0 \implies \tau_0^+ > 0$ a.s.)

$$(1.6)\qquad t \in [0, \tau_T^+) \longmapsto u_f(T - \Phi_t, X_t, Y_t) \in \mathbb{R} \textit{ is a continuous local martingale.}$$

In particular,

$$(1.7)\qquad \text{If } \mathbb{P}(\lim_{t \nearrow \tau_T^+} u_f(T-\Phi_t, X_t, Y_t) = 0 | X_0 = x, Y_0 = y, \tau_T^+ = \infty) = 1$$

$$\text{and } u_f \text{ is bounded, then } u_f(T, x, y) = \mathbb{E}[f(X_T^+, Y_T^+), \tau_T^+ < \infty | X_0 = x, Y_0 = y].$$

Similarly,

$$(1.8)\qquad \text{If } u_f \ge 0, \text{ then } u_f(T, x, y) \ge \mathbb{E}[f(X_T^+, Y_T^+), \tau_T^+ < \infty | X_0 = x, Y_0 = y].$$

Remark: It should be emphasized that, although the process(X_t^+, Y_t^+) is a familiar, continuous diffusion when $\sigma \ge 0$, it is highly discontinuous when $\sigma < 0$. Indeed, when $\sigma < 0$, although (X_t^+, Y_t^+) behaves just like (X_t, Y_t) as long as it stays away from $\{0\} \times (-\infty, +\infty)$, upon approaching $\{0\} \times (-\infty, +\infty)$, (X_t^+, Y_t^+) either jumps back inside or gets sent to ∂. In particular, even though it is right-continuous and has left limits, (X_t^+, Y_t^+) is not a Hunt process because its jump times are totally accessible.

In order to make the connections between (X_t^+, Y_t^+) and the function J in Theorem 1.3, we will need the following result from [12] about the behavior of Φ_t as $t \to \infty$.

Lemma 1.5 *Assume that $\sigma < 0$, then almost surely,*

$$(1.9) \qquad \lim_{t\to\infty} \Phi_t = \begin{cases} = +\infty & if\ \sigma < \mu_0 \\ = -\infty & if\ \sigma > \mu_0, \end{cases}$$

and

$$(1.10) \qquad \sigma = \mu_0 \Longrightarrow \overline{\lim_{t\to\infty}} \pm\Phi_t = \infty.$$

As a consequence of Lemma 1.5, we can now make the connection alluded to above.

Theorem 1.6 *Assume $\sigma < 0$. For all bounded, measurable $\varphi : E^+ \longrightarrow \mathbb{R}$,*

$$(1.11) \qquad \mathbb{E}[\varphi(X_{\tau_0^+}, Y_{\tau_0^+}), \tau_0^+ < \infty | X_0 = 0, Y_0 = y_0] = \int_0^{+\infty} (\varphi(x,\cdot) * J(x,\cdot))(y_0)dx,$$

$$for\ \forall\ y \in (-\infty, \infty).$$

In particular, $\mathbb{P}(\tau_0^+ < \infty | X_0 = 0, Y_0 = y_0) = \frac{\sigma}{\mu_0 \wedge \sigma}$ and $\frac{\mu_0\wedge\sigma}{\sigma} J^{\sigma,\mu_0,\mu_1}(\cdot, y_0 - \cdot)$ is the density function for the distribution of $(X_{\tau_0^+}, Y_{\tau_0^+})$ given that $X_0 = 0, Y_0 = y_0$ and $\tau_0^+ < \infty$.

3.2 Proof of Theorems and The Riccati Equation

Let $g(t,x) = (2\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{2t}}$ be the centered Gauss kernel with variance t, and set

$$(2.1) \quad Q^0(t,x,y,x',y') = e^{\mu_0(x'-x)+\mu_1(y'-y)-\frac{\mu_0^2}{2}t-\frac{\mu_1^2}{2}t}[g(t,x'-x) - g(t,x'+x)]g(t,y'-y)$$

$$for (t,x,y,x',y') \in (0,\infty) \times E^2.$$

As one can easily check, Q^0 is the fundamental solution to $\dot{u} = \frac{1}{2}\Delta u + \mu_0 u'_x + \mu_1 u'_y$ in $[0,\infty) \times E^+$ with boundary condition 0 at E^-. Equivalently, if ρ denotes $inf\{t \geq$

$0 : X_t = 0\}$, then

$$\mathbb{P}\left[(X_t, Y_t) \in (dx', dy') \ \& \ \rho > t | (X_0, Y_0) = (x, y)\right] = Q^0(t, x, y, x', y')dx'dy'.$$

Next, set

$$q(t, x, y, y') = \frac{1}{2}\frac{d}{dx'}Q^0(t, x, y, x', y') \mid_{x'=0} \tag{2.2}$$

Then by Green's Theorem, for $h \in C_b([0, \infty) \times (-\infty, \infty); \mathbb{R})$,

$$w(t, x, y) = \int_0^t \int_{-\infty}^{\infty} q(t - \tau, x, y, y')h(\tau, y')d\tau dy'$$

is the solution to $\dot{u} = \frac{1}{2}\triangle u + \mu_0 u'_x + \mu_1 u'_y$ in $[0, \infty) \times E^+$ satisfying $lim_{t \searrow 0} u(t, \cdot, \cdot) = 0$ and $lim_{x \to 0} u(t, x, y) = h(t, y)$. Equivalently,

$$\mathbb{P}\left[Y_\rho \in dy' \ \& \ \rho \in dt \mid (X_0, Y_0) = (x, y)\right] = q(t, x, y, y')dtdy'.$$

In particular, this leads to $q \geq 0$ and

$$Q^0(s + t, x, y, x', y') = \int\int_{E^+} Q^0(s, x, y, x'', y'')Q^0(t, x'', y'', x', y')dx''dy'' \tag{2.4a}$$

$$q(s + t, x, y, y') = \int\int_{E^+} Q^0(s, x, y, x'', y'')q(t, x'', y'', y')dx''dy'', \tag{2.4b}$$

$$\int\int_{E^+} Q^0(t, x, y, x', y')dx'dy' + \int_0^t \int_{-\infty}^{+\infty} q(\tau, x, y, y')d\tau dy' = 1 \tag{2.4c}$$

Moreover,

$$\begin{aligned} q(t, x, y, y') &= \frac{x}{\sqrt{2\pi t^3}}e^{-\frac{(x+\mu_0 t)^2}{2t}} \frac{1}{\sqrt{2\pi t}}e^{-\frac{(y'-y+\mu_1 t)^2}{2t}} \\ &= -e^{-\mu_0 x - \frac{\mu_0^2}{2}t} g'(t, x)g(t, y' - y + \mu_1 t) \end{aligned} \tag{2.5a}$$

$$\int_0^{\infty} \int_{-\infty}^{+\infty} q(t, x, y, y')dxdy \leq Ct^{-\frac{1}{2}} \tag{2.5b}$$

Proof of Theorem 1.1 In what follows, we will be using the notation $\overline{f}(\cdot) = \int_0^\infty f(x,\cdot)e^{-x}dx$ for $f \in F$ and set $\beta = \frac{1}{2} + \mu_0$. Note that if $u \in U_\Theta$ ($\Theta \in \mathbb{R}^+ \bigcup\{\infty\}$) satisfies (1.1), then after integrating by parts, one finds that

$$\dot{\overline{u}}(t,y) = \beta\overline{u}(t,y) - \tfrac{1}{2}u'_x(t,0,y) - \beta u(t,0,y) + \tfrac{1}{2}\overline{u}''_{yy}(t,y) + \mu_1\overline{u}'_y(t,y)$$

and therefore

$$\frac{d}{dt}(u(t,0,y)) = 2\sigma\beta\overline{u}(t,y) - 2\sigma\beta u(t,0,y) + \sigma\overline{u}''_{yy}(t,y) + 2\sigma\mu_1\overline{u}'_y(t,y) - 2\sigma\dot{\overline{u}}(t,y)$$

Solving this, we see that

$$e^{2\sigma\beta t}u(t,0,y) - e^{2\sigma\beta\delta}u(\delta,0,y) = -2\sigma(e^{2\sigma\beta t}\overline{u}(t,y) - e^{2\sigma\beta\delta}\overline{u}(\delta,y))$$

$$+2\sigma(2\sigma+1)\beta\int_\delta^t e^{2\sigma\beta\tau}\overline{u}(\tau,y)d\tau + \sigma\int_\delta^t e^{2\sigma\beta\tau}[\overline{u}''_{yy}(\tau,y) + 2\mu_1\overline{u}'_y(\tau,y)]d\tau$$

But since $\overline{u}(\delta,y) \longrightarrow \overline{f}(y)$ as $\delta \searrow 0$, it now follows that $u(0+,0,y) := \lim_{t\searrow 0} u(t,0,y)$ exists. Thus Theorem 1.1 is proved. □

Because, for any $u \in U_\Theta$ ($\Theta \in \mathbb{R}^+ \bigcup\{\infty\}$) satisfying $\dot{u} = \frac{1}{2}\triangle u + \mu_0 u'_x + \mu_1 u'_y$ and, as $t \searrow 0$, $u(t,\cdot,\cdot) \longrightarrow f$ pointwise on E^+,

$$u(t,x,y) = \mathbb{E}\left[f(X_t), \rho > t | (X_0,Y_0) = (x,y)\right] + \mathbb{E}\left[u(t-\rho,0,Y_\rho), \rho \le t | (X_0,Y_0) = (x,y)\right]$$

(2.6)
$$= \int\int_{E^+} Q^0(t,x,y,x',y')f(x',y')dx'dy' + \int_0^t\int_{-\infty}^{+\infty} q(\tau,x,y,y')u(t-\tau,0,y')d\tau dy'.$$

Theorem 2.1 *For $f \in F$, suppose there exists $u_f \in U_\infty$ which satisfies (1.1). We define $Df \equiv f(0,y) - \int_0^{+\infty} f(x,\cdot) * J(x,\cdot)dx$. Then we have*

(2.7)
$$Df(\cdot) = e^{t\sigma(-\sigma+\mu_0+K)}(Du_f(t,\cdot)),$$

where $K = \sqrt{-\partial^2 - 2\mu_1\partial + (\sigma - \mu_0)^2}$. *So* $Df = 0 \Longrightarrow Du_f(t) = 0$ *for all* $t > 0$. *In particular, if* $m(J) = \int_0^{+\infty}\int_{-\infty}^{+\infty} J(x,dy)dx = \frac{\sigma}{\mu_0 \wedge \sigma} \le 1$, *then* $Df = 0$ *implies that* $\|u_f\|_{\mathrm{u}} \le \|f\|_{\mathrm{u}}$. *If* $m(J) < 1$, *then* $Df = 0$ *implies* $|u_f(t,x,y)| \longrightarrow 0$ *as* $t \to \infty$ *uniformly over* x *in compact subsets of* $[0,\infty)$ *and over* y *in* $(-\infty,+\infty)$. *Finally, for any non-negative function* $f \in F$ *with the property that* $Df \equiv C \in [0,\infty)$, *we have* $u_f \ge 0$.

Proof. First, we notice $J(x,\cdot) = (-2\sigma)e^{(\mu_0+\sigma-|\sigma-\mu_0|)x}M_x(\cdot)$. Since J^{σ,μ_0,μ_1} satisfies (1.2), thus $\lim_{x\to\infty} J(x,\mathbb{R}) = 0$ and $\lim_{x\to\infty}\partial_x J(x,\mathbb{R}) = 0$.
By Theorem 1.3, Corollary 1.3 and $\mathcal{J}(x)(\varphi)(y) = (\varphi(\cdot) * J(x,\cdot))(y)$, we have

$$\frac{d}{dt}\int_0^{+\infty}\langle u_f(t,x,\cdot), J(x,y-\cdot)\rangle dx = \int_0^{\infty}\langle(\tfrac{1}{2}\Delta u_f(x,\cdot)+\mu_0(u_f)'_x(x,\cdot)+\mu_1(u_f)'_y(x,\cdot)), J(x,y-\cdot)\rangle dx$$

$$= -\tfrac{1}{2}\langle (u_f)'_x(t,0,\cdot), J(0,y-\cdot)\rangle + (\sigma(K-\sigma-\mu_0)u_f(t,0,\cdot))(y) - \mu_0(u_f(t,0,\cdot)*J(0,\cdot))(y)$$

$$+\int_0^{\infty}(u_f(t,x,\cdot)*(\tfrac{1}{2}\partial_x^2 J^{\sigma,\mu_0,\mu_1}(x,\cdot) - \mu_0\partial_x J^{\sigma,\mu_0,\mu_1}(x,\cdot)))(y)dx + \int_0^{\infty}(\tfrac{1}{2}\partial_y^2 u_f(t,x,\cdot)$$

$$+\mu_1\partial_y u_f(t,x,\cdot)) * J(x,\cdot)))(y)dx$$

$$= \dot{u}_f(t,0,y) + \sigma(\mu_0-\sigma)u_f(t,0,y) - \sigma(\mu_0-\sigma)\mathcal{J}(x)(u(t,x,\cdot))(y)$$

$$+\sigma K(u_f(t,0,\cdot))(y) - \sigma\mathcal{J}(x)K(u_f(t,0,\cdot))(y)$$

and so $\frac{d}{dt}Du_f(t,y) = \sigma(\sigma-\mu_0)Du_f(t,y) - \sigma K(Du_f(t,\cdot))(y)$, which is equivalent to $Df(\cdot) = e^{t\sigma(-\sigma+\mu_0+K)}(Du_f(t,\cdot))$.

Define $v(t,y) = (\tilde{J}(t+\tau,\cdot) * Du_f(\tau,\cdot))(y)$ if $t \ge -\tau$, where $\tau \ge 0$. The definition of $v(t,y)$ is consistent with any $\tau \ge 0$, so it's well-defined. Note that

$$v(0,y) = Df(y) \textit{ and } v(-t,y) = Du_f(t,y), \textit{ for } t > 0.$$

Fix y, $v(t,y)$ is analytic w.r.t. t. If $Df = 0$, then

$$v(t,y) = \tilde{J}(t,\cdot) * Df(\cdot) = 0 \textit{ for } t \ge 0.$$

So $v(t,y) = 0$ for any $t \in \mathbb{R}$. We have $Df = 0 \Longrightarrow Du_f = 0$.

Now assume that $m(J) \le 1$ and that $Df = 0$. We have $|u_f(t,0,y)| = |\int_0^{+\infty}(u_f(t,x,\cdot) * J(x,\cdot))(y)dx| \le \|u_f(t)\|_{\mathrm{u,E^+}}$. To see that $\|u_f\|_{\mathrm{u}} \le \|f\|_{\mathrm{u}}$, let $\epsilon > 0$ be given and suppose that $\|u_f(t)\|_{\mathrm{u},\Omega} \ge \|f\|_{\mathrm{u}}+\epsilon \ge \|f\|_{\mathrm{u},\Omega}+\epsilon$ for some $t > 0$ and some bounded domain

$\Omega \in E^+$ such that the closure $\overline{\Omega} \in E^+$. ($\|f\|_{\mathrm{u},\Omega}$ means the uniform norm of f on domain Ω.) We can then find a $T > 0$ such that $\|u_f(T)\|_{\mathrm{u},\Omega} = \|f\|_{\mathrm{u}} + \epsilon > \|u_f(t)\|_{\mathrm{u},\Omega}$ for $0 \le t < T$. Clearly there exists a point $P \in \overline{\Omega}$ for which $|u_f(t, P)| = \|f\|_{\mathrm{u}} + \epsilon$. Since $P \notin E^-$, it is contradictive to the weak maximum principle for the parabolic operator $\partial_t - \frac{1}{2}\Delta - \mu_0\partial_x - \mu_1\partial_y$.

Next assume that $m(J) < 1$ and that $Df = 0$. By (2.6), to see that $|u_f(t, x, y)| \longrightarrow 0$ as $t \to \infty$ uniformly over x in compact subsets of $[0, \infty)$ and over y in $(-\infty, +\infty)$, it suffices to show that $|u_f(t, 0, y)| \longrightarrow 0$ as $t \to \infty$ uniformly over y in $(-\infty, +\infty)$. Indeed, by the semigroup property, (2.6), and the fact that $Du_f(t) = 0$ for all $t \ge 0$, one would then know that $\|u_f(t, 0)\|_{\mathrm{u}} \le B_f(t) + M(J) \int_0^t \|u_f(\tau, 0)\|_{\mathrm{u}} d\tau$ where $B_f(t) = sup_{y\in\mathbb{R}} \int_0^\infty \int\int_{E^+} (Q^0(t, x, \cdot, x', y') * J^{\sigma,\mu_0,\mu_1}(x, \cdot))(y) f(x', y') dx dx' dy' \longrightarrow 0$ as $t \searrow 0$. By Gronwall's inequality, $\|u_f(t, 0)\|_{\mathrm{u}} \longrightarrow 0$ as $t \to \infty$.

Finally, assume that $Df \equiv C \in [0, \infty)$. If $f > 0$ and u_f ever becomes negative, then there exists a $T > 0$ such that $u_f(t) > 0$ for $t \in [0, T)$ and $u_f(T, x, y) = 0$ for some $(x, y) \in E$. Again, from the weak maximum principle, we get a contradiction if $(x, y) \in E^+$. At the same time, because $u_f(T, 0, y) \ge \int_0^\infty (u_f(T, x, \cdot) * J^{\sigma,\mu_0,\mu_1}(x, \cdot)(y) dx$, we see that the only way that $u_f(T)$ can vanish somewhere on E is if vanishes somewhere on E^+. Thus, when $f > 0$, $u_f \ge 0$. To handle the case $f \ge 0$, define $g \in F$ so that $g = 1$ on E^+ and $g(0, y) = \int_0^\infty (\mathbf{1} * J(x, \cdot))(y) dx$ for $y \in \mathbb{R}$. Next, apply the preceding result to see that $u_f + \epsilon u_g = u_{f+\epsilon g} \ge 0$ for all $\epsilon > 0$, and conclude that $u_f \ge 0$. □

Theorem 2.2 *Let J be the same as Theorem 2.1. Then*

$$\text{(2.8)} \qquad \int_0^{+\infty} (f(x, \cdot) * J(x, \cdot))(y) dx = \mathbb{E}[f(X_{\tau_0^+}, Y_{\tau_0^+}), \tau_0^+ < \infty | X_0 = 0, Y_0 = y]$$

for $f \in C_b(E^+)$ and all $y \in (-\infty, \infty)$.

In particular, $\int_0^\infty (\mathbf{1} * J(x, \cdot))(y) dx = \mathbb{P}(\tau_0^+ < \infty | (X_0, Y_0) = (0, y))$ for any $y \in \mathbb{R}$.

Proof. Given such an $f \in C_b(E^+)$, define $\tilde{f} \in F$ such that $\tilde{f} \restriction_{E^+} = f$ and $\tilde{f}(0, y) = \int_0^{+\infty} (f(x, \cdot) * J(x, \cdot))(y) dx$ for $y \in \mathbb{R}$. Then by Theorem 2.1, $u_{\tilde{f}}$ is bounded and, as $t \to \infty$, $|u_{\tilde{f}}(t, 0, y)| \longrightarrow 0$ uniformly over y in $(-\infty, +\infty)$ when $m(J) < 1$. By

(1.6), Lemma 1.5 and note that X_t keeps reaching $\{0\}$, we have $\mathbb{P}(\tau_0^+ = \infty \Longrightarrow \lim_{t \nearrow \tau_0^+} u_f(-\Phi_t, X_t, Y_t) = 0 | X_0 = x, Y_0 = y) = 1$. Hence by (1.7) and Lemma 1.5,

$$\int_0^{+\infty} (f(x,\cdot) * J(x,\cdot))(y)dx = \tilde{f}(0,y) = \mathbb{E}[f(X_{\tau_0^+}, Y_{\tau_0^+}), \tau_0^+ < \infty | X_0 = 0, Y_0 = y]. \qquad \square$$

Let $\phi(x,\xi) = \mathcal{F}(J(x,\cdot))(\xi)$. In order to find solutions to (R), we look for solutions to the Fourier transform(w.r.t. variable y) of (R):

$$\frac{1}{2}\phi''_{xx} + \frac{1}{2}\phi'_x(0,\xi)\phi - \frac{1}{2}\xi^2\phi + 2\sigma\mu_0\phi - \mu_0\phi'_x + i\mu_1\xi\phi = 0 \tag{2.9}$$

$$with \;\; \phi(0,\xi) \equiv -2\sigma$$

Notice there is no derivative term for ξ, so we can fix ξ and consider (2.9) as an ODE for variable x. It's easy, if somewhat tedious, to verify that

$$\phi = (-2\sigma) e^{\left(\mu_0 + \sigma - \sqrt{(\sigma-\mu_0)^2 + \xi^2 - 2i\mu_1\xi}\right)x}. \tag{2.10}$$

satisfies (2.9).
By subordination,

$$\mathcal{F}^{-1}(\phi) = \frac{-2\sigma x}{2\pi} e^{(\mu_0+\sigma)x - 2\mu_1 y} \int_0^\infty \exp\{-\tfrac{[(\sigma-\mu_0)^2 + 2\mu_1^2]}{\xi} - \tfrac{(x^2+y^2)\xi}{2}\} d\xi. \tag{2.11}$$

So let $J(0,\cdot) = -2\sigma\delta_0(\cdot))$ and $J(dx,dy) = J^{\sigma,\mu_0,\mu_1}(x,y)dxdy, \; for \; x > 0$, where $J^{\sigma,\mu_0,\mu_1}(x,y) = \frac{-2\sigma x}{2\pi} e^{(\mu_0+\sigma)x - 2\mu_1 y} \int_0^\infty \exp\{-\frac{[(\sigma-\mu_0)^2+2\mu_1^2]}{\xi} - \frac{(x^2+y^2)\xi}{2}\} d\xi$ is our desired solution to (R), and obviously is nonnegative finite measure when $\sigma < 0$.
Furthermore, $\int_0^{+\infty}\int_{-\infty}^{+\infty} J^{\sigma,\mu_0,\mu_1}(x,y)dxdy = \int_0^{+\infty} \phi(x,0)dx = \frac{\sigma}{\mu_0 \wedge \sigma} \begin{cases} = 1 & if \; \sigma \le \mu_0 \\ < 1 & if \; \sigma > \mu_0, \end{cases}$
Combining with Theorem 2.2, Theorem 1.3 is therefore proved. $\square$

Proof of Theorem 1.6: Clearly, it suffices to treat the case when φ is continuous as well as bounded. Since if $\sigma > \mu_0$, $m(J) < 1$, and if $\sigma \le \mu_0$, we have $m(J) = 1$. there is no difference between the proof of this result and the proof given earlier of Theorem 2.2. $\square$

We need the following simple lemma to prove Theorem 1.2 later:

Lemma 2.3 *Suppose $M(t) : C(\mathbb{R}) \longrightarrow C(\mathbb{R})$, where $0 < t \leq T$ are linear operators, which is continuous over t, with the property that*

$$L(T) = sup_{t\in(0,T)} t^{\frac{1}{2}} \|M(t)\|_{op} < \infty$$

and that $v^0 : (0,T] \longrightarrow C(\mathbb{R})$ is continuous and $\|v^0\|_{\alpha,T} = sup_{t\in(0,T]} t^{\alpha}|v^0(t)| < \infty$, where $\alpha \in [0,1)$. If $\{v^n : n \geq 1\}$ is defined inductively by

$$v^n(t) = v^0(t) + \int_0^t M(t-\tau)(v^{n-1}(\tau))d\tau, \quad t \in (0,T], \tag{2.12}$$

then

$$sup_{t\in[0,T]}|v^n(t) - v^{n-1}(t)| \leq \frac{(L(T)\sqrt{\pi})^n \Gamma(1-\alpha)\|v^0\|_{\alpha,T}}{\Gamma(\frac{n}{2}+1-\alpha)} T^{\frac{n}{2}-\alpha}. \tag{2.13}$$

In particular, $\{v^n - v^0 : n \geq 1\}$ converges uniformly on $(0,T]$ to a contiguous function which tends to 0 as $t \searrow 0$. Finally, if $v^{\infty} = v^0 + lim_{n\to\infty}(v^n - v^0)$, then v^{∞} is the unique $v : (0,T] \longrightarrow C(\mathbb{R})$ satisfying

$$v(t) = v^0(t) + \int_0^t M(t-\tau)(v(\tau))d\tau, \ \textit{with} \ \|v\|_{\alpha,T} < \infty. \tag{2.14}$$

In fact, there is a $C_\alpha < \infty$ such that $\|v^{\infty}\|_{\alpha,T} \leq C_\alpha L(T)\|v^0\|_{\alpha,T} e^{C_\alpha L(T) T}$.

3.3 Proof of Theorem 1.4

By combining Theorems 1.3, 2.1, 2.2, (1.8) and (2.10), we have a proof of the "if" part of Theorem 1.4. Now, we need to show the "only if" part.

Let $\tilde{J}(t,\cdot) = \frac{1}{-2\sigma} e^{2t\sigma\mu_0} J(-\sigma t, \cdot)$. Then by Corollary 1.3 and (2.7), we have

$$Df = \tilde{J}(t,\cdot) * Du_f(t,\cdot). \tag{3.1}$$

For any $\varphi \in C_b^2(\mathbb{R})$, let $\tilde{\mathcal{J}}(t)(\varphi) = \tilde{J}(t,\cdot) * \varphi(\cdot)$, then we have $\tilde{\mathcal{J}}(t) = e^{t\sigma(-\sigma+\mu_0+K)}$ and

$$\partial_t \tilde{J}(t,\cdot) * \varphi(\cdot) = \sigma(-\sigma+\mu_0+K)\tilde{J}(t,\cdot) * \varphi(\cdot), \tag{3.2}$$

$$\partial_t^2 \tilde{J}(t,\cdot) * \varphi(\cdot) = \sigma^2(\mu_0-\sigma)^2\tilde{J}(t,\cdot) * \varphi(\cdot)$$
$$+2\sigma^2(\mu_0-\sigma)K\tilde{J}(t,\cdot) * \varphi(\cdot) + \sigma^2K^2\tilde{J}(t,\cdot) * \varphi(\cdot). \tag{3.3}$$

Combining (3.2) and (3.3) and notice $K^2 = -\partial_y^2 + 2\mu_1\partial_y + (\mu_0-\sigma)^2$, we have

$$\partial_t^2(\tilde{J}(t,\cdot) * \varphi(\cdot)) - 2\sigma(\mu_0-\sigma)\partial_t(\tilde{J}(t,\cdot) * \varphi(\cdot)) + \sigma^2\partial_y^2(\tilde{J}(t,\cdot) * \varphi(\cdot))$$
$$-2\mu_1\sigma^2\partial_y(\tilde{J}(t,\cdot) * \varphi(\cdot)) = 0. \tag{3.4}$$

Define $\tilde{v}(t,y) = (\tilde{J}(t+\tau,\cdot) * Du_f(\tau,\cdot))(y)$ if $t \geq -\tau$, where $\tau \geq 0$. By (3.1), we know if we fix τ, $\tilde{v}(t,y)$ is well-defined for any $t \geq -\tau$. Now let $\tau \to \infty$, we get a well-defined $\tilde{v}(t,y)$ for $(t,y) \in \mathbb{R}^2$. By (3.4), we have

$$\partial_t^2\tilde{v}(t,y) - 2\sigma(\mu_0-\sigma)\partial_t\tilde{v}(t,y) + \sigma^2\partial_y^2\tilde{v}(t,y) - 2\mu_1\sigma^2\partial_y\tilde{v}(t,y) = 0.$$

with $\tilde{v}(0,y) = Df(y)$.

Let $v(t,y) = \tilde{v}(-\frac{t}{\sigma},y)$, then

$$\partial_t^2 v(t,y) + 2(\mu_0-\sigma)\partial_t v(t,y) + \partial_y^2 v(t,y) - 2\mu_1\partial_y v(t,y) = 0. \tag{3.5}$$

with $v(0,y) = Df(y)$.

Notice $\tilde{J}(t,\mathbb{R}) = e^{\sigma t(\sigma-\mu_0)^-}$, by the definition of $v(t,y)$, we have

$$\sup_{t \geq T} \|v(t,\cdot)\|_{\mathrm{u}} < \infty \quad for\ any\ T \in (-\infty,+\infty), \tag{3.6}$$

$$\lim_{t\to+\infty} \|v(t,\cdot)\|_{\mathrm{u}} = 0 \quad if\ \mu_0 > \sigma. \tag{3.7}$$

Let $w(t,y) = e^{(\mu_0-\sigma)t-\mu_1 y}v(t,y)$, then

$$\triangle w = ((\mu_0-\sigma)^2 + \mu_1^2)w, \tag{3.8}$$

with $w(0,y) = e^{-\mu_1 y} Df(y)$.

To prove the "only if" part of Theorem 1.4, we need the following representation theorem from [4]:

Theorem 3.1[4] *If w is a positive solution of the equation $\Delta w - w = 0$, there exists a unique measure μ defined on the unit sphere of $\mathbb{R}^n$, such that*

$$w(x) = \int_{|\lambda|=1} e^{\lambda \cdot x} d\mu(\lambda).$$

If $Du_f(t,\cdot) > 0$, then $w(t,y)$ is positive. By Theorem 3.1, there exists a unique measure ν on the sphere of $\mathbb{R}^2$ with radius $\sqrt{(\mu_0-\sigma)^2+\mu_1^2}$, such that

$$w(t,y) = \int_{\lambda_1^2+\lambda_2^2=(\mu_0-\sigma)^2+\mu_1^2} e^{\lambda_1 t+\lambda_2 y} d\nu((\lambda_1,\lambda_2)).$$

By (3.6), $sup_{y\in\mathbb{R}} e^{-(\sigma-\mu_0)t+\mu_1 y} w(t,y) < \infty$, thus ν must concentrate at two points: $(\pm(\mu_0-\sigma), -\mu_1)$. Assume ν have nonnegative mass C_1 at $(-(\mu_0-\sigma),-\mu_1)$ and C_2 at $((\mu_0-\sigma),-\mu_1)$ then $w(t,y) = C_1 e^{-(\mu_0-\sigma)t-\mu_1 y} + C_2 e^{(\mu_0-\sigma)t-\mu_1 y}$. In particular, $Df(y) = e^{\mu_1 y} w(0,y) \equiv C_1 + C_2 \geq 0$.

For $u_f \geq 0$,(1.8) says that $f(0,y) \geq \mathbb{E}\left[f(X_{\tau_0^+}, Y_{\tau_0^+}), \tau_0^+ < \infty | (X_0,Y_0) = (0,y)\right]$ and Theorem 2.2 says that $\mathbb{E}\left[f(X_{\tau_0^+}, Y_{\tau_0^+}), \tau_0^+ < \infty | (X_0,Y_0) = (0,y)\right] = \int_0^{+\infty} (f(x,\cdot) * J(x,\cdot))(y)dx$. Hence, we now know that $u_f \geq 0 \Longrightarrow Df \geq 0$, and by semigroup property, this self-improves to $u_f \geq 0 \Longrightarrow Du_f(t) \geq 0$ for all $t \geq 0$.

Assume Df is not nonnegative constant. Let

$$f_0(x,y) = \begin{cases} 1 & if \ \ x = 0, \\ 0 & otherwise. \end{cases}$$

By the "if" part of Theorem 1.4 and Theorem 2.1, we have $u_{f_0} \geq 0$ and $Du_f = e^{-\sigma(-\sigma+\mu_0+|\sigma-\mu_0|)t} > 0$. By the linearity of our PDE, we have $Du_{f+f_0} = Du_f + Du_{f_0} > 0$ and $D(f+f_0) = Df + Df_0 = Df + 1$ is not constant. This is contradictive to our previous argument. So the "only if" part of Theorem 1.4 is true. □

3.4 Proof of Theorem 1.2

$(i) \Longrightarrow (iii)$ is already shown at the proof of Theorem 1.4.

$(iii) \Longrightarrow (ii)$: Let B_t^1 and B_t^2 be two independent standard Brownian motions. Then $U_t = U_0 + B_t^1 + (\mu_0 - \sigma)t$ and $V_t = V_0 + B_t^2 - \mu_1 t$ are two independent Brownian motions with drift. Let $U_0 = t_0 > \sigma\Theta$, $V_0 = y_0 \in \mathbb{R}$ and $\eta_\tau = \inf\{t \geq 0 : U_t = \tau\}$. By a simple application of Itô's formula, $v(U_t, V_t)$ is a continuous martingale w.r.t. for $t \in [0, \eta_{\sigma\Theta})$. Since $\lim_{t\to\infty} \|v(t,\cdot)\|_u = 0 \ \ if \ \mu_0 > \sigma$, thus we have

$$\mathbb{P}(\eta_{\sigma\Theta} < \infty) = 1 \ \ if \ \mu_0 \leq \sigma,$$

$$\mathbb{P}(\lim_{t \nearrow \eta_{\sigma\Theta}} \|v(t,\cdot)\|_u = 0 \mid \eta_{\sigma\Theta} = \infty) = 1 \ \ if \ \mu_0 > \sigma.$$

So

$$v(t_0, y_0) = \mathbb{E}\left[v(\sigma\Theta, V_{\eta_{\sigma\Theta}}), \eta_{\sigma\Theta} < \infty | U_0 = t_0, V_0 = y_0\right], \tag{4.1}$$

which means $v(t,y)$ $(for \ t > \sigma\Theta)$ is uniquely determined by $v(\sigma\Theta, \cdot)$. Therefore,

$$v(t,y) = \tilde{J}(t - \sigma\Theta, \cdot) * v(\sigma\Theta, \cdot), \tag{4.2}$$

from which it is clear that, for each $y \in \mathbb{R}$, $v(\cdot, y)$ admits a holomorphic extension to $(\sigma\Theta, \infty)$. Assume $Df(y) = 0$, then for $t \geq -\sigma\Theta$, $\tilde{J}(t,\cdot) * v(\sigma\Theta, \cdot) = v(t + \sigma\Theta, \cdot) = \tilde{J}(t + \sigma\Theta, \cdot) * v(0, \cdot) = 0$. So $v(t, \cdot) \equiv 0$. for $t \in [\sigma\Theta, \infty)$.

By linearality, we now know that there is at most one v with $v(0,\cdot) = Df(\cdot)$. By (4.2), we have

$$Df(y) = v(0,y) = \tilde{J}(-\sigma\Theta, \cdot) * v(\sigma\Theta, \cdot),$$

where $v(\sigma\Theta, \cdot) \in C_b(\mathbb{R})$. Statement (ii) is true.

$(ii) \Longrightarrow (i)$:

By (2.6) and $u(\tau,0,y) = Du(\tau,y) + \int_0^\infty (u_f(\tau,x,\cdot) * J(x,\cdot)(y)dx$, we have

$$u(t,x,y) = r_f(t,x,y) + \int_0^t \int_{-\infty}^{+\infty} q(t-\tau,x,y,y')(\int_0^\infty (u_f(\tau,x',\cdot) * J(x',\cdot))(y')dx')d\tau dy'.$$

where

$$r_f(t,x,y) = \int\int_{E^+} Q^0(t,x,y,x',y')f(x',y')dx'dy' + \int_0^t \int_{-\infty}^{+\infty} q(t-\tau,x,y,y')Du(\tau,y')d\tau dy'$$

Let $\hat{u}(t,y) = \int_0^\infty (u_f(t,x',\cdot) * J(x',\cdot)(y)dx'$, then

$$u(t,x,y) = r_f(t,x,y) + \int_0^t \int_{-\infty}^{+\infty} q(t-\tau,x,y,y')\hat{u}(\tau,y')d\tau dy'. \tag{4.3}$$

Our proof of the existence and uniqueness statements (i) in Theorem 1.2 will be based on an analysis of the integral equation (4.3). Clearly, given $f \in F$, finding a solution to (4.3) for $t \in [0,T]$ comes down to finding a $t \in [0,T] \longmapsto v(t,y)$ which satisfies

$$v(t,y) = \hat{r}_f(t,y) + \int_0^t \hat{K}(t-\tau)(v(\tau,\cdot))(y)d\tau. \tag{4.4}$$

where

$$K(t,x)(g)(y) = \int_{-\infty}^\infty q(t,x,y,y')g(y')dy' \quad for\ any\ g \in C^1(\mathbb{R}).$$

Indeed, if v solves (4.4) and u is defined by

$$u(t,x,y) = r_f(t,x,y) + \int_0^t K(t-\tau,x)(v(\tau))(y)d\tau,$$

then u satisfies (4.3). Conversely, if u satisfies (4.3) and $v(t,y) = \hat{u}(t,y)$, then v solves (4.4). Thus, existence and uniqueness for solutions to (4.3) is equivalent to existence and uniqueness for solutions to (4.4).

By the expression of q, (3)c in [12], and (2.10), we can easily know that

$$t^{1/2}\|\hat{K}(t)\|_{op} < \infty,$$

If there exists $h \in C_b(\mathbb{R})$ such that $Df = \tilde{J}(\Theta,\cdot) * h(\cdot)$, then we can define

$$Du(t,\cdot) = \tilde{J}(\Theta - t,\cdot) * h(\cdot), \tag{4.5}$$

which is consistent with (2.7). So by (2.4) and notice that $|\int\int_{E+} Q^0(t,x,y,x',y')f(x',y')dx'dy'| \le \|f\|_u$ and $\tilde{J}(t,\mathbb{R}) = e^{\sigma t(\sigma-\mu_0)^-} \le 1$, we have $\hat{r}_f(t,y)$ is bounded. By applying Lemma 2.3 with $\alpha = 0$, we now know that, there is precisely one solution to (4.4), which, in view of the preceding discussion, means that there is precisely one solution to (4.3). Moreover, because every solution to (1.1) with initial data f is a solution to (4.3); and, for this reason, in spite of our not having shown yet that every solution to (4.3) is an admissible solution tp (1.1), we will use u_f to denote this solution. Note that, from the last part of Lemma 2.3 and our construction,

$$\|u_f(t,\cdot,\cdot)\|_u \le C\|f\|_u e^{Ct}, \tag{4.6}$$

for a suitable $C < \infty$.

Now we need to show that solutions to (4.3) have sufficient regularity to be an admissible solutions to (1.1) and that their dependence on f is sufficiently continuous. To this end, return to (4.4), set $v^0(t) = \hat{r}_f(t)$, and

$$v^n(t) = v^0(t) + \int_0^t \hat{K}(t-\tau)(v^{n-1}(\tau))d\tau. \quad t \in (0, \Theta - \epsilon], \ \epsilon > 0$$

Then

$$\dot{v}^n(t,y) = \dot{v}^0(t) + \hat{K}(t)(\hat{f})(y) + \int_0^t \hat{K}(t-\tau)(\dot{v}^{n-1}(\tau))(y)d\tau$$

$$= \int\int_{E+} \dot{\hat{Q}}^0(t,y,x',y')f(x',y')dx'dy' + \int_{-\infty}^{\infty} \hat{q}(t,y,y')Df(y')dy'$$

$$+ \int_0^t \int_{-\infty}^{+\infty} \hat{q}(t-\tau,y,y')\dot{D}u(\tau,y')d\tau dy' + \hat{K}(t)(\hat{f})(y) + \int_0^t \hat{K}(t-\tau)(\dot{v}^{n-1}(\tau))(y)d\tau.$$

By integration by parts, (R) and (2.2), we have

$$\dot{\hat{Q}}^0(t,y,x',y') = 2\sigma\frac{x'}{\sqrt{2\pi t^3}}e^{-\frac{(x'-\mu_0 t)^2}{2t}}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(y'-y+\mu_1 t)^2}{2t}} - 2\sigma\mu_0\hat{Q}^0(t,y,x',y')$$

$$-\frac{1}{2}(\hat{Q}^0(t,\cdot,x',y') * J'_x(0,\cdot))(y)$$

By (2.1) and (1.3), one can easily check that the last two terms are bounded. We also can easily check that

$$|\int_0^{\infty}\int_{-\infty}^{+\infty} \frac{x'}{\sqrt{2\pi t^3}}e^{-\frac{(x'-\mu_0 t)^2}{2t}}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(y'-y+\mu_1 t)^2}{2t}}dx'dy'| \le Ct^{-\frac{1}{2}}, \ \textit{some } C > 0.$$

Combining with (2.5) and (4.5), we have

$$|\dot{v}^0(t)| \leq Ct^{-\frac{1}{2}}\|f\|_u.$$

Lemma 2.3 guarantee that $\hat{u}_f$ is continuously differentiable with respect to $t \in (0, \Theta - \epsilon)$ and that

$$|\dot{\hat{u}}_f(t)| \leq C(\epsilon)t^{-\frac{1}{2}}\|f\|_u e^{C(\epsilon)t}. \tag{4.7}$$

for some $0 < C(\epsilon) < \infty$. Combining this with (4.3), it follows that u_f is continuously differentiable with respect to $t \in (0, \Theta - \epsilon)$ and that

$$\dot{u}_f(t,x,y) = \int\int_{E^+} \dot{Q}^0(t,x,y,x',y')f(x',y')dx'dy' + \int_{-\infty}^{\infty} q(t,x,y,y')Df(y')dy'$$

$$+\int_0^t\int_{-\infty}^{+\infty} q(t-\tau,x,y,y')\dot{D}u(\tau,y')d\tau dy' + K(t,x)(\hat{f})(y) + \int_0^t K(t-\tau,x)(\dot{\hat{u}}_f(\tau))(y)d\tau.$$

Some element estimates show that $sup_{t>0}|t^2\dot{Q}^0(t,x,y,x',y')| < \infty$, we have shown that

$$\|\dot{u}_f(t,\cdot,\cdot)\|_u \leq C(\epsilon)t^{-2}\|f\|_u e^{C(\epsilon)t} \tag{4.8}$$

for a suitable $C(\epsilon) < \infty$.

It is clear from (4.3) that u_f is differentiable on $(0, \Theta - \epsilon) \times (0, \infty) \times (-\infty, +\infty)$ and that

$$\frac{\partial}{\partial x}u_f(t,x,y) = \frac{\partial}{\partial x}r_f(t,x,y) + \int_0^t\int_{-\infty}^{+\infty}\frac{\partial}{\partial x}q(t-\tau,x,y,y')\hat{u}(\tau,y')d\tau dy',$$

and

$$\frac{\partial}{\partial y}u_f(t,x,y) = \frac{\partial}{\partial y}r_f(t,x,y) + \int_0^t\int_{-\infty}^{+\infty}\frac{\partial}{\partial y}q(t-\tau,x,y,y')\hat{u}(\tau,y')d\tau dy'.$$

The contribution of $\int\int_{E^+} Q^0(t,x,y,x',y')f(x',y')dx'dy'$ to $\frac{\partial}{\partial x}r_f$ and $\frac{\partial}{\partial y}r_f$ poses no difficulty and can extends to $(0, \Theta - \epsilon) \times [0, \infty) \times (-\infty, +\infty)$. And $\frac{\partial}{\partial y}q(t)$ also poses no difficulty as $x \to 0$. Instead, the problems come from the appearance of integrals of the form $\int_0^t \frac{\partial}{\partial x}q(t-\tau)\psi(\tau)d\tau$ as $x \to 0$. To handle such terms, we use (2.5) to write

$$\int_0^t \frac{\partial}{\partial x} q(t,x,y,y') = -\mu_0 q(t,x,y,y') - e^{-\mu_0 x - \frac{\mu_0^2}{2}t} g''(t,x) g(t,y'-y+\mu_1 t)$$

$$= -\mu_0 q(t,x,y,y') - 2e^{-\mu_0 x - \frac{\mu_0^2}{2}t} g(t,x)\frac{d}{dt}(g(t,y'-y+\mu_1 t)) + 2e^{-\mu_0 x - \frac{\mu_0^2}{2}t}\frac{d}{dt}(g(t,x)g(t,y'-y+\mu_1 t))$$

The first two terms ause no problem. As for the last term, we can integrate by parts to see that

$$\int_0^t \frac{d}{dt}(g(t-\tau,x)g(t-\tau,y'-y+\mu_1(t-\tau)))\psi(\tau)d\tau = g(t,x)g(t,y'-y+\mu_1 t)\psi(0)$$

$$+\int_0^t g(t-\tau,x)g(t-\tau,y'-y+\mu_1(t-\tau))\dot{\psi}(\tau)d\tau.$$

Hence by (4.7), the preceding expression for $\frac{\partial}{\partial x}u_f$ and $\frac{\partial}{\partial y}u_f$ on $(0,\Theta-\epsilon)\times(0,\infty)\times(-\infty,+\infty)$ admits a continuous extension to $(0,\Theta-\epsilon)\times[0,\infty)\times(-\infty,+\infty)$. In addition, one can easily check from our earlier estimates, especially (4.7), that

$$\text{(4.9)} \qquad max\{\|\frac{\partial}{\partial x}u_f\|_u, \|\frac{\partial}{\partial y}u_f\|_u\} \leq C(\epsilon)t^{-1}\|f\|_u e^{C(\epsilon)t}$$

for an appropriate $C(\epsilon) < \infty$. It's also easy to check that $\frac{\partial^2}{\partial^2 y}u_f$ is continuous on $(0,\Theta-\epsilon)\times(0,\infty)\times(-\infty,+\infty)$ and can extends continuously to $(0,\Theta-\epsilon)\times[0,\infty)\times(-\infty,+\infty)$. Finally, because u_f is smooth and satisfies $\dot{u}_f = \frac{1}{2}\Delta u_f + \mu_0\frac{\partial}{\partial x}u_f + \mu_1\frac{\partial}{\partial y}u_f$, we now see that $\frac{\partial^2}{\partial^2 x}u_f$ extends as a continuous function on $(0,\Theta-\epsilon)\times[0,\infty)\times(-\infty,+\infty)$. Then by letting $\epsilon\longrightarrow 0$, we established the desired regularity.

In view of the preceding, all that we have to do is to check that $\dot{u}(t,0,y) = \sigma u'(t,0,y)\ \ for\ t\in(0,\Theta), y\in(-\infty,+\infty)$. To this end, observe that (4.3) is designed so that its solutions will satisfy

$$\dot{u}(t,0,y) = \dot{D}u_f(t,y) + \dot{u}_f(t,y)$$

and because $\dot{u} = \frac{1}{2}\Delta u + \mu_0 u'_x + \mu_1 u'_y$,(R), and (4.5), we have

$$\dot{u}_f(t,y) = \sigma u'(t,0,y) - \dot{D}u_f(t,y).$$

So $(ii) \Longrightarrow (i)$ is true.

For $\Theta\in\mathbb{R}^+$, if $Df\in H^p(\sigma\Theta,-\sigma\Theta)$ for some $p\in[1,2]$, then by Theorem 10.4.1 in

[10], we can say $\mathcal{F}(Df)(\xi) = e^{\sigma\Theta|\xi|}h(\xi)$, where $h \in L^{p'}(\mathbb{R})$(if $p \in (1,2]$, $1/p+1/p' = 1$; if $p = 1$, $p' = \infty$). Then by (2.14), in the sense of distribution, we have

$$\mathcal{F}(Du_f(t,\cdot))(\xi) = e^{\left[-\sigma\mu_0+\sigma^2-\sigma\sqrt{(\sigma-\mu_0)^2+\xi^2-2i\mu_1\xi}\right]t+\sigma\Theta|\xi|}h(\xi)$$

By Hölder's inequality and Theorem 10.4.1 in [10], $\mathcal{F}(Du)(t,\xi)$ exists and is bounded if $t < \Theta$. Now we have $Df = \tilde{J}(T,\cdot) * Du(T,\cdot)$ for any $0 < T < \Theta$. Since $(ii) \Longrightarrow (i)$, Theorem 1.2 is proved. □

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