Graph Polynomials and Statistical Physics

by

Jae Ill Kim

B.S., Mathematics (2005)
Korea Advanced Institute of Science and Technology

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Master of Science in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2007

Author ....
Department of Mathematics
May 11, 2007

Certified by ....
Igor Pak
Associate Professor of Applied Mathematics
Thesis Supervisor

Accepted by ....
Pavel Etingof
Chairman, Department Committee on Graduate Students

Graph Polynomials and Statistical Physics

by

Jae Ill Kim

Submitted to the Department of Mathematics
on May 11, 2007, in partial fulfillment of the
requirements for the degree of
Master of Science in Mathematics

Abstract

We present several graph polynomials, of which the most important one is the Tutte polynomial. These various polynomials have important applications in combinatorics and statistical physics. We generalize the Tutte polynomial and establish its correlations to the other graph polynomials. Finally, our result about the decomposition of planar graphs and its application to the ice-type model is presented.

Thesis Supervisor: Igor Pak
Title: Associate Professor of Applied Mathematics

Acknowledgments

I would like to thank my advisor Igor Pak for guiding me in graph theory. It is with great appreciation that I acknowledge his help and expertise in advising me during my study. I could not have completed this thesis without him.

I am deeply grateful to Michael Sipser for advising and mentoring me throughout my time in grad school. During the work on this thesis, Daniel Kleitman advised and helped me to understand aspects and applications of this subject better. I thank Shan-Yuan Ho and Laura Leon for their elaborate comments about my work.

Finally, I would like to thank my parents, Unmook Kim and Okja Bae for their help and encouragement. Their love has supported me every minute of my life. This thesis is dedicated to them.

Contents

1 Introduction **9**

1.1 Motivation and context 9

1.2 Basic definitions 10

2 Graph polynomials **11**

2.1 The Tutte polynomial 11

2.1.1 Definition 11

2.1.2 Properties 12

2.2 Other graph polynomials 15

2.2.1 The chromatic polynomial 15

2.2.2 The monochrome polynomial 17

2.2.3 The dichromatic polynomial 19

2.2.4 The flow polynomial 22

2.2.5 The acyclic orientation 23

2.3 Proof of Theorem 2 24

3 Statistical physics **29**

3.1 The percolation process 29

3.2 The Potts model 31

4 Connections between graph polynomials and statistical physics **35**

4.1 The universal form of the Tutte polynomial 35

4.2 Relations between graph polynomials 38

4.2.1 The chromatic polynomial 38
4.2.2 The monochrome polynomial 39
4.2.3 The flow polynomial 40
4.2.4 The acyclic orientation 40
4.2.5 The percolation process 42
4.2.6 The dichromatic polynomial and the Potts model 42

5 Other results **45**

5.1 Planar lattices 45
5.1.1 Generalization of the partition function 45
5.1.2 Planar lattice and its surrounding lattice 47
5.1.3 Polygon decomposition 48
5.2 The ice-type model 50

Chapter 1

Introduction

1.1 Motivation and context

In recent decades, the *chromatic number* $\chi(G)$ of a given graph G became an important subject in combinatorics and complexity theory. Since it is one of the NP-hard problems and it also has strong connections to the *clique number* $\omega(G)$ and the *independence number* $\alpha(G)$ which are NP-hard problems, we would like to solve the problems in polynomial time. The chromatic polynomial in Chapter 2 is one approach to the study of the chromatic number. Besides the chromatic polynomial, we define other graph polynomials of graph G. Each of them has interesting combinatorial interpretations and important connections to statistical physics. With several graph operations, we derive surprising results. They all can be expressed in terms of the Tutte polynomial.

In Chapter 2, we will define graph polynomials including the Tutte polynomial and the chromatic polynomial. And we will show their important and useful properties.

In Chapter 3, we will present new approaches in statistical physics. We will introduce the percolation process and the Potts model and will show that the percolation probability of the percolation process and partition function of the Potts model defined on graph G have similar properties of graph polynomials in Chapter 2.

In Chapter 4, in order to make correlations between graph polynomials, we will first generalize the Tutte polynomial to the universal polynomial. We will then present their correlations with the Tutte polynomial.

In Chapter 5, we will present another result about decompositions of planar graphs and its application to the ice-type model.

1.2 Basic definitions

For a given multiple graph $G = (V, E)$, where V is a set of vertices and E is a set of edges of graph G, we define

- $k(G)$ to be the number of connected components of G,
- *rank* of G to be $r(G) = |V| - k(G)$,
- *nullity* of G to be $n(G) = |E| - |V| + k(G)$.

For $F \subset E$, define $\langle F \rangle$ to be a graph $\langle F \rangle = (V, F)$. Let $k(F)$, $r(F)$, $n(F)$ be the number of connected components, *rank* and *nullity* of the graph $\langle F \rangle$. For example, $k(E) = k(G)$, $r(E) = r(G)$ and $n(E) = n(G)$.

We define the two basic operations on graph $G = (V, E)$, *deletion (cutting)* and *contraction (fusing)*. For $e = (i, j) \in E$, $G - e$ is defined as $(V, E - \{e\})$. That is, $G - e$ is simply obtained by deleting the edge e from G. Similarly, G/e is obtained by contracting the edge e. In other words, we can obtain the graph G/e from G by deleting the edge e and identifying incident vertices i and j.

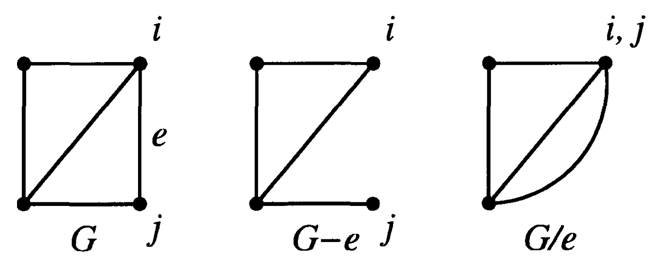

Figure 1-1: An example of the two basic operations on graph G.

Chapter 2

Graph polynomials

Graph polynomials defined on the graph G provide important information. For example, the chromatic polynomial with variable x gives us the number of proper colorings with x colors of graph G. If we want to know how many proper colorings of graph G exist with n colors, we simply plug n into the variable x of the chromatic polynomial of G, then the value is the number of proper colorings. In this chapter, we will present several graph polynomials and their properties.

2.1 The Tutte polynomial

In 1954, Tutte constructed a remarkable generalization of the chromatic polynomial [10]. This 2-variable graph polynomial gives us more information about graphs than many other graph polynomials. The Tutte polynomial is also important in statistical physics applications.

2.1.1 Definition

We define the Tutte polynomial via rank, nullity and number of components of subgraphs of G.

Definition 1 *The Tutte polynomial $T(G; x, y)$ of non-empty graph G with variable x*

and y is defined as follows

$$T(G;x,y) = \sum_{F \subset E} (x-1)^{r(E)-r(F)} \cdot (y-1)^{n(F)}.$$

It is obvious that $T(E_n;x,y) = 1$, where E_n is an empty graph with n vertices.

2.1.2 Properties

We now derive basic but very important properties of the Tutte polynomial. Theorem 2 gives us the recurrence relations for the Tutte polynomial. We will use this theorem to calculate the Tutte polynomial for certain families of graphs.

Theorem 2 *Let $G = (V,E)$ be a graph with $e \in E$. Then*

$$T(G;x,y) = \begin{cases} x \cdot T(G-e;x,y) & \text{if } e \text{ is a bridge,} \\ y \cdot T(G-e;x,y) & \text{if } e \text{ is a loop,} \\ T(G-e;x,y) + T(G/e;x,y) & \text{otherwise.} \end{cases}$$

Furthermore, $T(E_n;x,y) = 1$ for the empty graph $E_n, n \geq 1$.

Proof of this theorem is given at the end of this chapter, [2.3]. The above theorem simply implies the following corollaries.

Corollary 3 *If B_1, B_2, B_3,..., B_k are the blocks of non-empty graph G. Then*

$$T(G;x,y) = \prod_{i=1}^{k} T(B_i;x,y).$$

This means that the Tutte polynomial is multiplicative.

Proof Since there are no graph edges between distinct blocks, we can derive this corollary with Theorem 2 by induction. ■

Corollary 4 *If G is obtained from graph H by adding i bridges and j loops, then*

$$T(G;x,y) = x^i \cdot y^j \cdot T(H;x,y).$$

Proof It can be easily seen with Theorem 2 by induction. ■

Corollary 5 *Let C_n be a cycle with n vertices. Then*

$$T(C_n;x,y) = y + x + x^2 + \ldots + x^{n-1}.$$

Proof The proof proceeds by induction.

In the case of $n = 1$, C_1 is a graph with 1 vertex and 1 loop, thus $T(C_1;x,y) = y$.

Suppose this corollary holds for all $n \leq k$, then it suffices to show that this corollary also holds for $n = k+1$. Since C_n is a cycle, every edge is neither a bridge nor a loop (for $n \geq 2$). By the third property of Theorem 2

$$T(C_{k+1};x,y) = T(C_{k+1} - e;x,y) + T(C_{k+1}/e;x,y)$$

$$= T(L_k;x,y) + T(C_k;x,y) = (x^k) + (y + x + x^2 + \ldots + x^{k-1}) = y + x + x^2 + \ldots + x^k,$$

where L_k is a graph with k consecutive bridges. ■

We define the *thick edge* I_n to be a graph which consists of two vertices and n edges such that the two vertices are joined by n edges.

Corollary 6 *Let I_n be a thick edge. Then*

$$T(I_n;x,y) = x + y + y^2 + \ldots + y^{n-1}.$$

Proof The proof follows by induction.

In the case of $n = 1$, I_1 is a graph with 1 bridge, so $T(I_1;x,y) = x$.

Suppose this corollary holds for all $n \leq k$, then it suffices to show that this corollary also holds for $n = k+1$. Since I_n is a thick edge, every edge is neither a bridge nor a loop(for $n \geq 2$). By the third property of Theorem 2, we have

$$T(I_{k+1}; x, y) = T(I_{k+1} - e; x, y) + T(I_{k+1}/e; x, y) = T(I_k; x, y) + T(M_k; x, y)$$

$$= (x + y + y^2 + \ldots + y^{k-1}) + (y^k) = x + y + y^2 + \ldots + y^k,$$

where M_k is a graph with 1 vertex and k loops. ∎

Corollary 7 *We have:*

$$T(G; 2, 2) = 2^{|E|},$$

$$T(G; 1, 1) = \textit{Number of spanning trees in } G,$$

$$T(G; 1, 2) = \textit{Number of connected subgraphs in } G,$$

$$T(G; 2, 1) = \textit{Number of spanning rooted forests in } G,$$

$$T(G; 2, 0) = \textit{Number of acyclic orientations in } G.$$

Proof When $x = y = 2$, $T(G; 2, 2) = \sum_{F \subset E} 1^{r(E)-r(F)} 1^{n(F)}$, that is, for every $F \subset E$, every term in the summation is 1. Thus, $T(G; 2, 2)$ is equal to the number of subgraphs of G, which is $2^{|E|}$.

If $x = y = 1$, $T(G; 1, 1) = \sum_{F \subset E} 0^{r(E)-r(F)} 0^{n(F)}$, that is, only $F \subset E$ such that $r(E) = r(F)$ and $n(F) = 0$ has its value 1 in the summation. Thus $T(G; 1, 1)$ is equal to the number of spanning trees of G.

If $x = 1, y = 2$, $T(G; 1, 2) = \sum_{F \subset E} 0^{r(E)-r(F)} 1^{n(F)}$, that is, only $F \subset E$ such that $r(E) = r(F)$ has its value 1 in the summation. Thus $T(G; 1, 2)$ is equal to the number of connected subgraphs of G.

If $x = 2, y = 1$, $T(G; 2, 1) = \sum_{F \subset E} 1^{r(E)-r(F)} 0^{n(F)}$, that is, only $F \subset E$ such that $n(F) = 0$ has its value 1 in the summation. Thus $T(G; 2, 1)$ is equal to the number of spanning rooted forests of G.

The last case $T(G; 2, 0)$ will be presented again in [2.2.5] and will be proved in [4.2.4]. ∎

We define G^*, a *dual graph* of a given planar graph G, to be a graph which has a vertex for each plane region of G and an edge for each edge joining two neighboring regions.

Corollary 8 *If G^* is the dual graph of G, then*

$$T(G;x,y) = T(G^*;y,x).$$

Proof For a given graph $G = (V,E)$ and its dual graph $G^* = (V',E')$, if $e \in E$ is a bridge then the edge e corresponds to the edge e' which is a loop of G^*. Similarly, if $e \in E$ is a loop then the edge e corresponds to the edge e' which is a bridge of G^*. Clearly, G and G^* have the same number of edges.

We now use the recurrence relation from Theorem 2 to get their polynomials. By the above observation, if e is a bridge, e' is a loop. Thus $T(G;x,y) = x \cdot T(G-e;x,y)$ and $T(G^*;x,y) = y \cdot T(G^* - e';x,y)$. If e is a loop, e' is a bridge. Thus $T(G;x,y) = y \cdot T(G-e;x,y)$ and $T(G^*;x,y) = x \cdot T(G^* - e';x,y)$. Since the other two cases have the same relations and their initial conditions are the same as 1 for empty graphs, we conclude that $T(G;x,y) = T(G^*;y,x)$. ■

2.2 Other graph polynomials

Here are other examples of graph polynomials. Although they look original, they can be expressed by the Tutte polynomial. We will discuss their basic properties here and we will show their relationships in Chapter 4.

2.2.1 The chromatic polynomial

The chromatic number and the chromatic polynomial are the most combinatoric concepts and have been studied for a long time. They concern not only the number of proper colorings but also the colorability of a graph. After G.D. Birkhoff first found the chromatic polynomial in 1912 [3], G.D. Birkhoff and D.C. Lewis tried to prove the four color problem with the chromatic polynomial in 1946 [2]. They examined the behavior of the function. The chromatic polynomial has been developed and applied to other graph polynomials, including in the field of physics.

We call ϕ a *proper coloring* of a given graph G, if no edge has the same color of its vertices. We define the chromatic number $\chi(G)$ of a graph G to be the minimum number of colors which colors G properly.

Definition 9 *The chromatic polynomial $C_G(x)$ of a graph G is the number of proper colorings of graph G with x colors.*

When $x < \chi(G)$, we have $C(G;x) = 0$. By the properties of proper coloring, here is a lemma for the chromatic polynomial.

Lemma 10 $C(E_n;x) = x^n$,

$$C(G;x) = \begin{cases} \frac{x-1}{x} \cdot C(G-e;x) & \textit{if } e \textit{ is a bridge,} \\ 0 & \textit{if } e \textit{ is a loop,} \\ C(G-e;x) - C(G/e;x) & \textit{otherwise.} \end{cases}$$

Proof For empty graph E_n, since each vertex can be colored by x colors, there are x^n proper colorings of E_n.

Suppose that $e = (u,v)$ is a bridge. Without loss of generality, we also assume that we color the vertex u the first time and color the vertex v next. In the original graph G, we have $(x-1)$ colors for vertex v. However, in the deleted graph $G-e$, we still have x colors for the vertex v. The other vertices are the same. Thus we have $\frac{x-1}{x}$ proper colorings of G compared to $G-e$.

If G has a loop, then there are no proper colorings of G.

Finally, if $e = (u,v)$ is neither a bridge nor a loop, we can separate proper colorings of $G-e$ into two classes. In one class the colors of u and v are the same, and in the other class their colors are different. In the first case, those colorings are the same as the proper colorings of G/e. In the other case, those colorings are the same as the proper colorings of G. Thus, $C(G-e;x) = C(G;x) + C(G/e;x)$. ■

2.2.2 The monochrome polynomial

Let us define $m(G; i, x)$ to be the number of proper colorings with x colors which allows i monochromatic edges. It is obvious that $m(G; 0, x)$ is the chromatic polynomial. We now define the *monochrome polynomial* with $m(G; i, x)$.

Definition 11 *For a given graph $G = (V, E)$, we define the monochrome polynomial $M(G; x, s)$ as the following*

$$M(G; x, s) = \sum_{i=0}^{|E|} s^i \cdot m(G; i, x).$$

From the definition, we derive the following theorem.

Theorem 12 $M(E_n; x, s) = x^n$,

$$M(G; x, s) = \begin{cases} \frac{s+x-1}{x} \cdot M(G-e; x, s) & \text{if } e \text{ is a bridge,} \\ s \cdot M(G-e; x, s) & \text{if } e \text{ is a loop,} \\ M(G-e; x, s) + (s-1) \cdot M(G/e; x, s) & \text{otherwise.} \end{cases}$$

Proof For an empty graph E_n, there are x^n proper colorings of E_n. There are no other $i \geq 1$ monochromatic colorings of E_n.

As we saw before, $m(G; 0, x)$ is the chromatic polynomial. We will use Lemma 10 in Chapter 2.

Suppose that $e = (u, v)$ is a bridge, then the following equality holds for all $i \geq 1$.

$$m(G; i, x) = \frac{x-1}{x} \cdot m(G-e; i, x) + \frac{1}{x} \cdot m(G-e; i-1, x).$$

This is simply because the bridge e is monochromatic with $\frac{1}{x}$ proportion. Thus we have

$$M(G; x, s) = \sum_{i=0}^{|E|} s^i \cdot m(G; i, x) = m(G; 0, x) + \sum_{i=1}^{|E|} s^i \cdot m(G; i, x)$$

$$= \frac{x-1}{x} \cdot m(G-e; 0, x) + \sum_{i=1}^{|E|} s^i \cdot [(x-1)/x \cdot m(G-e; i, x) + 1/x \cdot m(G-e; i-1, x)]$$

$$= \frac{x-1}{x} \cdot \sum_{i=0}^{|E|} s^i \cdot m(G-e;i,x) + \frac{1}{x} \cdot \sum_{i=1}^{|E|} s^i \cdot m(G-e;i-1,x)$$

$$= \frac{x-1}{x} \cdot \sum_{i=0}^{|E|-1} s^i \cdot m(G-e;i,x) + \frac{1}{x} \cdot s \cdot \sum_{i=0}^{|E|-1} s^i \cdot m(G-e;i,x)$$

$$= \frac{s+x-1}{x} \cdot \sum_{i=0}^{|E|-1} s^i \cdot m(G-e;i,x) = \frac{s+x-1}{x} \cdot M(G-e;x,s).$$

If e is a loop, $m(G;i,x) = m(G-e;i-1,x)$ for all $i \geq 1$. This is because every loop is monochromatic. Thus we have

$$M(G;x,s) = \sum_{i=0}^{|E|} s^i \cdot m(G;i,x) = m(G;0,x) + \sum_{i=1}^{|E|} s^i \cdot m(G;i,x)$$

$$= \sum_{i=1}^{|E|} s^i \cdot m(G-e;i-1,x) = s \cdot \sum_{i=0}^{|E|-1} s^i \cdot m(G-e;i,x) = s \cdot M(G-e;x,s).$$

Finally, if $e = (u,v)$ is neither a bridge nor a loop, the following equality holds for all $i \geq 1$.

$$m(G-e;i,x) = m(G/e;i,x) + [m(G;i,x) - m(G/e,i-1,x)].$$

This is because the i monochromatic colorings of graph $G-e$, where u and v have the same color, are the same as the i monochromatic colorings of graph G/e. We also use here that the number of i monochromatic colorings of graph $G-e$ where u and v have different colors are the same as the number i monochromatic colorings of graph G minus the number of $i-1$ monochromatic colorings of graph G/e. Since the i monochromatic colorings of graph G include the case that the colors of u and v are the same. From here we have:

$$M(G;x,s) = \sum_{i=0}^{|E|} s^i \cdot m(G;i,x) = m(G;0,x) + \sum_{i=1}^{|E|} s^i \cdot m(G;i,x)$$

$$= m(G-e;0,x)-m(G/e;0,x)+\sum_{i=1}^{|E|} s^i\cdot[m(G-e;i,x)-m(G/e;i,x)+m(G/e;i-1,x)]$$

$$= \sum_{i=0}^{|E|-1} s^i \cdot m(G-e;i,x) - \sum_{i=0}^{|E|-1} s^i \cdot m(G/e;i,x) + s \cdot \sum_{i=0}^{|E|-1} s^i \cdot m(G/e;i,x)$$

$$= M(G-e;x,s) + (s-1)\cdot M(G/e;x,s).$$

This completes the proof. ■

2.2.3 The dichromatic polynomial

This is one of the most important graph polynomials in statistical physics applications. The partition function in the Potts model will prove to be a dichromatic polynomial in Chapter 3.

Definition 13 *The dichromatic polynomial $D(G;x,y)$ is defined as*

$$D(E_n;x,y) = x^n,$$

$$D(G;x,y) = D(G-e;x,y) + y\cdot D(G/e;x,y),$$

for every edge e in G.

We will show that $D(G;x,y)$ is well defined. There are several other definitions of a dichromatic polynomial.

Theorem 14 *Let $\tilde{D}(G;x,y)$ be defined as follows:*

$$\tilde{D}(G;x,y) = \sum_{F\subset G} x^{k(F)}y^{n(F)} = \sum_{n,k} d_{n,k}x^k y^n,$$

where $d_{n,k}$ is the number of spanning subgraphs of G with k components and nullity n. Then $\tilde{D}(G;x,y) = D(G;x,y)$.

Proof For the first condition,

$$\tilde{D}(E_n;x,y) = \sum_{F\subset E_n} x^{k(F)}y^{n(F)} = x^n y^0 = x^n.$$

Now, let us show the second condition.

$$\tilde{D}(G;x,y) = \sum_{F\subset G} x^{k(F)}y^{n(F)}$$

$$= \sum_{F\subset G-e} [x^{k(F)}y^{n(F)} + x^{k(F\cup e)}y^{n(F\cup e)}]$$

$$= \sum_{F\subset G-e} x^{k(F)}y^{n(F)} + y\cdot \sum_{F\subset G/e} x^{k(F)}y^{n(F)}$$

$$= \tilde{D}(G-e;x,y) + y\cdot\tilde{D}(G/e;x,y).$$

This completes the proof. ■

Here is a useful property of the dichromatic polynomial. We will need this property in Chapter 3 to find the relation between the dichromatic polynomial and the Tutte polynomial.

Theorem 15 *The dichromatic polynomial $D(G;q,v)$ has the following properties*

$$D(G;q,v) = \begin{cases} \frac{q+v}{q}\cdot D(G-e;q,v) & \textit{if } e \textit{ is a bridge,}\\ (v+1)\cdot D(G-e;q,v) & \textit{if } e \textit{ is a loop,}\\ D(G-e;q,v)+v\cdot D(G/e;q,v) & \textit{otherwise.}\end{cases}$$

Proof The third equation is immediate from the definitions.

When e is a loop, $G-e=G/e$. So we have that

$$D(G;q,v) = D(G-e;q,v)+v\cdot D(G/e;q,v)$$

$$= D(G-e;q,v)+v\cdot D(G-e;q,v) = (v+1)\cdot D(G-e;q,v).$$

Assume that e is a bridge. Then it suffices to show that

$$v \cdot D(G/e; q, v) = \frac{v}{q} \cdot D(G - e; q, v).$$

The proof proceeds by induction according to the number of edges. Suppose G has one edge e and $n \geq 1$ vertices. Then e is a bridge, $G - e$ is an empty graph with n vertices and G/e is an empty graph with $n - 1$ vertices. Thus we have

$$D(G - e; q, v) = q^n = q \cdot q^{n-1} = q \cdot D(G/e; q, v).$$

We now assume that $q \cdot D(G/e; q, v) = D(G - e; q, v)$ holds for all graph $G - e$ which has $k \geq 1$ edges and and $n \geq 1$ vertices, where e is a bridge in the original graph G. We will show the case of $k + 1$. Let $H - e_1$ be a graph with $k + 1$ edges and n vertices. Let e_1 be a bridge in H. Since $(H - e_1) - e_2 = (H - e_2) - e_1$ and $(H - e_1)/e_2 = (H/e_2) - e_1$ have k edges, we have

$$D((H - e_1) - e_2; q, v) = D((H - e_2) - e_1; q, v) = q \cdot D((H - e_2)/e_1; q, v)$$

and

$$D((H - e_1)/e_2; q, v) = D((H/e_2) - e_1; q, v) = q \cdot D((H/e_2)/e_1; q, v)$$

The following equalities hold from definition.

$$D(H - e_1; q, v) = D((H - e_1) - e_2; q, v) + v \cdot D((H - e_1)/e_2; q, v)$$

$$= q \cdot D((H - e_2)/e_1; q, v) + v \cdot q \cdot D((H/e_2)/e_1; q, v)$$

$$= q \cdot [D((H - e_2)/e_1; q, v) + v \cdot D((H/e_2)/e_1; q, v)]$$

$$= q \cdot [D((H/e_1) - e_2; q, v) + v \cdot D((H/e_1)/e_2; q, v)] = q \cdot D(H/e_1; q, v).$$

This completes the proof. ■

2.2.4 The flow polynomial

For a directed graph G, and a finite additive abelian group A, we define A-$flow$ of G if it satisfies the Kirchhoff's current law[1] for all vertices. An A-$flow$ of G is nowhere-zero if it has non-zero flows along every edge.

Definition 16 *$F(G; A)$ is the number of nowhere-zero A-$flows$ of G.*

Later, we will see that $F(G; A)$ is a polynomial in $|A|$, the order of the group A.

Theorem 17 $F(E_n; A) = 1^n = 1$,

$$F(G;A) = \begin{cases} 0 & \textit{if e is a bridge,} \\ (|A|-1) \cdot F(G-e;A) & \textit{if e is a loop,} \\ F(G/e;A) - F(G-e;A) & \textit{otherwise.} \end{cases}$$

Proof For empty graph E_n, it is obvious the there is only one A-$flow$, (the empty flow).

First, if G has a bridge, then there is no nowhere-zero A-$flow$ of G. To satisfy the Kirchhoff's current law, the flow in the bridge should be zero.

Second, if e is a loop, the flow through the edge e can be anything but zero. This does not affect the rest of the flows. Thus, the second equation holds.

In the case of the third equation, it suffices to show that

$$F(G/e; A) = F(G; A) + F(G - e; A).$$

Suppose that e is neither a bridge nor a loop, then the flow assignments for G is considered as the case that the flow along e is not zero and the flow assignments for $G - e$ is considered as the case that the flow along e is zero. Regardless of whether the flow along e is zero or not, both flow assignments are available for G/e. Therefore the above equation holds. ■

[1]At any vertex, the sum of currents flowing towards the vertex is equal to the sum of currents flowing away from the vertex.

2.2.5 The acyclic orientation

For a given undirected graph G, we assign any direction to every edge. If there are no directed cycles in this orientation, we call it the *acyclic orientation.*

Definition 18 *Let $a(G)$ denote the number of acyclic orientations of G.*

In fact, $a(G)$ is not a polynomial but a constant. Richard Stanley found the relation between $a(G)$ and the Tutte polynomial [8]. We will show this in Chapter 4. We need the following theorem to prove this relation.

Theorem 19 $a(E_n) = 1^n = 1,$

$$a(G) = \begin{cases} 2 \cdot a(G-e) & \textit{if } e \textit{ is a bridge,} \\ 0 & \textit{if } e \textit{ is a loop,} \\ a(G-e) + a(G/e) & \textit{otherwise.} \end{cases}$$

Proof For empty graph E_n, it is obvious that there is only one orientation, (the empty orientation).

If e is a bridge, any direction of e does not create a cycle. So any acyclic orientations of $G - e$ with both directions of e are also acyclic orientations of G. Thus, $a(G) = 2 \cdot a(G - e)$.

If G has a loop, any direction of e creates a cycle. Thus, there are no acyclic orientations of G.

Now, suppose e is neither a bridge nor a loop. For an acyclic orientation ξ of G, if ξ is still an acyclic orientation for the other direction of e in G then both ξ/e and $\xi - e$ are also acyclic orientations for G/e and $G - e$. If ξ is not an acyclic orientation for the other direction of e of G then ξ/e is not an acyclic orientation of G/e but $\xi - e$ is still an acyclic orientation of $G - e$. ■

Corollary 20 (Stanley) *For a given graph $G = (V, E)$ and $|V| = n$,*

$$C(G; -1) = (-1)^n \cdot a(G).$$

Proof The proof follows the induction according to the number of edges.

For an empty graph E_n,

$$C(E_n; -1) = (-1)^n = (-1)^n \cdot 1 = (-1)^n \cdot a(G).$$

Let us suppose that the equality holds for k or less than k edges. Our goal is to show that it also holds for $k+1$ edges.

For any graph $G = (V, E)$ with n vertices and $k+1$ edges, we pick any edge $e \in E$. By Lemma 10 and Theorem 20, if e is a bridge, then

$$C(G; -1) = 2 \cdot C(G - e; -1) = 2 \cdot (-1)^n \cdot a(G - e) = a(G).$$

If e is a loop, then

$$C(G; -1) = 0 = (-1)^n \cdot a(G).$$

And lastly, if e is neither a bridge nor a loop, then

$$C(G; -1) = C(G - e; -1) - C(G/e; -1) = (-1)^n \cdot a(G - e) - (-1)^{n-1} \cdot a(G/e)$$

$$= (-1)^n \cdot \{a(G - e) + a(G/e)\} = (-1)^n \cdot a(G).$$

This completes the proof. ■

2.3 Proof of Theorem 2

We first need to define the *rank generating polynomial* $S(G; x, y)$ on a graph $G = (V, E)$ as follows

$$S(G; x, y) = \sum_{F \subset E} x^{r(E) - r(F)} \cdot y^{n(F)},$$

or simply $S(G; x, y) = T(G; x + 1, y + 1)$.

Then, it suffices to show that

$$S(G;x,y) = \begin{cases} (x+1)\cdot S(G-e;x,y) & \text{if e is a bridge,} \\ (y+1)\cdot S(G-e;x,y) & \text{if e is a loop,} \\ S(G-e;x,y)+S(G/e;x,y) & \text{otherwise,} \end{cases}$$

and $S(E_n;x,y)=1$ for the empty graph E_n $(n \geq 1)$.

Let us define $G' = G-e$, $G'' = G/e$, and also define r' and n' as the *rank* and *nullity* functions of graph G', and r'' and n'' as those of G''. The following lemma includes the properties of the functions.

Lemma 21 *If $e \in E$ and $F \subset E-e$, then*

$$r(F) = r'(F),$$

$$n(F) = n'(F),$$

$$r(E) - r(F \cup e) = r''(E-e) - r''(F) = r(G'') - r''(F),$$

$$r(E) = \begin{cases} r'(E-e)+1 & \textit{if e is a bridge,} \\ r'(E-e) & \textit{otherwise,} \end{cases}$$

and

$$n(F \cup e) = \begin{cases} n''(F)+1 & \textit{if e is a loop,} \\ n''(F) & \textit{otherwise.} \end{cases}$$

Proof of this lemma is given at the end of the proof of Theorem 2.

We now define

$$S_0(G;x,y) = \sum_{F\subset E, e\notin F} x^{r(E)-r(F)} \cdot y^{n(F)},$$

$$S_1(G;x,y) = \sum_{F\subset E, e\in F} x^{r(E)-r(F)} \cdot y^{n(F)}.$$

Then, obviously, $S(G;x,y) = S_0(G;x,y) + S_1(G;x,y)$. From Lemma 21, we have

$$S_0(G;,x,y) = \sum_{F\subset E-e} x^{r(E)-r(F)} \cdot y^{n(F)}$$

$$= \begin{cases} \sum_{F\subset E(G')} x^{r'(E-e)+1-r'(F)} \cdot y^{n'(F)} & \text{if e is a bridge,} \\ \sum_{F\subset E(G')} x^{r'(E-e)-r'(F)} \cdot y^{n'(F)} & \text{otherwise,} \end{cases}$$

$$= \begin{cases} x \cdot S(G'; x, y) & \text{if e is a bridge,} \\ S(G'; x, y) & \text{otherwise.} \end{cases}$$

and

$$S_1(G; , x, y) = \sum_{F\subset E-e} x^{r(E)-r(F\cup e)} \cdot y^{n(F\cup e)}$$

$$= \begin{cases} \sum_{F\subset E(G'')} x^{r(G'')+1-r''(F)} \cdot y^{n''(F)+1} & \text{if e is a loop,} \\ \sum_{F\subset E(G'')} x^{r(G'')-r''(F)} \cdot y^{n''(F)} & \text{otherwise,} \end{cases}$$

$$= \begin{cases} y \cdot S(G''; x, y) & \text{if e is a loop,} \\ S(G''; x, y) & \text{otherwise.} \end{cases}$$

By adding S_0 and S_1 together,

$$S(G; x, y) = S_0(G; x, y) + S_1(G; x, y) = \begin{cases} x \cdot S(G'; x, y) + S(G''; x, y) & \text{if e is a bridge,} \\ y \cdot S(G''; x, y) + S(G'; x, y) & \text{if e is a loop,} \\ S(G'; x, y) + S(G''; x, y) & \text{otherwise.} \end{cases}$$

If e is a loop, it is obvious that $G'' = G'$. Thus $S(G''; x, y) = S(G'; x, y)$.

If e is a bridge, then $S(G''; x, y) = S(G'; x, y)$ also holds. Because $r''(E-e) - r''(F) = r'(E-e) - r'(F)$ and $n''(F) = n'(F)$ for all $F \subset E - e$.

Finally, it is obvious that $S(E_n; x, y) = 1$. Since an empty graph has only one subgraph, the empty graph. This completes the proof. ■

Proof of Lemma 21. Since $F \subset E - e$, we have

$$r(F) = |V| - k(F) = |V| - k'(F) = r'(F),$$

$$n(F) = |F| - |V| + k(F) = |F| - |V| + k'(F) = n'(F),$$

$$r(E) - r(F \cup e) = (|V| - k(E)) - (|V| - k(F \cup e)) =$$

$$(|V|-1-k(E))-(|V|-1-k(F\cup e))=r''(E-e)-r''(F)=r''(G)-r''(F).$$

If e is a bridge, then $k(E)=k(E-e)+1$. Otherwise, $k(E)=k(E-e)$.

If e is a loop, $|V|=|V''|$. Otherwise, $|V|=|V''|+1$. These two properties imply the last two equations. ■

Chapter 3

Statistical physics

In statistical physics, we work with random models in which states of sites and their local connections are determined by probabilities. Our goal is to measure the probability of certain states of the system. If we construct a graph by considering the sites as vertices and the local connections as edges, the graph describes a physical system and therefore is useful in mathematical modelings.

3.1 The percolation process

In the percolation process, we assume that in each component there are special sites which are called origins, that fluids are coming from the origins, and that each edge allows fluids to pass through it with probability p, $(0 < p < 1)$. Let us call the edge *open* when the edge allows fluids to pass through it, and *closed* when it does not allow fluids to pass through it. Let us define $P(G,p)$ to be the probability that every site gets wet. We want to estimate the probabilities $P(G,p)$. The following proposition is straightforward.

Proposition 22 *The percolation probability*

$$P(G;p) = \sum_F p^{|F|} \cdot (1-p)^{|G-F|},$$

where the summation is over all spanning subgraphs F of G.

With this proposition, we derive the following theorem.

Theorem 23 *We have*

$$P(G;p) = \begin{cases} p \cdot P(G-e;p) & \textit{if } e \textit{ is a bridge,} \\ P(G-e;p) & \textit{if } e \textit{ is a loop,} \\ (1-p) \cdot P(G-e;p)) + p \cdot P(G/e;p) & \textit{otherwise,} \end{cases}$$

and $P(E_n;p) = 1$.

Proof The case $P(E_n;p) = 1$ is obvious, since $|F| = 0$, $|G - F| = 0$.

In the first case, since e is a bridge, e should be an open edge. Therefore F is a spanning subgraph of G if and only if $F - e$ is a spanning subgraph of $G - e$ and e is an open edge.

Regardless of whether the subgraph has a loop or not, the connectivity of G does not change. Therefore the second equation holds.

If e is neither a loop nor a bridge, we have

$$P(G;p) = \sum_{F \subset G} p^{|F|} \cdot (1-p)^{|G-F|}$$

$$= \sum_{F \subset G-\{e\}} p^{|F|} \cdot (1-p)^{|G-F|} + \sum_{F \subset G-\{e\}} p^{|F \cup \{e\}|} \cdot (1-p)^{|G-(F \cup \{e\})|}$$

$$= \sum_{F \subset G-\{e\}} p^{|F|} \cdot (1-p)^{|(G-e)-F|+1} + \sum_{F \subset G-\{e\}} p^{|F|+1} \cdot (1-p)^{|G/e-F|}$$

$$= (1-p) \cdot P(G-e;p)) + p \cdot P(G/e;p).$$

This completes the proof of the third equation. ■

We will show the connection between the percolation probability and the Tutte polynomial in Chapter 4.

3.2 The Potts model

The *Potts model* is a mathematical model in statistical physics and a generalization of the Ising model. In the Ising model, we assume each site has only two states–such as positive or negative–and adjacent sites interact with each other regarding their states. In the Potts model, we assume that each site has q states.

Let $G = (V, E)$ be any graph such that each vertex i has a spin σ_i with values $1, 2, \ldots, q$. We define the *Hamiltonian* $H(\sigma)$ to be

$$H(\sigma) = \sum_{(i,j)\in E} (1 - \delta(\sigma_i, \sigma_j)),$$

where $\delta(a, b)$ is one when a is equal to b or is zero when a and b are different.

We define the *partition function* of q-state Potts model $O(G; q, \beta)$ as follow

$$O(G; q, \beta) = \sum_{\sigma} e^{-\epsilon\beta H(\sigma)},$$

where $\beta = \frac{1}{k \cdot T}$ is the *inverse temperature* such that k is the *Boltzmann constant* and T is the temperature of the system. And ϵ is the interactive energy between neighbor spins.

The probability of a state σ is $e^{-\epsilon\beta H(\sigma)}/O(G; q, \beta)$. Thus we can see that, if the temperature of the system is high (that is, if β is low), then the probability distribution of states are more uniformly distributed. But if the temperature of the system is low (that is β is high), then low energy states have greater probabilities than high energy states.

By the definition of the Hamiltonian, this model above is based on the assumption that the nearest neighbor spins have interactive energy 0 if they have the same spin, and energy ϵ if they are different.

Now, let us define another model which is based on the inverse assumption, that is, the nearest neighbor spins have interactive energy ϵ if they have the same spin, and 0 if they have different spin. Then we have the partition function

$$Z(G;q,v) = \sum_{\sigma} \prod_{(i,j)\in E} (1 + v \cdot \delta(\sigma_i, \sigma_j)),$$

where $v = e^{\epsilon\beta} - 1$.

This partition function $Z(G;q,v)$ is often used in the context of statistical physics. Here is an interesting theorem which shows the relation between partition function of the Potts model and the dichromatic polynomial.

Theorem 24

$$Z(G;q,v) = D(G;q,v).$$

Proof Since there are q^n possible states in any graph with n vertices

$$Z(E_n;q,v) = \sum_{\sigma} \prod_{(i,j)\in E} (1 + v \cdot \delta(\sigma_i, \sigma_j)) = \sum_{\sigma} 1 = q^n.$$

Let e be an edge from k to l, so that

$$Z(G;q,v) = \sum_{\sigma_k \neq \sigma_l} \prod_{(i,j)\in E} (1 + v \cdot \delta(\sigma_i, \sigma_j)) + \sum_{\sigma_k = \sigma_l} \prod_{(i,j)\in E} (1 + v \cdot \delta(\sigma_i, \sigma_j))$$

$$= \sum_{\sigma_k \neq \sigma_l} \prod_{(i,j)\in E-e} (1 + v \cdot \delta(\sigma_i, \sigma_j)) + (1+v) \cdot \sum_{\sigma_k = \sigma_l} \prod_{(i,j)\in E-e} (1 + v \cdot \delta(\sigma_i, \sigma_j))$$

$$= \sum_{\sigma} \prod_{(i,j)\in E-e} (1 + v \cdot \delta(\sigma_i, \sigma_j)) + v \cdot \sum_{\sigma_k = \sigma_l} \prod_{(i,j)\in E-e} (1 + v \cdot \delta(\sigma_i, \sigma_j))$$

$$= Z(G-e;q,v) + v \cdot Z(G/e;q,v).$$

This completes the proof. ■

The following theorem shows the relationship between two other partition functions.

Theorem 25 *Let G=(V,E) be a multigraph,* $q \geq 1$ *an integer and* β *an inverse* *temperature. Then the partition functions are*

$$O(G;q,\beta) = e^{-\epsilon\beta|E|} \cdot Z(G;q,v),$$

where $v = e^{\epsilon\beta} - 1$.

Proof We will show $e^{\epsilon\beta|E|} \cdot O(G;q,\beta) = Z(G;q,v)$ in the following.

$$e^{\epsilon\beta|E|} \cdot O(G;q,\beta) = e^{\epsilon\beta|E|} \cdot \sum_{\sigma} e^{-\epsilon\beta H(\sigma)}$$

$$= \sum_{\sigma} e^{\epsilon\beta(|E|-H(\sigma))} = \sum_{\sigma} e^{\epsilon\beta\{|E|-\sum_{(i,j)\in E}(1-\delta(\sigma_i,\sigma_j))\}}$$

$$= \sum_{\sigma} e^{\epsilon\beta(\sum_{(i,j)\in E}\delta(\sigma_i,\sigma_j))} = \sum_{\sigma} \prod_{(i,j)\in E} (1 + v \cdot \delta(\sigma_i,\sigma_j))$$

$$= Z(G;q,v).$$

This completes the proof. ■

Chapter 4

Connections between graph polynomials and statistical physics

We will show that every graph polynomial we mentioned, including those from statistical physics, can be expressed by the Tutte polynomial.

4.1 The universal form of the Tutte polynomial

In order to prove that other graph polynomials can be expressed by the Tutte polynomial, we present the more general form of the Tutte polynomial. We will next apply it to other graph polynomials.

Theorem 26 *Let $U(G;x,y,\alpha,\sigma,\tau)$ be a graph polynomial for graph $G=(V,E)$. Suppose that, for every $n \geq 1$,*

$$U(E_n;x,y,\alpha,\sigma,\tau)=\alpha^n,$$

and for every $e \in E$,

$$U(G;x,y,\alpha,\sigma,\tau)=\begin{cases} x \cdot U(G-e;x,y,\alpha,\sigma,\tau) & \textit{if } e \textit{ is a bridge,} \\ y \cdot U(G-e;x,y,\alpha,\sigma,\tau) & \textit{if } e \textit{ is a loop,} \\ \sigma \cdot U(G-e;x,y,\alpha,\sigma,\tau)+\tau \cdot U(G/e;x,y,\alpha,\sigma,\tau) & \textit{otherwise,} \end{cases}$$

then

$$U(G; x, y, \alpha, \sigma, \tau) = \alpha^{k(G)} \cdot \sigma^{n(G)} \cdot \tau^{r(G)} \cdot T(G; \alpha \cdot x/\tau, y/\sigma).$$

Proof Since $k(E_n) = n$, $r(E_n) = n(E_n) = 0$ and $T(E_n; \alpha \cdot x/\tau, y/\sigma) = 1$, it is immediate that

$$U(E_n; x, y, \alpha, \sigma, \tau) = \alpha^n \cdot \sigma^0 \cdot \tau^0 \cdot T(E_n; \alpha \cdot x/\tau, y/\sigma) = \alpha^n.$$

If e is a bridge, then $k(G) = k(G-e)-1$, $n(G) = n(G-e)$ and $r(G) = r(G-e)+1$. Thus we have

$$U(G; x, y, \alpha, \sigma, \tau) = \alpha^{k(G)} \cdot \sigma^{n(G)} \cdot \tau^{r(G)} \cdot T(G; \alpha \cdot x/\tau, y/\sigma)$$

$$= \alpha^{k(G-e)-1} \cdot \sigma^{n(G-e)} \cdot \tau^{r(G-e)+1} \cdot (\alpha \cdot x/\tau) \cdot T(G - e; \alpha \cdot x/\tau, y/\sigma)$$

$$= x \cdot U(G - e; x, y, \alpha, \sigma, \tau).$$

If e is a loop, then $k(G) = k(G - e)$, $n(G) = n(G - e) + 1$ and $r(G) = r(G - e)$. Thus we have

$$U(G; x, y, \alpha, \sigma, \tau) = \alpha^{k(G)} \cdot \sigma^{n(G)} \cdot \tau^{r(G)} \cdot T(G; \alpha \cdot x/\tau, y/\sigma)$$

$$= \alpha^{k(G-e)} \cdot \sigma^{n(G-e)+1} \cdot \tau^{r(G-e)} \cdot (y/\sigma) \cdot T(G - e; \alpha \cdot x/\tau, y/\sigma)$$

$$= y \cdot U(G - e; x, y, \alpha, \sigma, \tau).$$

Lastly, if e is neither a bridge nor a loop, then $k(G) = k(G-e)$, $n(G) = n(G-e)+1$, $r(G) = r(G - e)$, $k(G) = k(G/e)$, $n(G) = n(G/e)$ and $r(G) = r(G/e) + 1$. Thus

$$U(G; x, y, \alpha, \sigma, \tau) = \alpha^{k(G)} \cdot \sigma^{n(G)} \cdot \tau^{r(G)} \cdot T(G; \alpha \cdot x/\tau, y/\sigma)$$

$$= \alpha^{k(G)} \cdot \sigma^{n(G)} \cdot \tau^{r(G)} \cdot \{T(G - e; \alpha \cdot x/\tau, y/\sigma) + T(G/e; \alpha \cdot x/\tau, y/\sigma)\}$$

$$= \alpha^{k(G-e)} \cdot \sigma^{n(G-e)+1} \cdot \tau^{r(G-e)} \cdot T(G - e; \alpha \cdot x/\tau, y/\sigma)$$

$$+\alpha^{k(G/e)} \cdot \sigma^{n(G/e)} \cdot \tau^{r(G/e)+1} \cdot T(G/e; \alpha \cdot x/\tau, y/\sigma)$$

$$= \sigma \cdot U(G-e; x, y, \alpha, \sigma, \tau) + \tau \cdot U(G/e; x, y, \alpha, \sigma, \tau).$$

This completes the proof. ■

Like the Tutte polynomial, this universal form also has multiplicative properties.

Corollary 27 *Let G_1 and G_2 be vertex disjoint graphs. Then*

$$U(G_1 \cup G_2; x, y, \alpha, \sigma, \tau) = U(G_1; x, y, \alpha, \sigma, \tau) \cdot U(G_2; x, y, \alpha, \sigma, \tau).$$

And let G_1 and G_2 share exactly one vertex. Then

$$U(G_1 \cup G_2; x, y, \alpha, \sigma, \tau) = \frac{U(G_1; x, y, \alpha, \sigma, \tau) \cdot U(G_2; x, y, \alpha, \sigma, \tau)}{\alpha}.$$

Proof It is easy to see the following:

$$k(G_1 \cup G_2) = \begin{cases} k(G_1) + k(G_2) & \text{if } G_1 \text{ and } G_2 \text{ are vertex disjoint graphs,} \\ k(G_1) + k(G_2) - 1 & \text{if } G_1 \text{ and } G_2 \text{ share exactly one vertex,} \end{cases}$$

$$r(G_1 \cup G_2) = r(G_1) + r(G_2),$$

$$n(G_1 \cup G_2) = n(G_1) + n(G_2).$$

The following equality holds from Theorem 26.

$$U(G_1 \cup G_2; x, y, \alpha, \sigma, \tau) = \alpha^{k(G_1 \cup G_2)} \cdot \sigma^{n(G_1 \cup G_2)} \cdot \tau^{r(G_1 \cup G_2)} \cdot T(G_1 \cup G_2; \alpha \cdot x/\tau, y/\sigma)$$

we know the Tutte polynomial is multiplicative from Corollary 3,

$$= \alpha^{k(G_1 \cup G_2)} \cdot \sigma^{n(G_1 \cup G_2)} \cdot \tau^{r(G_1 \cup G_2)} \cdot T(G_1; \alpha \cdot x/\tau, y/\sigma) \cdot T(G_2; \alpha \cdot x/\tau, y/\sigma)$$

if G_1 and G_2 are vertex disjoint graphs,

$$= \alpha^{k(G_1)+k(G_2)} \cdot \sigma^{n(G_1)+n(G_2)} \cdot \tau^{r(G_1)+r(G_2)} \cdot T(G_1; \alpha \cdot x/\tau, y/\sigma) \cdot T(G_2; \alpha \cdot x/\tau, y/\sigma)$$

$$= \alpha^{k(G_1)} \cdot \sigma^{n(G_1)} \cdot \tau^{r(G_1)} \cdot T(G_1; \alpha \cdot x/\tau, y/\sigma) \cdot \alpha^{k(G_2)} \cdot \sigma^{n(G_2)} \cdot \tau^{r(G_2)} \cdot T(G_2; \alpha \cdot x/\tau, y/\sigma)$$

$$= U(G_1; x, y, \alpha, \sigma, \tau) \cdot U(G_2; x, y, \alpha, \sigma, \tau).$$

and if G_1 and G_2 share exactly one vertex,

$$= \alpha^{k(G_1)+k(G_2)-1} \cdot \sigma^{n(G_1)+n(G_2)} \cdot \tau^{r(G_1)+r(G_2)} \cdot T(G_1; \alpha \cdot x/\tau, y/\sigma) \cdot T(G_2; \alpha \cdot x/\tau, y/\sigma)$$

$$= \alpha^{k(G_1)} \cdot \sigma^{n(G_1)} \cdot \tau^{r(G_1)} \cdot T(G_1; \alpha \cdot x/\tau, y/\sigma) \cdot \alpha^{k(G_2)} \cdot \sigma^{n(G_2)} \cdot \tau^{r(G_2)} \cdot T(G_2; \alpha \cdot x/\tau, y/\sigma) \cdot \alpha^{-1}$$

$$= \frac{U(G_1; x, y, \alpha, \sigma, \tau) \cdot U(G_2; x, y, \alpha, \sigma, \tau)}{\alpha}.$$

This completes the proof. ■

4.2 Relations between graph polynomials

In Chapters 2 and 3, we discussed several graph polynomials. With the aid of Theorem 26, We will now show their relationships.

4.2.1 The chromatic polynomial

In Chapter 2, we present the definition and basic properties of the chromatic polynomial. We begin with the chromatic polynomial to show connections with the Tutte polynomial.

Theorem 28 *For the chromatic polynomial $C(G; x)$ and the Tutte polynomial $T(G; x, y)$ of graph G,*

$$C(G; i) = (-1)^{r(G)} \cdot i^{k(G)} \cdot T(G; 1 - i, 0).$$

Proof From Lemma 10 in Chapter 2, we know that $C(E_n; i) = i^n$ and

$$C(G; i) = \begin{cases} \frac{i-1}{i} \cdot C(G - e; i) & \text{if } e \text{ is a bridge,} \\ 0 & \text{if } e \text{ is a loop,} \\ C(G - e; i) - C(G/e; i) & \text{otherwise.} \end{cases}$$

We get $x = \frac{i-1}{i}$, $y = 0$, $\alpha = i$, $\sigma = 1$, $\tau = -1$.

Hence, according to Theorem 26, it is immediate that

$$C_G(i) = U(G; (i-1)/i, 0, i, 1, -1) = (-1)^{r(G)} \cdot i^{k(G)} \cdot T(G; 1-i, 0).$$

This completes the proof. ■

4.2.2 The monochrome polynomial

Theorem 29 *For the monochrome polynomial $M(G; x, s)$ and the Tutte polynomial $T(G; x, y)$ of graph G,*

$$M(G; i, s) = i^{k(G)} \cdot (s-1)^{r(G)} \cdot T(G; (s+i-1)/(s-1), s).$$

Proof From Theorem 12 in Chapter 2, we know that $M(E_n; i, s) = i^n$ and

$$M(G; i, s) = \begin{cases} \frac{s+i-1}{i} \cdot M(G-e; i, s) & \text{if } e \text{ is a bridge,} \\ s \cdot M(G-e; i, s) & \text{if } e \text{ is a loop,} \\ M(G-e; i, s) + (s-1) \cdot M(G/e; i, s) & \text{otherwise.} \end{cases}$$

We get $x = \frac{s+i-1}{i}$, $y = s$, $\alpha = i$, $\sigma = 1$, $\tau = s-1$.

Hence, according to Theorem 26, it is immediate that

$$M(G; i, s) = U(G; (s+i-1)/i, s, i, 1, s-1) = i^{k(G)} \cdot (s-1)^{r(G)} \cdot T(G; (s+i-1)/(s-1), s).$$

This completes the proof. ■

4.2.3 The flow polynomial

Theorem 30 *For the flow polynomial $F(G;A)$, where $|A|$ is the order of group A and the Tutte polynomial $T(G;x,y)$ of graph G,*

$$F(G;A) = (-1)^{n(G)} \cdot T(G;0,1-|A|).$$

Proof From Theorem 17 in Chapter 2, we know that $F(E_n;A) = 1^n = 1$ and

$$F(G;A) = \begin{cases} 0 & \text{if } e \text{ is a bridge,} \\ (|A|-1) \cdot F(G-e;A) & \text{if } e \text{ is a loop,} \\ F(G/e;A) - F(G-e;A) & \text{otherwise.} \end{cases}$$

We get $x = 0$, $y = |A| - 1$, $\alpha = 1$, $\sigma = -1$, $\tau = 1$.
Hence, according to Theorem 26, it is immediate that

$$F(G;x) = U(G;0,|A|-1,1,-1,1) = (-1)^{n(G)} \cdot T(G;0,1-|A|).$$

This completes the proof. ■

4.2.4 The acyclic orientation

We are now able to prove the last equality of Corollary 7 in Chapter 2.

Theorem 31 *For the acyclic orientation $a(G)$ and the Tutte polynomial $T(G;x,y)$ of graph G,*

$$a(G) = T(G;2,0).$$

We present two different proofs of this theorem.

Proof 1

From Theorem 19 in Chapter 2, we know that $a(E_n) = 1^n = 1$ and

$$a(G) = \begin{cases} 2 \cdot a(G-e) & \text{if } e \text{ is a bridge,} \\ 0 & \text{if } e \text{ is a loop,} \\ a(G-e) + a(G/e) & \text{otherwise.} \end{cases}$$

We get $x = 2$, $y = 0$, $\alpha = 1$, $\sigma = 1$, $\tau = 1$.

Hence, according to Theorem 26, it is immediate that

$$a(G) = T(G; 2, 0).$$

This completes the proof. ■

Proof 2

From Corollary 20 in Chapter 2, we know that

$$C(G; -1) = (-1)^n \cdot a(G).$$

From Theorem 28 in Chapter 4, we also know that

$$C(G; x) = (-1)^{r(G)} \cdot x^{k(G)} \cdot T(G; 1-x, 0).$$

Thus we have

$$a(G) = (-1)^{-n} \cdot C(G; -1) = (-1)^{-n} \cdot (-1)^{r(G)} \cdot (-1)^{k(G)} \cdot T(G; 2, 0)$$

$$= (-1)^{-n+r(G)+k(G)} \cdot T(G; 2, 0) = (-1)^0 \cdot T(G; 2, 0) = T(G; 2, 0).$$

This completes the proof. ■

4.2.5 The percolation process

Theorem 32 *For the percolation probability $P(G;p)$ and the Tutte polynomial $T(G;x,y)$ of graph G,*

$$P(G;p) = (1-p)^{n(G)} \cdot p^{r(G)} \cdot T(G;1,1/(1-p)).$$

Proof From Theorem 23 in Chapter 3, we know that $P(E_n;p) = 1^n = 1$ and

$$P(G;p) = \begin{cases} p \cdot P(G-e;p) & \text{if } e \text{ is a bridge,} \\ P(G-e;p) & \text{if } e \text{ is a loop,} \\ (1-p)\cdot P(G-e;p)) + p \cdot P(G/e;p) & \text{otherwise.} \end{cases}$$

We get $x = p$, $y = 1$, $\alpha = 1$, $\sigma = 1-p$, $\tau = p$.
Hence, according to Theorem 26, it is immediate that

$$P(G;p) = U(G;p,1,1,1-p,p) = (1-p)^{n(G)} \cdot p^{r(G)} \cdot T(G;1,1/(1-p)).$$

This completes the proof. ■

4.2.6 The dichromatic polynomial and the Potts model

Theorem 33 *For the partition function of the q-state Potts model $Z(G;q,v)$ and the Tutte polynomial $T(G;x,y)$ of graph G,*

$$Z(G;q,v) = q^{k(G)} \cdot v^{r(G)} \cdot T(G;(q+v)/v, v+1).$$

Proof From Theorem 24 in Chapter 3 and Theorem 15 in Chapter 2, we know that $Z(G;q,v)$ is a dichromatic polynomial and has the following properties.

$$Z(E_n;q,v) = q^n,$$

and

$$Z(G;q,v) = \begin{cases} \frac{q+v}{q} \cdot Z(G-e;q,v) & \text{if } e \text{ is a bridge,} \\ (v+1) \cdot Z(G-e;q,v) & \text{if } e \text{ is a loop,} \\ Z(G-e;q,v) + v \cdot Z(G/e;q,v) & \text{otherwise.} \end{cases}$$

We get $x = \frac{q+v}{q}$, $y = v+1$, $\alpha = q$, $\sigma = 1$, $\tau = v$.

Hence, according to Theorem 26, it is immediate that

$$Z(G;q,v) = U(G;(q+v)/q, v+1, q, 1, v) = q^{k(G)} \cdot v^{r(G)} \cdot T(G;(q+v)/v, v+1).$$

This completes the proof. ■

Chapter 5

Other results

In this chapter, we consider only *planar graphs*, defined as graphs drawn on a plane such that no two edges cross. First, we will generalize the dichromatic polynomial and define a surrounding lattice L' according to a planar lattice L. Next, we will decompose the graph L' and will apply it to the ice-type model.

5.1 Planar lattices

In 1971, Temperley and Lieb established a remarkable relation between the Tutte polynomial on a square lattice graph L and the partition function on a related square lattice polygon surrounding graph L' [9]. And in 1975, R. J. Baxter, S. B. Kelland and F. Y. Wu developed the relation between the partition function and the Whitney polynomial which is the same as the dichromatic polynomial [1].

5.1.1 Generalization of the partition function

In Chapter 2, we found that the partition function

$$Z(G;q,v) = \sum_{\sigma} \prod_{(i,j)\in E} (1 + v \cdot \delta(\sigma_i, \sigma_j))$$

is the dichromatic polynomial. From Theorem 14, we also know that

$$\tilde{D}(G;x,y) = \sum_{F \subseteq G} x^{k(F)} y^{n(F)}$$

is also the dichromatic polynomial. In short, $D(G;x,y) = \tilde{D}(G;x,y) = Z(G;x,y)$. Let us now modify the polynomial $\tilde{D}(G;x,y)$ and define a new polynomial *Lattice polynomial*

$$L(G;x,y) = \sum_{F \subseteq G} x^{k(F)} y^{|F|}.$$

The next theorem shows that $L(G;q,v)$ is also the dichromatic polynomial and demonstrates the partition function.

Theorem 34

$$L(G;x,y) = Z(G;x,y) = D(G;x,y).$$

Proof It suffices to show that $L(G;x,y)$ is the dichromatic polynomial.

For the first condition of Definition 13,

$$L(E_n;x,y) = \sum_{F \subseteq E_n} x^{k(F)} y^{|F|} = x^n y^0 = x^n.$$

Now, let us show the second condition.

$$L(G;x,y) = \sum_{F \subseteq G} x^{k(F)} y^{|F|}$$

$$= \sum_{F \subseteq G-e} [x^{k(F)} y^{|F|} + x^{k(F \cup e)} y^{|F \cup e|}]$$

$$= \sum_{F \subseteq G-e} x^{k(F)} y^{|F|} + y \cdot \sum_{F \subseteq G/e} x^{k(F)} y^{|F|}$$

$$= L(G-e;x,y) + y \cdot L(G/e;x,y).$$

This completes the proof. ■

Until now, we have assumed that all interactive energies are the same and equal to ϵ. Let us now make it more general. We group the edges of the graph into certain classes. For example, a square gird has two classes of edges, one is vertical and the other is horizontal. Let us define the interactive energy as ϵ_r and $v_r = e^{\epsilon_r \beta} - 1$ for class r. Then the generalized partition function is expressed as

$$L'(G; q, v_1, v_2, \ldots) = \sum_{F \subseteq G} q^{k(F)} \cdot v_1^{l_1} \cdot v_2^{l_2} \cdot \ldots,$$

where l_i is the number of edges in the class i in F.

5.1.2 Planar lattice and its surrounding lattice

Though the partition function $L'(G; q, v_1, v_2, \ldots)$ defined in 5.1.1 applies to any graph, we only consider planar lattices in this chapter. For every planar lattice L, we associate another planar lattice L' according to L. The following are the definitions of the *surrounding lattice* L' of L [1].

Definition 35 *We associate with L another planar lattice L′, as follows.*

Draw simple polygons surrounding each site of L such that

1. no polygons overlap, and no polygon surrounds another,

2. polygons of non-adjacent sites have no common corner,

3. polygons of adjacent sites i and j have one and only one common corner. This corner lies on the edges (i, j).

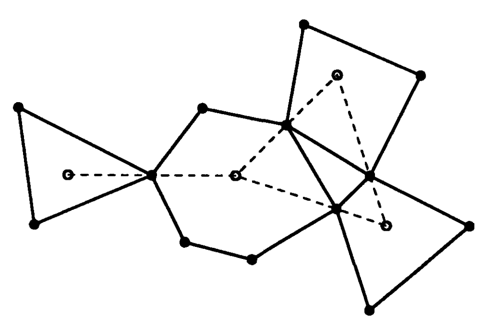

Figure 5-1: A planer lattice L (a dotted structure) and its surrounding lattice L'.

The surrounding lattice L' is not unique for given L. But the general argument in the following section should apply to any choice of L'.

5.1.3 Polygon decomposition

We now have a lattice $L = (V, E)$ and its surrounding lattice L'. For a subgraph $G = (V, F)$ of L where $F \subset E$, we will make a bijection between the subgraph G and the decomposition of L'.

If G does not contain an edge (i, j), then we separate the polygons of L' corresponding to the each vertex i and j, as in figure 5-2(a). if G contains an edge (i, j), then we join the polygons of L' corresponding to the each vertex i and j, as in figure 5-2(b).

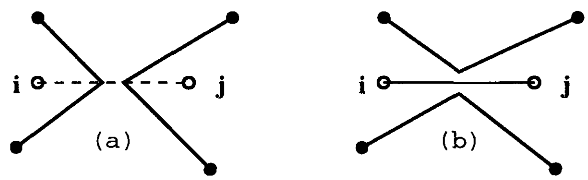

Figure 5-2: The two possible cases for an edge (i, j)

Here, figure 5-3 is an example of decomposition of L' shown in figure 5-1.

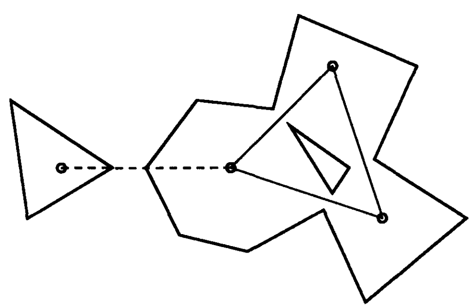

Figure 5-3: A subgraph G of L, and the corresponding decomposition of L'.

We use the term *island* for a connected large polygon in the decomposition of L'. And if one polygon contains another polygon, we use the term *lake* to describe the smaller one. So every polygon in the decomposition of L' is one of two types, an island without lake or an island with lake. Clearly, any connected component of G corresponds to an island in the decomposition of L'.

We define p to be the number of polygons in the decomposition of L', C to be the number of connected components in G or equivalent to the number of islands in the decomposition of L', and S to be the number of circuits of G or equivalent to the number of lakes in the decomposition of L'. It is obvious that

$$(1)\ p = C + S.$$

If we also define N to be the number of vertices of G, $|F| = l_1 + l_2 + \ldots$ to be the number of edges of G, and $|R| = S + 1$ to be the number of regions of G, then using the following equation from the Euler characteristic

$$|N| - |F| + |R| - |C| = 1,$$

we obtain:

$$N - (l_1 + l_2 + \ldots) + (S + 1) - C = 1.$$

Then we have

$$(2)\ S = C - N + l_1 + l_2 + \ldots,$$

where l_i is the number of edges of i-th component in G. If we eliminate S from the equation (2) using (1), we get

$$(3)\ C = 1/2 \cdot (p + N - l_1 - l_2 - \ldots).$$

If we plug the equation (3) into the generalized partition function from [5.1.1], we get

$$L'(G; q, v_1, v_2, \ldots) = \sum_{H \subseteq G} q^{k(H)} \cdot v_1^{l_1} \cdot v_2^{l_2} \cdot \ldots = \sum q^C \cdot v_1^{l_1} \cdot v_2^{l_2} \cdot \ldots$$

$$= \sum q^{1/2 \cdot (p+N-l_1-l_2-\ldots)} \cdot v_1^{l_1} \cdot v_2^{l_2} \cdot \ldots = q^{N/2} \cdot \sum q^{p/2} \cdot q^{-l_1/2} \cdot v_1^{l_1} \cdot q^{-l_2/2} \cdot v_2^{l_2} \ldots$$

$$= q^{N/2} \cdot \sum q^{p/2} \cdot x_1^{l_1} \cdot x_2^{l_2} \ldots,$$

where $x_i = q^{-1/2} \cdot v_i$.

Now we can take the summation over all polygon decompositions of G, where p is the number of polygons in the decomposition and l_i is the number of internal sites of the i-th island in the decomposition.

5.2 The ice-type model

In this section, we will present the definition of the ice-type model on the lattice L' and will find the relation between its partition function and the partition function defined in [5.1.3].

The following definitions are originally from [1].

Definition 36 *Let θ and z be two parameters given by*

$$q^{1/2} = 2 \cdot \cosh\theta, \; z = \exp(\theta/2\pi).$$

Then the ice-type model is defined as follows.

1. Place arrows on the edges of L' so that at each site an equal number of arrows points in and out.

2. With each external site associate a weight z^{α} if an observer moving in the direction of the arrows turns through an angle α to his left, or an angle $-\alpha$ to his right, as he goes through the site. This angle α is shown in figure 5-4.

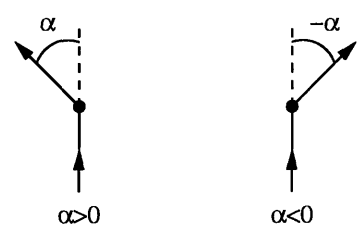

Figure 5-4: Directions regarding α

3. There are six possible arrangements of arrows at an internal site, as shown in figure 5-5. With arrangement k on a site of class r associate a weight ω_k, where

$$\omega_1, \ldots, \omega_6 = z\alpha - \gamma, z^{\gamma-\alpha}, x_r \cdot z^{\beta-\delta}, x_r \cdot z^{\delta-\beta}, z^{-\beta-\delta} + x_r \cdot z^{\alpha+\gamma}, z^{\beta+\delta} + x_r \cdot z^{-\alpha-\gamma},$$

and the angle α, β, γ, δ are those shown in figure 5-5.

The partition function of this ice-type model is

$$Z' = \sum \prod (weights),$$

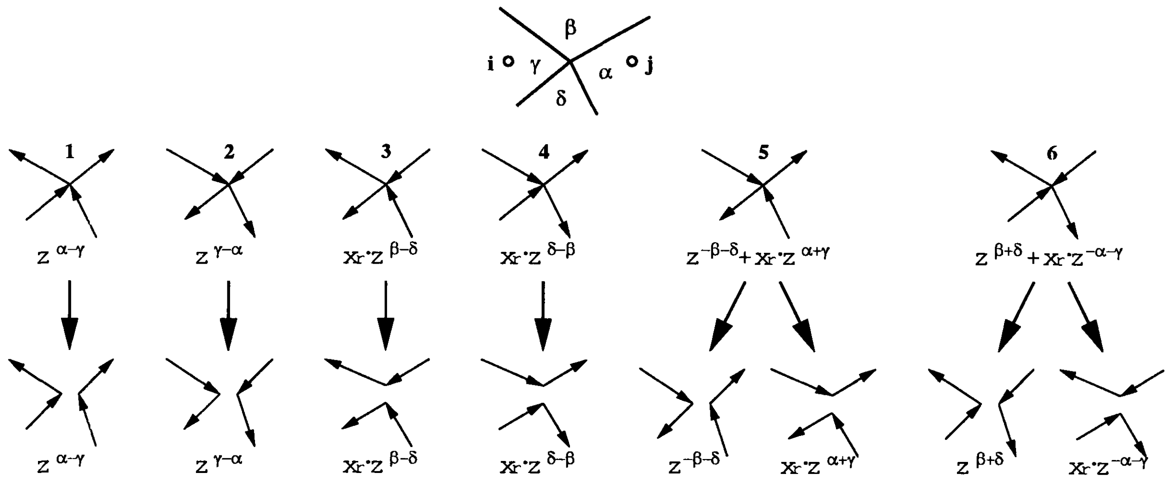

Figure 5-5: Arrow configurations and decomposition

where the summation over all allowed arrow configurations on L' and the product is over all sites of L'.

We now derive an interesting theorem. The next theorem shows the relation between the partition function of the ice-type model and the partition function of the Potts model defined in the preceding section.

Theorem 37

$$L'(G; q, v_1, v_2, \ldots) = q^{N/2} \cdot Z',$$

where L' is the partition function of the Potts model.

Proof It suffices to show that

$$\sum q^{p/2} \cdot x_1^{l_1} \cdot x_2^{l_2} \ldots = \sum \prod (\text{weights}).$$

From the above equation, it also suffices to show that

$$q^{p/2} \cdot x_1^{l_1} \cdot x_2^{l_2} \ldots = \prod (\text{weights}),$$

for a given polygon decomposition of L'.

We will calculate the product of weights by moving around a polygon in the direction of the arrow and multiplying the weight whenever we get to the sites of the polygon. After we finish moving around one polygon, we use the same procedure for the next polygon. Since every polygon is a simple closed curve, if we move around the polygon, then we make a $\pm 2\pi$ turn. The sign of the angle depends on the directions of the arrows of the polygon, clockwise or counterclockwise. Since both cases occur, the total weight for a polygon is $z^{2\pi} + z^{-2\pi}$.

For a r-th polygon which has l_r internal sites, the total weight is $x_r^{l_r} \cdot (z^{2\pi} + z^{-2\pi})$. Since every internal site has the term x_r in its weight. Therefore, if there are p components in L', the total product of weight is

$$\prod x_1^{l_1} \cdot (z^{2\pi} + z^{-2\pi}) \cdot x_2^{l_2} \cdot (z^{2\pi} + z^{-2\pi}) \cdot x_3^{l_3} \cdot (z^{2\pi} + z^{-2\pi}) \cdot \ldots = \prod (z^{2\pi} + z^{-2\pi})^p \cdot x_1^{l_1} \cdot x_2^{l_2} \cdot x_3^{l_3} \cdot \ldots$$

$$= \prod q^{p/2} \cdot x_1^{l_1} \cdot x_2^{l_2} \cdot x_3^{l_3} \cdot \ldots ,$$

since $q^{1/2} = 2 \cdot cosh\theta = exp(\theta) + exp(-\theta) = z^{2\pi} + z^{-2\pi}$. This completes the proof. ■

Bibliography

[1] R. J. Baxter, S. B. Kelland and F. Y. Wu. Equivalence of the Potts model or Whitney polynomial with an ice-type model. *J. Phys. A: Math. Gen.*, 9:397-406, 1976.

[2] G. D. Birkhoff and D. C. Lewis. Chromatic Polynomials. *Trans. Amer. Math. Soc.*, 60:355-451, 1946.

[3] G. D. Birkhoff. A Determinant Formula for the Number of Ways of Coloring a Map. *Ann. Math.*, 14:42-46, 1912.

[4] Béla Bollobás and Oliver Riordan. A Tutte Polynomial for Colored Graphs. *Combinatorics, Probability and Computing.*, 8:45-93, 1999.

[5] Béla Bollobás. *Modern graph theorey*, volume 184 of *Graduate Texts in Mathematics.* Springer, New York, 1998.

[6] Reinhard Diestel. *Graph theorey*, volume 173 of *Graduate Texts in Mathematics.* Springer, New York, 2000.

[7] Michael Korn and Igor Pak. Combinatorial evaluations of the Tutte polynomial. Preprint, 2003.

[8] Richard P. Stanley. Acyclic orientations of graphs. *Discrete Math.*, 5:171-178, 1973.

[9] H. N. V. Temperley and E. H. Lieb. Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular

planar lattice: some exact results for the percolation problem. *Proc. Royal Soc. London A.*, 322:251-280, 1971.

[10] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Can. J. Math.*, 6:80-91, 1954.

[11] D. J. A. Welsh. *Complexity: knots, colourings and counting*, colume 186 of *London Math. Soc. Lecture Note Series.* Cambridge U. Press, Cambridge, 1993.