ELECTROHYDRODYNAMIC AND MAGNETOHYDRODYNAMIC SURFACE WAVES AND INSTABILITIES

by

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ABSTRACT

The concern of this thesis is with purely superficial magneto- and electrohydrodynamic interactions with liquid-liquid or liquid-gas interfaces supporting free, polarization or magnetic charge; free, Korteweg or Amperian current. The problems considered are so arranged as to exclude any coupling between fields and fluid in the volume. Those effects resulting from the gravitational field and the action of cohesion are included.

Surface problems in six configurations are analyzed and compared. Use is made of a perturbation analysis and of a "space rate" expansion of the dependent variables. Surface waves are classified according to whether the steady electric or magnetic fields are perpendicular (type I) or tangential (type II) to the interface.

Electrohydrodynamic examples of type I and type II waves are experimentally verified by studying surface instability, radiation and resonance. Conditions for resonance in rectangular geometry are given for each of the six waves. The type I waves are characterized by a phase velocity that decreases with increasing field strength, the possibility of negative group velocity simultaneously with a positive phase velocity and by a discrete field-strength and wave-length for impending instability.

Type II waves theoretically may propagate more, or less rapidly along field lines, depending on the strictions at the interface. Although the Clausius-Mossotti equation is commonly used to illustrate the effect of strictions at dielectric interfaces, its validity is clearly contradicted by experimental evidence that the waves are speeded up.

A discussion of the non-linear behavior of two of the waves is given, in the limit where the interface interacts strongly with external boundaries. The growth of magneto- and electrohydrodynamic shocks and anti-shocks from compression and depression waves is discussed, with transition electrohydrodynamic waves originating from time-like data given to illustrate waves partly controlled by gravity and partly by the electric field. Waves initiated from space-like data are also described and the integral shock and anti-shock relations derived.

Thesis Supervisor: Herbert H. Woodson
Title: Associate Professor of Electrical Engineering

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The author has been privileged to make contact with people of widely varying viewpoint and interest who have strongly influenced his thinking. Discussions with Professors H.A. Haus and W.P. Allis have injected useful ideas not ordinarily connected with fluid dynamics, while work with Professor D. Young of Iowa State University provided an introduction to some unusual aspects of fluid dynamics.

This work is the outgrowth of a project concerning magnetohydrodynamic bulk interactions suggested by Professor H.H. Woodson, who provided valuable criticism and comments throughout the research effort. More than this, however, the environment of the Energy Conversion Group has been conducive to an efficient use of research time.

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CHAPTER 1

INTRODUCTION

A. Purpose

Water or gravity waves have been of interest since at least the time of Lagrange. As a result of this interest, problems involving the dynamics of a free fluid interface essentially controlled by superficial forces have received considerable attention. More recently, problems concerning free surfaces supporting free surface currents have appeared in the context of magnetohydrodynamic stability theory.[1,2,*] On the other hand, even the casual observer has seen the effect of electric surface charge on the surface of a liquid jet or the surface of a water drop; problems that had received some mathematical attention even before the turn of the century.[3]

This thesis represents an effort to go back to the beginning. Field coupled surface waves, as they appear in the literature, are complicated by geometry, until the basic features of the dynamics are obscure and the possibilities of experimental verification unnecessarily remote. An effort is made here to compare several basic types of waves that can exist at fluid interfaces stressed by either magnetic or electric fields, under conditions made as simple as is consistent with a laboratory investigation. No bulk forces are involved, so that the complicating features of coupling between bulk and surface interactions are removed. The philosophy used here is to begin with as simple a set of models as is physically plausible and proceed to a detailed mathematical description and comparison. The major classifications of waves (those stressed by perpendicular or tangential

*The superscript numerals refer to the Bibliography that begins in Appendix D.

fields) are then illustrated experimentally by two of the three electrohydrodynamic wave types. The experimental results are accurate enough to serve to justify the mathematical model used.

Two of the wave types discussed, those involving free currents and free charges, illustrate a concept that will be termed anti-duality, as opposed to duality. Among the six types of waves investigated, four have complete duals while two have complete anti-duals. One system is the dual of another if the defining equations for one system result from substituting analogous or dual variables and constants in the equations describing the dual system. An anti-dual involves a change of sign of a constant of one set of equations, in order to obtain the equations of the anti-dual.

In the same sense as for duality, the worth of recognizing anti-duality arises from the considerable economy of effort that results, because computations need only be extended to include the reversal of sign in one of the parameters. Unlike systems that satisfy conditions of duality, however, anti-duals have antithetic behavior. Hence, wave like properties are anti-dual to unstable behavior. A surface shock on a compression wave is the anti-dual of a shock (or anti-shock as it is dubbed here) on a depression wave.

This work is not meant to be a complete investigation, but rather a starting point from which work in the area of linear and non-linear magneto- and electrohydrodynamics of surface interactions may proceed in an efficient and physically meaningful way. The equations of motion and boundary conditions tabulated here indicate for example the consistent sets of equations that may be used to analyze problems for each wave type in more complicated geometries.

B. Background

The literature of surface waves is one of the oldest of mathematical physics. Yet, there are many of the well-developed concepts that appear but rarely in the context of modern-day investigations of field coupled surface waves. Here, emphasis is placed on this work by putting the field coupled waves into the nomenclature of the more common gravity and capillary surface waves. The exposition on Water Waves[4] by Stoker is an invaluable reference, and of course Lamb's Hydrodynamics[5] is of great assistance.

Several of the wave types investigated here have precedent because of their connection with the theory of stability. The role of the free surface current in various configurations involving perfect conductors has been rather exhaustively investigated as it applies to the thermonuclear effort.[6] An early interest in the dynamics of charged jets has already been indicated. More recently some work has been done, by means of energy principles, on the stability of a dielectric jet in a longitudinal electric field.[7]

The area of electrohydrodynamics has received some attention recently[8] concerning electro-convection of surfaces, an outgrowth of work done by Avsec and Luntz in the late 30's.[9] The phenomenon they considered is of some importance in the present experimental work because it appears as an interfering effect in the observation of one of the wave types.

After forty years of virtual stagnation, interest in electrohydrodynamics appears to be on the up-swing. In this connection it is important that the force density on a dielectric liquid be accurately known. In the chapters which follow, the stress tensor of Korteweg and Helmholtz will be used. It is common practice in well-known texts[10] to illustrate the use of this force density by computing the traction at the dielectric interface with the assumed density dependence of the permittivity given by the Clausius-Mossotti equation, and to arrive at results that are directly contradicted by certain of the experimental observations of Chapter 4. Hence, one of the functions of this work is to emphasize the fact that this equation is not valid when applied to a liquid interface.

Non-uniform electric field effects have been investigated experimentally at some length.[11] Perhaps too much has been attributed to the effect of the non-uniformity, however, as will be shown by the investigations of free charge interactions in Chapters 3 and 4.

C. Areas of Interest

The emphasis throughout this work is on the electrohydrodynamic waves. Aside from the fact that these represent interactions about which the least is known, they are the logical wave types to investigate first. This is true because they are most easily produced in the laboratory, where high electric fields are more easily obtained than high magnetic fields.

In fact, one is at more of an advantage than this, since the natural tendency of currents is to diffuse throughout the bulk of a conducting fluid while charges relax to the boundaries with time.

CHAPTER 2

FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

A. Bulk Equations

This thesis will be devoted to the low frequency electromechanical interactions between polarized or magnetized incompressible fluids and electric or magnetic fields. Hence, the equations that will be used to describe volume dynamics are the non-relativistic Maxwell equations for moving media,* Euler's equation, and the appropriate continuity conditions. The nomenclature used conforms to that commonly used in the literature. A list of symbols is given in Appendix A.

$$\rho \frac{D\bar{V}}{Dt} + \nabla p = \bar{F} \tag{2-1}$$

$$\nabla \cdot \bar{V} = 0 \tag{2-2}$$

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \tag{2-3}$$

$$\nabla \cdot \mu \bar{H} = 0 \tag{2-4}$$

$$\nabla \times \bar{H} = \bar{j} + \bar{V} q + \frac{\partial \varepsilon \bar{E}}{\partial t} + \nabla \times ((\varepsilon - \varepsilon_o)\bar{E} \times \bar{V}) \tag{2-5}$$

$$\nabla \cdot \varepsilon \bar{E} = q \tag{2-6}$$

The form of the body force $\bar{F}$ will depend on the problem considered, and will be introduced when needed. Equation (2-3) implies the Minkowski model of magnetization.

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*See Panofsky and Phillips,[10] p. 147.

The problems that will be considered have in common the following properties:

1. The bulk of the fluid does not support free current or charges.
2. Disturbances communicate with a celerity much less than that of sound in the fluid.

The first of these properties excludes free current or charge interactions from $\bar{F}$ in Eq. (2-1), while the second justifies the use of Eq. (2-2). Together they eliminate bound current or charge interactions from $\bar{F}$ and show that Eq. (2-1) is independent of the electromagnetic fields. It follows also that Eqs. (2-5) and (2-6) are homogeneous (the effects of polarization or displacement current are not significant unless the fluid is compressible).

In the work which follows, four types of problems will be considered that concern interactions between:

1. electric fields and bound surface currents or charges;
2. magnetic fields and bound surface currents or charges;
3. electric fields and both free and bound surface charges;
4. magnetic fields and both free and bound surface currents.

Each of these problem types corresponds to a further simplification of the equations (2-1) through (2-6). The fact that the currents of Eq. (2-5) are dropped from the problems involving electric fields leads to a dynamics governed by Eqs. (2-1), (2-2), (2-3) with $\frac{\partial \bar{B}}{\partial t} = 0$ and (2-6). The problems involving magnetic fields but no free surface currents are predicted by the similar Eqs. (2-1), (2-2), (2-4) and (2-5) while the addition of a free surface current requires the use of Eq. (2-3).

All of these problems may be described by a single set of equations of motion (hybrid equations) if it is recognized that the equations imbody, in general, more information than is necessary to solve individual problems. A list of the equations of motion for each problem is given below.

$$\rho \frac{D\bar{V}}{Dt} + \nabla p = \bar{F} \qquad \text{(2-1a)}$$

$$\nabla \cdot \bar{V} = 0 \qquad \text{(2-2a)}$$

Electric Fields		Magnetic Fields		Hybrid Equations	
no free charge	free charge	no free current	free current		
$\nabla x E = 0$	$\nabla x \bar{E} = 0$		$\nabla x \bar{E} = - \frac{\partial \bar{B}}{\partial t}$	$\nabla x \bar{E} = - \frac{\partial \bar{B}}{\partial t}$	(2-3a)
$\nabla \cdot \varepsilon \bar{E} = 0$	$\nabla \cdot \varepsilon \bar{E} = q$			$\nabla \cdot \varepsilon \bar{E} = q$	(2-4a)
		$\nabla x \bar{H} = 0$	$\nabla x \bar{H} = \bar{j}$	$\nabla x \bar{H} = \bar{j}$	(2-5a)
		$\nabla \cdot \mu \bar{H} = 0$	$\nabla \cdot \mu \bar{H} = 0$	$\nabla \cdot \mu \bar{H} = 0$	(2-6a)

It may be seen that each of the sets of equations is a simplification of the hybrid equations.

In order to provide an economy of effort, six different problems will be solved simultaneously in

Chapter 3, using as a starting point the same set of hybrid equations. It is important to note that although these equations are not consistent (the neglect of displacement current leads to an inconsistent continuity of charge equation), the equations used in each particular problem are consistent.

B. Surface Conditions

As has been pointed out, the electric and magnetic fields will satisfy bulk equations that are elliptic in character, i.e. the static field equations. The fact that the present concern is with wave-like interactions or phenomenon predicted by hyperbolic equations points to the extreme importance of the boundary conditions.

The solutions to the bulk equations will differ according to the properties of the material occupying the region of interest. The boundary conditions, at the same time fit these solutions together so as to define field quantities throughout the region of interest, and define the domain of validity of the volume solutions.

The interface between two fluids is physically a three dimensional surface always composed of the same particles. If $F(x,y,z,t) = 0$ is the equation defining this surface in space and time, it follows from the physical definition that

$$\frac{DF}{Dt} = 0 \qquad (2\text{-}7)$$

The fact that particles on the surface defined by (2-7) remain on the surface is evident by a simple argument originating with Lagrange.* If (x,y,z) are the space coordinates of a particle with an initial position a,b,c the solution for the position of that particle at time t may be written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x(a,b,c,t) \\ y(a,b,c,t) \\ z(a,b,c,t) \end{bmatrix} \qquad (2\text{-}8)$$

The particle will always remain on a surface defined by

$$F(a,b,c,t) = 0 \qquad (2\text{-}9)$$

since a given point (a,b,c) always corresponds to the same particle. Equation (2-9) may be solved for a,b,c as functions of (x,y,z). Hence differentiation of Eq. (2-9) with respect to time and the fact that $(V_x,V_y,V_z) = (\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t})$ gives Eq. (2-7).

Boundary conditions are commonly formulated in terms of a unit normal vector $\bar{n}$. Because this vector intimately depends on the orientation of the surface, but not on dynamical considerations, it is purely a function of $F(x,y,z,t) = 0$. That is,[12]

$$\bar{n} = \frac{\nabla F}{[\nabla F \cdot \nabla F]^{1/2}} \qquad (2\text{-}10)$$

Equations (2-7) and (2-10) are quite general in that they hold for any interface, free or not.

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*See Lamb,[5] p. 7.

The boundary conditions are a result of integrating Eqs. (2-1a) through (2-6a) across the interface. It is convenient, in this connection, to consider the body force $\bar{F}$ as derived from the gradient of a stress tensor.

$$F_\alpha = \frac{\partial M_{\alpha\beta}}{\partial x_\beta} \qquad (2\text{-}11)$$

where $M_{\alpha\beta}$ includes the Maxwell stress tensor (superficial forces of electromagnetic origin) and mechanical stresses, such as those due to surface tension. Superscripts will designate the region in which the variable is to be evaluated while single subscripts will indicate the axis directions. The normal vector is directed from region (1) to (2), when evaluated at the fluid-fluid interface. The conditions that correspond to the hybrid Eqs. (2-1a) through (2-4a) are

$$n_\alpha[p^{(2)} - p^{(1)}] - [M_{\alpha\beta}^{(2)} - M_{\alpha\beta}^{(1)}]n_\beta = 0 \qquad (2\text{-}12)$$

$$\bar{n} \cdot [\bar{v}^{(2)} - \bar{v}^{(1)}] = 0 \qquad (2\text{-}13)$$

$$\bar{n} \times [\bar{E}^{(2)} - \bar{E}^{(1)}] - \bar{n} \cdot \bar{v}[\mu^{(2)}\bar{H}^{(2)} - \mu^{(1)}\bar{H}^{(1)}] = 0 \qquad (2\text{-}14)$$

$$\bar{n} \cdot (\mu^{(2)}\bar{H}^{(2)} - \mu^{(1)}\bar{H}^{(1)}) = 0 \qquad (2\text{-}15)$$

Because the electromagnetic body force is written in terms of the field variables (not the currents or charges) the conditions of Eqs. (2-5a) and (2-6a) are not needed unless the free charges or currents on the interface vanish.

In that case

$$n \times (\bar{H}^{(2)} - \bar{H}^{(1)}) = 0 \quad (2\text{-}16)$$

$$n \cdot (\varepsilon^{(2)} \bar{E}^{(2)} - \varepsilon^{(1)} \bar{E}^{(1)}) = 0 \quad (2\text{-}17)$$

PART I

CHAPTER 3

CLASSIFICATION OF WAVES; PERTURBATION THEORY

A. Classification

By far the most commonly used model for analyzing surface interactions is the "small signal" or "linearized" representation. This approximation is used by virtually all workers concerned with magnetohydrodynamic stability problems, whether they approach the problem by use of an energy principle or by means of direct dynamical analysis. It has the virtue of reducing the problem, not only to a linear one, but to one of fixed domain as well. As a result, solutions can be produced by superimposing normal modes that individually represent waves and instabilities.

As was pointed out in Chapter 1, the literature includes a considerable number of experimental observations on interactions between charges and electric fields of a highly inhomogeneous nature. Magnetohydrodynamic stability problems are mainly concerned with the useful interactions between currents and non-uniform magnetic fields. Yet, the normal mode analysis used, shows that each of these problems is ultimately a complicated set of waves and instabilities. It would seem worthwhile, in order to understand the basic character of the possible waves and instabilities, to consider them as they are supported in a situation of greatest possible simplicity, i.e. uniform fields and plane geometry. It will be seen that certain phenomena, previously associated with non-uniform fields, are accounted for by the dynamics of a surface stressed by a uniform field.

This chapter will be concerned with demonstrating theoretically six possible types of surface interactions in plane geometry. These are defined in Table (3-1).

Table (3-1)

Type	Static Field	Static Surface Sources or Vortices
EH-If	E perpendicular to interface	free charges
EH-Ip	E perpendicular to interface	polarization charges
EH-II	E tangential to interface	---
MH-I	B perpendicular to interface	magnetic charge
MH-IIf	B tangential to interface	free current
MH-IIa	B tangential to interface	amperian current

Just as volume Alfvén waves couple to acoustic and electromagnetic waves, so also incompressible surface waves couple to gravity and capillary waves. A physically realistic theory must include, therefore, the effects of energy storages in the gravitational field and in the molecular formation of the interface, as well as interactions between external boundaries and the interface. The problems under consideration are shown in Figs. 3-1 and 3-2. They involve two non-miscible fluids separated by a plane interface of tension T parallel to two rigid boundaries. The fluids are presumed inviscid, incompressible and perfect dielectrics. A gravitational field g is taken as acting in the negative x-direction with the other axes lying in the plane of the static interface. The circumstance of the free surface current of case MH-IIf might be created by passing a current through a conducting film at the interface. This problem is similar to the incompressible limit of the gravitational instability considered by M. Kruskal and M. Schwarszchild,[1] in the limit where $\mu^{(1)} = \mu^{(2)} =$ permeability of free space, surface tension is zero and $a = b \rightarrow \infty$.

B. Equations of Motion

The bulk equations consistent with the present problems are Eqs. (2-1a) through (2-6a) with $\bar{F} = -g\rho\bar{a}_x$, $\bar{j} = q = 0$.

The interface is conveniently described by letting

$F(x,y,z,t) = x - \xi(y,z,t) = 0$

Hence Eqs. (2-7) and (2-10) become

$$\frac{\partial\xi}{\partial t} - V_x + V_y\frac{\partial\xi}{\partial y} + V_z\frac{\partial\xi}{\partial z} = 0 \qquad (3\text{-}1)$$

$$\bar{n} = [\bar{a}_x - \bar{a}_y\frac{\partial\xi}{\partial y} - \bar{a}_z\frac{\partial\xi}{\partial z}][1 + (\frac{\partial\xi}{\partial y})^2 + (\frac{\partial\xi}{\partial z})^2]^{-1/2} \qquad (3\text{-}2)$$

The boundary conditions that apply at $x = \xi(y,z,t)$ are Eqs. (2-12) through (2-17). For the present set of problems the stress tensor $M_{\alpha\beta}$ is*

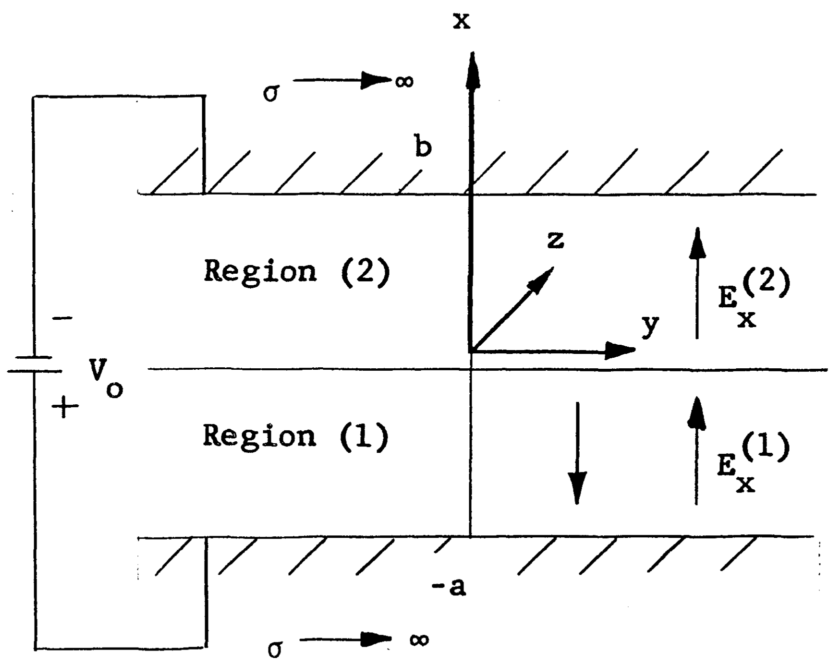
$$M_{\alpha\beta} = T_{\alpha\beta} + \delta_{\alpha\beta}T\left\{\frac{\partial^2\xi}{\partial y^2}[1 + (\frac{\partial\xi}{\partial y})^2]^{-3/2} + \frac{\partial^2\xi}{\partial z^2}[1 + (\frac{\partial\xi}{\partial z})^2]^{-3/2}\right\} \qquad (3\text{-}3)$$

where $T_{\alpha\beta}$ will be taken as the Maxwell stress tensor consistent with the surface traction derived by Korteweg and Helmholtz.[13]

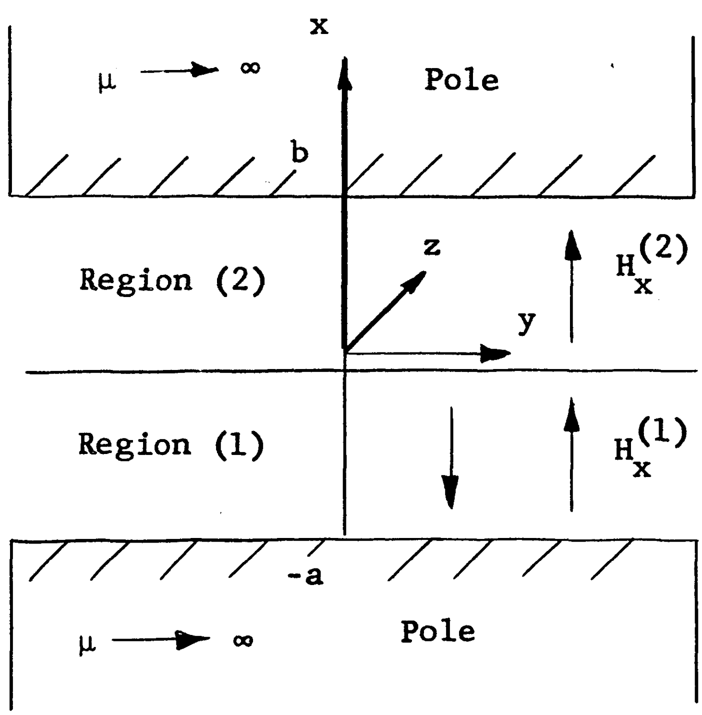
$$T_{\alpha\beta} = \varepsilon E_\alpha E_\beta - \frac{1}{2}\varepsilon\delta_{\alpha\beta}(1-c)E_\gamma E_\gamma + \mu H_\alpha H_\beta - \frac{1}{2}\mu\delta_{\alpha\beta}(1-d)H_\gamma H_\gamma \qquad (3\text{-}4)$$

---

*See Lamb,[5] p. 455.

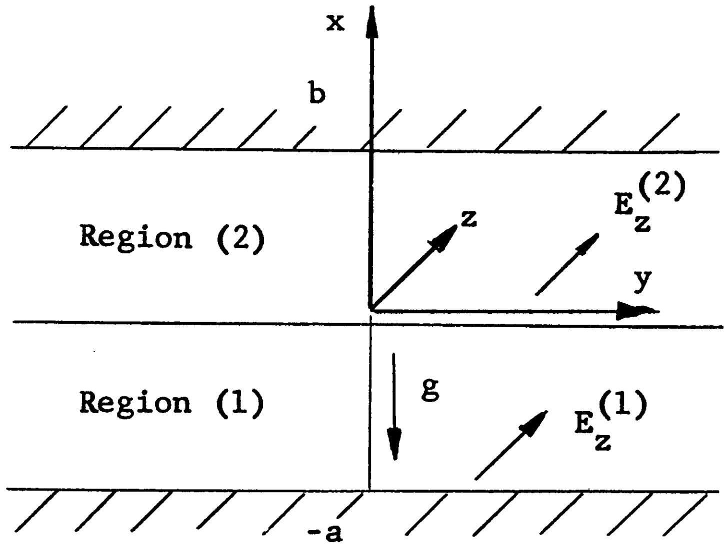


EH-If ; EH-Ip

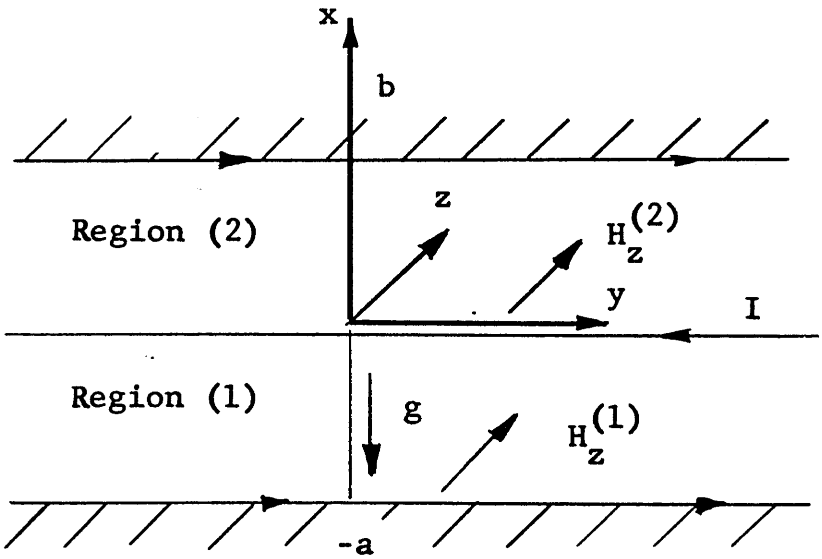


MH-I

Fig. 3-1 Type I Configuration



EH-II



MH-IIf ; MH-IIa

Fig. 3-2 Type II Configuration

Basic to the derivation of this tensor is the assumption that ε and μ are only functions of the density. The striction constants are defined by

$$d = \frac{\rho}{\mu}\frac{d\mu}{d\rho} \; ; \quad c = \frac{\rho}{\varepsilon}\frac{d\varepsilon}{d\rho}$$

In problems EH-If, EH-Ip and MH-IIf, the boundaries at $x = -a,b$ are perfectly conducting and in MH-I infinitely permeable, so that

$$\begin{aligned} &\text{EH-If: EH-Ip: MH-IIf;} \quad \bar{n} \times \bar{E} = 0 \\ &\text{MH-I;} \quad \bar{n} \times \bar{H} = 0 \end{aligned} \tag{3-5}$$

In problems EH-II and MH-IIa, it is required that as $x \rightarrow \infty$ the perturbation fields vanish. Finally, in all problems the boundaries are rigid so that at $x = -a,b$

$$\bar{n} \cdot \bar{V} = 0 \tag{3-6}$$

C. Perturbation Analysis

It would seem worthwhile to make a few comments about the formalism of making an amplitude parameter expansion while carrying it out for the problems at hand. This discussion is an adaptation of one given by Stoker[4] for gravity waves and is the mathematical justification for what is commonly termed linearization.

Each of the dependent variables, $\bar{V}, p, \bar{E}, \bar{B}$ and $\bar{n}$ are assumed to have an expansion in a parameter Δ of the form

$$p = \overset{o}{p}(x,y,z,t) + \Delta \overset{1}{p}(x,y,z,t) + \Delta^2 \overset{2}{p}(x,y,z,t) + \ldots$$

If these expansions are substituted into the bulk equations, as well as into Eq. (3-1), an infinite set of conditions is generated from the requirement that each equation be satisfied for each power of Δ. For example Eq. (2-1a) is written

$$\Delta^{o}\Big[\frac{\partial \overset{o}{V}_x}{\partial t} + \overset{o}{V}_x \frac{\partial \overset{o}{V}_x}{\partial x} + \overset{o}{V}_y \frac{\partial \overset{o}{V}_x}{\partial y} + \overset{o}{V}_z \frac{\partial \overset{o}{V}_x}{\partial z} + \frac{\partial \overset{o}{p}}{\partial x} + \rho g\Big]$$

$$+\Delta^{1}\Big[\frac{\partial \overset{1}{V}_x}{\partial t} + \overset{o}{V}_x \frac{\partial \overset{1}{V}_x}{\partial x} + \overset{1}{V}_x \frac{\partial \overset{o}{V}_x}{\partial x} + \overset{o}{V}_y \frac{\partial \overset{1}{V}_x}{\partial y} + \overset{1}{V}_y \frac{\partial \overset{o}{V}_x}{\partial y} + \overset{o}{V}_z \frac{\partial \overset{1}{V}_x}{\partial z} + \overset{1}{V}_z \frac{\partial \overset{o}{V}_x}{\partial z} + \frac{\partial \overset{1}{p}}{\partial x}\Big]$$

$$+\Delta^{2}\Big[\frac{\partial \overset{2}{V}_x}{\partial t} + \overset{o}{V}_x \frac{\partial \overset{2}{V}_x}{\partial x} + \overset{1}{V}_x \frac{\partial \overset{1}{V}_x}{\partial x} + \overset{2}{V}_x \frac{\partial \overset{o}{V}_x}{\partial x} + \overset{o}{V}_y \frac{\partial \overset{2}{V}_x}{\partial y} + \overset{1}{V}_y \frac{\partial \overset{1}{V}_x}{\partial y} + \overset{2}{V}_y \frac{\partial \overset{o}{V}_x}{\partial y}$$

$$+ \overset{o}{V}_z \frac{\partial \overset{2}{V}_x}{\partial z} + \overset{1}{V}_z \frac{\partial \overset{1}{V}_x}{\partial z} + \overset{2}{V}_z \frac{\partial \overset{o}{V}_x}{\partial x} + \frac{\partial \overset{2}{p}}{\partial x}\Big] + \ldots = 0$$

and each term in brackets is required to vanish. A "linearizable" problem is one where the zero order equations can be satisfied by solutions that in turn generate a first order set of equations that are linear. In the present problems, these are $\overset{o}{V}_x = \overset{o}{V}_y = \overset{o}{V}_z = 0$ and $\overset{o}{p} = -\rho g x$ + constant, or the conditions for static equilibrium.

The resulting first order equations have constant coefficients and admit solutions of the form $\overset{1}{p} = \tilde{p}\, e^{\alpha t + \beta_y y + \beta_z z}$. All first order variables may be written in terms of the first order fields $\tilde{p}$, $\tilde{e}_z$ and $\tilde{h}_z$ which are defined by

$$\Big[\frac{d^2}{dx^2} - k^2\Big] \begin{bmatrix} \tilde{p} \\ \tilde{e}_z \\ \tilde{h}_z \end{bmatrix} = 0 \qquad (3\text{-}7)$$

where $k^2 = -\beta_y^2 - \beta_z^2 = -\beta^2$

Now the assumption of negligible displacement and conduction currents is justified by the fact that for all of the fluids considered here $|\beta_y^2| + |\beta_z^2| >> |\mu\sigma\alpha| + |\mu\varepsilon\alpha^2|$. Equation (3-7) has solutions

$$\begin{bmatrix} \tilde{p} \\ \tilde{e}_z \\ \tilde{h}_z \end{bmatrix} = \begin{bmatrix} A_1 \\ C_1 \\ D_1 \end{bmatrix} e^{kx} + \begin{bmatrix} A_2 \\ C_2 \\ D_2 \end{bmatrix} e^{-kx} \tag{3-8}$$

where A's, C's and D's are coefficients determined by the boundary conditions. The remaining variables are written in terms of the A's, C's and D's

$$\tilde{V}_x = -\frac{1}{\rho\alpha}\frac{d\tilde{p}}{dx} = -\frac{k}{\rho\alpha}[A_1 e^{kx} - A_2 e^{-kx}] \tag{3-9}$$

$$\tilde{V}_y = -\frac{\beta_y}{\rho\alpha}\tilde{p} = -\frac{\beta_y}{\rho\alpha}[A_1 e^{kx} + A_2 e^{-kx}] \tag{3-10}$$

$$\tilde{V}_z = -\frac{\beta_z}{\rho\alpha}\tilde{p} = -\frac{\beta_z}{\rho\alpha}[A_1 e^{kx} + A_2 e^{-kx}] \tag{3-11}$$

$$\tilde{h}_x = \frac{1}{\beta_z}\frac{d\tilde{h}_z}{dx} = \frac{k}{\beta_z}[D_1 e^{kx} - D_2 e^{-kx}] \tag{3-12}$$

$$\tilde{h}_y = \frac{\beta_y}{\beta_z}\tilde{h}_z = \frac{\beta_y}{\beta_z}[D_1 e^{kx} + D_2 e^{-kx}] \tag{3-13}$$

$$\tilde{e}_x = \frac{1}{\beta_z}\frac{d\tilde{e}_z}{dx} - \frac{\alpha\beta_y\mu\tilde{h}_z}{\beta_z^2} = \frac{k}{\beta_z}[C_1 e^{kx} - C_2 e^{-kx}] \tag{3-14}$$

$$-\frac{\alpha\mu\beta_y}{\beta_z^2}[D_1 e^{kx} + D_2 e^{-kx}]$$

$$\tilde{e}_y = \frac{\beta_y}{\beta_z}\tilde{e}_z + \frac{\alpha\mu}{\beta_z^2}\frac{d\tilde{h}_z}{dx} = \frac{\beta_y}{\beta_z}[C_1 e^{kx} + C_2 e^{-kx}] \tag{3-15}$$

$$+ \frac{\alpha k\mu}{\beta_z^2}[D_1 e^{kx} - D_2 e^{-kx}]$$

These solutions may now be used to generate driving functions for the second order equations, and so on.

The arbitrary coefficients are determined from the boundary conditions which must also be satisfied to all orders of Δ. Care must be taken that the boundary conditions are evaluated at the position of the interface, i.e. at $x = \xi\,(y,z,t)$, $x = -a$ or $x = b$ as the case may be. Since a series solution has been assumed for $\xi\,(y,z,t)$, each of the other dependent variables is evaluated at $x = \xi(y,z,t)$ by making a further expansion of its x dependence in powers of Δ and the functions $\overset{o}{\xi}, \overset{1}{\xi}, \overset{2}{\xi}$ .... Equation (3-1) then defines these ξ's in terms of the arbitrary constants introduced by the bulk equations. To first order terms:

$$\tilde{\xi} = \frac{1}{\alpha}\tilde{V}_x\Big/_{x=o} = -\frac{k}{\rho\alpha^2}[A_1 - A_2] \tag{3-16}$$

and the first order normal vector is defined,

$$\tilde{n}_x = 0$$

$$\tilde{n}_y = -\frac{\beta_y\tilde{V}_x}{\alpha}\Big/_{x=o} \qquad \tilde{n}_z = -\frac{\beta_z\tilde{V}_x}{\alpha}\Big/_{x=o} \tag{3-17}$$

It is now clear that the domain of validity of solutions to the bulk equations is fixed. The problem has been reduced to the usual fixed boundary value problem.

The boundary conditions for first order terms in Δ (Eqs. 3-18 through 3-36) are shown in Table B-1 of Appendix B. Not all of these equations are independent or pertinent for each problem. The eqüations defining the A's, C's and D's, together with comments on the consistency of the remaining equations are tabulated in Table B-2.

D. Dispersion Equations

For a given surface perturbation, Eqs. (3-30) through (3-36) define the first order relations among the dependent variables. Which perturbations are allowed is determined by the compatibility condition of the system equations. The reduced form of these conditions is,

$$\text{Prob. 1} \begin{bmatrix} \text{EH-If} \\ \text{EH-Ip} \\ \text{MH-I} \end{bmatrix}; \quad \omega^2 = -\alpha^2 = k^2 \left[V_g^2 + V_c^2 - \begin{pmatrix} V_a^2 \\ V_b^2 \end{pmatrix} \right] \tag{3-37}$$

$$\text{Prob. 2} \begin{bmatrix} \text{EH-II} \\ \text{MH-IIf} \\ \text{MH-IIa} \end{bmatrix}; \quad \omega^2 = -\alpha^2 = k^2 \left[V_g^2 + V_c^2 + \frac{k_z^2}{k^2} \begin{pmatrix} V_a^2 \\ V_b^2 \end{pmatrix} \right] \tag{3-38}$$

where $$V_g^2 = g(\rho^{(1)} - \rho^{(2)}) / \rho_{eq} k \tag{3-39}$$

$$V_c^2 = Tk / \rho_{eq} \tag{3-40}$$

$$\rho_{eq} = \rho^{(2)}\coth(kb) + \rho^{(1)}\coth(ka) \tag{3-41}$$

The parameters V_g and V_c are the phase velocities of gravity and capillary waves respectively* while V_a and V_b are the effective Alfven velocity and its electrohydrodynamic dual as given below,

1a - free charge on interface (EH-If)

$$V_b^2 = \frac{1}{\rho_{eq}}[\varepsilon^{(2)}(E_x^{(2)})^2(1+c^{(2)})\coth(kb)+ \varepsilon^{(1)}(E_x^{(1)})^2(1+c^{(1)})\coth(ka)] \tag{3-42}$$

1a - no free charge on interface (EH-Ip)

$$V_b^2 = \frac{1}{\rho_{eq}}[\frac{(\varepsilon^{(2)}- \varepsilon^{(1)})(\varepsilon^{(2)}(1+c^{(1)})-\varepsilon^{(1)}(1+c^{(2)}))E_x^{(2)}E_x^{(1)}}{\varepsilon^{(2)}\tanh(ka) + \varepsilon^{(1)}\tanh(kb)}] \tag{3-43}$$

2a - EH-II

$$V_b^2 = E_z^2 \frac{(\varepsilon^{(2)}-\varepsilon^{(1)})(\varepsilon^{(2)}(1-c^{(2)})-\varepsilon^{(1)}(1-c^{(1)}))}{(\varepsilon^{(2)} + \varepsilon^{(1)})\ \rho_{eq}} \tag{3-44}$$

1b - (MH-I)

V_a^2 is given by Eq. (3-43) after the substitution

$\varepsilon \rightarrow \mu$, $E \rightarrow H$, $c \rightarrow d$

2b - free current on interface (MH-IIf)

$$V_a^2 = [(1-d^{(1)})\mu^{(1)}(H_z^{(1)})^2\coth(ka)+ (1-d^{(2)})\mu^{(2)}(H_z^{(2)})^2\coth(kb)]\left(\frac{1}{\rho_{eq}}\right) \tag{3-45}$$

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*See Lamb,[5] p. 461.

2b - no free current on interface (MH-IIa)
V_a^2 is given by Eq. (3-44) after the substitution

$\varepsilon \rightarrow \mu, \quad E \rightarrow H, \quad c \rightarrow d$.

E. Interpretation of Results

The phase velocities of the six types of surface waves are given by Eqs. (3-37) through (3-45) as they depend on the perturbation wave-length $\frac{2\pi}{k}$.

For reasonable values of the electrostriction constants, the velocities squared V_a^2 and V_b^2 are positive. It follows from Eqs. (3-37) through (3-41) that the effect of an increased field intensity in Problem 1 (type I waves) is to slow down a surface wave of a given wave length until the effective phase velocity vanishes and the surface becomes unstable. The wave-length of the instability is that value of $\frac{2\pi}{k}$ that first fulfills this condition. If the electro- or magnetostriction is assumed to be proportional to the permittivity or permeability, the striction terms act to enhance the unstabilizing influence of the fields for the EH-If waves, while tending to stabilize the EH-Ip and MH-I waves.

As would be expected, in all of the cases considered, there is an instability at some sufficiently long wave length if the heavier fluid is placed on top ($\rho^{(2)} > \rho^{(1)}$). This is the familiar gravitational instability.

If the effect of the strictions are presumed small in the EH-II, MH-IIf and MH-IIa problems, the results indicate that transverse incompressible surface waves propagate along the field lines, which for high magnetic or electric field intensities $[(\begin{smallmatrix}|V_a|\\|V_b|\end{smallmatrix}) \gg |V_c| + |V_g|]$ may be termed Alfven or electrohydrodynamic surface waves respectively.

The parallel field problems are apparently more strongly dependent on the strictive terms, however. In fact, Eqs. (3-44) and (3-45) show that the effective velocities squared, V_a^2 and V_b^2, may be positive or negative depending on the values of the d's and c's.

The dependence of ε on ρ, for example, is sometimes computed from the Clausius-Mossotti equation* which gives

$$c = [\frac{(k_e - 1)(k_e + 2)}{3k_e}] \tag{3-46}$$

It follows that,

$$(\varepsilon^{(2)} - \varepsilon^{(1)})[\varepsilon^{(2)}(1-c^{(2)}) - \varepsilon^{(1)}(1-c^{(1)})] \tag{3-47}$$
$$= \frac{\varepsilon_o^2}{3}(k_e^{(2)}-k_e^{(1)})[2 - (k_e^{(1)}+k_e^{(2)})]$$

or, according to the Clausius-Mossotti equation, for $k_e^{(1)}$ and $k_e^{(2)}$ greater than unity, V_b^2 of Eq. (3-43) is negative and the electric field slows down surface waves propagating in the z direction. In Chapter 4, independent experimental means will be used to show that such large

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*See Stratton,[13] p. 140.

electrostrictions do not occur at interfaces of common liquids, but rather, that V_b^2 is positive for waves of type EH-II.

With the assumption that V_a^2 and V_b^2 are positive, certain characteristics that are common to the wave types may be pointed out. If the interactions with the external boundaries are small, the electric and magnetic parts of the waves are dispersionless. Waves propagating in the z direction then are dispersed according to the equation

$$\begin{bmatrix} \mathrm{I} \\ \mathrm{II} \end{bmatrix}; \quad \omega^2 = k^2\left[\frac{g(\rho^{(1)} - \rho^{(2)})}{k(\rho^{(1)} + \rho^{(2)})} + \frac{Tk}{(\rho^{(1)} + \rho^{(2)})} \mp U^2\right] \tag{3-48}$$

where the U^2's are given for the respective problems by the limit $a \to \infty$, $b \to \infty$ of Eqs. (3-42) through (3-45), and are independent of k.

Observe that:

1. Although at a fixed k the field intensity can always be made large enough to make U^2 larger than the square of the gravity or capillary phase velocities, there is always a wavelength such as to make either the capillary wave or gravity wave phase velocity the largest of the three terms.

2. The wave number of minimum phase velocity corresponding to

$$k = \sqrt{\frac{g(\rho^{(1)} - \rho^{(2)})}{T}} \tag{3-49}$$

remains unchanged with increasing field intensity.

3. For type II waves the group and phase velocities are always positive, while for type I waves there may be values of k for which the group velocity is negative, while the phase velocity is positive. This will happen if

$$U^2 > \sqrt{\frac{3Tg(\rho^{(1)} - \rho^{(2)})}{(\rho^{(1)} + \rho^{(2)})^2}} \tag{3-50}$$

The nomenclature of capillary and gravity waves may be generalized to define waves as capillary, gravity or of type I or II in accordance with the phase velocity that is largest in magnitude. The regions of each wave are illustrated in Fig. 3-3. The least value of k that can give a type I or type II wave is

$$K = \frac{g(\rho^{(1)} - \rho^{(2)})}{U^2(\rho^{(1)} + \rho^{(2)})}$$

while the largest value of k to give these waves is

$$k = U^2(\rho^{(1)} + \rho^{(2)})/T$$

Hence, U^2 must exceed a minimum value for the type I or II wave to exist as defined. That value is

$$U^2 = \sqrt{\frac{gT(\rho^{(1)} - \rho^{(2)})}{(\rho^{(1)} + \rho^{(2)})^2}}$$

Interactions with the parallel plates occur at long wavelengths (small values of k). The effect is illustrated by considering the following limits:

EH-If $\sigma^{(1)} \gg \sigma^{(2)}$, $E^{(1)} \to 0$, $\rho^{(2)} \to 0$

EH-Ip $\varepsilon^{(1)} \gg \varepsilon^{(2)}$, $\rho^{(2)} \to 0$

MH-I $\mu^{(1)} \gg \mu^{(2)}$, $\rho^{(2)} \to 0$

MH-IIf currents returned on upper plate:
$H_z^{(1)} \to 0$, $\rho^{(2)} \to 0$

These problems then have the same dispersion relation

$$\begin{bmatrix} \mathrm{I} \\ \mathrm{II} \end{bmatrix} \quad \omega^2 = k^2\left[\frac{g}{k}\tanh ka + \frac{Tk}{\rho}\tanh ka \mp U^2 \frac{\tanh ka}{\tanh kb}\right] \tag{3-51}$$

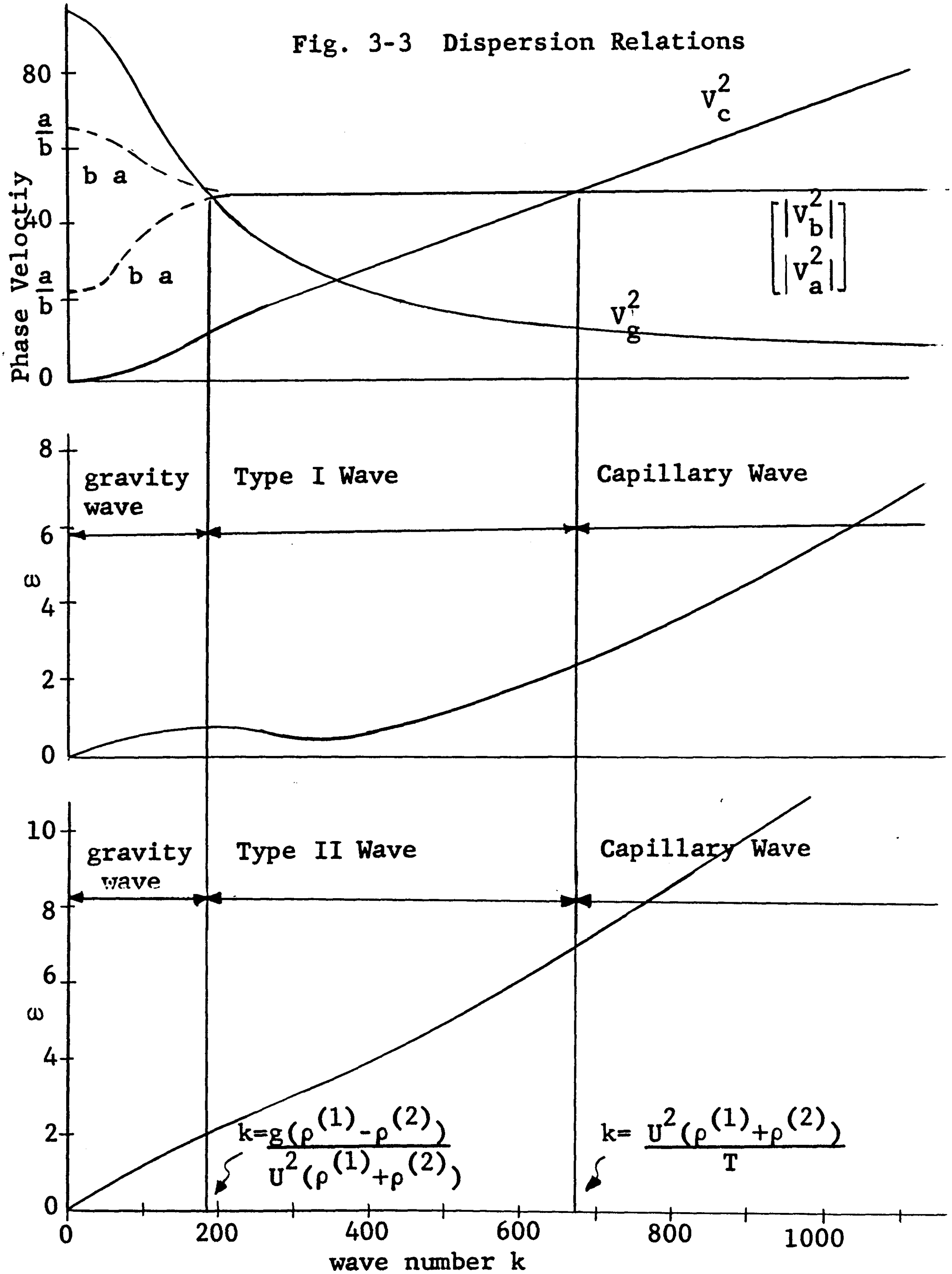
This effect of the boundaries at low k has been included in Fig. 3-3. (This figure was drawn using data for water at a depth of 1 cm. and $U^2 = 50$.) In the limit where both ka and kb are much less than one (wavelengths long compared to both plate spacings), the dispersion relation is

$$\begin{bmatrix} \mathrm{I} \\ \mathrm{II} \end{bmatrix} \quad \omega^2 = k^2\left[ag + \frac{aTk^2}{\rho} \mp U^2 \frac{a}{b}\right] \tag{3-52}$$

Again, the type I and II phase velocities are independent of k. However, now the gravity wave propagates without dispersion.

It will be observed that the EH-Ip and MH-I, as well as the MH-IIa and EH-II waves are complete duals. Also, if $\frac{\partial}{\partial y} = 0$, the MH-IIf and EH-If waves are anti-duals, even including coupling to the external plates. That is V_b^2 is of the same form as V_a^2 for these cases and the dispersion relations are given by the substitution $V_b^2 \to -V_a^2$.

Fig. 3-3 Dispersion Relations



F. Field Distributions

The physical picture of the fundamental dynamics of each wave type is worthy of mention. This is especially true of the polarization and magnetization surface waves, since their prediction depends on a force density derived by means of energetic rather than dynamic principles.

The stress tensor of Korteweg and Helmholtz has, as its volume force counterpart, a force density given by

$$\bar{f}_m = \bar{j}_f \times \bar{B} - \frac{1}{2} \bar{H}\cdot\bar{H}\nabla\mu + \frac{1}{2} \nabla(\bar{H}\cdot\bar{B}d) \tag{3-53}$$

$$\bar{f}_e = q_f\bar{E} - \frac{1}{2}(\bar{E}\cdot\bar{E})\nabla\varepsilon + \frac{1}{2} \nabla(\bar{D}\cdot\bar{E}c) \tag{3-54}$$

In four of the cases considered, the free currents and charges are assumed to vanish. A few remarks will be made at this point about the force densities $\bar{f}_m$ and $\bar{f}_e$ as applied to these cases where $\bar{j}_f = q_f = 0$.

A manipulation of Eqs. (3-53) and (3-54) renders them more physically interesting. Consider Eq. (3-54) for example in the case where c is small enough that the last term can be ignored. Then,

$$\bar{f}_e = - \frac{1}{2}(\bar{E}\cdot\bar{E})\nabla\varepsilon = - \frac{1}{2}\nabla[\bar{E}\cdot\bar{D}] + \frac{1}{2} \varepsilon\nabla (\bar{E}\cdot\bar{E}) \tag{3-55}$$

Because $\nabla \times E = 0$ this can be rewritten as

$$\bar{f}_e = - \frac{1}{2}(\bar{E}\cdot\nabla)D + \frac{1}{2}(\bar{D}\cdot\nabla)E + \frac{1}{2} \bar{E} \times (\nabla \times \bar{D}) \tag{3-56}$$

The first two terms of this equation combine to $\frac{1}{2} \bar{D}(\nabla\cdot\bar{E})$, since $\bar{D}$ is parallel to $\bar{E}$ and $\nabla\cdot\bar{D} = 0$. Hence

$$\bar{f}_e = \frac{1}{2}\frac{\bar{D}}{\varepsilon_o} q_p + \frac{1}{2}\bar{j}_p \times \varepsilon_o\bar{E} \tag{3-57}$$

where

$$q_p = \varepsilon_o \nabla\cdot\bar{E}$$

$$\bar{j}_p = [\nabla \times \bar{D}]/\varepsilon_o$$

Analogously, Eq. (3-48), for small d's and $\bar{j}_f = 0$ is

$$\bar{f}_m = \frac{1}{2}\frac{\bar{B}}{\mu_o} q_m + \frac{1}{2}\bar{j}_m \times \mu_o\bar{H} \tag{3-58}$$

where

$$q_m = \mu(\nabla\cdot\bar{H})$$

$$\bar{j}_m = [\nabla \times \bar{B}]/\mu_o$$

The force density, according to the constrained form of the model used by Korteweg and Helmholtz, can apparently be divided into interactions between the macroscopic fields and equivalent field sources and vortices. The quantities $\bar{j}_m$ and q_m will be recognized as in the form of what are sometimes called the amperian current and magnetic charge. Although q_p is commonly termed the polarization charge, there does not seem to be a precedent for $\bar{j}_p$. (Note that the static surface vortices for type EH-II waves have not been specified in Table B-1.) In this work, $\bar{j}_p$ will be termed the Korteweg current, for lack of another name. The terminology here has no more significance for the microscopic dynamics than does the magnetic charge. The variable $\bar{j}_p$ signifies the vorticity of polarization only.

In an actual fluid, the force densities of Eqs. (3-57) or (3-58) would revise the material microstructure, and the resulting equilibrium (assuming there is one) would be accounted for by the force densities corrected for strictions.

Finally, it is useful to note that since $\bar{j}_m$ and $\bar{j}_p$ are wholly rotational,

$$\nabla\cdot\bar{j}_m = \nabla\cdot\bar{j}_p = 0 \qquad (3\text{-}59)$$

The physical interactions that account for the six types of surface waves may now be pointed out by making use of the preceding concepts.

The static equilibria corresponding to the unperturbed problems are of two types; interfaces stressed by perpendicular fields (type I, Prob. 1), or interfaces stressed by tangential fields (type II, Prob. 2). The type I waves are dynamically dominated by surface charge interactions (free, polarization or magnetic) while the type II waves are more reasonable if viewed as surface current interactions (free, Korteweg or Amperian). It is essential that, according to this model, dynamical interactions can not, in general, be predicted by use of either sources or vortices alone. Both must be included.

The equations of Table B-1 make possible a discussion of the field distributions that correspond to the various wave types. As an example, consider the fields that accompany waves traveling in the negative or positive z direction. (For this discussion the motions will be assumed to have no y dependence.) The surface could be distorted so that

$$\xi = \xi_o \cos\,(\omega t \pm kz); \quad \tilde{\xi} = \xi_o \qquad (3\text{-}60)$$

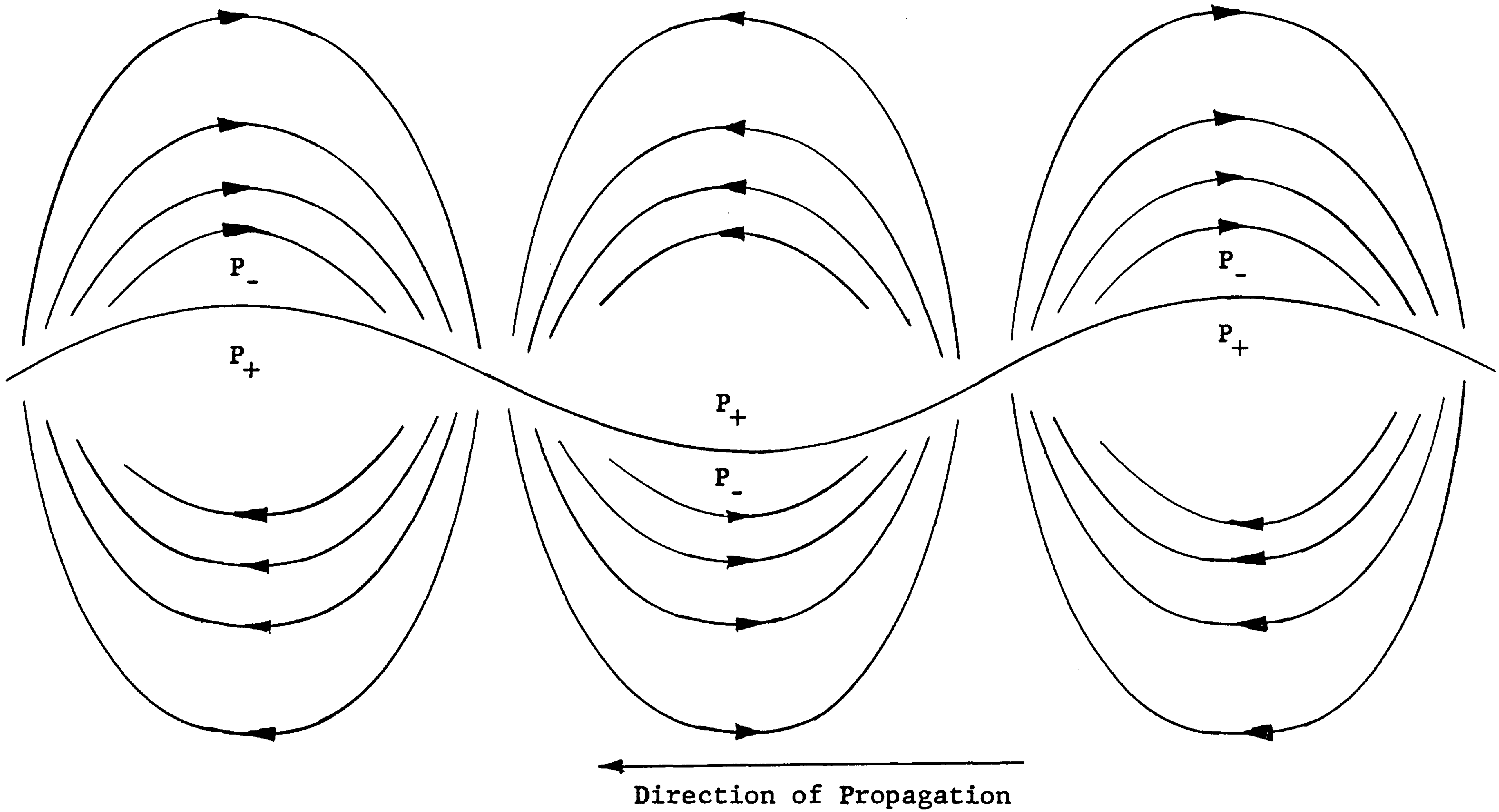
where k and ω are related by the dispersion relation (3-37) or (3-38).

The perturbation fields consistent with this traveling wave would appear as shown in Figs. 3-4a through 3-4e and are described by the equations of Table C-1. Superimposed on the electric or magnetic field intensities shown are zero order field intensities. The first order fields are, in general, distributed sinusoidally along the z axis and exponentially perpendicular to the surface. Field lines, stream lines and points of maximum pressure are shown. Note that Fig. 3-4a gives the pressure and velocity distributions for all of the wave types and that the fluid particle velocities are such as to make the pressure peaks propagate from right to left, i.e. the wave traveling to the left is shown.

In Table C-1, the sources and vortices of the previous sections are given as defined. They do not include the effect of strictions.

The most obvious difference between the field distributions of the type I and type II waves, is that the primary field intensities (perturbations of the static field intensities) are 90 degrees out of space phase with the surface displacement for type I waves, while they are in phase for the type II waves. This result is directly related to the unstabilizing influence of the type I fields as compared to the stabilizing influence of type II fields, i.e. to the decrease and increase of the phase velocity of type I and II waves respectively with increasing field intensities. To see this, note that the gravitational field and surface tension produce wholly wave-like interactions, and always tend to force the interface toward its center of curvature. The perturbations

Fig. 3-4a General Velocity Distribution



velocity field

P_+, P_- points of peak positive and negative pressure

Fig. 3-4b EH-If Wave

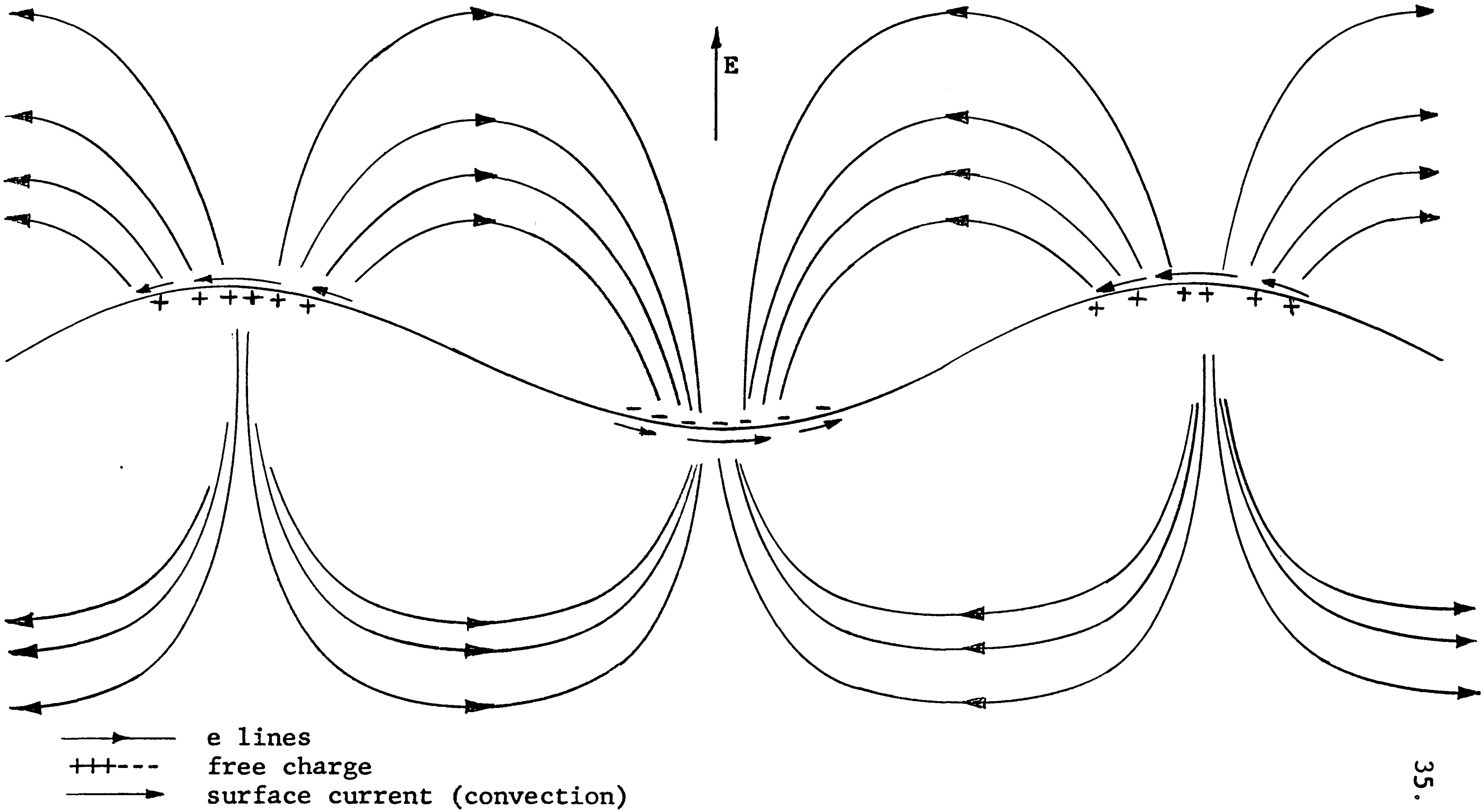


Fig. 3-4c EH-Ip or MH-I Waves

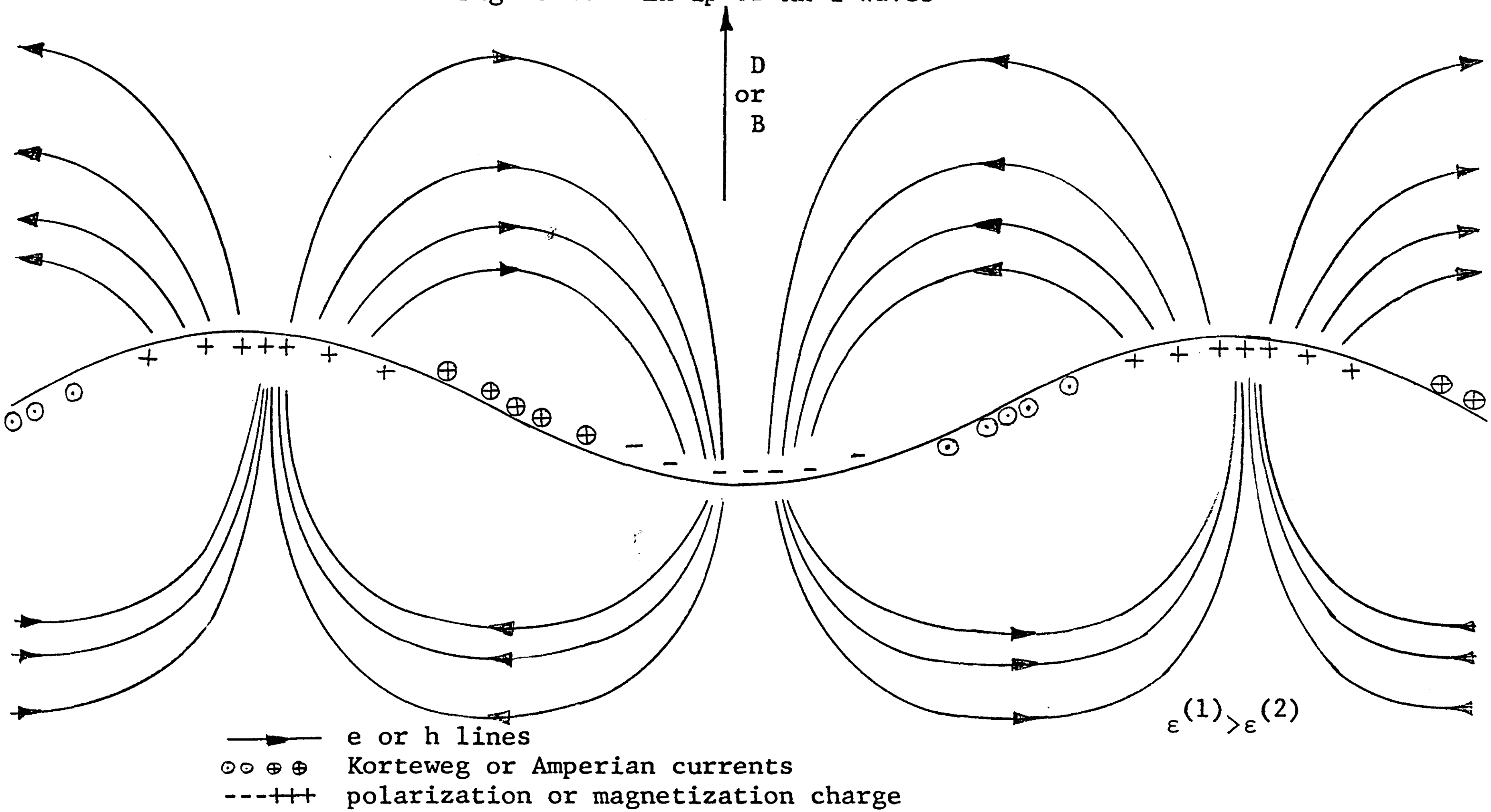


Fig. 3-4d MH-IIf Wave

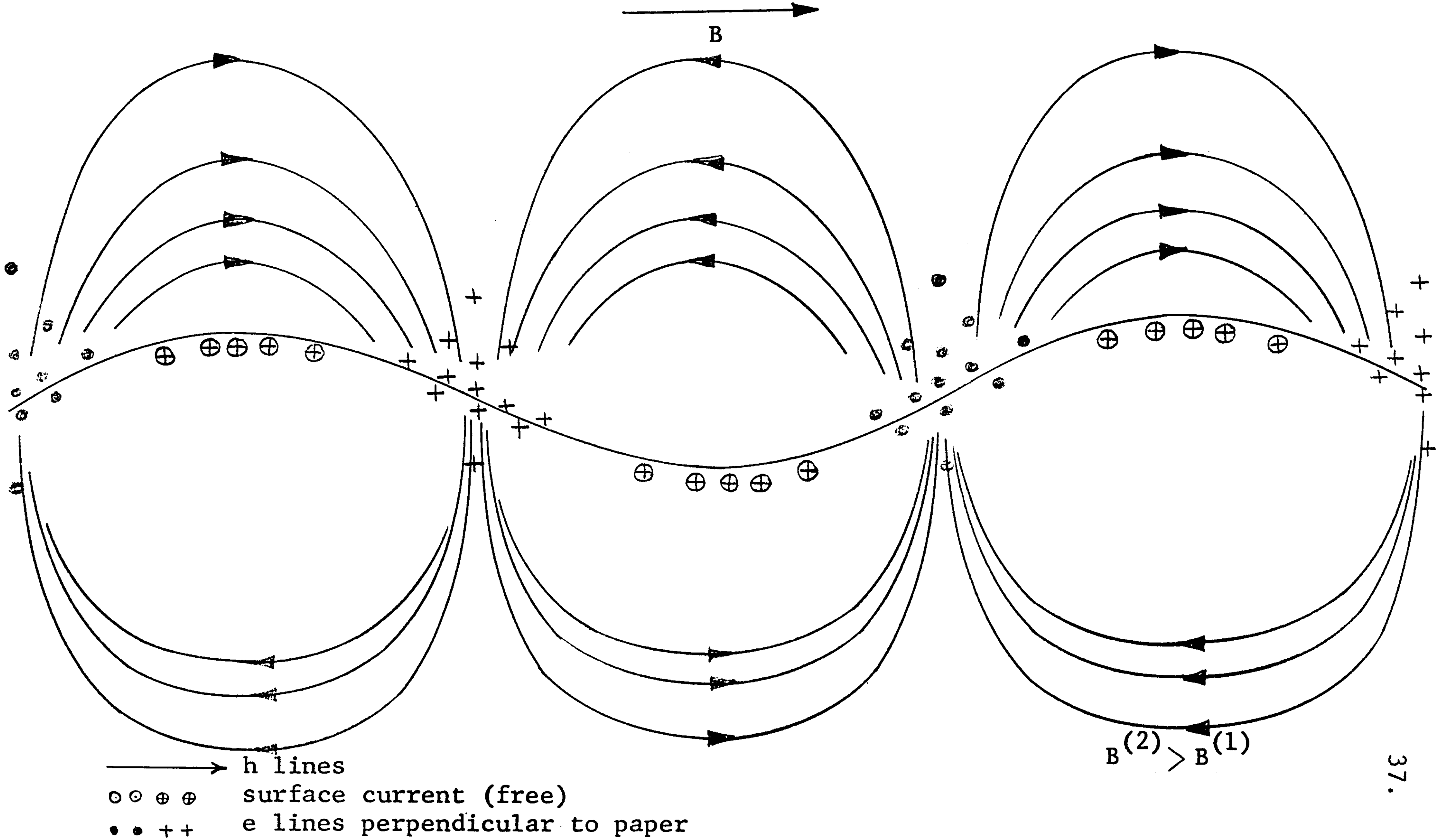
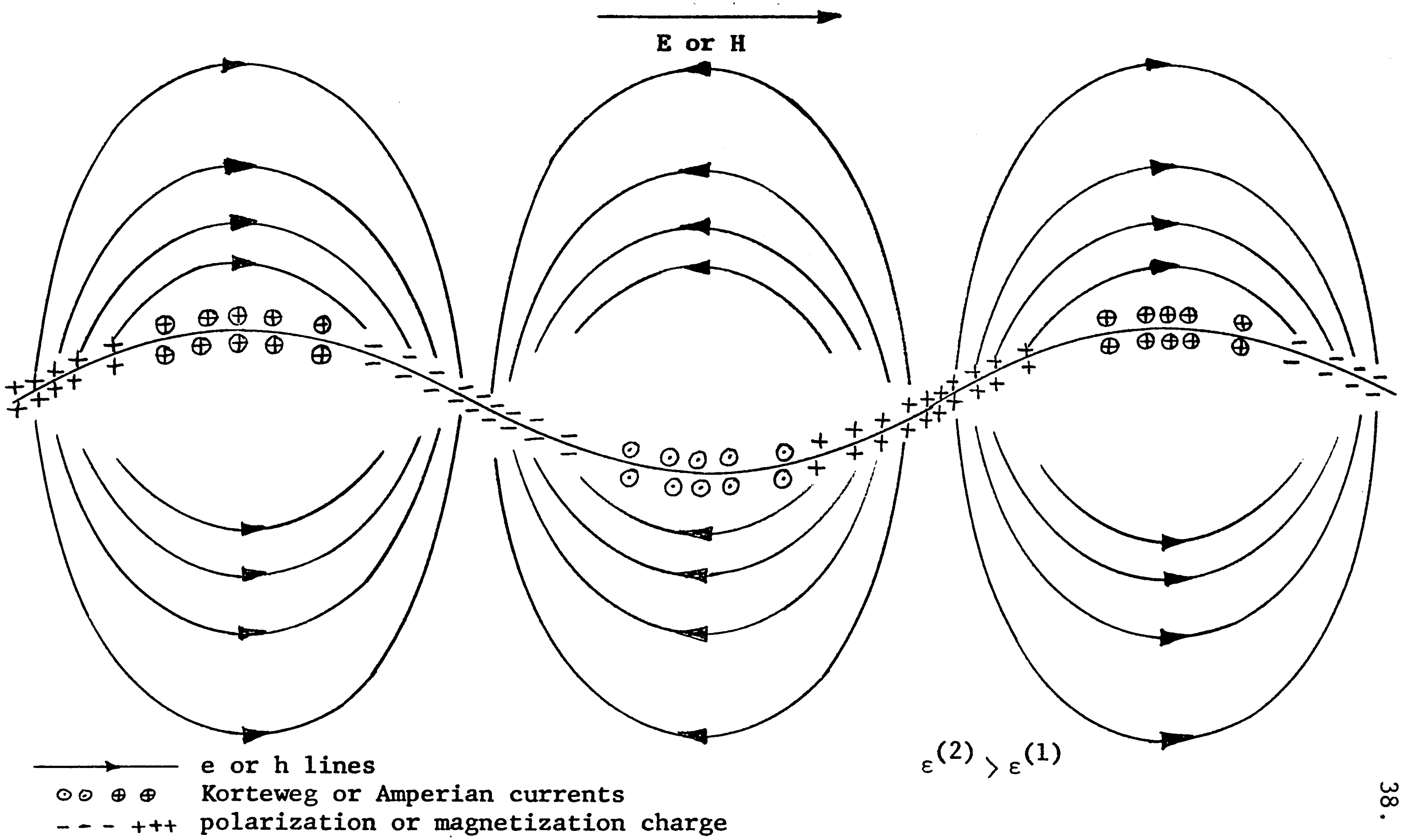


Fig. 3-4e EH-II or MH-IIa Waves



that play a dominant role in stressing the interface are made from finite zero order field variables. Hence, for type I waves, the surface traction resulting from surface charges and the perpendicular fields is always in a direction opposite the center of curvature of the surface, while for type II waves the traction resulting from surface currents and horizontal fields is toward the center of curvature. From these statements the predicted dynamics would be expected. More physical feeling for the wave types will follow from the consideration of standing waves and surface resonators given in Chapter 4.

G. EH-If and MH-IIf Waves

Insight into the physical meaning of the previous free surface charge and free surface current problems may be gained by considering a pair of simple problems that illustrate the role of conductivity.

Consider first of all the problem illustrated by Fig. 3-5.

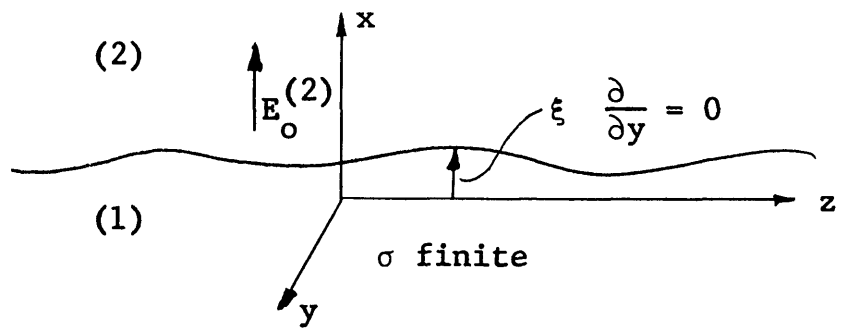


Figure 3-5
EH-If Configuration

A fluid of finite conductivity σ in region (1) forms a horizontal interface with a fluid of zero conductivity.

This is stressed by a perpendicular electric field. After sufficient time has elapsed for the charge to relax to the interface,

$$\sigma^{(1)}E^{(1)} = \sigma^{(2)}E^{(2)} \tag{3-61}$$

Since $\sigma^{(1)}$ is finite, $\sigma^{(2)} = 0$ implies that $E_o^{(1)} = 0$. the static electric field is confined to region (2), where it terminates on a surface charge $Q = \varepsilon_o E_o^{(2)}$. The fact that there is a no static conduction current makes it possible to treat this problem exactly, to first order perturbation terms. The equations of motion are Eqs. (2-1a) with $\bar{F} = -\rho g \bar{a}_x + q\bar{E}$, (2-21), (2-3), (2-4) and (2-6). Displacement current will be included so that Eq. (2-5) is

$$\nabla \times \bar{H} = \sigma\bar{E} + \varepsilon_o \frac{\partial \bar{E}}{\partial t} \tag{3-62}$$

It is assumed that $\varepsilon = \varepsilon_o$, $\mu = \mu_o$.
In the bulk of the fluid

$$\nabla \cdot \sigma\bar{E} = -\frac{\partial q}{\partial t} \tag{3-63}$$

so that in region (1) Eqs. (3-63) and (2-6) give

$$\frac{\partial q}{\partial t} + \frac{\sigma}{\varepsilon_o} q = 0 \tag{3-64}$$

This is identically satisfied by the steady state condition that $q = (\nabla \cdot \bar{E})\varepsilon_o = 0$. It follows that Eqs. (2-3) and (3-62) define the z components of $\bar{B}$ and $\bar{E}$. If perturbations are taken of the form

$$e_z = \tilde{e}_z \, e^{\alpha t + \beta_z z}$$

these equations are

$$\left[\frac{d^2}{dx^2} - k'^2\right] \begin{bmatrix} \tilde{e}_z \\ \tilde{b}_z \end{bmatrix} = 0 \tag{3-65}$$

where now

$$k'^2 = (\alpha\mu_o\sigma + \alpha^2\mu_o\varepsilon_o - \beta^2)$$

In the same way as demonstrated in section C, Chapter 3, the remaining variables are defined in terms of solutions to Eq. (3-65).

If
$$\begin{bmatrix} \tilde{e}_z \\ \tilde{b}_z \end{bmatrix} = \begin{bmatrix} C_1 \\ D_1 \end{bmatrix} e^{k'x} + \begin{bmatrix} C_2 \\ D_2 \end{bmatrix} e^{-k'x} \tag{3-66}$$

then these variables are

$$\tilde{e}_y = \frac{\alpha k'}{\delta^2}\left[D_1 e^{k'x} - D_2 e^{-k'x}\right] \tag{3-67}$$

$$\tilde{e}_x = \frac{\alpha\beta_z}{\delta^2}\left[C_1 e^{k'x} - C_2 e^{-k'x}\right] \tag{3-68}$$

$$\tilde{b}_x = \frac{k'\beta_z}{\delta^2}\left[D_1 e^{k'x} - D_2 e^{-k'x}\right] \tag{3-69}$$

$$\tilde{b}_y = -\frac{[\mu_o\sigma + \mu_o\varepsilon_o\alpha]k'}{\delta^2}\left[C_1 e^{k'x} - C_2 e^{-k'x}\right] \tag{3-70}$$

where

$$\delta^2 = (\beta_z^2 - \mu_o\sigma\alpha - \mu_o\varepsilon_o\alpha^2)$$

and the perturbed pressure and velocity fields are, as defined by Eqs. (3-8), (3-9), and (3-11).

The present purposes are served if the external boundaries of the system are assumed to be very far away. Then,

$$D_1^{(2)} = D_2^{(1)} = A_1^{(2)} = A_2^{(1)} = C_1^{(2)} = C_2^{(1)} = 0$$

so that the fields are finite at infinity. The boundary conditions evaluate the remaining coefficients. These conditions are Eqs. (3-18), (3-19a) through (3-24a). It is now possible to see the crucial role played by the static electric fields. Since $E_o^{(1)} = 0$, Eqs. (3-21a) and (3-22a); Eqs. (3-20a) and (3-23a) show that $e_z^{(1)} = e_y^{(1)} = 0$. This implies that $C_1^{(1)} = D_1^{(1)} = 0$, or that all electric and magnetic field components are excluded from region (1). Since the boundary conditions and the remaining field solutions are independent of the conductivity, the dispersion relation and field distributions are also independent of σ. Hence the solution is meaningful for fluids that vary in conductivity from $\sigma^{(1)} \to \infty$ to the σ's of good insulators, so long as $\sigma^{(2)} = 0$ (roughly speaking $\sigma^{(1)} \gg \sigma^{(2)}$) and sufficient time is allowed for the charge to relax to the interface. Although it is not physically interesting, the limit can in fact be taken where both regions have a vanishing conductivity and hence a static electric field, but where charge has been placed on the boundary by some external means. The previous solutions for the EH-If waves include this possibility. It is clear that in this limit the surface currents necessary to account for perturbations of the surface charge must be convective. The previous results make it apparent however, that the currents are convective whether one fluid is a perfect conductor or a perfect insulator.

To see this, consider once again a wave where $\tilde{\xi} = \xi_o$. Then $\tilde{V}_x \Big|_{x=\xi} \alpha\xi_o$ and Eqs. (3-9) and (3-11) give $\tilde{V}_z = \frac{\beta_z \alpha \xi}{k}$. Eq. (3-20) shows that $e_y^{(2)} = 0$, because $\beta_y = 0$. It follows from (3-67) that $D_2^{(2)} = 0$. Boundary condition (3-22a) gives $-C_2^{(2)} = \xi_o \beta_z E_o^{(2)}$ and hence the magnetic field at the interface is

$$\tilde{b}_y \Big|_{x=\xi} = - \frac{\mu_o \varepsilon_o \alpha k \xi_o \beta_z E_o^{(2)}}{(\beta_z^2 - \mu_o \varepsilon_o \alpha^2)} \quad ; \quad \tilde{b}_z = \tilde{b}_x = 0 \tag{3-71}$$

and the surface current density is

$$\tilde{k}_z = \frac{1}{\mu_o} \bar{n} \times \tilde{B}^{(2)} \Big|_{x=\xi} \tag{3-72}$$

$$= -\varepsilon_o \alpha k \xi_o \beta_z E_o^{(2)} \Big/ (\beta_z^2 - \mu_o \varepsilon_o \alpha^2)$$

On the other hand the surface charge density is

$$\tilde{q} = \varepsilon_o \tilde{n} \cdot \tilde{E}^{(2)} \Big|_{x=\xi} = \frac{\varepsilon_o k \beta_z^2 \xi_o E_o^{(2)}}{(\beta_z^2 - \mu_o \varepsilon_o \alpha^2)} \tag{3-73}$$

Note that the continuity relation is satisfied on the surface.

The term δ^2 is within relativistic corrections of β_z^2 so that the current i_z, given by

$$\tilde{i}_z = Q\tilde{V}_z = - \varepsilon_o E_o^{(2)} \frac{k\xi_o}{\beta_z} \alpha \tag{3-74}$$

where Q is the static surface charge, and agrees with (3-72).

The migration of charge along the interface is therefore accounted for by the convection of the zero order charge as carried by the first order particle motions. No conduction process is involved. The charges convect at the interface in just such a way as to satisfy simultaneously the conditions that the interface can not support shear and that the tangential component of the electric field be continuous. This is why, for example, the problem of an interface supporting a charge that is constrained to remain fixed is not well posed for the model used here.

The MH-IIf problem has a somewhat similar property

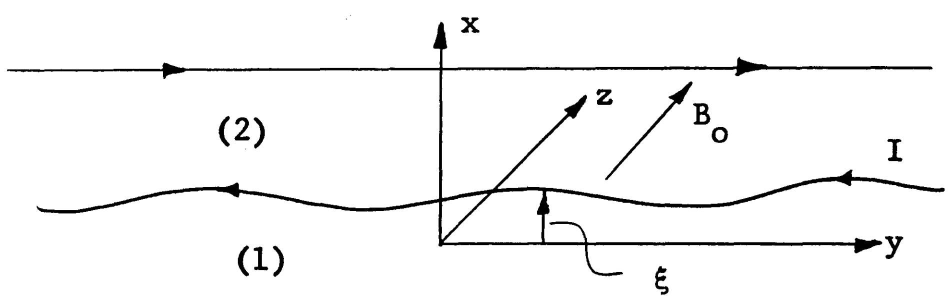


Figure 3-6
MH-IIf Configuration

If the surface current is returned as shown in Fig. 3-6, and there are no external currents, the zero order magnetic field is excluded from region (1). In the same way as in the previous EH-IIf problem, the magnetic and electric field perturbations are then excluded from region (1) also. As a result, the wave properties are again independent of the conductivity in region (1), where $\sigma^{(1)}$ may range from infinity to zero.

However, a stronger statement can be made. At the interface, the boundary conditions require that $\bar{E} = \bar{B} \times \bar{V}$, a condition satisfied by a perfectly conducting interface. As a consequence, the fields throughout the bulk of regions (1) and (2) satisfy this condition (see Eqs., Table C-1). If the problem were to be solved with finite conductivity in regions (1) and (2), the boundary conditions would be the same as for the problem with zero conductivity. The difference would arise in the solutions to the bulk equations, which would couple to the magnetic field through a finite conductivity. The bulk equations are (2-1a) through (2-6a) with $\bar{F} = \bar{j} \times \bar{B}$, where $\bar{j} = \sigma(\bar{E} + \bar{V} \times \bar{B})$. Since $\bar{E} = \bar{B} \times \bar{V}$ everywhere, $\bar{F} = 0$ and the original equations for $\sigma = 0$ are retained. The solutions to the problem with zero conductivity are admissible with finite bulk conductivity. The wave properties are independent of finite conductivities in both regions (1) and (2). The unphysical limit of σ infinite in both regions is analogous to the limit where $\sigma = 0$ in the previous EH-If problem.

CHAPTER 4

INSTABILITY, RADIATION AND RESONANCE

A. Introduction

A party trick of supporting water in an inverted glass by a coarse-meshed cloth placed over the open end may have embarrassing consequences if too coarse a mesh is used. Everyone has seen the radiating wavefront resulting from a rock thrown into a quiet pond, or the standing surface waves in a cup of coffee tapped on the side with a spoon. These are demonstrations of surface wave instability, resonance and radiation. It will be the purpose of this chapter to indicate experimentally the nature of the type I and type II waves with these three common phenomena.

B. Type I Waves - Instabilities

i. Theory

The type I waves have in common the fact that V_a^2 or V_b^2 can be made large enough to lead to a vanishing phase velocity, i.e. an instability. The type I waves will be considered here in the limit where the field intensities are confined to region (1) or region (2). For the EH-IIf waves, this means that $\sigma^{(2)} \gg \text{ or } \ll \sigma^{(1)}$ so that $E_x^{(2)} \ll \text{ or } \gg E_x^{(1)}$ while for EH-IIp or MH-II waves $\varepsilon^{(2)} \gg \text{ or } \ll \varepsilon^{(1)}$ and $\mu^{(2)} \gg \text{ or } \ll \mu^{(1)}$ so that $E_x^{(2)} \ll \text{ or } \gg E_x^{(1)}$ and $H_x^{(2)} \ll \text{ or } \gg H_x^{(1)}$. In these cases, the conditions for stability (see Eq. (3-37)) are given by

$\eta > 0$ where

$$\eta = (kf)^2 - kf(\coth kf)W + G \tag{4-1}$$

$$W = (1 + c)\varepsilon\begin{bmatrix} E_x \\ H_x \end{bmatrix}^2 f/T$$

$$G = (\rho^{(1)} - \rho^{(2)})gf^2/T$$

$$f = \begin{cases} b \text{ when } & E_x = E_x^{(2)} \\ & H_x = H_x^{(2)} \\ a \text{ when } & E_x = E_x^{(1)} \\ & H_x = H_x^{(1)} \end{cases}$$

The dimensionless number W is proportional to the ratio of electric energy stored per unit volume to surface energy per unit volume, while G is proportional to the ratio of energy density stored in the gravitational field to energy density stored in the surface formation. By analogy with ordinary fluid mechanics, $\sqrt{W}$ might be called the electric Weber number and $\sqrt{\frac{W}{G}}$ the electric Froude number.* For increasing values of field intensity, i.e. W, there is one value of the normalized wave-number $(kf)_m$ at which instability will first occur. The combination of W and $(kf)_m$ that will first produce instability simultaneously, satisfy Eq. (4-1) written as an equality and the condition that $(kf)_m$ give the least possible value of η;

$$\frac{d\eta}{d(kf)} = 0$$

*See Rouse,[14] p. 104.

It follows that

$$\sinh 2(kf)_m \, ((kf)_m^2 - G) + 2(kf)_m((kf)_m^2 + G) = 0 \qquad (4\text{-}2)$$

$$W = [\frac{(kf)_m^2 + G}{(kf)_m \coth (kf)_m}] \qquad (4\text{-}3)$$

A general numerical evaluation of the problem is effected by solving Eq. (4-2) for $(kf)_m$ as a function of G. (This is shown in Fig. 4-1.) The corresponding values of W follow from Eq. (4-3). If $(kf)_m$ is very large or very small (corresponding to weak or strong interactions with the charges on the plate), Eqs. (4-2) and (4-3) reduce to simple forms.

$$(kf)_m \gg 1 \qquad W = 2\sqrt{G} \qquad (kf)_m = \sqrt{G} \qquad (4\text{-}4)$$

$$(kf)_m \ll 1 \qquad W = G \qquad (kf)_m = 0 \qquad (4\text{-}5)$$

As would be expected, the instability condition is independent of the surface tension for long wave-length instabilities. It may be surprising, however, that this condition occurs for low values of G, since this corresponds to low values of the density difference at the interface or high values of surface tension. This is a consequence of short wavelength stabilization by the surface tension, a phenomenon that forces the initial instability wave-lengths to infinity for G less than 3. In this limit the electric and gravitational fields have the same dependence on λ and exactly cancel at the point of instability.

For large values of G, the initial instability occurs at the wavelength corresponding to the minimum phase velocity

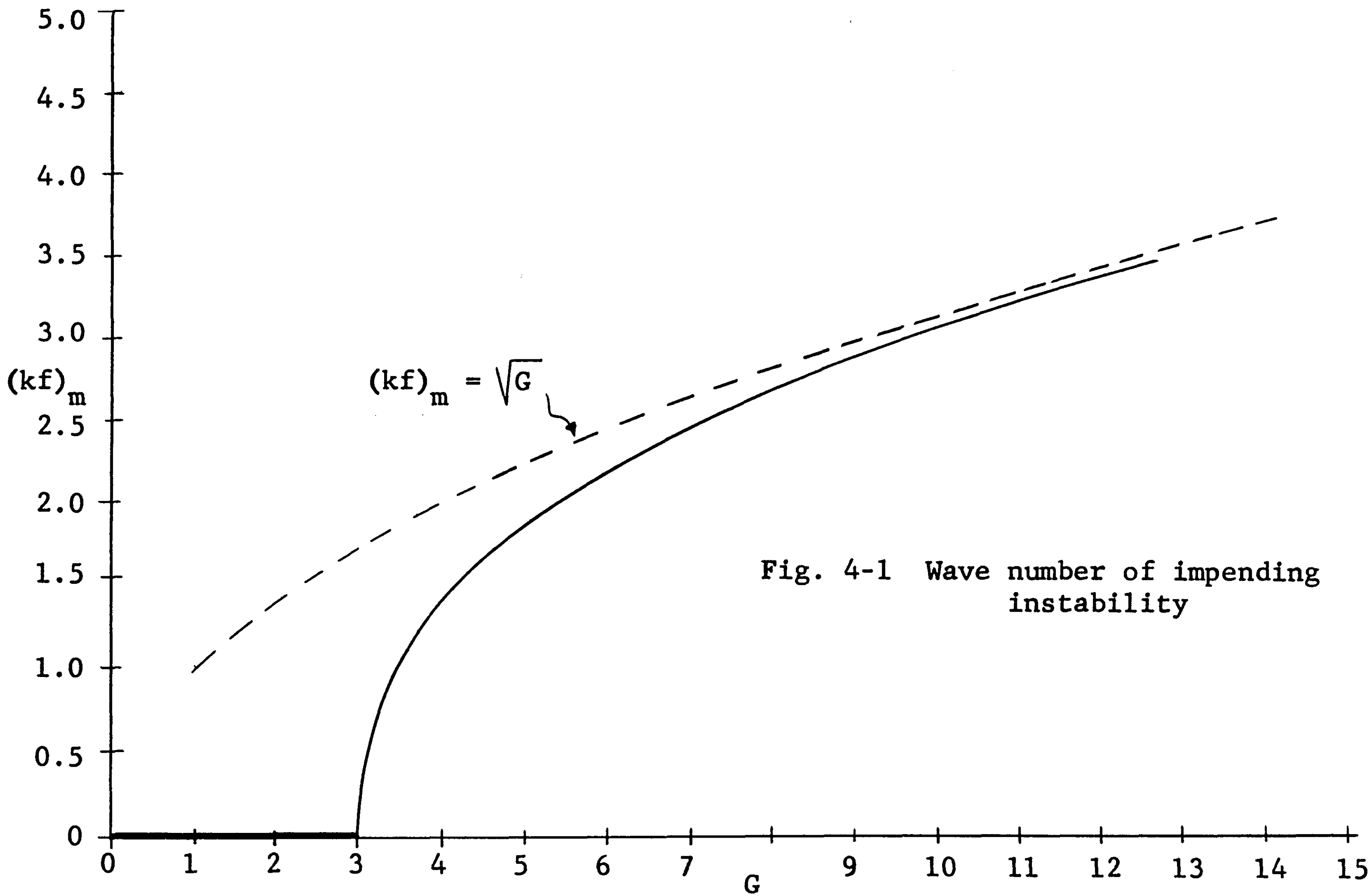


Fig. 4-1 Wave number of impending instability

given by Eq. (3-49). The dispersionless character of the field coupling is responsible for the fact that this is also the upper bound on the wavelength for ordinary gravitational instability.

ii. Type EH-If Waves - Experiment

For the EH-If problem, the conditions of Eqs. (4-4) and (4-5), written in terms of the voltage V_o are:

$$(kf)_m \gg 1 \qquad V_o = f\left[\frac{4(\rho^{(1)} - \rho^{(2)})gT}{[(1+c)\,\varepsilon]^2}\right]^{1/4} \tag{4-6}$$

$$(kf)_m \ll 1 \qquad V_o = f^{3/2}\sqrt{\frac{(\rho^{(1)} - \rho^{(2)})g}{(1+c)\,\varepsilon}} \tag{4-7}$$

Equations (4-6), (4-7) and (3-49) provide an experimental test of the previous theory. The first two of these equations give the theoretical dependence of the voltage for instability on the spacing of the plate and the interface. The experimental arrangement of Fig. 4-2 gives a check on this result.

Several liquids were used with water to obtain curves of the general form shown in Fig. 4-3. In general, it is found that the curves follow the 3/2 power law for low values of b and are linear at high values, with the transition in the region of $kf = 1$. The plot shown is for a Xylene-water interface,* so that $b = f$. The two solid curves in Fig. 4-3

*For constants used for theoretical results see Lange,[15].

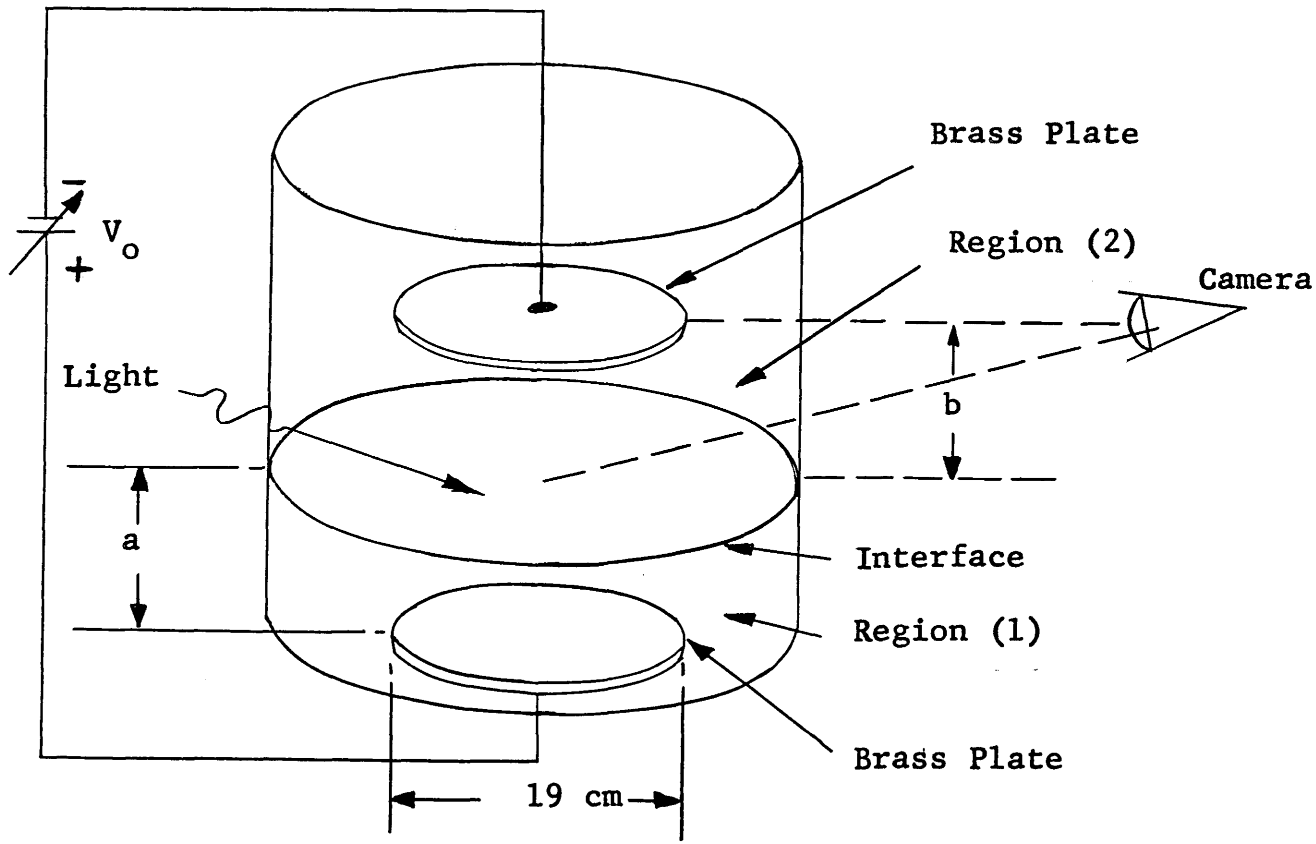


Fig. 4-2 EH-If Instability Apparatus

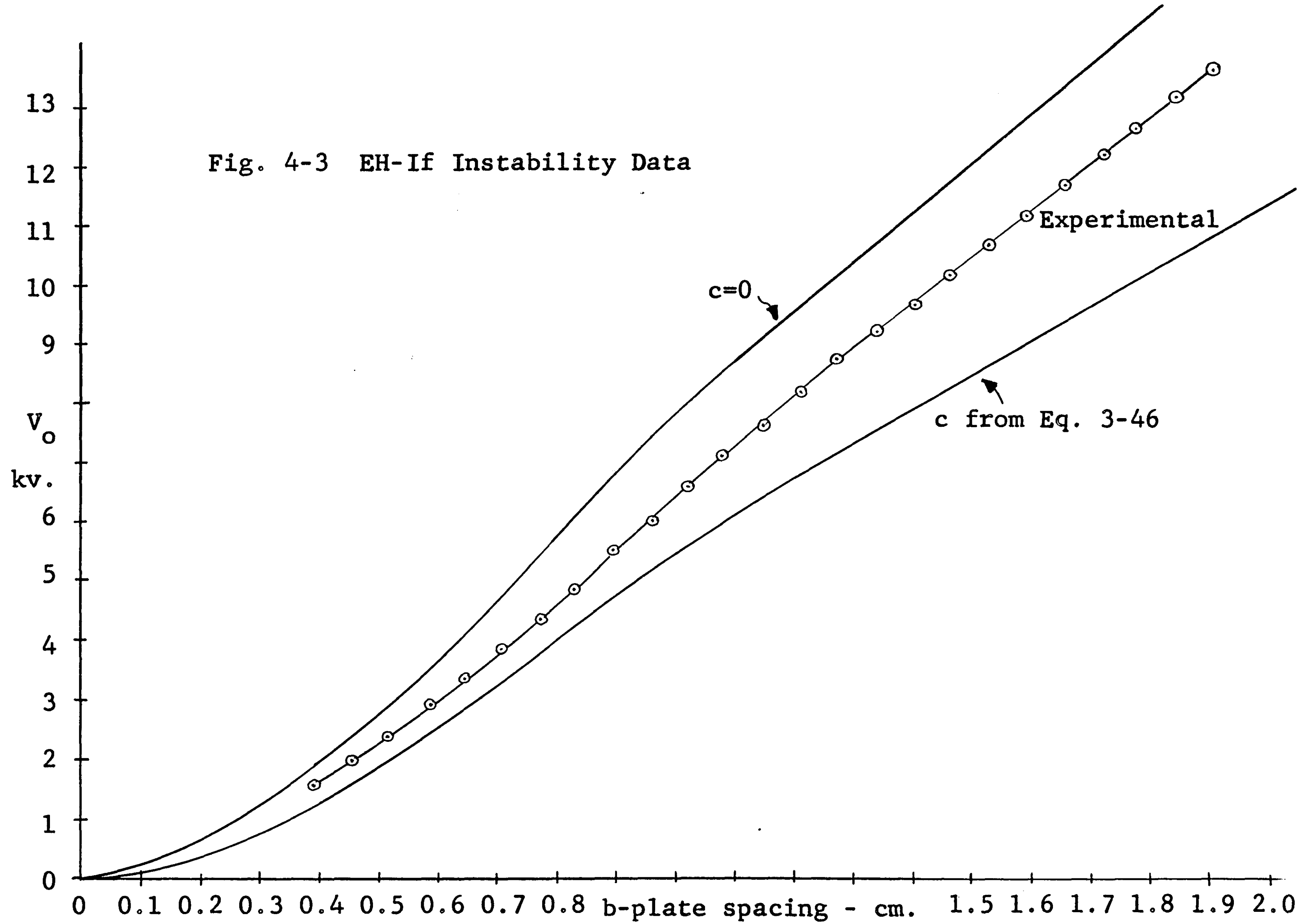


Fig. 4-3 EH-If Instability Data

indicate the theoretical results if electrostriction is ignored and if it is taken into account. The discrepancy between the curve for $c = 0$ and the experimental results may be used to infer a value for c of 0.40 as compared to a value of 0.85 from Eq. (3-46).

The instability consists of a disturbance on the interface which, although initially sinusoidal in nature, quickly grows into a sharply pointed spout that extends toward the upper plate. The point of instability is taken as the voltage at which this spout appears.

A sequential photograph of one section of the interface during the instability growth is shown in Fig. 4-4. The position of the viewer in these photographs is shown in Fig. 4-2. Note that the initial sinusoidal disturbance quickly grows into a sharp peak with a much shorter base than the initial disturbance wavelength. As may be seen, the instability is in a more advanced stage in the background.

The sharp point of the non-linear disturbance peak leads to a considerably increased local field intensity. If this intensity is sufficient to break down the dielectric, the instability criterion gives also a prediction of the voltage breakdown between a liquid interface and a solid boundary.

The approximate disturbance wavelength may be taken from Fig. 4-4. In this experiment $b = 5.22$ cm. Using a value of the wavelength from the picture of 3.3 cm, the

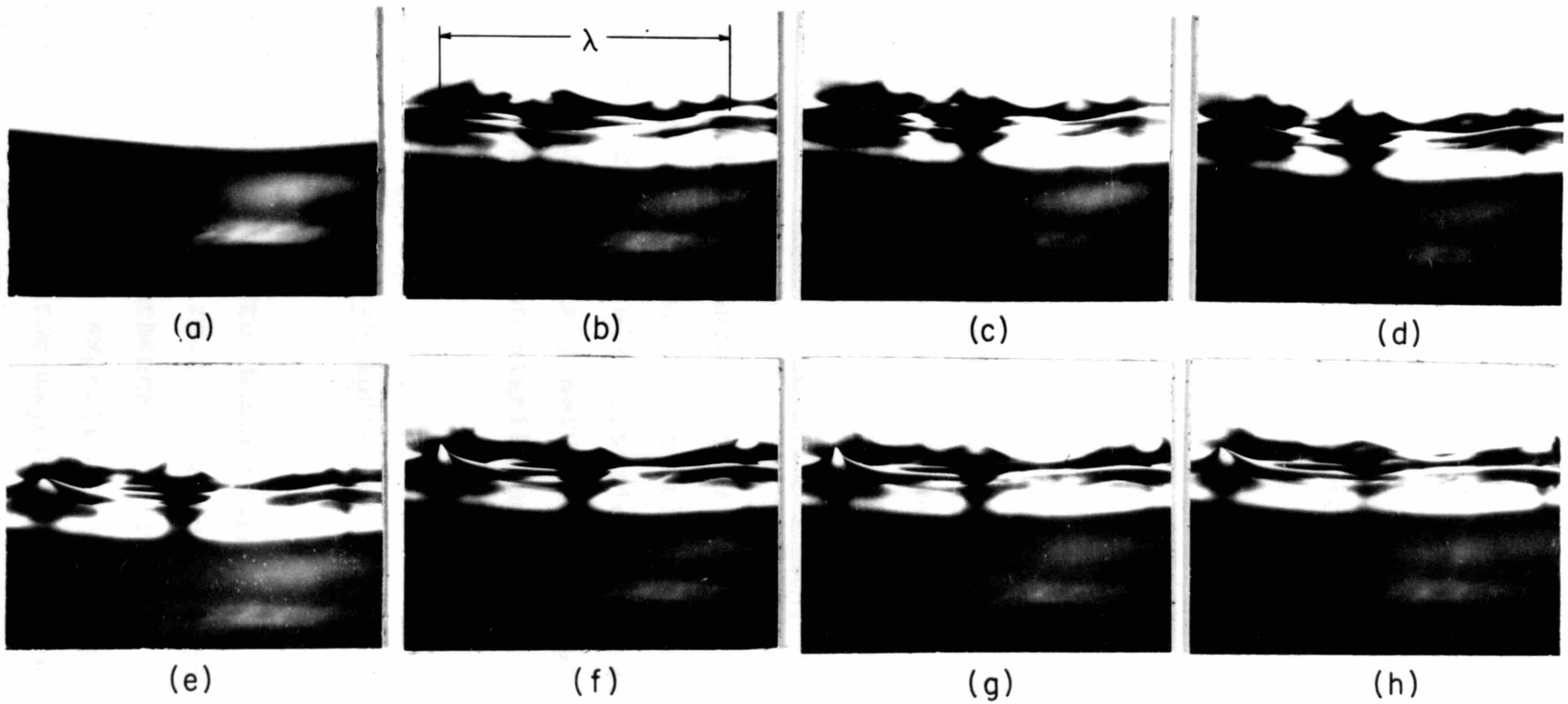


Figure 4-4

Sequential Photograph of EH-If Instability

Picture (a) at no field, following pictures at relative times of $0,2,4,6,7,8,9 \times 10^{-2}$ sec.

value of kb is about 5. Hence, Eq. (3-49) gives the initial instability wavelength as 3.5 cm. This agreement is better than would be expected from the experimental error involved.

iii. Critique

The instability experiment described here has at least two basic difficulties. The fringing fields resulting from the finite dimensions of the plate lead to an enhanced unstabilizing influence at the outer edges of the experimental surface area. At the same time, the steady state or zero order condition of a flat, parallel interface becomes extremely difficult to maintain as the point of instability is approached. Both of these effects lead to a measured point of instability that is lower than the true one. For this reason, it can only be concluded from this experiment that the electrostriction constant c is less than 0.4. This certainly serves to indicate that the Claussius-Mossotti equation significantly over-estimates the value of c for Xylene.

C. <u>Type II Waves - Surface Wave Radiation</u>

i. Theory

The most obvious way to demonstrate type II waves, is to take advantage of the direction dependence of the propagation velocity. A disturbance on a surface stressed by a tangential field would be expected to lead to "elliptical" wave fronts with the major axis in the direction of highest propagation velocity. For the EH-II waves and $V_b^2 > 0$, propagation on a dispersionless surface (i.e. at a constant wavelength) would lead qualitatively to wave fronts as shown in Fig. 4-5, where the ratio of the major and minor axes would give a check on the magnitude of V_b^2.

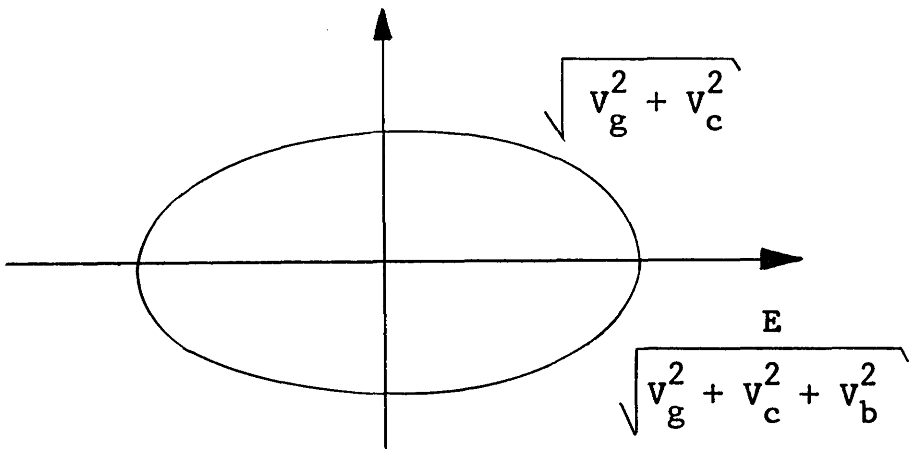


Figure 4-5
EH-II Wave Pattern

The major difficulty that arises in an actual experiment comes in determining the influence of dispersion on the wave fronts, since any disturbance excites a spectrum of wavelengths. For example, in the problem considered here, an essentially point disturbance is used. In two space dimensions, the wave resulting from an initial elevation of the fluid surface at the origin (an impulse S) are described by*

$$\xi = \lim_{x \to o} \int_o^\infty e^{kx} [\cos(kz+\omega t) + \cos(kz-\omega t)] dk (\frac{S}{2\pi}) \qquad (4\text{-}8)$$

$$\omega = [kg + \frac{k^3 T}{\rho} + k^2 V_b^2]^{1/2}$$

*See Sneddon,[16] p. 285.

where replacing $z \rightarrow y$ and $V_b \rightarrow 0$ gives the wave propagating across the E lines. Although this Fourier representation ignores the effect of curvature of the wave front, i.e. one of the three space dimensions, solutions in three dimensions for the gravity wave alone indicate that this is consistent with further approximations that will be made here.*

Integrals of the type of (4-8) are difficult to complete, because of the complicated interference phenomena that they describe. The integrand is a rapidly oscillating function of k. This is the basis of a well-known approximation technique called Kelvin's Principle of the Stationary Phase** which says that the main contribution to the integral arises when one or the other of the phases is stationary in k. That is, those wave numbers on a positive traveling wave that make a major contribution to the integral satisfy the relation

$$\frac{d}{dk}[kz - \omega t] = 0 \qquad (4\text{-}9)$$

Physically, these are the wave numbers that would predominate at a position z and time t. Equation (4-9) is then recognized as the familiar statement that

$$z = t\,\frac{d\omega}{dk} \qquad (4\text{-}10)$$

---

*See Sneddon,[16] p. 290.

**See Stoker,[4] p. 163.

This says that a group of waves characterized by the wave number k will be found at a distance z from the origin after an elapsed time t.

ii. Type EH-II Waves, Experiment

The experiment is best described by Fig. 4-6. An interface of air and nitrobenzene is stressed by a tangential electric field applied simultaneously with a disturbance impulse of air. After a time interval sufficient to allow the resulting waves to develop, the radiating disturbance is momentarily projected by a point source of light onto enlarging paper. Examples of the resulting photographs are shown in Fig. 4-7. These pictures were taken using nitrobenzene. Similar photographs have been made using water, transformer oil, acetaphenone and xylene. The resulting "ellipses" have in common the fact that the major axis of the "ellipse" extends in the direction of the electric field, never perpendicular to it as would be predicted by use of the Claussius-Mossotti equation (Eq. (3-46)).

Equation (4-10) makes it possible to obtain somewhat more quantitative information from the photographs. The ratio of the major and minor axes is given by this equation as

$$R = \frac{z_{\text{major axis}}}{y_{\text{minor axis}}} = \frac{\left[1 + \dfrac{V_b^2}{[\frac{g}{2k} + \frac{3}{2}\frac{kT}{\rho}]}\right]}{\left[1 + \dfrac{V_b^2}{[\frac{g}{k} + \frac{kT}{\rho}]}\right]^{1/2}} \qquad (4\text{-}11)$$

where k is the wave number corresponding to the waves at the points z and y. Unfortunately the theory requires

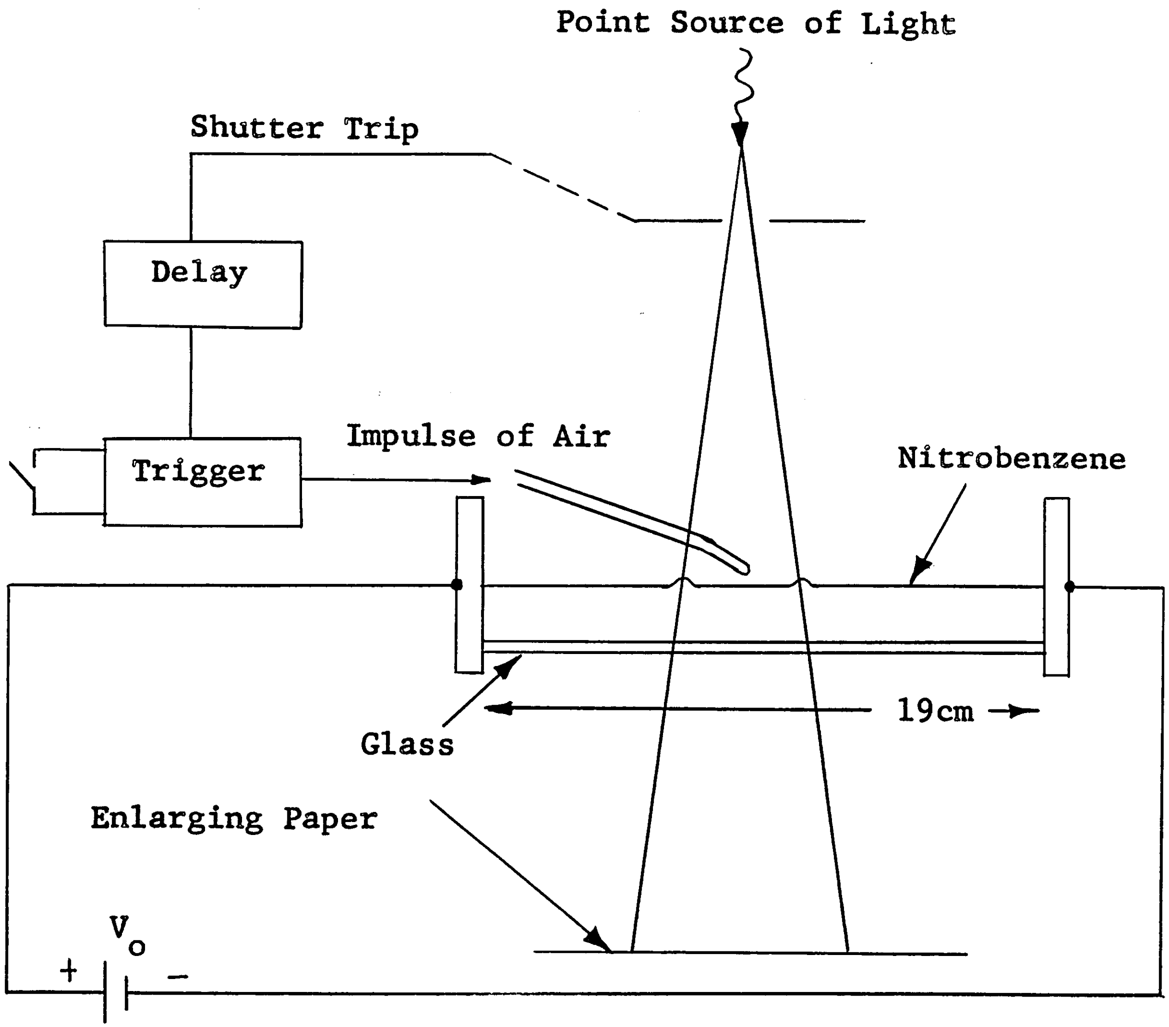
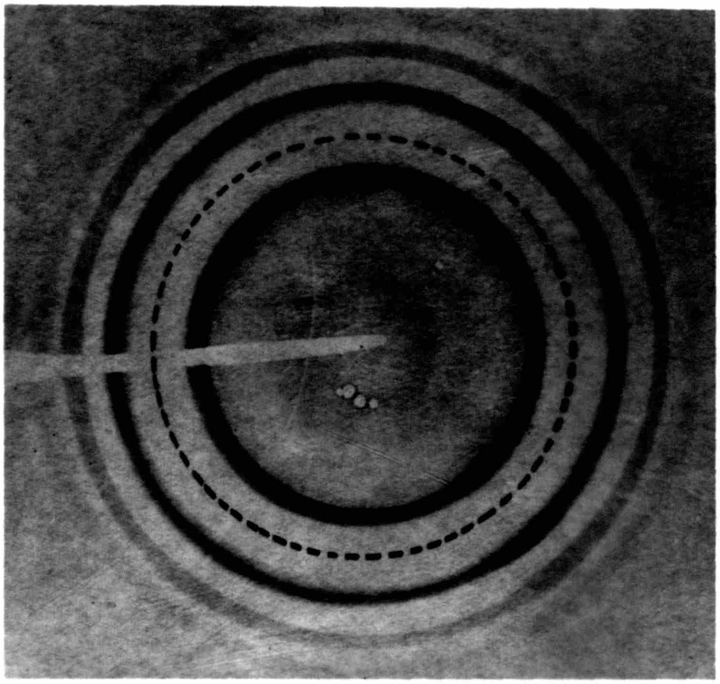
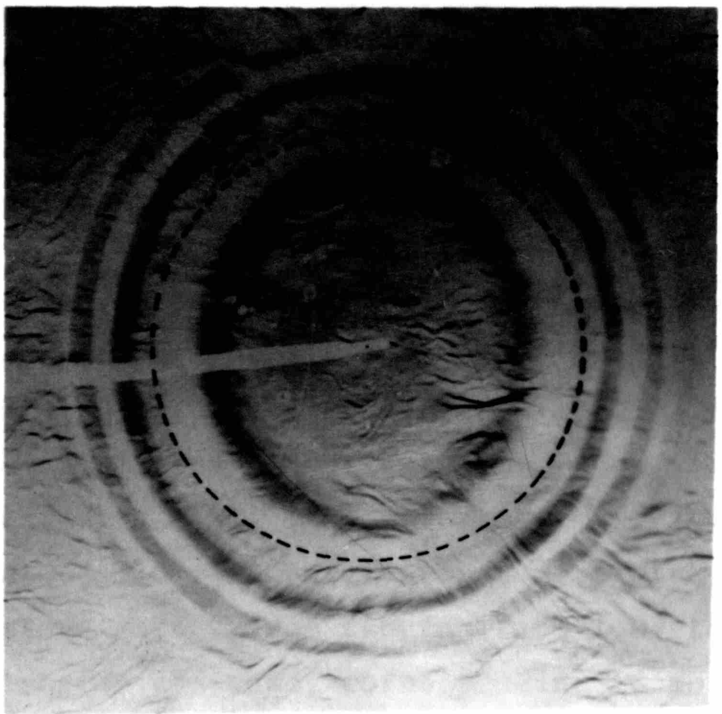


Fig. 4-6 EH-II Surface Wave Radiation Experiment



No E Field



E Field Top to Bottom

Figure 4-7
EH-II Wave Patterns Photographed on the Interface of Nitro-Benzene

that z and y are the distances along the axes to groups of waves corresponding to a given k. The distance may be determined by this prescription only if there are many waves in a distance on the order of z from which to select. Fortunately, the photographs indicate that the wavelength is not essentially altered along a given phase. This is a direct consequence of the fact that the waves are very nearly at a wavelength where the phase velocity is equal to the group velocity, so that a dispersionless addition to the phase velocity in one direction does not essentially alter the wavelength.

For the present purposes z and y are measured to points of equal phase, it being recognized that an error is incurred that appears to be proportional to the difference between the phase and group velocities.

The initial wavefronts of Fig. 4-7 have wavelengths of about 0.78 cm. If the electrostriction constant is ignored (this will be shown to be an excellent assumption in the next section), at an electric field of 1.57×10^5 v/m, $V_b = 7.77$ cm/sec, Eq. (4-11) gives $R = 1.05$. If the ratio of the phase velocities rather than the group velocities is used, this ratio is 1.07. The wavefronts of Fig. 4-7 give a ratio of about 1.07. Several consecutive pictures under these conditions gave values ranging from 1.07 to 1.08. If longer wavelengths are used, corresponding to points nearer the origin of the wavefronts, better agreement of theory and experiment is obtained.

Eccentricities as high as 1.25 have been observed at higher field intensities. However, there seems to be a close connection between the dynamics of electrohydrodynamic waves and the propagation of electroconvective surface instabilities.[8] It becomes increasingly difficult to observe distinct wave-fronts as the field is increased, even if, as was done in taking the pictures of Fig. 4-7, the field is applied simultaneously with the impulse of air. In some pictures there was a marked increase in electroconvection behind the wave fronts. The appearance of the surface convection may be noted in Fig. 4-7.

iii. Critique

The qualitative nature of the previous experiment is evident. Although the measurements support the theory of chapter 3, they are not definitive. However, it is clear that:

1. The waves propagate more rapidly along the $\bar{E}$ lines indicating that V_b^2 is positive.

2. The propagation is independent of the sense of the field as indicated by the symmetry of the disturbance rings. The predicted dependence is on $\bar{E}^2$.

3. The propagation velocity is in fair agreement with theory if $c \ll 1$.

D. Type I and II Waves, Resonances

i. Theory

The previous experiments have in common the complications resulting from a wave number that is free to change. A resonator has the advantage that for any given mode, the wave number is fixed. Even analytically this is an advantage, since the dispersion relations, Eqs. (3-37) and (3-38) are conveniently written as explicit functions of k. More important, however, is the basically elliptic character of the bulk equations, which makes it possible to write down the total wave number k, independent of dynamical considerations.

The results of chapter 3 lead directly to the resonator conditions. Equation (3-7) shows that p satisfies the equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \tag{4-12}$$

If a rectangular resonator is formed by rigid perpendicular walls at $y = 0$, $y = L_y$ and $z = 0$, $z = L_z$, the perpendicular velocities must vanish on each of the boundaries. Hence, from Eqs. (3-10) and (3-11)

$$\left.\frac{\partial p}{\partial y}\right|_{y=o,L_2} = 0 \; ; \quad \left.\frac{\partial p}{\partial z}\right|_{z=o,L_3} = 0 \tag{4-13}$$

Solutions to Eq. (4-12) have been found such that for each wave type,

$$p^{(1)} = \underline{p}^{(1)}[e^{kx} + e^{-kx}\, e^{-2ka}] \tag{4-14}$$

so that Eq. (4-12) becomes

$$\frac{\partial^2 \underline{p}}{\partial y^2} + \frac{\partial^2 \underline{p}}{\partial z^2} + k^2 \underline{p} = 0 \tag{4-15}$$

The boundary conditions (4-13) and Eq. (4-15) lead to the requirement that

$$k_y^2 = \left(\frac{n\pi}{L_y}\right)^2 \quad n = 0,\ 1,\ 2,\ 3\ \ldots$$

$$k_z^2 = \left(\frac{m\pi}{L_z}\right)^2 \quad m = 0,\ 1,\ 2,\ 3\ \ldots \qquad (4\text{-}16)$$

where $k^2 = k_y^2 + k_z^2$; m and n $\neq 0$ at the same time.

For any given mode, specified by m and n, k_y, k_z and hence k are fixed. The corresponding resonant frequency for each particular wave type is given by Eqs. (3-37) through (3-45).

It will be observed that no stipulation has been made on the electrical properties of the rigid vertical walls; that is, on the values taken by the electric and magnetic fields at the walls. This makes clear the point that single modes may be excited only if the conditions implied by the eigenvalues of Eq. (4-16) and the dispersion relations are satisfied. Furthermore, the walls must have properties consistent with the static fields.

It follows that the walls must be insulating for the type I waves. If there are to be no sources or vortices in the problem not accounted for by the bulk equations and the boundary conditions, it follows that at the vertical walls,

$$\bar{n} \cdot \bar{E} = 0 \qquad (4\text{-}17)$$

$$\bar{n} \cdot \bar{j}_f = 0 \qquad (4\text{-}18)$$

The boundary conditions of Table B-1 show that these conditions are implied by the requirement that $\frac{\beta_y \tilde{V}_x}{\alpha}$ or $\frac{\beta_z \tilde{V}_x}{\alpha} = 0$ at the vertical boundaries. That is, that $\frac{\partial\xi}{\partial y}$ or $\frac{\partial\xi}{\partial z} = 0$ at the boundaries; a direct consequence of the continuity condition and the irrotational character of $\bar{V}$. The pertinent equations to show this are:

EH-If; Eqs. (3-20a) and (3-23a); Eqs. (3-21a) and (3-22a) show that

$e_y = 0$ at $y = 0, L_y$ $\qquad$ $e_z = 0$ at $z = 0, L_z$

$j_f = 0$ when V_y or V_z vanish.

EH-Ip; Eqs. (3-22a), (3-23a), (3-27a), (3-14) and (3-15) show that

$e_y = 0$ at $y = 0, L_y$ $\qquad$ $e_z = 0$ at $z = 0, L_z$

MH-I; Eqs. (3-24a), (3-25a), (3-26a), (3-12) and (3-13) show that

$h_y = 0$ at $y = 0, L_y$ $\qquad$ $h_z = 0$ at $z = 0, L_z$

The type II resonators are not symmetrical in y and z. The rigid plates at $y = 0, L_y$ and $z = 0, L_z$ may therefore imply different boundary conditions. Those listed below are allowed by the static and first order fields if $k_y = 0$. (The modes resulting from propagation in the z direction are the interesting ones.)

EH-II Perfect conductor or infinite permittivity at $z = 0, L_z$
Perfect insulator at $y = 0, L_y$.

MH-IIa Infinite permeability at $z = 0, L_z$
Perfect insulator at $y = 0, L_y$.

MH-IIf Perfect insulator at $z = 0, L_z$ but
infinite permeability
Infinite conductivity at $y = 0, L_y$.

This follows from the conditions of Table B-1.

EH-II Since $\beta_y = 0$, $e_y = 0$ everywhere
Eqs. (3-21b) and (3-27b) show that when $\frac{\partial \xi}{\partial z} = 0$,
$e_x = 0$.

MH-IIa Since $\beta_y = 0$, $h_y = 0$ everywhere
Eqs. (3-21b) and (3-24b) show when $\frac{\partial \xi}{\partial z} = 0$,
$h_x = 0$.

MH-IIf Eqs. (3-22b) and (3-21b) show $e_x = e_z = 0$ everywhere
Since $\beta_y = 0$, $h_y = 0$.

It must be recognized that the resonator solutions determined in this way are idealizations, for the cases with electrically transparent boundaries. The equations of motion are not satisfied for the fields that must fringe out of the resonator region, so that the problems are similar, in these cases, to any of the "open ended" wave structures that support standing waves. The solutions are questionable at the edges of the resonator, in the same way that solutions to a transmission line problem are questionable at an open end. This is a meaningful approximation if the sources and vortices of the fields are accounted for within the resonator volume.

ii. Type EH-If, EH-II Waves, Experiment

For a given mode, the wave numbers k_y and k_z are fixed by Eq. (4-16). Equations (3-37) and (3-38) then give a linear relationship between the square of the resonant angular frequency (ω^2) and the square of the electric field intensity, (or the applied electrode voltage squared V_o^2). The type I and II problems are characterized respectively by negative and positive resonant frequency shifts as a function of an increasing applied voltage.

Experiments using the EH-If and EH-II resonators are depicted by Fig. 4-8. A transparent shake-table driven by a synchronous motor was used to create standing waves. The resonant frequency and mode were found by varying the driving frequency or applied voltage while observing the mode patterns as projected onto a screen. For the type I resonator the best accuracy was obtained by tuning the frequency while for type II resonators it was found best to tune the voltage.

The resonant condition was defined by assuming that the resonance "Q" curve was symmetrical as a function of frequency or voltage, the desired condition being the one for which there was no adjacent condition that would result in the same pattern. The driving mechanism favored the excitation of even n and odd m modes. The resonant frequency was on the order of 7 cps., so that numbers on the order of 350 were recorded as proportional to the driving frequency. With no electric fields applied, the resonant condition could be repeated within one part in 350. This is better than the dimensional accuracy of the two resonators.

Data for the type I resonator using water and air are plotted in Fig. 4-9. In this experiment the electrostriction did not enter, because the electric fields are virtually confined to a region where the permittivity is very nearly that of free space. The frequency squared as a function of applied

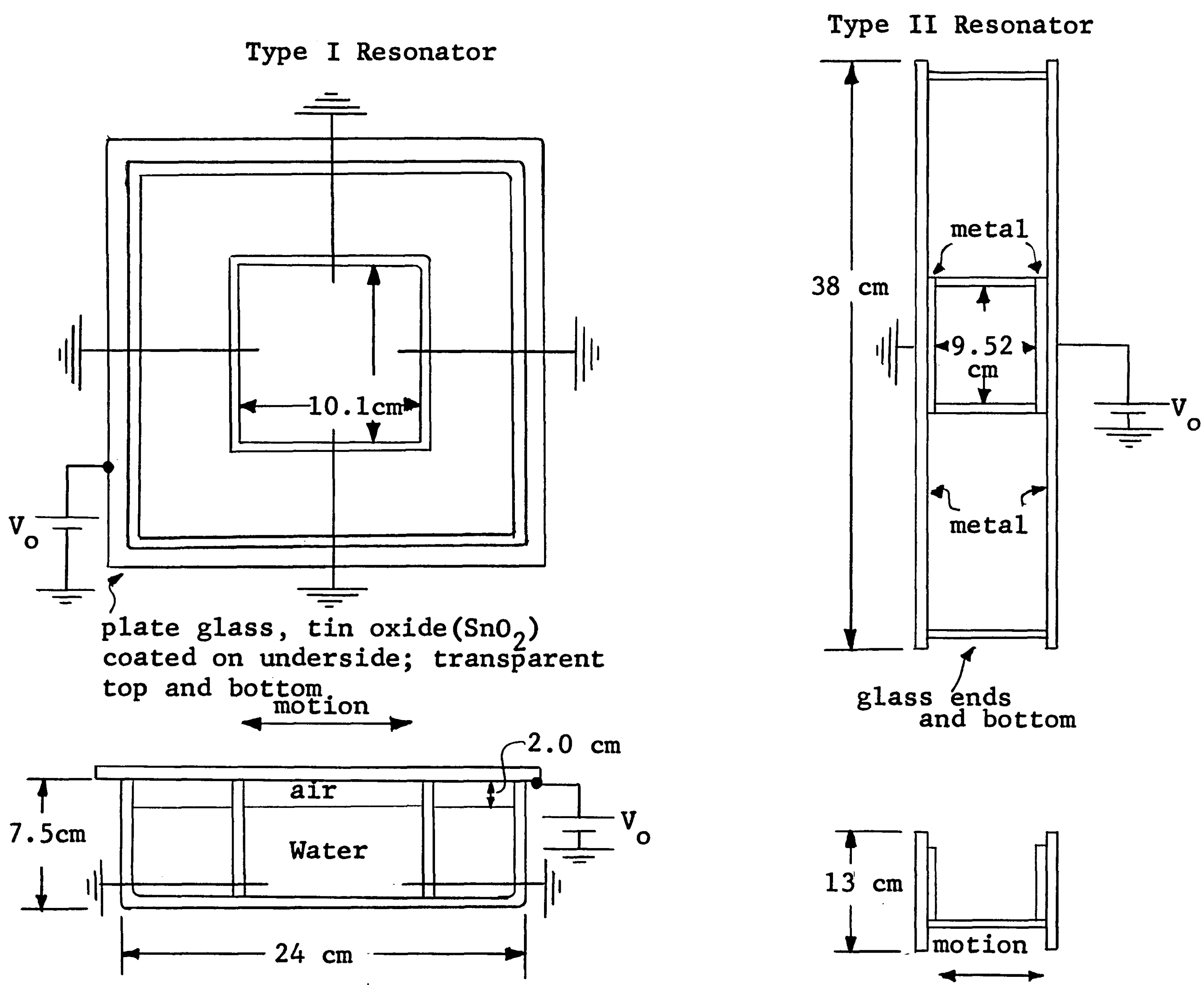


Fig. 4-8a EH-If and EH-II Resonator Experiments

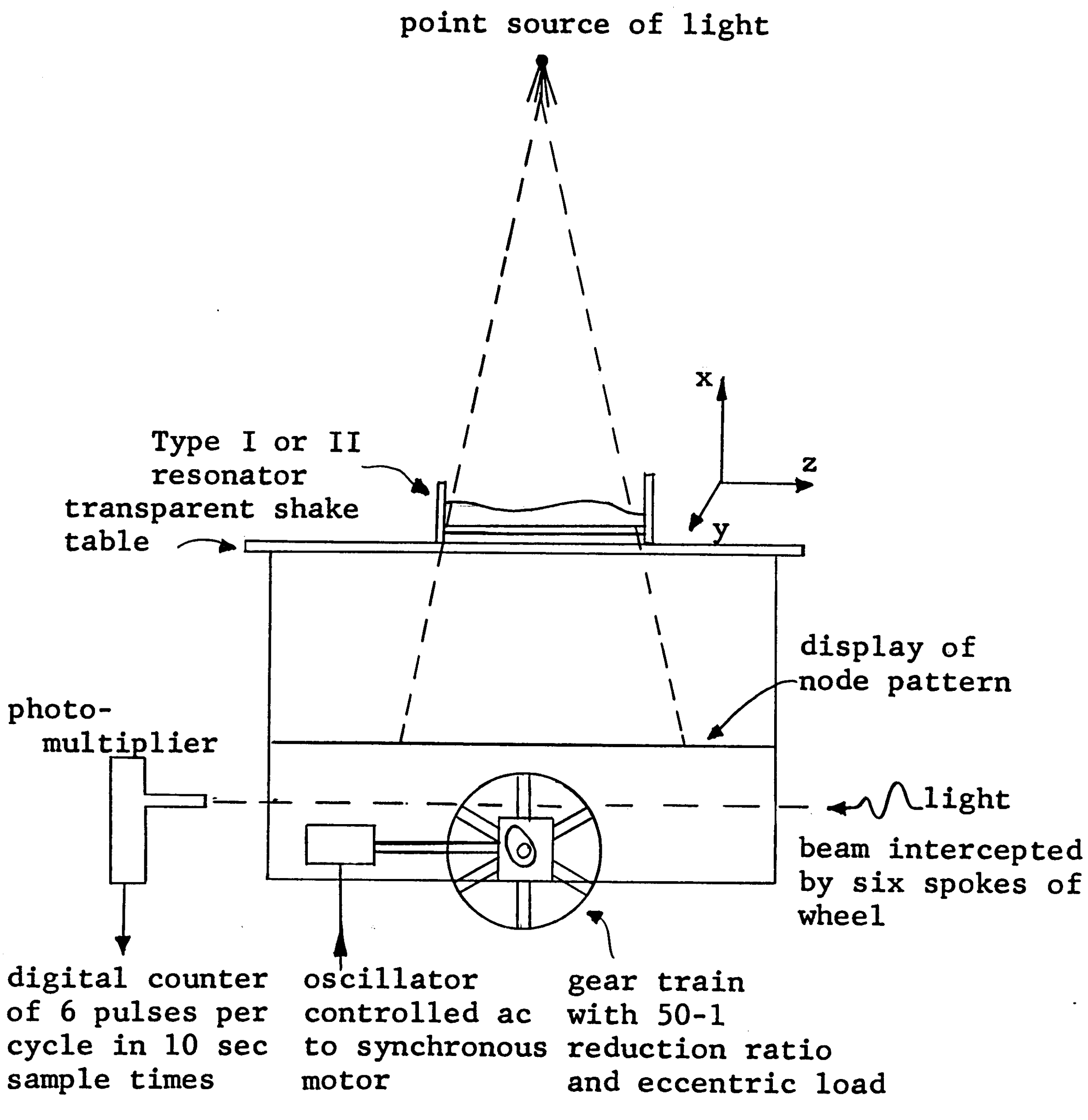


Fig. 4-8b EH-If and EH-II Resonator Experiments

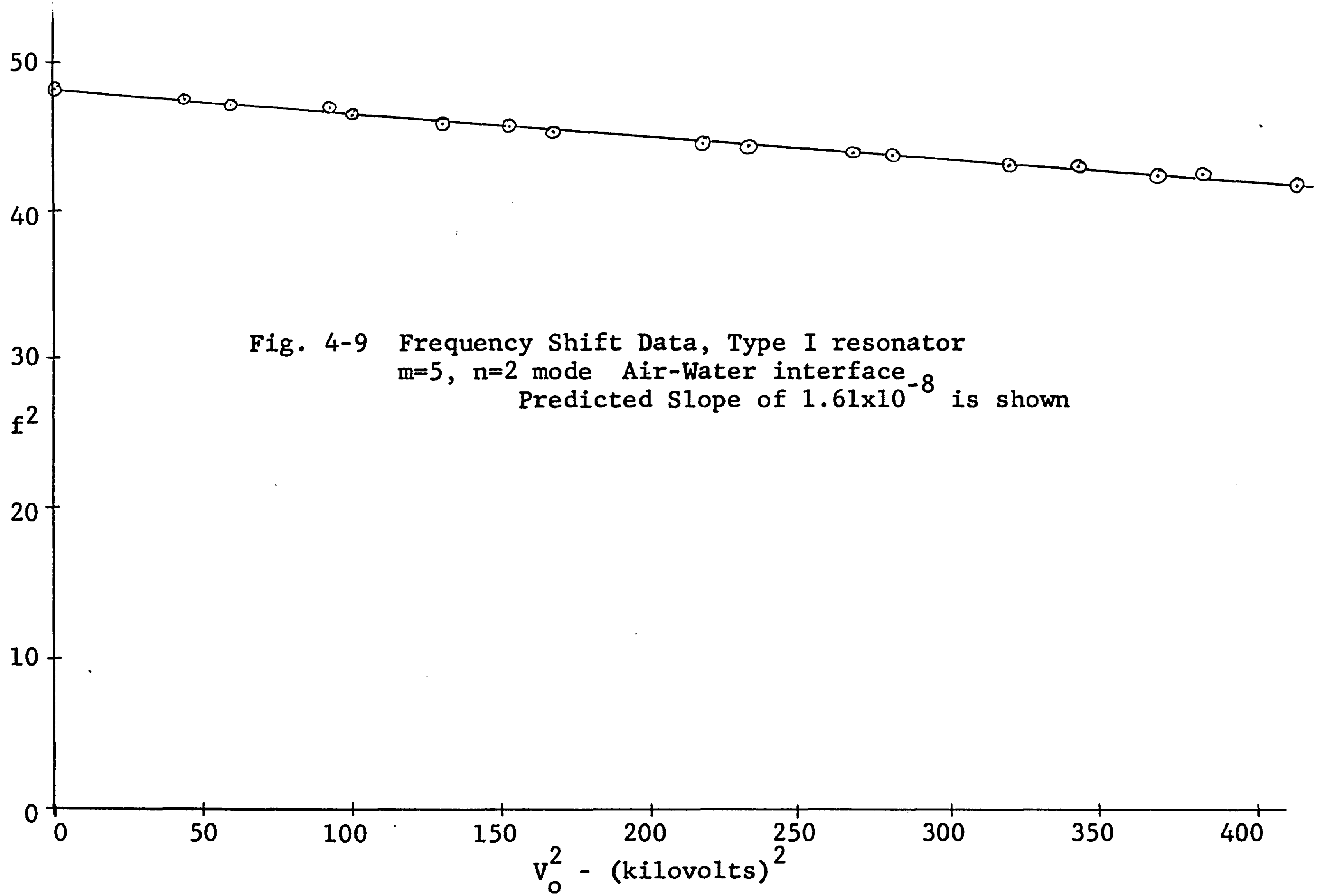


Fig. 4-9 Frequency Shift Data, Type I resonator
m=5, n=2 mode Air-Water interface
Predicted Slope of 1.61×10^{-8} is shown

voltage squared is clearly a straight line. Moreover, the slope of the line, as predicted by Eq. (3-37), is within about 5% of straight lines that can reasonably be drawn through the points. The line drawn has the theoretical slope.

The upper limit of voltage represents a practical limit on the resistance to break-down of the apparatus used. It must further be mentioned that hysteresis effects due to accumulating free charges may produce a considerable error unless care is taken with insulating surfaces. In the system used here for example, this meant making the bottom surface of the glass plate the conducting one.

The experimental results of measuring the frequency shift of a resonance on an air-nitrobenzene interface in a type II resonator are shown in Fig. 4-10. Here, only m modes were excited. The upper limit on the curve represents the highest field intensity at which the mode could be reasonably recognized in spite of the interfering electroconvection. Again, the curve appears to be a straight line. The value of c inferred from this curve and Eq. (3-38) is 0.03.* The value of c must again be interpreted as predicting c to be on the order of 0.03 or less, since this represents the limit of the accuracy involved in the measurements. It is much lower than would be expected from the Claussius-Mossotti equation (3-46) which gives a value of 12.0. This latter value would indicate a negative rather than a positive slope for the curve of Fig. 4-10.

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*For constants used for theoretical results see Lange.[15]

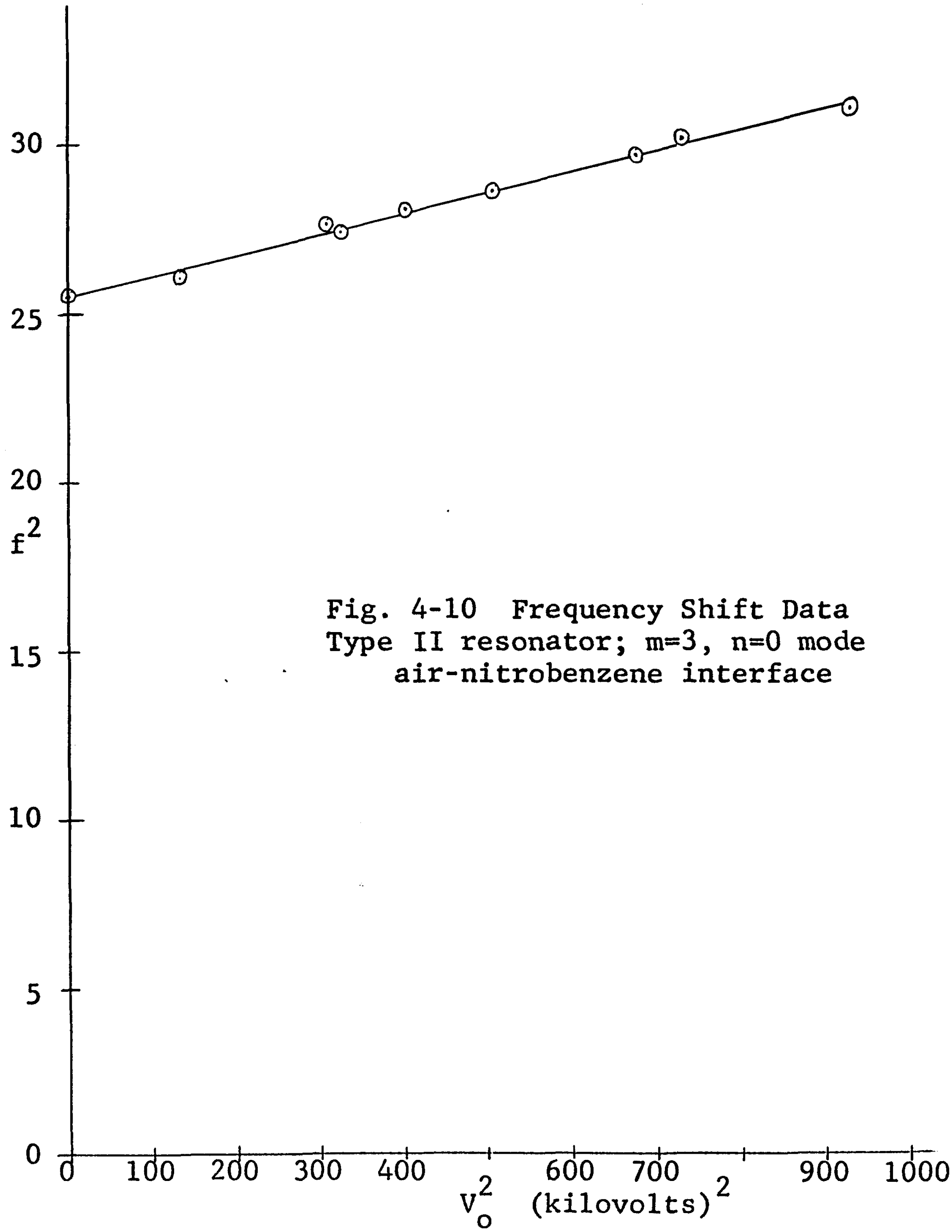

Fig. 4-10 Frequency Shift Data
Type II resonator; m=3, n=0 mode
air-nitrobenzene interface

iii. Critique

The resonator experiments are, undoubtedly, the best of the three methods of observing electro-hydrodynamic surface waves. They are dynamical in nature and do not depend on the maintenance of a uniform interface at high field intensities, as the instability experiment does. At the same time, they fix the wave length and hence make the dispersion relations directly applicable.

The main errors arise from:

1. First order fringing fields, especially in the "open circuit" peripheries of the type I resonator. This may be minimized by making L_y and L_z much larger than b or a.

2. The meniscus formed at boundaries by the surface tension and enhanced by the electric field. This leads to an error in the effective positions of the rigid walls, an effect that is minimized by making the dimensions of the resonator large.

3. The dielectric boundaries give a distortion of the zero order fields, an effect that is minimized by making these walls very thin.

4. The experiments are steady state and hence are even more subject to the noisy conditions of electroconvection.

E. Conclusions

The previous experiments include both type I and type II configurations and serve to indicate the behavior that may be expected dynamically with waves other than EH-If and EH-II.

The experiments have verified:

EH-If

1. The dispersion relation at high and low field intensities (Sections B and D). This followed from the observation of an interface in the extreme of impending instability and at constant k and low field intensities.

2. The dependence of the wave velocity on interactions with a close spaced conductor, (Section B). This is an indication of the validity of the electrical part of the "long wave" approximation used in chapter 5.

EH-II

1. The dispersion relation at low field intensities (Sections C and D). The value of V_b^2 was clearly positive and dependent on E^2. The fact that experiments using several different liquids always made c on the order of the accuracy of the experiment, indicates not only that c is very much smaller than is often supposed, but that the value of V_b^2 is correctly given by the dispersion relation.

The experiments of this chapter strongly indicate that the theoretical treatment of EH-If and EH-II waves given in chapter 3 is physically meaningful in situations easily produced in the laboratory.

PART II

NON-LINEAR ANALYSIS

CHAPTER 5

MAGNETOHYDRODYNAMIC AND ELECTROHYDRODYNAMIC SURFACE SHOCKS AND ANTI-SHOCKS

A. Introduction

There is a large class of problems, of interest mainly to hydraulic engineers, concerned with free surface flows dominated by gravity, and hence dynamically dominated by gravity waves. The effects of interest in these problems are often grossly non-linear in character and are meaningfully approximated by the so-called long-wave approximation. Many of the most interesting phenomena observed on fluid surfaces are essentially non-linear in character. Breakers at the beach or hydraulic jumps at the foot of a dam are waves strongly influenced by a velocity of propagation which depends on the fluid depth. The breaker, for example, is a form of surface shock. As a wave progresses into shallow water it is reduced in velocity only to be overtaken by faster moving fluid that follows in deeper water.

The purpose of this chapter is to indicate the theoretical basis for a class of free channel electro- and magnetohydrodynamic flows that are, to linear terms, the EH-If and MH-IIf waves as they interact strongly with external boundaries. EH and MH surface waves have phase velocities that are directly dependent on the electric and magnetic field intensities, just as long gravity waves directly depend for their velocity on the square root of the depth. It would be expected that non-linear stages of the surface interactions of the previous chapters could be discussed in a way similar to that commonly used in the literature of gravity waves. Thus, the problem may be approached by proceding to higher order terms in the perturbation analysis, or by reducing the equations to a non-linear set that may be handled by known techniques. The first sections of this chapter follow the latter course, and arrive at a tractable, but non-linear formulation of EH-If and MH-IIf configurations.

B. Electrohydrodynamic Equations

In ordinary bulk interactions, much can be said about non-linear plane waves by taking advantage of the well-known theory of characteristics. The power of this technique is available if the defining equations are hyperbolic and have two independent variables. A set of equations is now developed that meet these conditions and define the dynamics of a shallow conducting fluid stressed by a perpendicular field and bounded above by a fluid of small density (water and air for example). It is assumed that surface tension is ignorable (note that T did not play an important role in the long wave limit ($kb < 1$) of the linear problem except near the point of instability. However, in this limit the point of instability was itself independent of T) (See Eqs. (3-52) and (4-7)). Also of note is the fact that there is an equipotential boundary close to the surface of the conducting fluid. The procedure of rationalization used here is similar to that used in formally arriving at the long-wave equations for gravity waves.[17]

The configuration under investigation is shown in Fig. 5-1.

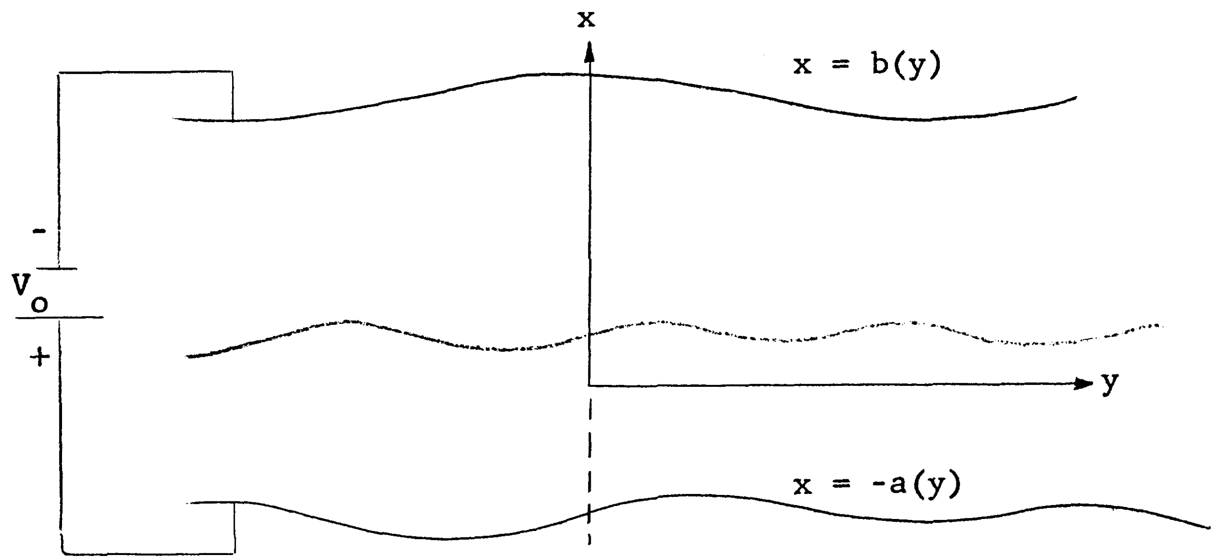

Figure 5-1
EH-If Configuration

The fluid is confined from below by a surface at $x = -a(y)$ and the potential at $x = b(y)$ is constrained to $-V_o$ with respect to the conducting fluid. A gravitational field acts in the negative x direction. The pertinent bulk equations are:

$$\rho[\frac{\partial \bar{V}}{\partial t} + (\bar{V}\cdot\nabla)\bar{V}] + \nabla p = -\rho g \bar{a}_x \tag{5-1}$$

$$\nabla \times V = 0; \quad \nabla\cdot\bar{V} = 0 \tag{5-2}$$

$$\nabla \times \bar{E} = 0; \quad \nabla\cdot\bar{E} = 0 \tag{5-3}$$

Equation (5-2) is valid if the motions studied proceed from irrotational flows. Boundary conditions consistent with these equations are:

$$[n_\alpha[p] - [T_{\alpha\beta}]n_\beta]\Big|_{x=\xi(y,t)} = 0 \tag{5-4}$$

where $T_{\alpha\beta}$ is defined by $T_{\alpha\beta} = \varepsilon E_\alpha E_\beta - \delta_{\alpha\beta}\varepsilon E_\gamma E_\gamma/2$

$$[\bar{n} \cdot \bar{V}]\Big/_{x=-a(y)} = 0 \qquad (5\text{-}5)$$

$$\int_{\xi}^{b} E_x \, dx = + V_o \qquad (5\text{-}6)$$

$$[\bar{n} \times \bar{E}]\Big/_{x=\xi} = 0 \qquad (5\text{-}7)$$

In addition the interface is defined dynamically and geometrically by

$$\frac{DF}{Dt} = - \frac{\partial \xi}{\partial t} + V_x - V_y \frac{\partial \xi}{\partial y}\Big/_{x=\xi} = 0 \qquad (5\text{-}8)$$

$$\bar{n} = \frac{\nabla F}{(\nabla F \cdot \nabla F)^{1/2}} = [\bar{a}_x - \bar{a}_y \frac{\partial \xi}{\partial y}][1 + (\frac{\partial \xi}{\partial y})^2]^{-1/2} \qquad (5\text{-}9)$$

where

$$F = x - \xi(y,t)$$

The last two equations may be used to restate the boundary conditions of Eqs. (5-5) and (5-7)

$$[V_x + V_y \frac{\partial a}{\partial y}]\Big/_{x=-a(y)} = 0 \qquad (5\text{-}10)$$

$$E_z\Big/_{x=\xi} = 0 \qquad (5\text{-}11)$$

$$[E_y + \frac{\partial \xi}{\partial y} E_x]\Big/_{x=\xi} = 0 \qquad (5\text{-}12)$$

Differentiate Eq. (5-6) with respect to y and use Eq. (5-12) to write

$$[E_y + \frac{\partial b}{\partial y} E_x]\Big/_{x=b} = 0 \qquad (5\text{-}13)$$

Note that this is equivalent to the condition that $(\bar{n} \times \bar{E})$ vanish at $x = b$.

Equation (5-4) gives two non-trivial relations:

$$p + \left[\frac{\varepsilon E_x^2}{2} - \frac{\varepsilon E_y^2}{2}\right] - \frac{\partial \xi}{\partial y} \varepsilon E_x E_y \Big|_{x=\xi} = 0 \qquad (5\text{-}14)$$

$$p \frac{\partial \xi}{\partial y} - \varepsilon E_y E_x + \frac{\partial \xi}{\partial y} \left[\frac{\varepsilon E_y^2}{2} - \frac{\varepsilon E_x^2}{2}\right] \Big|_{x=\xi} = 0 \qquad (5\text{-}15)$$

Equations (5-12), (5-14) and (5-15) combine

$$p + \frac{\varepsilon E_x^2}{2} \left(1 + \left(\frac{\partial \xi}{\partial y}\right)^2\right) \Big|_{x=\xi} = 0 \qquad (5\text{-}16)$$

The pertinent Equations are now (5-1), (5-2), (5-3), (5-8), (5-10), (5-12), (5-13) and (5-16).

An approximation to these equations is now formulated based upon an expansion in a "space rate" parameter. This parameter, which has the effect of scaling the width to depth ratio, will be taken as $\lambda = (\frac{w}{s})^2$ where dimensionless variables are defined as:

$$\begin{aligned}
&y = \bar{y}s && V_1 = \bar{V}_1\sqrt{gw}\,(\tfrac{s}{w}) && b = \bar{b}w \\
&x = \bar{x}w && V_2 = \bar{V}_2\sqrt{gw} && a = \bar{a}w \\
&t = \frac{\bar{t}s}{\sqrt{gw}} && p = \bar{p}(\rho g)w && E_x = E_o\bar{E}_x \\
& && \xi = \bar{\xi}w && E_y = E_o(\tfrac{s}{w})\bar{E}_y \\
& && && wE_o\bar{V}_o = V_o
\end{aligned} \qquad (5\text{-}17)$$

The time has been purposely scaled to horizontal distances while the velocity is scaled to the velocity of gravity waves.

The vertical velocity and y directed electric field have been intentionally surpressed by the ratio $(\frac{s}{w})$. The equations of motion and their boundary conditions summarized in dimensionless form are: (For convenience, the bar is omitted from the dimensionless variables in the following equations.)

$$\lambda[\frac{\partial V_x}{\partial t} + V_y \frac{\partial V_x}{\partial y} + \frac{\partial p}{\partial x} + 1] + V_x \frac{\partial V_x}{\partial x} = 0 \qquad (5\text{-}18)$$

$$\lambda[\frac{\partial V_y}{\partial t} + V_y \frac{\partial V_y}{\partial y} + \frac{\partial p}{\partial y}] + V_x \frac{\partial V_y}{\partial x} = 0 \qquad (5\text{-}19)$$

$$\frac{\partial V_x}{\partial x} + \lambda \frac{\partial V_y}{\partial y} = 0 \qquad (5\text{-}20)$$

$$\frac{\partial V_y}{\partial x} = \frac{\partial V_x}{\partial y} \qquad (5\text{-}21)$$

$$\frac{\partial E_y}{\partial x} - \lambda \frac{\partial E_x}{\partial y} = 0 \qquad (5\text{-}22)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \qquad (5\text{-}23)$$

$$[p + \frac{U_b^2 E_x^2}{2} (1 + \lambda (\frac{\partial \xi}{\partial y})^2)] \Big/_{x=\xi(y,t)} = 0 \qquad (5\text{-}24)$$

where

$$U_b^2 = [\frac{\varepsilon E_o^2}{\rho g w}]$$

$$[E_y + \lambda \frac{\partial \xi}{\partial y} E_x] \Big/_{x=\xi(y,t)} = 0 \qquad (5\text{-}25)$$

$$[V_x - \lambda(V_y \frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial t})]\Big|_{x=\xi(y,t)} = 0 \tag{5-26}$$

$$[E_y + \lambda \frac{\partial b}{\partial y} E_x]\Big|_{x=b(y)} = 0 \tag{5-27}$$

$$[V_x + \lambda V_y \frac{\partial a}{\partial y}]\Big|_{x=-a(y)} = 0 \tag{5-28}$$

A series expansion of each of the dependent variables is taken to be of the form,

$$V_x = \overset{o}{V}_x + \lambda \overset{1}{V}_x + \lambda^2 \overset{2}{V}_x + \ldots \tag{5-29}$$

and substituted into Eqs. (5-18) - (5-28). The problem is formally described by the equations obtained by equating coefficients of like power in λ. Hence, the zero order terms are:

$$\overset{o}{V}_x \frac{\partial \overset{o}{V}_x}{\partial x} = 0 \tag{5-30}$$

$$\overset{o}{V}_x \frac{\partial \overset{o}{V}_y}{\partial x} = 0 \tag{5-31}$$

$$\frac{\partial \overset{o}{V}_x}{\partial x} = 0 \tag{5-32}$$

$$\frac{\partial \overset{o}{V}_y}{\partial x} = \frac{\partial \overset{o}{V}_x}{\partial y} \tag{5-33}$$

$$\frac{\partial \overset{o}{E}_y}{\partial x} = 0 \tag{5-34}$$

$$\frac{\partial \overset{o}{E}_x}{\partial x} + \frac{\partial \overset{o}{E}_y}{\partial y} = 0 \tag{5-35}$$

$$[\overset{o}{p} + \overset{o}{E}{}_x^2 \frac{U_b^2}{2}]\Big|_{x=\xi(y,t)} = 0 \tag{5-36}$$

$$\overset{o}{E}_y\Big|_{x=\xi(y,t)} = 0 \tag{5-37}$$

$$[\overset{o}{V}_x]\Big|_{x=\xi(y,t)} = 0 \tag{5-38}$$

$$[\overset{o}{E}_y]\Big|_{x=b(y,t)} = 0 \tag{5-39}$$

$$[\overset{o}{V}_x]\Big|_{x=-a(y,t)} = 0 \tag{5-40}$$

Equations (5-32) and (5-38) or (5-40) imply $\overset{o}{V}_x = 0$, Eq. (5-33) shows that $\overset{o}{V}_y = \overset{o}{V}_y(y,t)$, Equations (5-34) and (5-37) or (5-39) give $\overset{o}{E}_y = 0$, while (5-35) states that $\overset{o}{E}_x = \overset{o}{E}_x(y,t)$. Use is made of these facts in writing the first order equations.

$$\overset{o}{p} = -x + F_1(y,t) \tag{5-41}$$

$$\frac{\partial \overset{o}{V}_y}{\partial t} + \overset{o}{V}_y \frac{\partial \overset{o}{V}_y}{\partial y} + \frac{\partial \overset{o}{p}}{\partial y} = 0 \tag{5-42}$$

$$\overset{1}{V}_x = -\frac{\partial \overset{o}{V}_y}{\partial y} x + F_2(y,t) \tag{5-43}$$

$$\frac{\partial \overset{1}{V}_y}{\partial x} = \frac{\partial \overset{1}{V}_x}{\partial y} \tag{5-44}$$

$$\overset{1}{E}_y = \frac{\partial \overset{o}{E}_x}{\partial y} x + F_3(y,t) \tag{5-45}$$

$$\frac{\partial \overset{1}{E}_x}{\partial x} + \frac{\partial \overset{1}{E}_y}{\partial y} = 0 \tag{5-46}$$

$$\left[\overset{1}{E}_y + \frac{\partial \overset{o}{\xi}}{\partial y} \overset{o}{E}_x\right]\Big/_{x=\xi(y,t)} = 0 \tag{5-47}$$

$$\left[\overset{1}{V}_x - \overset{o}{V}_y \frac{\partial \overset{o}{\xi}}{\partial y} - \frac{\partial \overset{o}{\xi}}{\partial t}\right]\Big/_{x=\xi(y,t)} = 0 \tag{5-48}$$

$$\left[\overset{1}{E}_y + \frac{\partial b}{\partial y} \overset{o}{E}_x\right]\Big/_{x=b(y)} = 0 \tag{5-49}$$

$$\left[\overset{1}{V}_x + \overset{o}{V}_y \frac{\partial a}{\partial y}\right]\Big/_{x=-a(y)} = 0 \tag{5-50}$$

$F_1(y,t)$ is evaluated using Eq. (5-36). Hence Eq. (5-41) becomes

$$\overset{o}{p} = (\xi - x) - \overset{o}{E}{}_x^2 \frac{U_b^2}{2} \tag{5-51}$$

$F_2(y,t)$ follows from Eq. (5-50) to make Eq. (5-43)

$$\overset{1}{V}_x = -(x + a) \frac{\partial \overset{o}{V}_y}{\partial y} - \overset{o}{V}_y \frac{\partial a}{\partial y} \tag{5-52}$$

$F_3(y,t)$ is given by Eq. (5-49) so that Eq. (5-45) is

$$\overset{1}{E}_y = \frac{\partial \overset{o}{E}_x}{\partial y} [x - b] - \overset{o}{E}_x \frac{\partial b}{\partial y} \tag{5-53}$$

The equations of motion consistent with an expansion to first order in λ are (5-42), (5-47) and (5-48) with $\overset{o}{p}$, $\overset{1}{V}_x$ and $\overset{1}{E}_y$ eliminated by Eqs. (5-51) - (5-53)

$$\frac{\partial V_y}{\partial t} + V_y \frac{\partial V_y}{\partial y} + \frac{\partial \xi}{\partial y} + \frac{U_b^2 V_o^2}{(b-\xi)^3} \frac{\partial (b-\xi)}{\partial y} = 0 \tag{5-54}$$

$$\frac{\partial \xi}{\partial t} + \frac{\partial (V_y(\xi+a))}{\partial y} = 0 \tag{5-55}$$

where

$$E_x = \frac{V_o(t)}{(b-\xi)} \tag{5-56}$$

In retrospect, these equations could be simply derived after making three assumptions, corresponding to the statements of Eqs. (5-50) - (5-52) and (5-53).

C. Magnetohydrodynamic Anti-Dual

The magnetohydrodynamic problem of a flux of magnetic field trapped between a rigid perfectly conducting wall at

$x = b(y)$ and a perfectly conducting fluid interface at $x = \xi(y,t)$ may be shown to be very similar to the problem outlined in the previous section. This problem is summarized by Fig. 5-2.

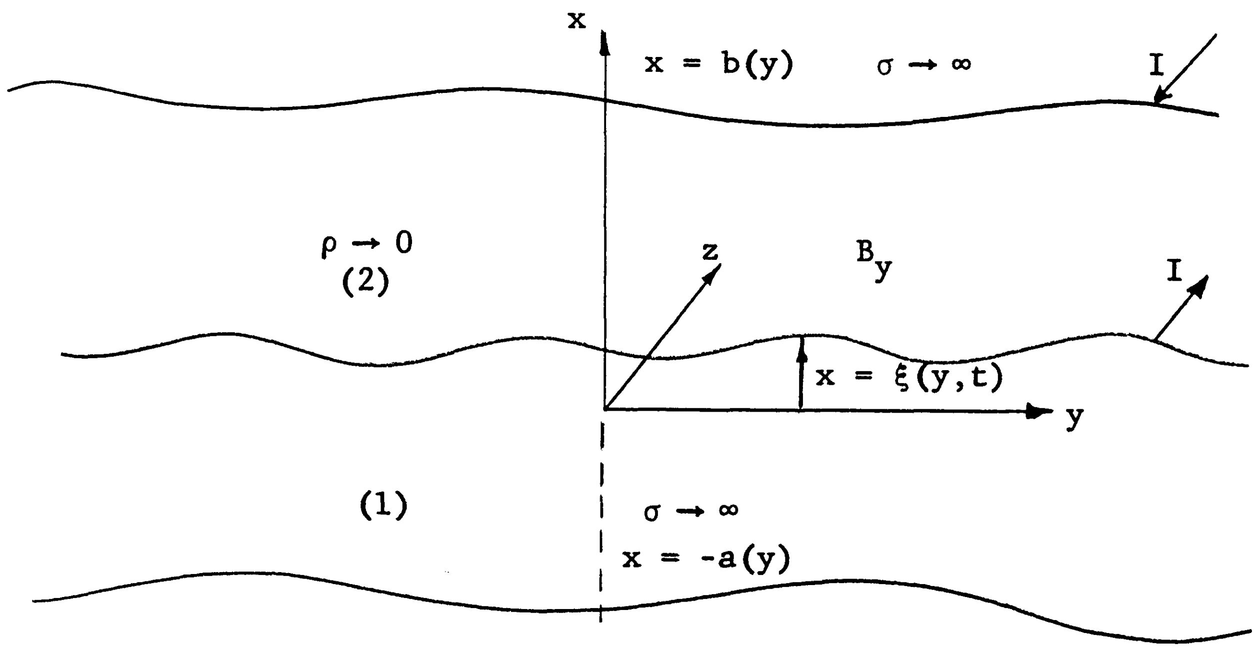

Figure 5-2
"Long-Wave" MH-IIf Configuration

Since the magnetic field is presumed confined to region (2), the fluid bulk equations are again given by Eqs. (5-1) and (5-2). However, induction is essential so that

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{5-57}$$

$$\nabla \cdot \bar{B} = 0 \tag{5-58}$$

$$\nabla \times \bar{B} = 0 \tag{5-59}$$

where these equations apply in region (2).

The boundary conditions of Equations (5-10) and (5-12) are applicable, while the conditions that replace Equations (5-6) and (5-7) are:

$$\int_{\xi(y,t)}^{b(y)} B_y dx = \Lambda_o \tag{5-60}$$

$$[\bar{n} \cdot \bar{B}]\Big/_{x=\xi(y,t)} = 0 \tag{5-61}$$

In addition Eq. (5-57) requires that

$$n \times [\bar{E}^{(2)} - \bar{E}^{(1)}]\Big/_{x=\xi(y,t)} = \bar{n}\cdot\bar{V}[\bar{B}^{(2)} - \bar{B}^{(1)}]_{x=\xi(y,t)} \tag{5-63}$$

The conditions of Eq. (5-12) and (5-61) show that for problems where the induced electric field E_z exists, the surface charge must vanish. Hence, the surface traction is accounted for by the magnetic counterpart of the stress tensor in Eq. (5-12).

Equation (5-59) shows that B_z may be taken as zero. Hence, Eq. (5-4) gives

$$[p - \frac{1}{2\mu} B_y^2[1 + (\frac{\partial\xi}{\partial y})^2]]\Big/_{x=\xi(y,t)} = 0 \tag{5-64}$$

The problem is now defined by the volume equations, (5-1), (5-2), (5-58), (5-59) together with the surface conditions of (5-8), (5-10), (5-60), (5-61) and (5-64).

In the same way as in section B, a solution in the form of a power series in λ is assumed and equations are found to first order in λ. However, here the* magnetic field is scaled so that $B_x = B_o\bar{B}_x(\frac{s}{w})$, $B_y = B_o\bar{B}_y$. Following the line of reasoning used previously the long wave normalized equations are:

---

*$wB_o\bar{\Lambda}_o = \Lambda_o$

$$\frac{\partial V_y}{\partial t} + V_y \frac{\partial V_y}{\partial y} + \frac{\partial \xi}{\partial y} - \frac{U_a^2 \Lambda_o^2}{(b-\xi)^3} \frac{\partial (b-\xi)}{\partial y} = 0 \tag{5-65}$$

$$\frac{\partial [V_y (a + \xi]}{\partial y} + \frac{\partial \xi}{\partial t} = 0 \tag{5-66}$$

$$B_y = \frac{\Lambda_o}{(b-\xi)} ; \quad U_a^2 = \frac{B_o^2}{\mu \rho g w} \tag{5-67}$$

The transformation of this set of equations to those defining the electrohydrodynamic system is completed by substituting $\Lambda_o \rightarrow V_o$ $U_a^2 \Lambda_o^2 \rightarrow -U_b^2 V_o^2$ and $B_y \rightarrow E_x$. The change of sign in going from U_a^2 to U_b^2 is the basis for terming the problems of sections B and C anti-duals. The effect of this sign change on physical problems will be apparent in the sections which follow.

D. Physical Meaning of the "Long-Wave" Approximation

If Eqs. (5-54) or (5-65) and (5-66) are linearized and solutions are taken of the form $V_y = V_y e^{j(\omega t + ky)}$, the dispersion relation

$$\omega^2 = k^2 \left[a + \frac{a}{b} \begin{bmatrix} U_a^2 \bar{\Lambda}_o^2 / b^2 \\ -U_b^2 \bar{V}_o^2 / b^2 \end{bmatrix} \right] \tag{5-68}$$

is obtained. This is identical with Eq. (3-52) when $T \rightarrow 0$.

A similar limitation exists on the non-linear equations resulting from λ expansion. The general nature of this limitation is demonstrated by considering the dependent variable $\bar{E}_y$. Since

$$E_y = \overset{o}{E}_y + \lambda \overset{1}{E}_y + \lambda^2 \overset{2}{E}_y + \ldots \tag{5-69}$$

a meaningful approximation taking terms to first order in λ requires that

$$\left|\overset{1}{E}_y\right| \gg \left|\lambda \overset{2}{E}_y\right| \tag{5-70}$$

Consider now, as an example, the problem where $\frac{\partial b}{\partial y} = 0$. Then Eqs. (5-53) and (5-46) generate $\overset{1}{E}_x$;

$$\overset{1}{E}_x = - \frac{\partial^2 \overset{o}{E}_x}{\partial y^2} \frac{(x-b)^2}{2} + \overset{1}{F}_4(y,t) \tag{5-71}$$

The second order form of Eq. (5-22) in turn determines $\overset{2}{E}_y$

$$\frac{\partial \overset{2}{E}_y}{\partial x} = \frac{\partial \overset{1}{E}_x}{\partial y} = - \frac{\partial^3 \overset{o}{E}_x}{\partial y^3} \frac{(x-b)^2}{2} + \frac{\partial \overset{1}{F}_4}{\partial y} (y,t) \tag{5-72}$$

or

$$\overset{2}{E}_y = - \frac{\partial^3 \overset{o}{E}_x}{\partial y^3} \frac{(x-b)^3}{6} + \frac{\partial \overset{1}{F}_4}{\partial y} (x-b)$$

where use has been made of the second order form of Eq. (5-27). The solution of the problem, to second order would require $\overset{1}{F}_4(y,t)$ to be evaluated using Eq. (5-26). This would introduce ξ and eventually the remaining dependent variables. However, interest is confined here only to determining the approximations involved in cutting the series off at first order terms, and not in solving the second order problem. Hence, consider $\overset{2}{E}_y$ to be proportional to terms like the first one of (5-73) and write Eq. (5-70) as, at worst

$$\left|\frac{\partial \overset{o}{E}_x}{\partial y}\right| \gg \left|\frac{\partial^3 \overset{o}{E}_x}{\partial y^3}\,\frac{(a+b)^2}{6}\right| \qquad (5\text{-}74)$$

If linear solutions of the form e^{jky} are considered a limitation similar to that arising in linear theory is obtained. That is

$$1 \gg k(a+b)/\sqrt{6}$$

However, Eq. (5-74) gives an indication of a more general limitation which may be used to test a solution to Eqs. (5-54) and (5-55) for validity. Since, at a given x, $\overset{1}{E}_y$ goes like $\frac{\partial \overset{o}{E}_x}{\partial y}$, Eq. (5-74) says that the first order approximation breaks down when $\overset{1}{E}_y$ is not much larger than the product of its second derivative (say the curvature) and the square of a characteristic dimension in the transverse direction. Hence, solutions become suspect at points where the interface is sharply distorted such as near the face of a shock. It will be seen later that this is not the only reason why the first order equations, or any order of equations for that matter, are not valid at a shock front. Nevertheless, they will predict the properties of a flow as it leads to a shock.

E. Characteristic Equations

The characteristic equations equivalent to Eqs. (5-65) and (5-66) are found by standard techniques.* (It is assumed

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*See Courant and Friedrichs,[18] p. 40.

now that a(y) = a, b(y) = b or that the boundaries are parallel.) Designating the families of characteristics in the y-t plane by C^+ and C^- these are:

$$C^+ \quad V' + R(\phi,K) = c_1$$
$$C^- \quad V' - R(\phi,K) = c_2 \qquad (5\text{-}75)$$

c_1 and c_2 constants

$$C^+ \quad \frac{dy'}{dt} = V' + \sqrt{(1-\phi)(1-K/\phi^3)}$$
$$C^- \quad \frac{dy'}{dt} = V' - \sqrt{(1-\phi)(1-K/\phi^3)} \qquad (5\text{-}76)$$

where

$$R(\phi,K) = -\int_{\text{constant}}^{\phi} \left[\frac{\phi^3-K}{\phi^3(1-\phi)}\right]^{1/2} d\phi \qquad (5\text{-}77)$$

$$V' = \bar{V}_y / \sqrt{\bar{a} + \bar{b}}$$

$$y' = \bar{y} / \sqrt{\bar{a} + \bar{b}}$$

$$\phi = \left[\frac{\bar{b}-\bar{\xi}}{\bar{b}+\bar{a}}\right]$$

$$K = U_b^2 \bar{V}_o^2 \,/\, (\bar{b} + \bar{a})^3 \quad \text{or} \quad -U_a^2 \Lambda_o^2 \,/\, (\bar{b} + \bar{a})^3$$

$$= \varepsilon_o V_o^2 \,/\, (b+a)^3 \rho g \quad \text{or} \quad -\Lambda_o^2/\mu_o \rho g (b+a)^3$$

Since Eq. (5-76) is complex if $k/\phi^3 > 1$ ($\phi > 1$ is not physically meaningful), this is the condition that real characteristics exist, and hence that the equations be hyperbolic. In the linear theory, this is also the condition that the interface be stable. The two theories agree on the definition of stability if, as will be done here, the non-linear unstable interface is taken as one which is no longer defined by hyperbolic differential equations.

Solutions starting from $\phi^3 > K$ will be shown to tend to remain in this condition. Hence, this is a plausible and consistent definition if used in connection with this first order theory.

F. The Growth of Shocks and Anti-Shocks

i. Simple Waves

The problem of a wave propagating into a region of constant state ($V' = 0$, $\phi = \phi_o$), defined by its height for time $t > 0$ at a fixed position, is easily investigated, since all of the characteristics of one family (say the C^+) intersect at some point characteristics which originate in a region of constant state. Equations (5-75) and (5-76) imply that ϕ and V' are constant along C^+ characteristics and that these are straight lines. Only $\phi/_{y=0}$ as a function of time need be given since from Eq. (5-75),

$$V/_{y=0} = R(\phi,K)/_{y=0} - R(\phi_o,K)$$

Hence the characteristics are easily drawn, using Eq. (5-76) to determine the slopes. That is, the C^+ characteristics intersect the time axis with a slope given by,

$$M = \frac{dy'}{dt} = \left[R(\phi,K) - R(\phi_o,K) + \sqrt{(1-\phi)(1-K/\phi^3)}\ \right]_{y=0} \qquad (5\text{-}78)$$

Clearly, if this slope increases with increasing time, the C^+ characteristics must cross and the wave must steepen into a shock. Hence, the condition

$$\left.\frac{dM}{d\phi}\frac{d\phi}{dt}\right/_{y=0} > 0 \qquad (5\text{-}79)$$

implies that a shock will form. Equations (5-77) and (5-78) show that $K \gtrless \phi^4$ implies $\frac{dM}{d\phi} \gtrless 0$

Four possible dynamical situations are of interest.

1. Magnetohydrodynamic compression waves; $\frac{d\phi}{dt} < 0$, $K < 0$, $\frac{dM}{d\phi} < 0$; a shock will always form.
2. Magnetohydrodynamic depression wave; $\frac{d\phi}{dt} > 0$, $K < 0$, $\frac{dM}{d\phi} < 0$; a shock will never form.
3. Electrohydrodynamic compression wave; $\frac{d\phi}{dt} < 0$, $K > 0$, $\phi^3 > K$ for stability. A shock may or may not form according to whether:
   a. $\phi^4 < K < \phi^3$ or $\frac{dM}{d\phi} > 0$ and a shock does not form;
   b. $\phi^4 > K$ or $\frac{dM}{d\phi} < 0$ in which case a shock forms.
4. Electrohydrodynamic depression wave; $\frac{d\phi}{dt} > 0$, $K > 0$; A shock may or may not form according to whether:
   a. $\phi^4 < K < \phi^3$ or $\frac{dM}{d\phi} > 0$, in which case a shock forms;
   b. $\phi^4 > K$ or $\frac{dM}{d\phi} < 0$, in which case a shock does not form.

These statements strictly apply only to waves with corresponding values of ϕ confined to a single regime. For waves that make a transition from one regime to another the negative statements should say that a shock does not tend to form since, while the C^+ characteristics may be spreading with increasing time, they may still cross characteristics in another regime, although at a later time than if the spreading had not occurred.

A transition compression wave would tend to form into a shock at its base while smoothing out at its top. Similarly, a transition depression wave would form a discontinuity near the initial wave front and smooth out at the wave back. These are reasonable consequences of the tendency of the wave velocity to increase with increasing depth due to the gravitational field and to decrease with increasing depth due to the electric field. The dynamics are roughly the same as for ordinary gravity waves so long as $\phi^4 > K$. Hence, this regime will be called gravity controlled, while $\phi^4 < K < \phi^3$ will be called EH controlled.

As examples of EH transition waves, Figs. 5-6 and 5-7 show the compression and depression waves of Fig. 5-3 propagating into regions of constant state. The associated characteristics are given in Figs. 5-4 and 5-5.

Since the dynamics of the Magnetohydrodynamic simple waves are not grossly different from those of ordinary gravity waves, it will simply be pointed out that the effect of a magnetic field is always to make a compression wave shock earlier in time, since increasing the magnetic field corresponds to making $\frac{dM}{d\phi}$ more negative.

Although the discontinuities formed by EH controlled waves may be called shocks, it would seem worthwhile to draw attention to the fact that their behavior is just the opposite of shocks normally formed in nature by referring

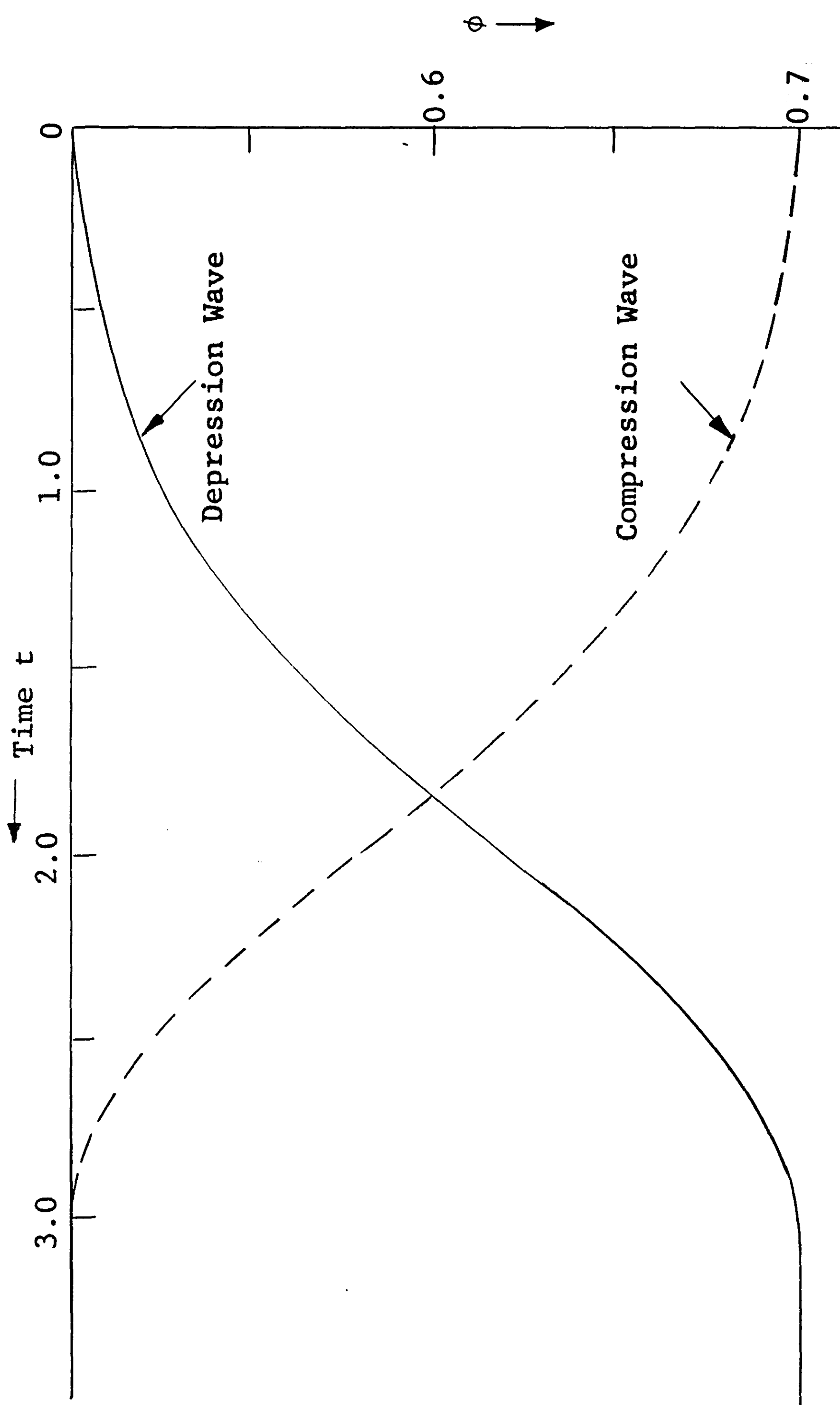

Fig. 5-3 Time-like Wave

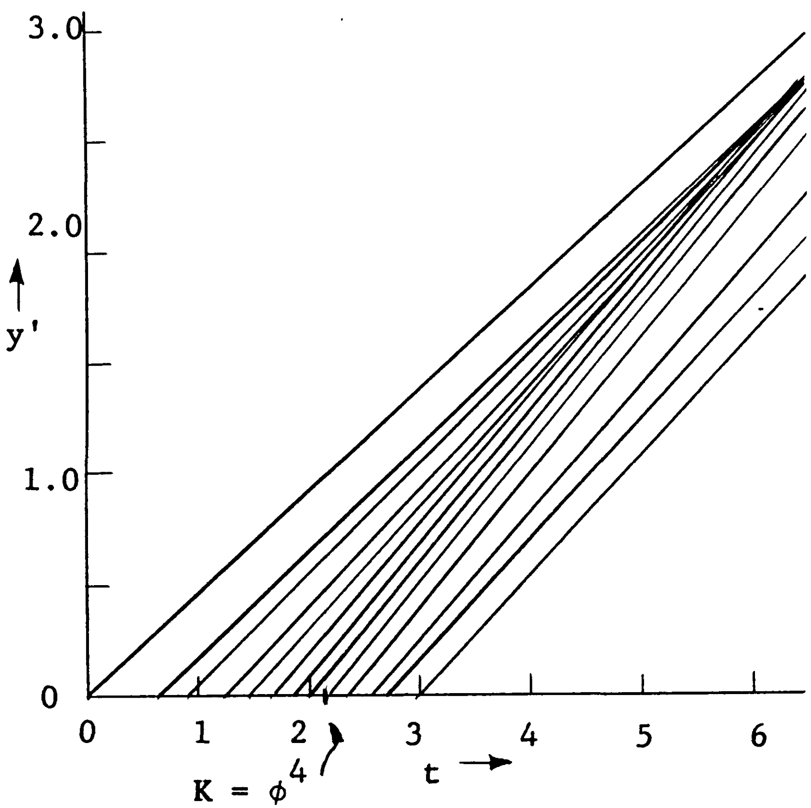

Fig. 5-4 Characteristics for Compression E-H Wave

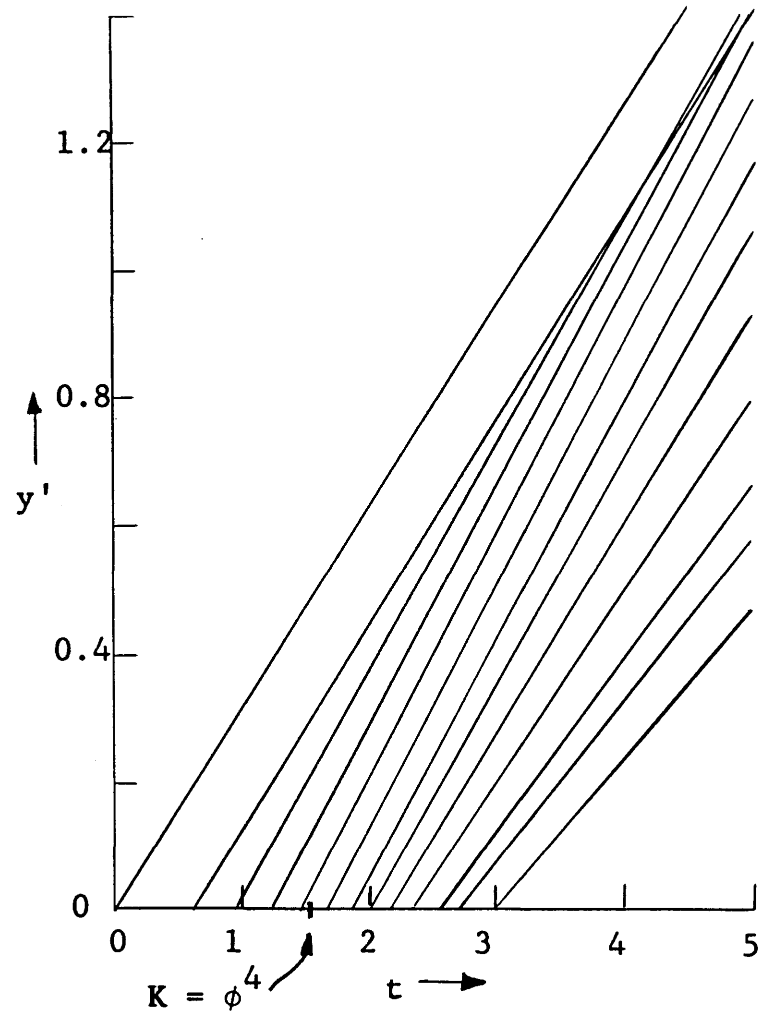

Fig. 5-5 Characteristics for Depression E-H Wave

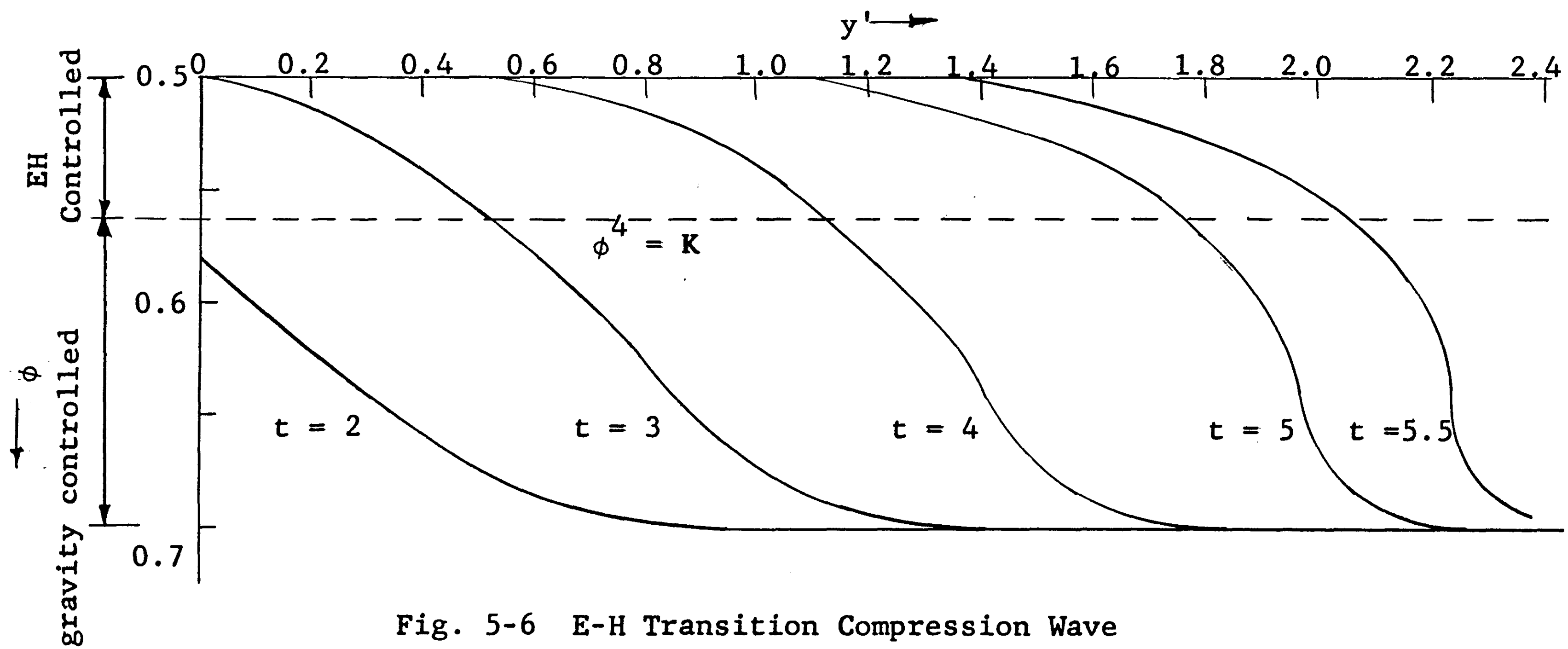

Fig. 5-6 E-H Transition Compression Wave

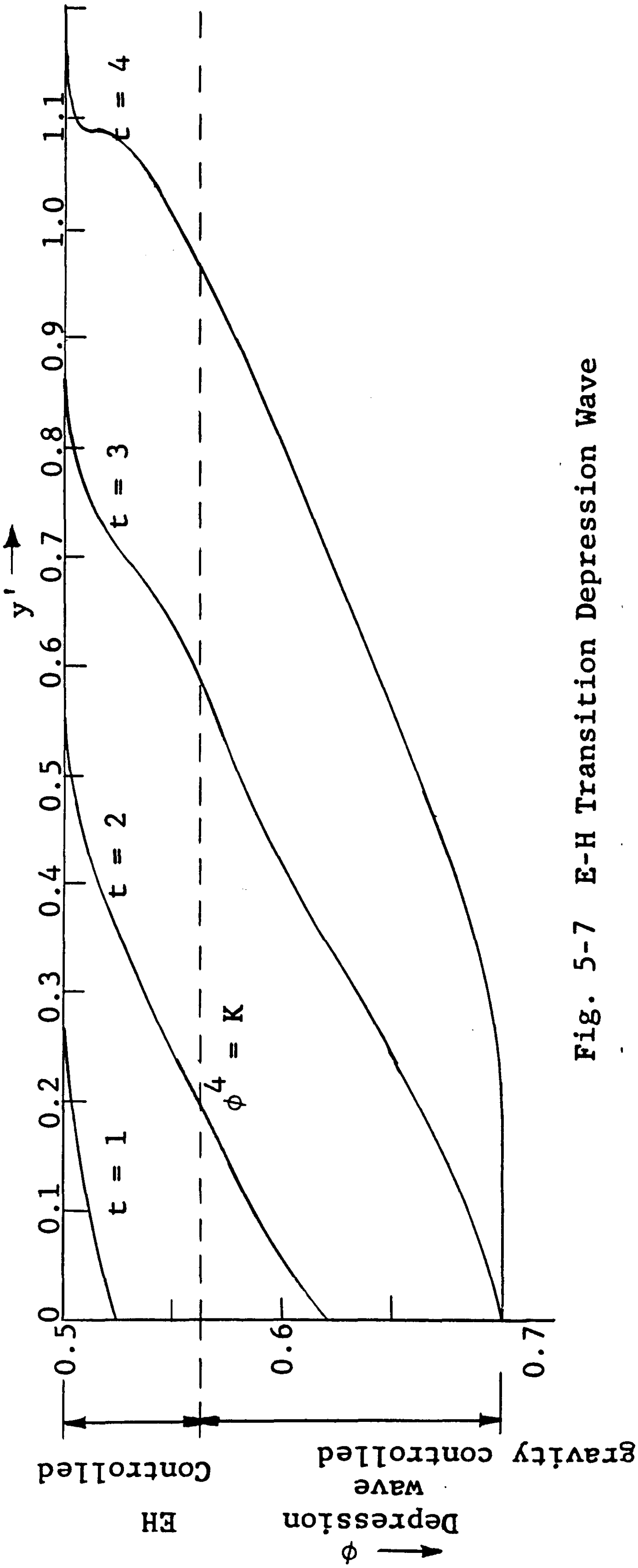

Fig. 5-7 E-H Transition Depression Wave

to them as anti-shocks. Here the shock is associated with the wave-like property of a system, while the anti-shock is a consequence of a tendency toward instability. The anti-shock should not, however, be confused with the manifestation of an instability.

ii. Waves from Space-Like Data

It is interesting to consider briefly certain disturbances which originate on initial static displacements of the interface. That is, given that $V = 0$ and $\xi = \xi(y)$ at $t = 0$, what is the ensuing dynamical motion.

Equations (5-75), (5-76) and (5-77) make possible in the usual way, a step-wise construction of the characteristics and hence of the solution. Interest is confined here to pulses that are symmetrical about the origin and bounded by regions of constant state. Hence, the problem is completely solved for $t \geq 0$ by:

1. Performing the integral of Eq. (5-77) to determine the values of c_1 and c_2 from the initial data for characteristics originating at intervals along the y axis in the region of the pulse.

2. Simultaneously solving Eqs. (5-75) and inverting $R(\phi,K)$ to determine ϕ and V at all characteristic intersections in a region bounded by the y axis and the C^+ and C^- characteristics originating from the negative and positive y extremities of the initial pulse.

3. Iteratively using the characteristic directions defined by Eq. (5-76) to determine the (y,t) position corresponding to each of the characteristic intersections of 2.

The problem is solved for all y and t once the characteristics have been determined in the "cone" of part 2, since all characteristics leaving the region are straight lines, (they intersect characteristics that originate in a region of constant state).

A problem, pertinent to the effect of finite surface displacements on impending instability, can be resolved from the simple fact that $R(\phi,K)$ is a negative monotonically decreasing function of ϕ. The fact that any initial static displacement of the interface that is stable will remain stable follows from Eq. (5-75). Let ϕ_o correspond to the minimum value of ϕ at $t = 0$ and corresponding to $y = y_o$. Then originating at ϕ there are C^+ and C^- characteristics along which

$$V' \pm R(\phi,K) = \pm R(\phi_o,K) \tag{5-80}$$

Similarly along characteristics originating at another arbitrary point y_1,

$$V' \pm R(\phi,K) = \pm R(\phi_1,K) \tag{5-81}$$

where $\phi_1 > \phi_o$. Therefore at any given point on the characteristics coming from y_o there is a ϕ_1, such that

$$R(\phi,K) = [R(\phi_o,K) + R(\phi_1,K)]/2 \tag{5-82}$$

or since $R(\phi_o,K) > R(\phi_1,K)$, $R(\phi,K) < R(\phi_o,K)$ and finally $\phi > \phi_o$ along the y_o characteristics originating at y_o. The argument is complete if it is observed from Eq. (5-82) that ϕ takes on the smallest possible values on the characteristics that cross at y_o. Hence, ϕ is greater than ϕ_o everywhere in the $(y,t > 0)$ plane and the interface remains stable.

As examples of symmetric waves originating on initially static pulses, Fig. 5-8 shows magnetohydrodynamic and electrohydrodynamic hump and depression pulses propagating into regions of constant state. Since the resulting waves are symmetric about the origin, only the positive y axis is shown. In these examples the electrohydrodynamic cases do not make a transition to the gravity controlled regime.

The waves behave as would be expected. The velocity of small disturbances increases with the depth of the fluid in the magnetohydrodynamic problem. The particles at the top of the hump catch up with those at the front while the particles at the back of a depression catch up with those at the bottom. Hence in a way similar to the ordinary gravity wave, a hump steepens into a shock at the front, a depression forms a shock at its back edge.

Conversely, the velocity of small disturbances can decrease with the fluid depth for an electrohydrodynamic problem. Hence, in the examples shown, anti-shocks form at the back and front edges of the hump and depression waves respectively.

G. Integral Conditions

The conditions which correspond to fully developed shocks may be written by integrating the first order equations over a volume which encloses the shock. The appropriate volume element is shown in Fig. 5-9 for the E-H and M-H cases.

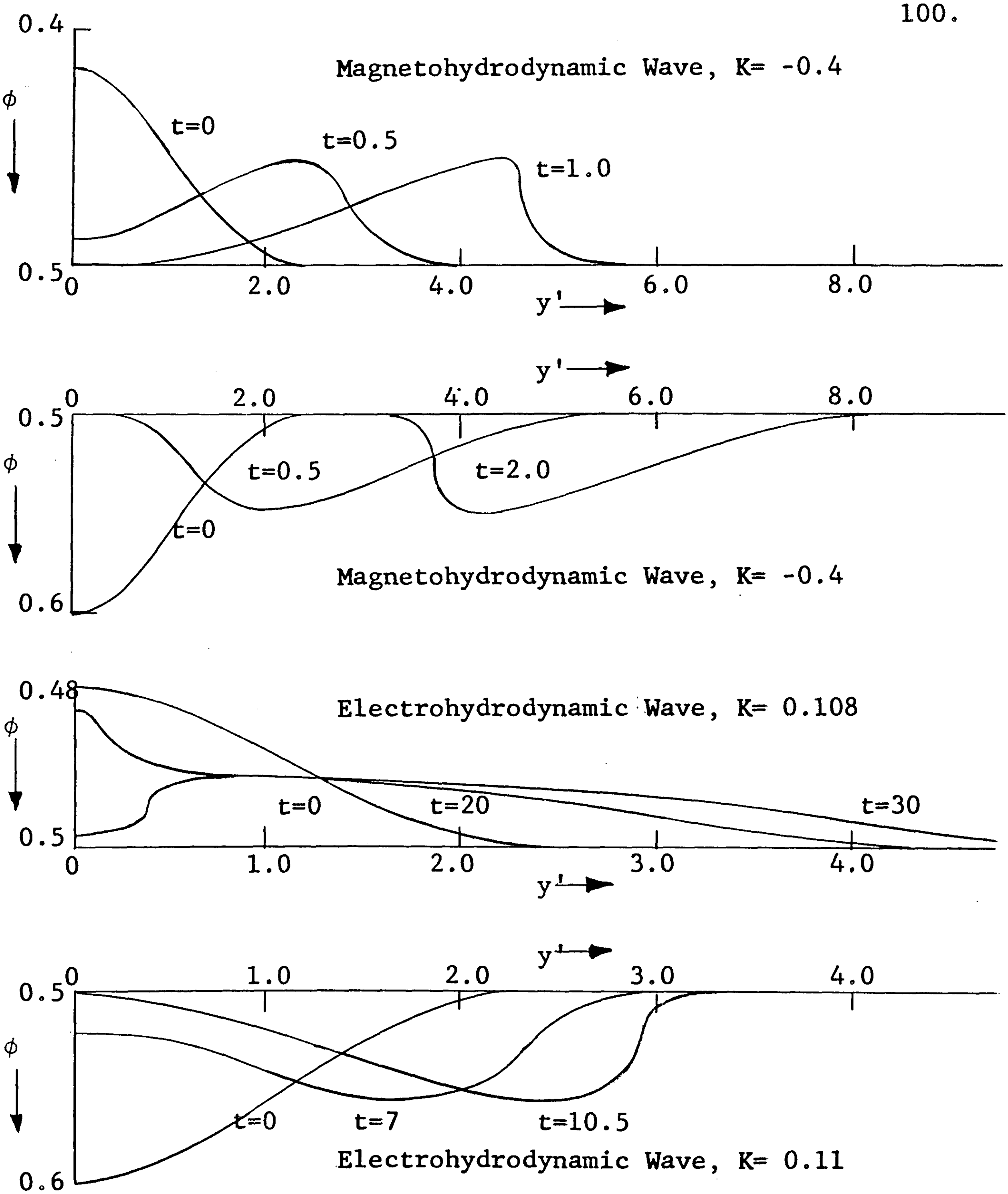

Fig. 5-8a Waves From Space-like Data

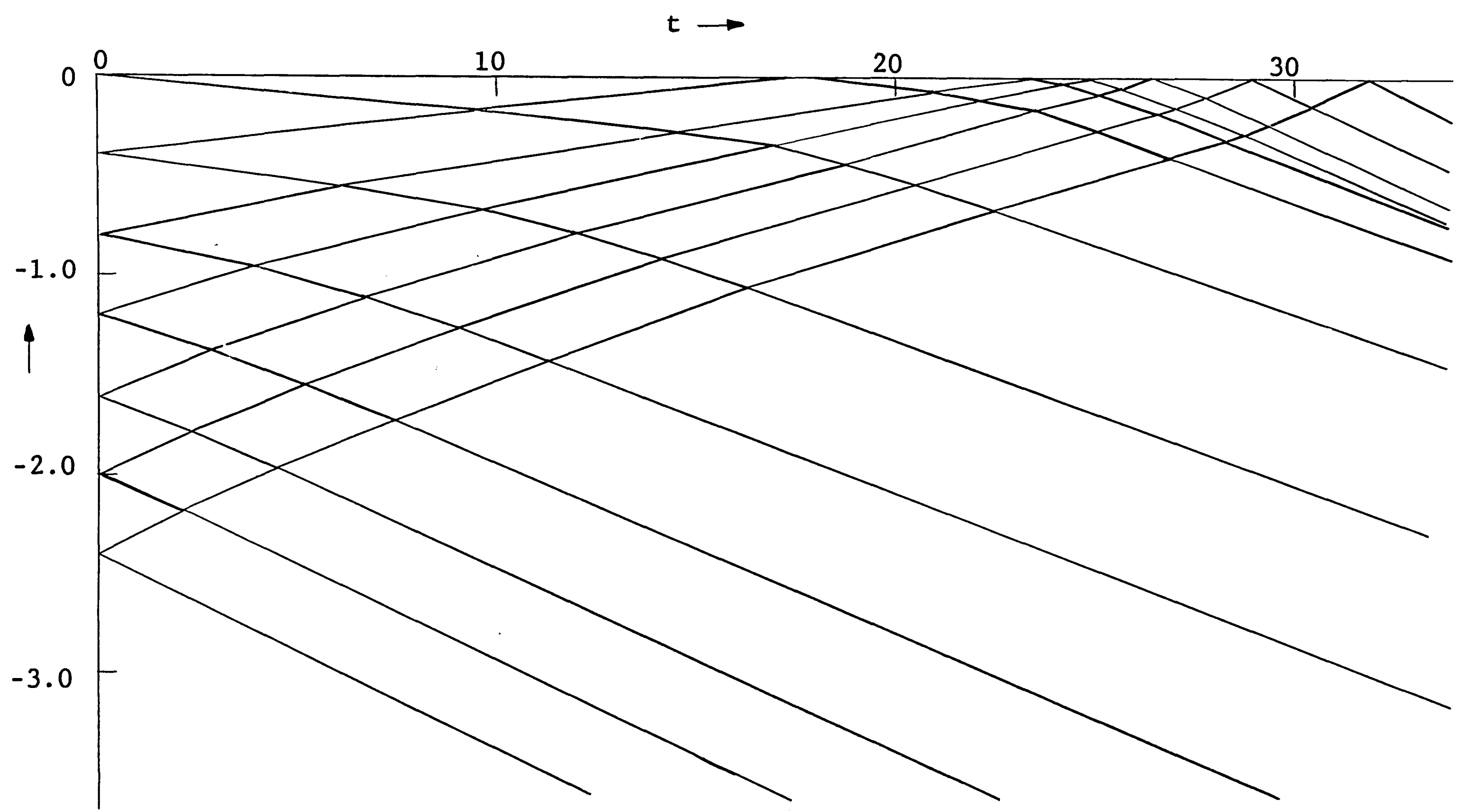

Fig. 5-8b E-H Characteristics; Compression Wave

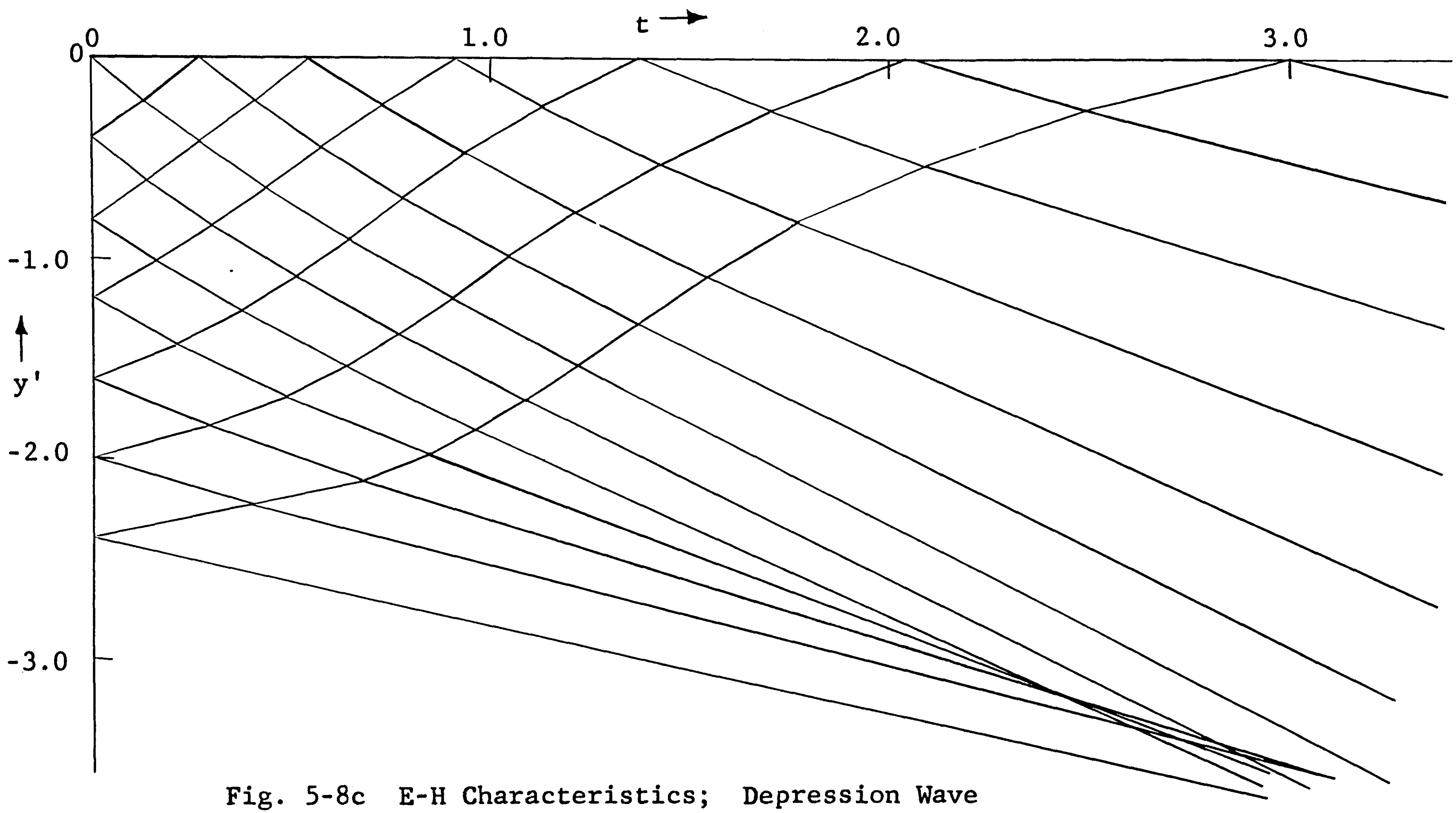

Fig. 5-8c E-H Characteristics; Depression Wave

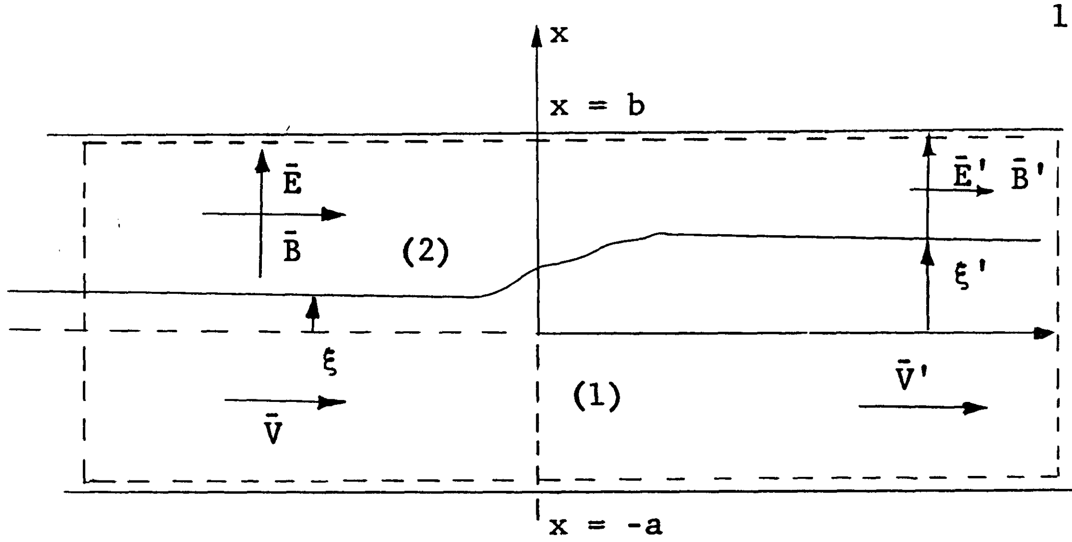

Figure 5-9

Shock Configuration

Note that the upper and lower boundaries of the volume are just inside the rigid parallel plates. The fluid is assumed to enter and leave the volume with velocities V and V' as shown. The problem is assumed to be steady state, i.e. the discontinuity is fixed in position. The equations of motion are invarient to a constant translation. Hence, the problem is easily generalized to give a moving discontinuity. Since $\overset{o}{V}_y = \overset{o}{\bar{V}}_y(y,t)$, the continuity condition requires that

$$V'(\xi' + a) = V(\xi + a) \qquad (5\text{-}81)$$

From Eq. (5-56), the electric fields are related by

$$E(b-\xi) = E'(b-\xi') = V_o \qquad (5\text{-}82)$$

The momentum equation, (5-1), may be generalized to apply throughout the volume, so long as it is agreed that when it is taken in region (2), the density vanishes. That is

$$\rho\left[\frac{\partial V_\alpha}{\partial t} + V_\beta \frac{\partial V_\alpha}{\partial x_\beta}\right] + \frac{\partial p}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}(-\rho g x\ ) + \frac{\partial T_{\alpha\beta}}{\partial x_\beta} \tag{5-83}$$

A detailed knowledge of the field in the vicinity of the discontinuity is not known. However, far from the discontinuity it would be expected that the first order field approximations would apply. Use may be made of these solutions to integrate Eq. (5-83) over the volume indicated in Fig. 5-9. First, the volume integral of Eq. (5-83) is converted to a surface integral

$$\frac{\partial}{\partial t}\int_v \rho V_\alpha dv + \int_s \rho V_\alpha (V_\beta n_\beta) ds + \int_s p n_\alpha ds = \int_s T_{\alpha\beta} ds_\beta \tag{5-84}$$

Since the volume was chosen so that there is no contribution to the integration of the y component of Eq. (5-84) on the upper and lower surfaces, the first order fields serve to evaluate Eq. (5-84), and give

$$\rho[(\xi' + a)V'^2 - (\xi+a)V^2] + \frac{\rho g}{2}[(\xi' + a)^2 - (\xi+a)^2] \tag{5-85}$$

$$- \frac{\varepsilon}{2}[E'^2[(\xi' + a)-(b-\xi')] - E^2[(\xi+a)-(b-\xi)]] = 0$$

Equations (5-81), (5-82) and (5-85) relate the velocity of the fluid into the volume to the discontinuity in the surface height.

$$V^2 = \frac{(\xi'+a)}{(\xi+a)}\left[\frac{g}{2}((\xi'+a)+(\xi+a)) - \frac{\varepsilon_o V_o^2}{2\rho}\left(\frac{(\xi+a)(b-\xi')+(\xi'+a)(b-\xi)}{(b-\xi')^2(b-\xi)^2}\right)\right] \tag{5-86}$$

The velocity of a wave-front moving into a region of constant state is numerically equal to V. Hence, Eq. (5-86) reduces, for small ξ' and ξ, to the phase velocity resulting from linear theory, i.e. Eq. (5-68).

The shock conditions are fully formulated only after account is taken of the energy balance associated with a given discontinuity in the interface position. Since $\nabla\cdot\bar{V} = 0$, the energy equation results from dotting $\bar{V}$ into Eq. (5-1) to obtain the condition:

$$\frac{dw}{dt} = -\int_v \nabla\cdot(\bar{V}w)\,dv = -\int_s \bar{V}w \cdot ds \qquad (5\text{-}87)$$

where

$$w = \rho\left(\frac{V\cdot V}{2}\right) + p + \rho g(x + a)$$

For a conservative system $\frac{dw}{dt} = 0$. At the shock front kinetic energy may not be conserved. However, it is certainly true that in the physical system energy must flow into the volume and not out of it. Hence the condition

$$\frac{dw}{dt} > 0$$

The integration of Eq. (5-87), carried out using the value of V^2 given by Eq. (5-86) results, after some algebraic manipulations, in Eq. (5-88).

$$\frac{dw}{dt} = \frac{V}{4(\xi' + a)}\left[(\xi'-\xi)^3\rho g - \varepsilon_o V_o^2\left[\frac{[(b-\xi')(\xi+a)+(\xi'+a)(b-\xi)][\xi'-\xi][(\xi'+a)+(\xi+a)]}{(b-\xi')^2(b-\xi)^2}\right]\right] \qquad (5\text{-}88)$$

If $V_o = 0$, this equation gives the well known result that $V > 0$ implies that $\xi' > \xi$ or that the step in fluid height is as shown in Fig. 5-9.

Equations (5-82), (5-86), and (5-88) hold for the magnetohydrodynamic problem of a trapped-magnetic field directed far from the discontinuity, along the y axis, if the substitution is made:

$$E \rightarrow B$$

$$\varepsilon_o V_o^2 \rightarrow -\Lambda_o^2 / \mu$$

From these equations it is now clear that:

1. A magnetohydrodynamic surface shock, like the gravitational shock, must propagate into a region of higher velocity and lower depth. That is, for $V > 0$ the energy balance requires $\xi' > \xi$.

2. An electrohydrodynamic surface shock, unlike the gravitational shock, may propagate into a region of lower velocity and greater depth. That is, for $V > 0$ the energy balance does not require $\xi' > \xi$. In fact, it follows from Eq. (5-88) that the existence of a discontinuity does not necessarily imply that energy is dissipated at the interface.

These results are a consequence of the fact that the magnetic field always increases the equilibrium velocity of the discontinuity while the electric field always decreases this velocity. It appears that the electric field may interact as an "electric weir", in that it may produce an abrupt decrease in the fluid depth in the stationary configuration of Fig. 5-9.

CHAPTER 6
CONCLUSION

The concern of this thesis has been with purely superficial magneto- and electrohydrodynamic interactions with liquid-liquid or liquid-gas interfaces. The problems considered have been so arranged as to exclude any coupling between fields and fluid in the volume. Those effects resulting from the gravitational field and the action of cohesion have been included and also will be recognized as characterized by a surface traction. Hence, the unifying theme of this work has been the delineation of the dynamics of interfaces in plane geometry, stressed by uniform magnetic or electric fields.

A. Summary

Surface problems in six configurations have been analyzed and compared. Use has been made of a perturbation analysis and of a space rate expansion of the dependent variables. The first of these techniques is common in field-coupled surface wave problems that appear in the literature of magnetohydrodynamic stability theory and leads to a linear approximation. The second, to the author's knowledge, is a new approach to field coupled surface wave problems, and leads to a non-linear approximation.

The waves studied have been classified according to whether the steady electric or magnetic fields are perpendicular (type I) or tangential (type II) to the interface. The relationship between the wave types has been clarified by viewing their similarities (duality) and by taking advantage of their antithetic behavior (anti-duality).

The problems considered in the linear analysis are summarized by Table 6-1. The first three columns give the shorthand designation used here for each of the waves, while the fourth column indicates the zero order surface charge or current. The role of each wave type may be clarified by considering the circumstance under which it might be observed. For example, if the EH-I problems were established by applying a step of voltage to parallel plates, the EH-Ip wave would define dynamics on an initially uncharged surface at times $t \ll T_r$ while EH-If waves would occur in the limit of $t \gg T_r$, where T_r is the time constant for relaxation of free charge to the interface;

$$T_r = \frac{\varepsilon_2 a + \varepsilon_1 b}{\sigma_2 a + \sigma_1 b} \qquad (6\text{-}1)$$

In an inverse way, if uniformly conducting fluids were used to establish the MH-II waves, the MH-IIf wave would be observed during the initial stages of a pulsed experiment, while an MH-IIa wave would result after a long time. This points to an inherent characteristic of the EH problems. The interactions in plane geometry are always at the surface. The EH-II wave is observable at all times, as is also the MH-I wave. This is a direct consequence of the homogeneous nature of two of Maxwell's equations.

The type I waves are characterized by a phase velocity that decreases with increasing field strength, the possibility of a negative group velocity simultaneously with a positive phase velocity and by a discrete field strength and wave-length for impending instability. The conditions corresponding to these characteristics have been given ranging from strong to weak interactions with the external boundaries.

Table 6-1

Field	Orientation	Designation	Surface Sources or Vortices	Validity in Time	External Boundaries	Comparison: Dual ——— / Anti-Dual - - - -
EH	I	f	free charge	$t \gg T_r$	$\sigma \rightarrow \infty$ or $\varepsilon \rightarrow \infty$	if $\sigma^{(1)}$ or $\sigma^{(2)} \rightarrow \infty$; if $\frac{\partial}{\partial y}=0$
	I	p	polarization charge	$t \ll T_r$	$\sigma \rightarrow \infty$ or $\varepsilon \rightarrow \infty$	if $\varepsilon^{(1)}$ or $\varepsilon^{(2)} \rightarrow \infty$
	II	-	Korteweg current	no limit because $\nabla x E = 0$	ε same as adjacent fluid	
MH	I	-	magnetic charge	no limit because $\nabla \cdot B = 0$	$\mu \rightarrow \infty$	if $\mu^{(1)}$ or $\mu^{(2)} \rightarrow \infty$
	II	f	free current	$t \ll T_d$	$\sigma \rightarrow \infty$	
	II	a	amperian current	$t \gg T_d$	μ same as adjacent fluid	

Type II waves theoretically may propagate more, or less rapidly along field lines, depending on the strictions at the interface. Although the Clausius-Mossotti equation is commonly used to illustrate the effect of strictions at dielectric interfaces, its use here indicates that a type EH-II wave would be slowed by an electric field, whereas it was clearly indicated experimentally that the wave is in fact speeded up.

The nature of both type I and II waves has been illustrated by a discussion of several experimental observations on the EH-If and EH-II waves.

The discrete nature of the field strength and wave-length corresponding to impending instability in an EH-If configuration made it possible to correlate theory with experiment in an unambiguous way. This experiment showed the expected behavior of voltage for instability as a function of plate spacing, and clearly indicated that the effect of electrostriction was considerably smaller than predicted by the Clausius-Mossotti equation; a result that correlated with the EH-II experiments. The results of this experiment were shown by mathematical similarity to be illustrative of the type of behavior to be expected from certain EH-Ip and MH-I experiments.

The conditions for surface resonance in rectangular geometry have been given for each of the wave types. Errors inherent to the instability experiment were avoided by using one of these resonators involving an interface where strictions could not play an important role (water-air). The resonant frequency shift as a function of field strength was correlated with theory to within errors expected from the dimensional uncertainty of the equipment.

One of the most intriguing aspects of free surface waves results from the fact that they can be seen with the aid of little or no optical aids. The anisotropic nature of the Type II waves makes it seem worthwhile to demonstrate these waves by simply taking a picture of radiating waves, as created by a disturbance. The dependence of the EH-II wave velocity along E lines on E^2 was demonstrated by shadow projections, two of which were given in chapter 4. However, the dispersive nature of surface wave propagation and the experimental difficulties involved (it is certainly not as easy as it looks) make this a poor experiment in terms of information for effort. The elliptical nature of the radiating disturbance made possible a rough correlation with experiment, but the accuracy was limited by ambiguities arising from dispersion.

A considerably more satisfactory correlation of theory and experiment was made using an EH-II resonator. Here correlation was within the dimensional accuracy of the equipment, if the electrostriction constant was ignored. The fact that this experiment, conducted using xylene, acetaphenone and nitrobenzene (materials of very different conductivities and polar characteristics) always indicated an electrostriction effect much smaller than would be expected from the Clausius-Mossotti equation, indicates not only that this equation is not valid at an interface, but that the theory predicts, at least at low field intensities, the correct surface dynamics. The effect of electro-convection is a limit on the electric field intensity that can be used in this experiment.

The concept of anti-duality may seem a needless definition of terminology if viewed in terms of the linear analysis given in chapter 3. However, the convenience that can result from observing this relationship between problems was made evident in chapter 5, where a discussion of non-linear surface dynamics of the type EH-If and MH-IIf waves was given. The "space rate" or "long wave" approximation used makes the results physically meaningful when the boundaries interact strongly with the interface. This work, at the same time, indicated a tractable method of getting at non-linear field-surface interactions and showed the non-linear role played by unstabilizing influences in strong contrast to stabilizing influences. The theoretical nature of MH-IIf and EH-If shocks and anti-shocks was investigated, showing the growth of MH shocks from compression waves and EH anti-shocks from depression waves. Transition EH-If waves originating from time-like data were given to illustrate the dynamics of waves that were partly controlled by gravity and partly by the electric field. As a further illustration, both MH-IIf and non-transition EH-If waves were given, illustrating hump and depression waves originating on space-like data.

The integral conditions for shocks and anti-shocks were given, demonstrating that the anti-shock could in theory exist without the dissipation of kinetic energy. This is in strong contrast to the usual gravity shock or the MH-IIf shock.

B. Areas of Active and Suggested Research

There are many directions in which research efforts can go, using as a starting point the wave properties that have been described. Several of these are presently being pursued by the author and his students so that preliminary observations and results can be given.

The role of losses in surface waves presents a problem of considerable difficulty. This is true because the losses provide coupling between the mechanical and electrical bulk equations. Surface wave problems are than complicated by the fact that solutions must be fabricated from bulk solutions that may satisfy linear differential equations, but with space varying coefficients. The role of viscosity is not this complicated. In the resonator problems it was noted that as the E field was increased, the apparent "Q" of the system increased also. Hence, the viscous damping may predominate the electrical losses in these problems. Certainly a Q curve could be measured for a resonator and it would then be possible to correlate what appears to be a tractable theory with experiment. In any case surface wave solutions with losses could provide a valuable addition to the physical picture.

Only two of the six types of waves have been experimentally investigated in this thesis. It should be possible to demonstrate each of the wave types. The EH-Ip wave could be observed using existing dielectrics if considerable care were exercised in handling the liquid. The high magnetic fields now available make possible experiments involving the MH-I and MH-IIa waves in spite of the low permeabilities of liquids (Ferric Chloride solution for example). The MH-IIf experiment could be conducted using an air-liquid metal interface and a transient current. However, it would seem more reasonable to float a light liquid metal (NaK) on the surface of a dielectric liquid to achieve the simulation of a steady surface current. The conjecture will be made here that systems that involve zero order currents interacting with zero order fields are much less subject to the effects of attenuation than are those involving first order currents interacting with zero order fields. Physically this would seem to be plausible, since MH pinches of mercury jets are observed to become unstable at modest currents. The existence of the instability would seem to imply the existence of a wave propagating with small enough attenuation that at some current it predominates the dynamics.

The problems considered in this thesis are physically meaningful at extremes in time. It would be interesting to know more about the dynamics of interfaces during the transient stages of type EH-Ip → EH-If and MH-IIf → MH-IIa waves.

It was pointed out in chapter 3 that the type I waves could have a negative group velocity at the same field strength and wave-length that would give a positive phase velocity. This suggests interesting interactions that may occur in convective systems. In fact, if the waves are considered in cylindrical geometry, it is possible for each of the wave-types to exhibit a convective instability. This is an area of active research. On the theoretical side, the cylindrical counterpart of the EH-If, EH-Ip and EH-II waves has been analyzed in a convective state of rigid body axial translation and rotation. Preliminary investigations show theoretically and experimentally that traveling wave amplification occurs at certain frequencies in a particular case of this problem--the EH-If configuration of a liquid jet. Here problems connected with the detection and excitation of surface waves by electrical means become of considerable interest.

While the name magnetohydrodynamics is a misnomer, the word electrohydrodynamics is not. As has been shown in this thesis, ordinary water can act like a dielectric or like a conductor, depending on the orientation of the applied electric field. In the EH-If configuration it is for all practical purposes a perfect conductor. The great abundance of water in the earth's atmosphere and on the earth's surface, together with the high electric fields that commonly accompany thunder storms, make it plausible that electrohydrodynamics is of some importance in the weather. Even gravity waves on the ocean are significantly affected by electric fields from atmospheric conditions (E fields as high as 2.8×10^5 v/m at the earth's surface are reported).[19] However, the fact that gravity waves are important in the earth's atmosphere also, where density gradients are small, leads to the suspicion that here electrohydrodynamic waves may be of considerable importance. If one can, for example, simply turn on a water faucet to convert steady kinetic energy to alternating electric energy, (the only mechanical motion that of the water), there must be a multiplicity of interesting dynamical situations that can occur in the atmosphere, where all of the ingredients are present.

If EH waves are of any importance in the atmosphere, it would be suspected that they, like the gravity waves, might prove to be most important as a result of non-linear motions.

The non-linear behavior of EH-If waves has been demonstrated for one approximate model in this thesis and work is presently proceeding to varify this model using a water-air interface. This work views the EH-If wave in the context of free surface flows and has the further purpose of determining the flow regimes and types of non-linear interactions that can occur with the boundaries. Gross interactions can be demonstrated experimentally that are roughly as would be expected from the theory. However, there is still much to be learned about the way in which, for example MH and EH shocks and anti-shocks may unambiguously be produced. The problem must still be taken from the kinematical description of chapter 5 to a dynamical one.

Compressibility in electrohydrodynamics has the effect of allowing for the possibility of bulk coupling. A class of waves that propagate along E lines with the velocity

$$V = \sqrt{a^2 + \frac{\varepsilon_o}{\varepsilon}\frac{\partial \varepsilon}{\partial \rho} E^2} \qquad (6\text{-}2)$$

a = velocity of sound

may be shown to exist. In interactions with a plasma sheath such effects may be of importance and remain to be considered, even in the simple continuum picture of the fluid. Incompressible bulk waves are not impossible in a dielectric, however, for if the dielectric is anisotropic, bulk waves may certainly exist. This may be shown by simply superimposing, in layers, the waves discussed in this thesis.

APPENDIX A

List of Symbols

ρ	Mass density
$\bar{V}$	Fluid velocity, Eulerian System
$\bar{F}$	Fluid body force
p	Fluid pressure
$\bar{E}, e$	Electric field intensity
$\bar{B}$	Magnetic flux density
$\bar{H}, h$	Magnetic field intensity
μ	Permeability
ε	Permittivity
$\bar{j}$	Free current density
g	Free charge density
F	Functional form of free surface
$\bar{n}$	Normal vector
$M_{\alpha\beta}$	Total body force stress tensor
ξ	Interface position
$T_{\alpha\beta}$	Maxwell stress tensor
$\delta_{\alpha\beta}$	Kronecker delta function
$\bar{a}$	Vector with unit components in axes directions
$\bar{n}$	Unit normal to interface
V_a	Effective Alfvén velocity
V_b	Effective electrohydrodynamic velocity
c	Electrostriction constant
d	Magnetostriction constant
$\bar{D}$	Electric displacement vector
T	Surface tension

$\bar{g}$	Gravitational constant
b	Upper plate distance from interface
a	Lower plate distance from interface
f	General distance standing for a or b
σ	Conductivity
η	
W	Defined by Equation (4-1)
C	
k	Wave number
ω	Angular frequency
L_y	y dimension of surface resonator
L_z	z dimension of surface resonator
λ	Wave-length or dimensionless "space-rate" parameter
ϕ	Dimensionless interface position
V'	Renormalized velocity
K	Coupling coefficient defined by Equation (5-77)
$R(\phi,K)$	Defined by Equation (5-77)
w,s	Characteristic dimensions
V_o	Conserved electric potential
Λ_o	Conserved magnetic flux
U_b, U_a	Defined by Equations (5-24) and (5-67)
M	Slope of straight line C^+ characteristics

APPENDIX B

Detailed Compatibility Conditions

Table B-1

At the Interface

(2-13)→ $$\tilde{V}_x^{(2)} - \tilde{V}_x^{(1)} = 0 \qquad (3\text{-}18)$$

(2-12)→

$$(\tilde{p}^{(2)} - \tilde{p}^{(1)}) - \frac{\tilde{V}_x}{\alpha}[(\rho^{(2)} - \rho^{(1)})g + \beta^2 T] + \begin{bmatrix} -N_{11} \\ N_{33} \end{bmatrix} = 0 \qquad (3\text{-}19_b^a)$$

$$\frac{V_x S_1}{\alpha}\begin{bmatrix} \beta_y \\ 0 \end{bmatrix} + \begin{bmatrix} N_{21} \\ 0 \end{bmatrix} = 0 \qquad (3\text{-}20_b^a)$$

$$\frac{V_x \beta_z}{\alpha}\begin{bmatrix} S_1 \\ S_3 \end{bmatrix} + \begin{bmatrix} N_{31} \\ -N_{13} \end{bmatrix} = 0 \qquad (3\text{-}21_b^a)$$

(2-14)→

$$(\tilde{e}_z^{(2)} - \tilde{e}_z^{(1)}) + \begin{bmatrix} \beta_z \tilde{V}_x (E_x^{(2)} - E_x^{(1)})/\alpha \\ 0 \end{bmatrix} = 0 \qquad (3\text{-}22_b^a)$$

$$(\tilde{e}_y^{(2)} - \tilde{e}_y^{(1)}) + \begin{bmatrix} \beta_y \tilde{V}_x (E_x^{(2)} - E_x^{(1)})/\alpha \\ -\tilde{V}_x(\mu^{(2)} H_z^{(2)} - \mu^{(1)} H_z^{(1)}) \end{bmatrix} = 0 \qquad (3\text{-}23_b^a)$$

(2-15)→ $$(\mu^{(2)}\tilde{h}_x^{(2)} - \mu^{(1)}\tilde{h}_x^{(1)}) + \begin{bmatrix} 0 \\ -\beta_z \tilde{V}_x (\mu^{(2)} H_z^{(2)} - \mu^{(1)} H_z^{(1)})/\alpha \end{bmatrix} = 0 \qquad (3\text{-}24)_b^a$$

If free currents and charges = 0

(2-16)→

$$(\tilde{h}_z^{(2)} - \tilde{h}_z^{(1)}) + \begin{bmatrix} \beta_z \tilde{V}_x (H_x^{(2)} - H_x^{(1)})/\alpha \\ 0 \end{bmatrix} = 0 \qquad (3\text{-}25_b^a)$$

$$(\tilde{h}_y^{(2)} - \tilde{h}_y^{(1)}) + \begin{bmatrix} \beta_y \tilde{V}_x (H_x^{(2)} - H_x^{(1)})/\alpha \\ 0 \end{bmatrix} = 0 \qquad (3\text{-}26_b^a)$$

$$(2\text{-}17)\quad (\varepsilon^{(2)}\tilde{e}_x^{(2)}-\varepsilon^{(1)}\tilde{e}_x^{(1)})+\begin{bmatrix} 0 \\ -\beta_z\tilde{V}_x(\varepsilon^{(2)}E_z^{(2)}-\varepsilon^{(1)}E_z^{(1)}/\alpha \end{bmatrix}=0 \quad (3\text{-}27{}^a_b)$$

AT x = -a, x = b

$$(3\text{-}5)\quad \begin{bmatrix} \text{EH-If,p};\tilde{e}_z=\tilde{e}_y=0:\ \text{MH-I};\tilde{h}_z=\tilde{h}_y=0 \\ \text{MH-IIf};\tilde{e}_z=\tilde{e}_y=0:\ \text{EH-II,MH-IIa};C_1^{(2)}=C_2^{(1)}=D_1^{(2)}=D_2^{(1)}=0 \end{bmatrix} \quad (3\text{-}28{}^a_b)$$

$$(3\text{-}6)\qquad \tilde{V}_1 = 0 \qquad (3\text{-}29)$$

where

$$S_n = \begin{vmatrix} \varepsilon^{(2)}(E_n^{(2)})^2 - \varepsilon^{(1)}(E_n^{(1)})^2 \\ \mu^{(2)}(H_n^{(2)})^2 - \mu^{(1)}(H_n^{(1)})^2 \end{vmatrix} \quad ; \begin{vmatrix} \text{EH} \\ \text{MH} \end{vmatrix}$$

$$N_{mn} = \begin{bmatrix} \varepsilon^{(2)}E_n^{(2)}\tilde{e}_m^{(2)}(1 \pm \delta_{mn}c^{(2)})-\varepsilon^{(1)}E_n^{(1)}\tilde{e}_m^{(1)}(1\pm\delta_{mn}c^{(1)}) \\ \mu^{(2)}H_n^{(2)}\tilde{h}_m^{(2)}(1 \pm \delta_{mn}d^{(2)})-\mu^{(1)}H_n^{(1)}\tilde{h}_m^{(1)}(1\pm\delta_{mn}d^{(1)}) \end{bmatrix} ; \begin{bmatrix} \text{EH} \\ \text{MH} \end{bmatrix}$$

in which the upper quantities are to be used in type I problems, lower quantities in type II problems except where indicated.

Table B-2

Problem 1	Problem 2
EH-If Eqs. 3-24a,-20a and -21a show D's=0 while Eqs. 3-24a and -22a give -23a. System Eqs. are 3-18,-19a,-21a,-22a,-28a and -29.	EH-II Eqs. 3-24b,-25b and -28b show D's=0. Eqs. 3-26b,-24b and -25b are identically satisfied. Eqs. 3-22b and -23b are the same. System Eqs. are 3-18,-19b,-21b -22b and 29.
EH-Ip Eqs. 3-25a,-24a, and -28a show D's=0 Eqs. 3-22a,-23a,-20a and -21a are equivalent. System Eqs. are 3-18,-19a,-22a,-27a,-28a and 29.	MH-IIf Eq. 3-22b eliminates C's while -24b and -23b are equivalent. System Eqs are 3-18,-19b,-21b,-24b and -29.
MH-I Eq. 3-22a can be used to eliminate all C's while Eqs. 3-24a and -23a,-20a and -21a are equivalent. System Eqs. are 3-18 -19a,-20a,-24a,-28a and 29.	MH-IIa Eq. 3-22b eliminates C's while 3-21b, -23b and -24b are equivalent. System Eqs. are 3-18,-19b,-24b,-25b and 29.
System Equations for EH-If,EH-Ip. (MH-I is obtained by replacing $E_x \rightarrow H_x$ $\varepsilon \rightarrow \mu$, C's → D's and ignoring Eq. 3-32a.)	System Equations for MH-IIf,MH-IIa. (EH-II is obtained by replacing $H_z \rightarrow E_z$ $\mu \rightarrow \varepsilon$, D's → C's and ignoring Eq. 3-32b.)

$$(A_2^{(2)} - A_1^{(2)})/\rho^{(2)} + (A_1^{(1)} - A_2^{(1)})/\rho^{(1)} = 0 \qquad (3\text{-}30)$$

$$Q - \varepsilon^{(2)}E_x^{(2)}(1+c^{(2)})(C_1^{(2)} - C_2^{(2)})k/\beta_z + \varepsilon^{(1)}E_x^{(1)}(1+c^{(1)})(C_1^{(1)} - C_2^{(2)})k/\beta_z = 0 \qquad (3\text{-}31a)$$

$$Q + \mu^{(2)}H_z^{(2)}(1-d^{(2)})(D_2^{(2)} + D_1^{(2)}) - \mu^{(1)}H_z^{(1)}(1-d^{(1)})(D_1^{(1)} + D_2^{(1)}) = 0 \qquad (3\text{-}31b)$$

where $Q = (A_1^{(2)} + A_2^{(2)}) - (A_1^{(1)} + A_2^{(1)}) + (g(\rho^{(2)} - \rho^{(1)}) + T\beta^2)(A_1^{(1)} - A_2^{(1)})k/\rho^{(1)}\alpha^2$

cont.

Table B-2 cont.

EH-If

$$[\varepsilon^{(2)}(E_x^{(2)})^2-\varepsilon^{(1)}(E_x^{(1)})^2](A_1^{(1)}-A_2^{(1)})\beta_z/\rho^{(1)}\alpha^2$$
$$-\varepsilon^{(2)}E_x^{(2)}(C_1^{(2)}+C_2^{(2)})+\varepsilon^{(1)}E_x^{(1)}(C_1^{(1)}+C_2^{(1)})=0 \quad (3\text{-}32a)$$

EH-Ip

$$\varepsilon^{(2)}(C_1^{(2)}-C_2^{(2)})-\varepsilon^{(1)}(C_1^{(1)}-C_2^{(1)})=0 \quad (3\text{-}33a)$$

$$\left.\begin{aligned}&(E_x^{(2)}-E_x^{(1)})(A_1^{(1)}-A_2^{(1)})k\beta_z/\rho^{(1)}\alpha^2 - \\ &\quad (C_1^{(2)}+C_2^{(2)})+(C_1^{(1)}+C_2^{(1)})=0\end{aligned}\right] \quad (3\text{-}34a)$$

$$\left[\begin{aligned}&C_1^{(1)}e^{-ka} + C_2^{(1)}e^{ka} = 0\\ &C_1^{(2)}e^{kb} + C_2^{(2)}e^{-kb} = 0\end{aligned}\right. \quad (3\text{-}35a)$$

MH-IIf

$$\mu^{(2)}H_z^{(2)}(D_2^{(2)}-D_1^{(2)})+\mu^{(1)}H_z^{(1)}(D_1^{(1)}-D_2^{(1)})-$$
$$[\mu^{(2)}(H_z^{(2)})^2-\mu^{(1)}(H_z^{(1)})^2] \cdot \quad (3\text{-}32b)$$
$$(A_1^{(1)}-A_2^{(1)})\beta_z^2/\rho^{(1)}\alpha^2$$

MH-IIa

$$(D_2^{(2)} - D_1^{(1)} = 0 \quad (3\text{-}33b)$$

$$\left.\begin{aligned}&\mu^{(2)}(D_2^{(2)}-D_1^{(2)})+\mu^{(1)}(D_1^{(1)}-D_2^{(1)})\\ &[\mu^{(2)}H_z^{(2)}-\mu^{(1)}H_z^{(1)}]\cdot\\ &(A_1^{(1)}-A_2^{(1)})\beta_z^2/\rho^{(1)}\alpha^2 = 0\end{aligned}\right] \quad (3\text{-}34b)$$

MH-IIf
$$\left[\begin{aligned}&D_1^{(1)}e^{-ka}-D_2^{(1)}e^{ka} = 0\\ &D_1^{(2)}e^{kb}- D_2^{(2)}e^{kb} = 0\end{aligned}\right.$$

MH-IIa
$$D_1^{(2)} = D_2^{(1)} = 0$$

$$\begin{aligned}&A_1^{(1)}e^{-ka} - A_2^{(1)}e^{ka} = 0\\ &A_1^{(2)}e^{kb} - A_2^{(2)}e^{-kb} = 0\end{aligned} \quad (3\text{-}36)$$

APPENDIX C

Traveling Wave Fields

Table C-1

Solutions for wave types illustrated by traveling waves propagating in the positive and negative z directions. The dispersion equations relate k and ω $+a \rightarrow b \rightarrow \infty$

For all wave types:

$$\xi = \xi_o \cos(\omega t \pm kz)$$

$$V_x^{(2)} = -\xi_o \omega e^{-kx} \sin(\omega t \pm kz)$$

$$V_x^{(1)} = -\xi_o \omega e^{kx} \sin(\omega t \pm kz)$$

$$V_z^{(2)} = \pm \xi_o \omega e^{-kx} \cos(\omega t \pm kz)$$

$$V_z^{(1)} = \mp \xi_o \omega e^{kx} \cos(\omega t \pm kz)$$

$$p^{(2)} = -\frac{\omega^2 \rho^{(2)} \xi_o}{k} e^{-kx} \cos(\omega t \pm kz)$$

$$p^{(1)} = \frac{\omega^2 \rho^{(2)} \xi_o}{k} e^{kx} \cos(\omega t \pm kz)$$

For EH-If waves:

$$\varepsilon = \varepsilon_o \quad e_z^{(1)} E_x^{(2)} - e_z^{(2)} E_x^{(1)} = 0$$

$$e_z^{(2)} = \pm E_x^{(2)} k \xi_o e^{-kx} \sin(\omega t \pm kz)$$

$$e_z^{(1)} = \pm E_x^{(1)} k \xi_o e^{kx} \sin(\omega t \pm kz)$$

$$e_x^{(2)} = E_x^{(2)} \xi_o k e^{-kx} \cos(\omega t \pm kz)$$

$$e_x^{(1)} = -E_x^{(2)} \xi_o k e^{kx} \cos(\omega t \pm kz)$$

Table C-1 cont.

$$q_f = \varepsilon_o k[E_x^{(2)} + E_x^{(1)}]\ \xi_o \cos(\omega t \pm kz)$$

$$k_f = \mp \varepsilon_o \xi_o \omega [E_x^{(1)} + E_x^{(2)}] \cos(\omega t \pm kz)\bar{a}_1$$

q_f = perturbation of free surface charge

k_f = first order surface current (convective)

For EH-Ip or MH-Im waves

$$e_z^{(2)} = \pm G\xi_o \varepsilon^{(1)} e^{-kx} \sin(\omega t \pm kz)$$

$$e_z^{(1)} = \mp G\xi_o \varepsilon^{(2)} e^{kx} \sin(\omega t \pm kz)$$

$$e_x^{(2)} = \xi_o G\varepsilon^{(1)} e^{-kx} \cos(\omega t \pm kz)$$

$$e_x^{(1)} = \xi_o G\varepsilon^{(2)} e^{kx} \cos(\omega t \pm kz)$$

$$k_p = \mp 2\xi_o G \sin(\omega t \pm kz)\bar{a}_y$$

$$q_p = -\xi_o G \cos(\omega t \pm kz)(\varepsilon^{(2)} - \varepsilon^{(1)})$$

$$G = (E_x^{(2)} - E_x^{(1)})k/(\varepsilon^{(2)} + \varepsilon^{(1)})$$

k_p = first order Korteweg surface current

q_p = perturbed surface polarization charge

for MH-Im waves replace

$E \rightarrow H$, $e \rightarrow h$, $\varepsilon \rightarrow \mu$, $k_p \rightarrow k_m$, $q_p \rightarrow q_m$

k_m = first order amperian surface current

q_m = perturbed surface magnetic charge

Table C-1 cont.

For MH-IIf waves

$$\mu = \mu_o \qquad H_z^{(1)} h_x^{(2)} - H_z^{(2)} h_x^{(1)} = 0$$

$$h_x^{(2)} = \mp H_z^{(2)} k \xi_o e^{-kx} \sin(\omega t \pm kz)$$

$$h_x^{(1)} = \mp H_z^{(1)} k \xi_o e^{kx} \sin(\omega t \pm kz)$$

$$h_z^{(2)} = H_z^{(2)} k \xi_o e^{-kx} \cos(\omega t \pm kz)$$

$$h_z^{(1)} = -H_z^{(1)} k \xi_o e^{kx} \cos(\omega t \pm kz)$$

$$e_y^{(2)} = -\omega\mu H_z^{(2)} \xi_o e^{-kx} \sin(\omega t \pm kz)$$

$$e_y^{(1)} = -\omega\mu H_z^{(1)} \xi_o e^{kx} \sin(\omega t \pm kz)$$

$$k_f = -k\xi_o [H_z^{(1)} + H_z^{(2)}] \cos(\omega t \pm kz)$$

k_f = free surface current

For EH-II or MH-IIa waves

$$e_x^{(2)} = \mp \xi_o L e^{-kx} \sin(\omega t \pm kz)$$

$$e_x^{(1)} = \pm \xi_o L\, e^{kx} \sin(\omega t \pm kz)$$

$$e_z^{(2)} = \xi_o L\, e^{-kx} \cos(\omega t \pm kz)$$

$$e_z^{(1)} = \xi_o L\, e^{kx} \cos(\omega t \pm kz)$$

$$k_p = -\frac{\xi_o L}{\varepsilon_o} \cos(\omega t \pm kz)[\varepsilon^{(2)} - \varepsilon^{(1)}]\bar{a}_y$$

Table C-1 cont.

$q_p = \mp 2\xi_o L \sin(\omega t \pm kz)$

$L = kE_z[\varepsilon^{(2)} - \varepsilon^{(1)}]/(\varepsilon^{(2)} + \varepsilon^{(1)})$

for MH-IIa waves replace

$e \rightarrow h$, $\varepsilon \rightarrow \mu$, $k_p \rightarrow k_m$, $q_p \rightarrow q_m$

APPENDIX D

Biographical Sketch

James Russell Melcher was born in Giard, Iowa on July 5, 1936. He was graduated from Lisbon High School, Lisbon, Iowa in 1954 after having attended Cornell College for a year. In 1957, he received a B.S. in Electrical Engineering and a year later an M.S. in Nuclear Engineering. In 1959 he entered M.I.T. on an Atomic Energy Commission Special Fellowship.

He spent summers at the Nuclear Division of the Martin Company and the Physics Group of the Boeing Aircraft Company. While in school he assumed the duties of staff announcer at WOI in Ames, Iowa; Research Assistant in the Ames Laboratory of the Atomic Energy Commission; Research Assistant in the Research Laboratory of Electronics, M.I.T.; and Instructor of Electrical Engineering, M.I.T. He is a member of Eta Kappa Nu, Tau Beta Pi, Phi Kappa Phi, I.R.E. and the A.P.S. and recipient of the American Nuclear Society's First Mark Mills Award. He is the author of:
"A Useful Analogy for Single-Group Neutron Diffusion Theory," Nuclear Science and Engineering 1,235-239 (1960) and "Electrohydrodynamic and Magnetohydrodynamic Surface Waves and Instabilities," Physics of Fluids, 4, 11, 1961.

APPENDIX E

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